Non-Trivial Pursuit

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Abstract

We explore the dynamics of a chase between a pursuer and an evader. We include various mathematical models that incorporate different strategies for the pursuer and the evader. We explore how these dynamics change when environmental factors are introduced such as a goal destination for the evader and terrain that increases or decreases the speed of the pursuer or evader in that region. We claim that an inverse distance weighting and exponential weighting intuitively describe effective strategies for evasion. The performance of the pursuer was improved by attempting to predict the behavior of the evader only when the evader was following a straight path, otherwise we saw little to no improvement.

1 Introduction and Background

Every species has specific instincts and abilities carefully honed through thousands of generations of natural selection. These instincts improve an individuals chances of survival, for example, a rabbit knows that it should run for cover if it is being chased by a fox, because that fox isn't looking for a friend – it's looking for a meal. Similarly, foxes have developed certain traits that help them catch rabbits and other tasty treats.

Our goal is to model different strategies for foxes and rabbits to adopt that would help them achieve their desired outcomes in the event of a chase. The ideas explored in this paper aren't just limited to adorable woodland creatures, however. These models could also be generalized for a variety of situations where two parties have contradictory goals.

The idea to model pursuit curves is nothing new, with published works describing the concept dating as early as the eighteenth century [1], so while the concepts discussed in this project may just be "reinventing the wheel," they still provide insight into how more complicated models are made and help develop skills we can take into our future careers.

2 Model

For our initial model, we used the model described by [1] and [2]. In this model, the rabbit (whose position is represented by the variable R) travels along a predetermined curve (a straight line, for example). The fox (whose position is represented by the variable F) travels directly toward the rabbit at a constant speed. This yields the nonlinear system of ODEs

$$\begin{aligned} F' &= \|F'\| \frac{R - F}{\|R - F\|} \\ R' &= \|R'\| \frac{f(F, R)}{\|f(F, R)\|}, \end{aligned} \tag{1}$$

where $\|F'\|$ and $\|R'\|$ are taken to be constant equal to 1 in our analysis and the choice of f determines the rabbit's strategy for avoiding the fox.

A solution to this ODE if $f(F, R) = [0, 1]$ (the rabbit travels along a vertical line) is plotted in Figure 1.

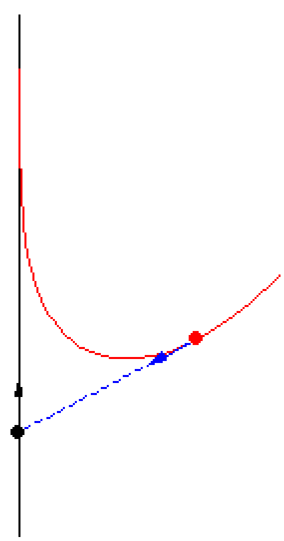

Figure 1: This figure from Wolfram shows the solution to Equation (1) for unspecified initial conditions.

To analyze and extend this model, we will look at things like what happens if the fox predicts the future position of the rabbit (based on the rabbit's position and velocity) and moves towards that position, what is the best evasion technique for the rabbit if the rabbit also has a goal location (such as a burrow), what happens if the speed of the fox or rabbit is not constant, what happens as the rabbit follows different paths, and what happens if the rabbit changes its trajectory based on the position of the fox.

We were able to use our initial model and a solver to see the dynamics of a rabbit being chased by a fox. This model incorporated two paths of the rabbit, first, the rabbit followed a straight path, and then we were able to revise it so that the rabbit's evasion path followed a cosine escape plan which meant the fox followed a similar trajectory. We created an animation for both scenarios.

Thus we report that the initial model and solutions are consistent with our intuition, and could be enhanced by using a 3D model or incorporating spatial restraints like obstacles.

Our model allows for the speed of the rabbit and the fox to be a parameter. We experimented with varying speeds. First when the rabbit and the fox had the same velocity, and second when the fox had a greater velocity than the rabbit; We did notice that when the fox had caught up to the rabbit the fox ran in a slightly different path than the rabbit. This is because the fox was moving faster than the rabbit.

2.1 Updates to the Fox Strategy

In Equation (1), the fox runs directly toward the rabbit. We call this fox method the *naive strategy.* The naive strategy is a good starting point for the model, but it doesn't incorporate all the information about the rabbit that the fox has; in addition to the rabbit's position, the fox also knows the rabbit's velocity.

One way to incorporate the rabbit's velocity into the fox's decision process is to predict where the rabbit will be if it follows the course it is currently on for some period of time and then head toward that point, like so:

$$F' = \|F'\| \frac{R + \alpha R' - F}{\|R + \alpha R' - F\|},$$

where the magnitude of $\alpha > 0$ determines how far into the future the fox should project the rabbit's position. (Note that setting $\alpha = 0$ yields Equation (1).)

We tried several constant values for α, but found that they all performed poorly when the fox is too close or too far away from the rabbit. To fix this, we let α be the distance from the rabbit to the fox multiplied by 0.8; this

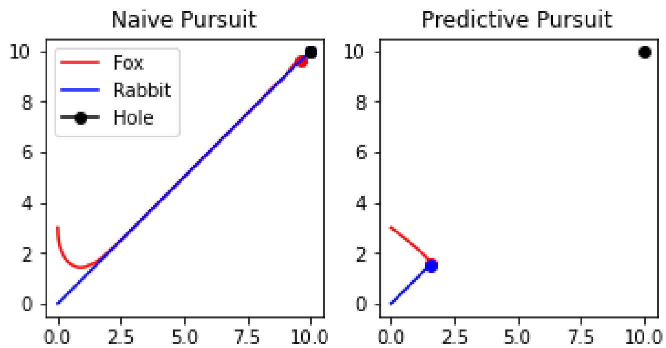

Figure 2: The naive strategy allows the rabbit to run past the fox, but the predictive strategy causes the fox to head off the rabit.

yields what we call the *predictive strategy*:

$$F' = \left\|F'\right\| \frac{R + 0.8 \left\|R - F\right\| R' - F}{\left\|R + 0.8 \left\|R - F\right\| R' - F\right\|}. \tag{2}$$

As is exemplified in Figure 2, a fox using the predictive strategy consistently outperforms a fox using the naive strategy when chasing a rabbit that follows a straight path.

2.2 Updates to the Rabbit Strategy

The choice of f in Equation (1) determines the strategy that the rabbit takes to avoid the fox. If the rabbit's only goal is to avoid the fox, then the best strategy is to run directly away from the fox, i.e. $f(F, R) = F - R$. However, this situation is unrealistic; in practice, the rabbit will most likely be trying to reach some location (its burrow) at which it will be safe from the fox. We examined this situation in more detail.

In this situation, the rabbit has two goals: to avoid the fox and to reach its burrow. Each goal corresponds to a direction that the rabbit should travel. We explored different methods of combining these two directions by assigning a weight to each. If the weight for one direction was too strong, then the rabbit either never reached its burrow or allowed the fox to catch it.

We explored five methods for weighting the goals: ignoring the fox altogether, assigning equal weight to the two directions, using weights of the form e^{-d} (where d is the distance from the rabbit to the fox or burrow), using weights of the form $\frac{1}{d}$, and using weights of the form $\frac{1}{\sqrt{d}}$.

2.2.1 Straight Line

We first consider the strategy where the rabbit simply travels in a straight line to the rabbit burrow, and the fox moves based on the current position of the rabbit. In this case it's apparent that the rabbit's success is dependent on the initial conditions, as well as the velocities of the rabbit and the fox.

For example, when the rabbit is running directly to the rabbit burrow, and the fox is between the rabbit and the rabbit burrow, intuitively we can see the rabbit will be eaten so long as the rabbit and the fox have similar speeds.

We next examined straight line strategies when the fox's direction is based on trying to predict the future location of the rabbit.

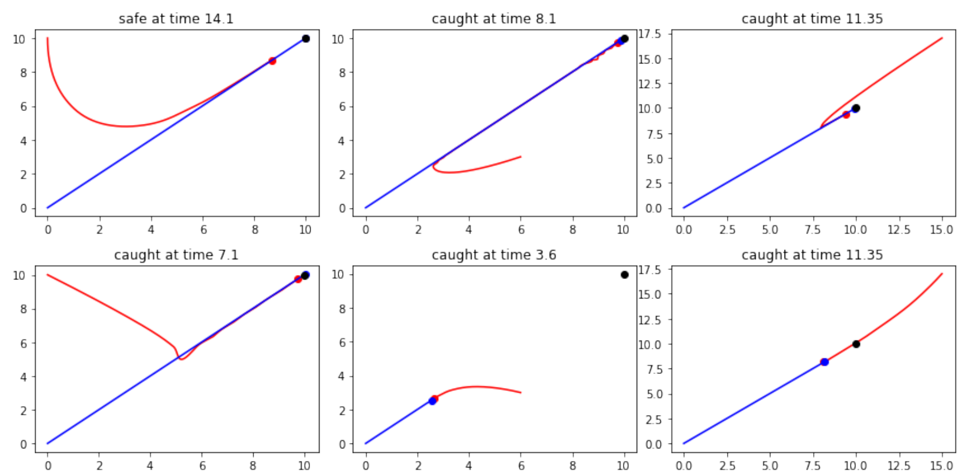

Figure 3: This shows the outcome for a rabbit that chooses to run directly to it's burrow, with the fox in 3 different starting positions. The top row shows the fox using a naive pursuit method, while the bottom row is the predictive pursuit method.

We modeled these scenarios in both 2D and 3D. Although 3D might not make as much physical sense for a rabbit, and fox, it would make sense for missiles. The "rabbit" object would then more likely be interpreted as the attacking missile with a target (which was the "rabbit burrow"), and the "fox" could be interpreted as the defending missile trying to hit the other missile before the "rabbit" missile reaches it's target.

2.2.2 Exponential Weights

Our first attempt at combining the directions looks like this, if G is the location of the burrow:

$$f(F,R) = e^{-0.01\|G-R\|}\frac{G-R}{\|G-R\|} - e^{-0.01\|F-R\|}\frac{F-R}{\|F-R\|}. \tag{3}$$

We chose the exponential weights because the limit of the weights as the distance approaches zero is 1, and the limit of the weights as the distance approaches infinity is zero. The coefficient of 0.01 in the exponential prevents the rabbit from prioritizing the nearer goal too heavily. We thought this might provide a decent weighting between the two directions, in order to prioritize whichever goal is more "urgent."

This strategy fared quite well in our tests; so long as the initial distance between the rabbit and the fox was not too small (if the speed of the rabbit and fox is equal, then then "too small" means roughly less than half the speed), the rabbit was always able to reach its goal. This is demonstrated in Figure 4.

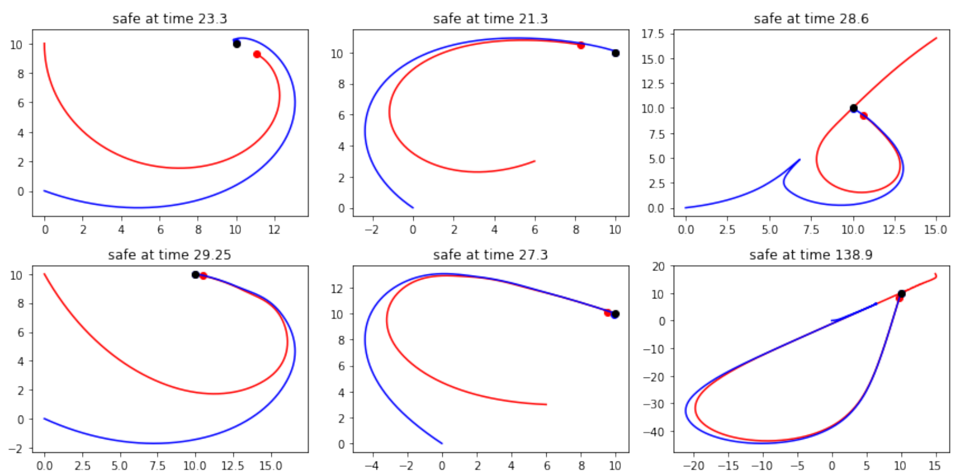

Figure 4: The exponential weight strategy allows the rabbit to reach its burrow under most conditions. The starting positions and strategies are as in Figure 3. The top row shows the fox using a naive pursuit method, while the bottom row is the predictive pursuit method.

2.2.3 Equal Weights

After the exponential weighting we thought it was natural to try weighting the rabbit's two goals (avoiding the fox and reaching home) equally:

$$f(F,R) = \frac{G-R}{||G-R||} - \frac{F-R}{||F-R||}. \tag{4}$$

On testing this strategy in the naive case we found that the rabbit followed similar paths to the exponential model. However, the rabbit did not reach home quite as quickly. In the predictive case the fox was able to catch up to the rabbit much quicker than in the naive case.

We see this with the initial rabbit position at (0,0) and the initial fox position at (6,3). See Figure 5. In the predictive case the fox is able to catch the rabbit. However in the naive case the rabbit makes it home safely.

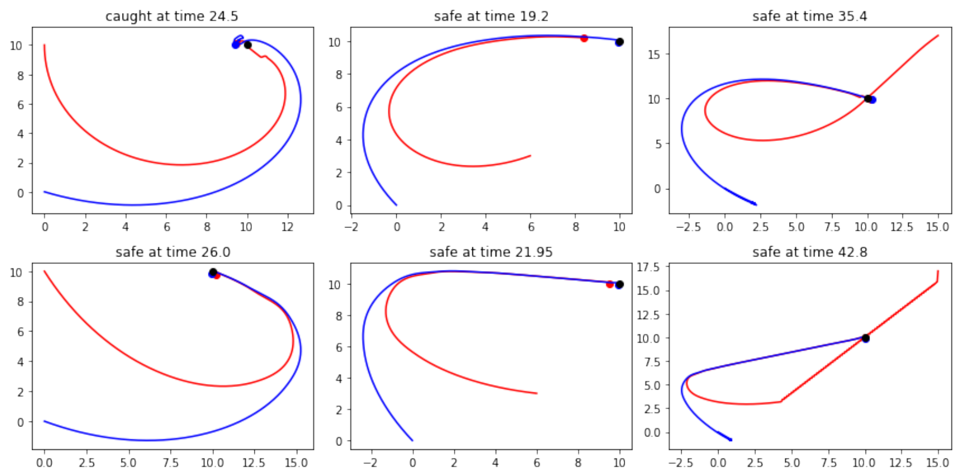

Figure 5: Equal Weight Method - The top row shows the fox using a naive pursuit method, while the bottom row is the predictive pursuit method.

We also note that with the equal weighting model the rabbit often makes a sudden change of direction. This happens especially often when the rabbit and fox are both close to the target destination.

2.2.4 Inverse Distance

Next we tried weighting the rabbits desires using the inverse distance. The goal here was that we wanted the rabbit's desire to reach home to grow as the rabbit was closer to it's home. We also wanted the rabbit's desire to avoid capture be stronger when the rabbit was closer to the fox. This led us to the following model for the rabbit's movement:

$$f(F, R) = \frac{G - R}{||G - R||} \frac{1}{G - R} - \frac{F - R}{||F - R||} \frac{1}{F - R}. \tag{5}$$

In both the naive and predictive cases this model did not perform very well: as soon as the rabbit and the fox got close to each other, the rabbit begins to almost completely ignore the goal and instead just runs away from the fox. We conclude that this weight system assigns too much weight to the objective that is nearer.

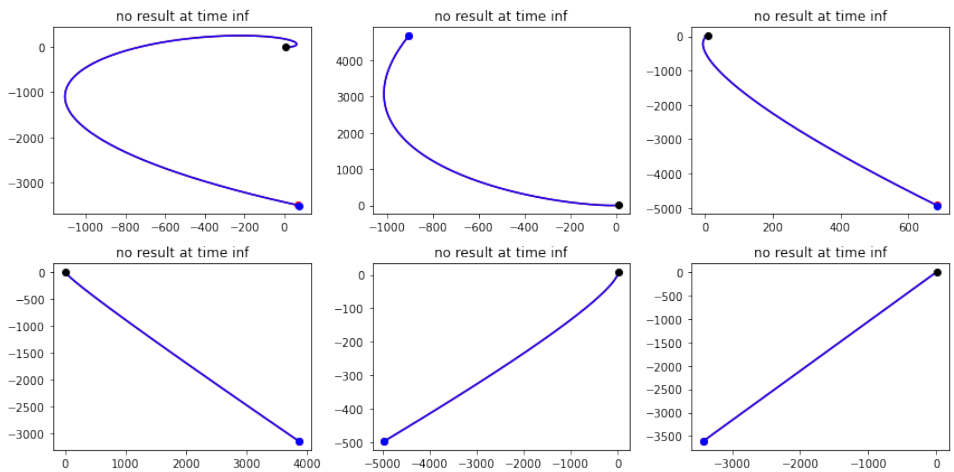

Figure 6: Inverse Distance Method - The top row shows the fox using a naive pursuit method, while the bottom row is the predictive pursuit method.

2.2.5 Inverse Square Root Distance

To improve our previous model, we tried the following:

$$f(F, R) = \frac{1}{\sqrt{\|G - R\|}} \frac{G - R}{||G - R||} - \frac{1}{\sqrt{\|F - R\|}} \frac{F - R}{||F - R||}. \tag{6}$$

This ended up being one of our most interesting models. In the naive case the rabbit was able to successfully make it to safety most of the time. In the predictive case the rabbit was still often able to make it to safety but had to follow wide swooping paths in order to evade the fox.

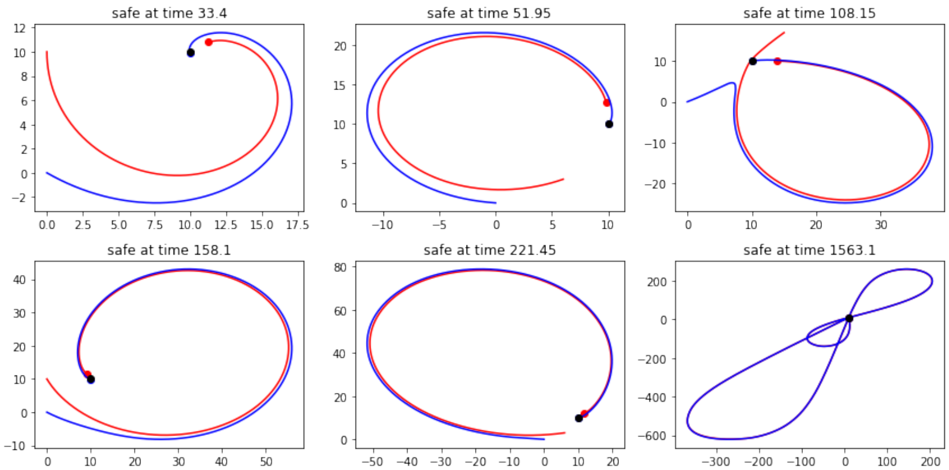

Figure 7: Inverse Square Root Distance Method - The top row shows the fox using a naive pursuit method, while the bottom row is the predictive pursuit method.

2.3 Updates to the Environment

Another factor that influences the outcome of the chase is the speed at which the animals move. For example, a rabbit might be faster when it is close to it's burrow because it knows the terrain, while a fox might run faster in the grass, where it has a clean line of sight. To model this in our simulation, we created additional parameters that allow the animals to adjust their speed depending on their location. Unfortunately, this added complexity introduced new problems into our model and couldn't be implemented at the same time as the predictive strategy for the fox. Several depictions of successful implementations with location dependent speed are included in the auxiliary code.

3 Results

Our experiments show that the two strategies that perform the best for the rabbit are the exponential weighting and the inverse square root distance weighting. Between these two, exponential weighting tends to reach the burrow significantly faster than inverse square root distance, and so we conclude that the best strategy for the rabbit is to use exponential weighting. When the fox attempted to predict the behavior of the rabbit it helped most when the rabbit was following a straight path to the rabbit burrow. We didn't see a dramatic difference in performance for the fox for either the naive or predictive strategy when the rabbit was not following a straight path to the goal.

4 Conclusion

There were a number of experiments where the solver exhibited minor instabilities which were apparent in the visualizations. To overcome this we explored another ODE solver. With more time, we could have identified the sources of the irregularities. In spite of infrequent and minor numerical mishaps, it was thrilling to see many of the experiments exhibit behavior consistent with our intuition of physical phenomena. We recognize several venues of exploration for future endeavors such as including obstacles, having multiple predators, and by increasing the number of strategies.

There are far reaching implications based upon this analysis and investigation. A modern application of these dynamics is the development of autonomous missile guidance systems for attack or defense. Simpler applications include strategies in sports like running a football down the field without getting tackled, or the classic game of Pac-Man.

In conclusion, while there is plenty of room for growth and additional investigation into this model, one of the largest impressions this project left on us was a deeper understanding of how marvelously complex and incredible the natural world is. After comparing the models we were able to create over the course of a semester to the abilities that these creatures have after honing their instincts through millennia of evolution, we are left to conclude that pursuit is anything but trivial.

References

[1] https://www.hsu.edu/uploads/pages/2006-7afpursuit.pdf

[2] https://mathworld.wolfram.com/PursuitCurve.html