Evidence Holonomy and Entropy Production: From Universal Coding to Irreversibility

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Abstract

We define an "evidence holonomy" functional on loops of representation transforms applied to sample paths. Using pointwise universality of code lengths for stationary ergodic processes, we prove two reductions: (i) **representation-space holonomy** converges (up to $o(n)$) to the entropy-rate difference $h(Q) - h(P)$; (ii) **KL-holonomy**—implemented by transporting a universal code trained under P to one trained under Q—converges to the relative-entropy rate $\mathsf{d}(P\|Q)$. For finite-state Markov processes, KL-holonomy along the time-reversal loop equals the entropy-production rate σ (bits/step). We show asymptotic code-invariance of KL-holonomy and validate the framework on Arrow-of-Time benchmarks (audio, sensors, finance).

1 Setup and Definitions

Alphabet and path space. Fix a finite alphabet $\mathcal{X}$. Let $\mathcal{X}^n$ denote length-n strings and $\mathcal{X}^{\mathbb{N}}$ the one-sided sequence space with its product σ-algebra. Let P be a stationary ergodic probability measure on $(\mathcal{X}^{\mathbb{N}}, \mathcal{F})$. Write $X_{0:n-1}$ for the length-n prefix of a sample from P and P_n for its law on $\mathcal{X}^n$.

Universal codes. A *universal code* on alphabet $\mathcal{A}$ is a map $\mathcal{E}_{\mathcal{A}} : \bigcup_{n\geq 1} \mathcal{A}^n \to \mathbb{R}_+$ assigning a code length in bits to any finite string, such that for every stationary ergodic law Q on $\mathcal{A}^{\mathbb{N}}$,

$$\frac{1}{n}\Big(\mathcal{E}_{\mathcal{A}}(Y_{0:n-1}) + \log_2 Q_n(Y_{0:n-1})\Big) \xrightarrow[n\to\infty]{Q\text{-a.s.}} 0, \tag{1}$$

with convergence also in $L^1(Q)$. Classical examples include LZ78, Krichevsky–Trofimov mixtures for finite-order Markov models, and CTW [1, 2, 3, 6, 7].

Representation transforms and loops. For each n, let $F_{i,n}$ be a measurable map $F_{i,n} : \mathcal{X}_{i-1}^{n_{i-1}(n)} \to \mathcal{X}_i^{n_i(n)}$, where the alphabets $\mathcal{X}_i$ may differ by step, and $n_0(n) = n$. Define successive images

$$x^{(0)} = x \in \mathcal{X}^n, \quad x^{(i)} = F_{i,n} \circ \cdots \circ F_{1,n}(x^{(0)}) \in \mathcal{X}_i^{n_i(n)}.$$

A finite list $\gamma = (F_{1,n}, \ldots, F_{m,n})$ is a *loop at scale* n if $n_m(n) = n$ and $\mathcal{X}_m = \mathcal{X}_0 = \mathcal{X}$. Let $L_n = F_{m,n} \circ \cdots \circ F_{1,n} : \mathcal{X}^n \to \mathcal{X}^n$ be the loop map and let $Q_n := (L_n)\#P_n$ (hence Q_n is a law on $\mathcal{X}^n$).

1.1 Two holonomy functionals

Definition 1.1 (Representation-space holonomy)**.** Given universal codes $\mathcal{E}_{\mathcal{X}_i}$ for intermediate alphabets, define

$$\begin{aligned}\mathrm{Hol}_n^{\mathrm{out},\gamma}(x) &= \sum_{i=1}^{m} \Big(\mathcal{E}_{\mathcal{X}_i}\big(x^{(i)}\big) - \mathcal{E}_{\mathcal{X}_{i-1}}\big(x^{(i-1)}\big) \Big) \\ &= \mathcal{E}_{\mathcal{X}}\big(L_n(x)\big) - \mathcal{E}_{\mathcal{X}}(x).\end{aligned}$$

Definition 1.2 (KL (observer-transported) holonomy)**.** Let $\mathcal{E}_{\mathcal{X}}^{(P)}$ and $\mathcal{E}_{\mathcal{X}}^{(Q)}$ be universal on $\mathcal{X}$ for P and for the stationary pushforward Q with marginals Q_n, respectively. Define the *code-based* estimator

$$\mathrm{Hol}_n^{\mathrm{KL},\gamma}(x) := \mathcal{E}_{\mathcal{X}}^{(Q)}(x) - \mathcal{E}_{\mathcal{X}}^{(P)}(x).$$

Operationally, $\mathcal{E}_{\mathcal{X}}^{(Q)}$ is the universal code trained on samples from Q (e.g., loop-transformed training sequences) but *evaluated on the original sequence* x; hence it estimates $-\log_2 Q_n(x)$, so that $\mathrm{Hol}_n^{\mathrm{KL},\gamma}$ estimates $\log_2 \frac{P_n(x)}{Q_n(x)}$.

2 Reductions via Universality

Lemma 2.1 (Pointwise reductions)**.** *Assume* (1) *for the relevant laws.*

1. *For* $\mathrm{Hol}^{\mathrm{out}}$*: with* $\mathcal{E}_{\mathcal{X}}$ *universal for both* P *and the pushforward process,*

$$\frac{1}{n}\Big(\mathrm{Hol}_n^{\mathrm{out},\gamma}(X_{0:n-1}) - \log_2 \frac{P_n(X_{0:n-1})}{Q_n(L_n(X_{0:n-1}))} \Big) \xrightarrow[n\to\infty]{P\text{-a.s.}} 0. \tag{2}$$

2. *For* $\mathrm{Hol}^{\mathrm{KL}}$*:*

$$\frac{1}{n}\Big(\mathrm{Hol}_n^{\mathrm{KL},\gamma}(X_{0:n-1}) - \log_2 \frac{P_n(X_{0:n-1})}{Q_n(X_{0:n-1})} \Big) \xrightarrow[n\to\infty]{P\text{-a.s.}} 0. \tag{3}$$

Both convergences also hold in $L^1(P)$*.*

Proof. Apply (1) (Barron's strong pointwise coding theorem) to each code/law pair and subtract the limits; see [5, 7, 6]. □

Averaging yields the two central identities.

Theorem 2.2 (Expectation-level reductions)**.** *Under* L^1 *universality,*

$$\frac{1}{n}\,\mathbb{E}_P\big[\mathrm{Hol}_n^{\mathrm{out},\gamma}\big] = h(Q) - h(P) + o(1), \tag{4}$$

$$\frac{1}{n}\,\mathbb{E}_P\big[\mathrm{Hol}_n^{\mathrm{KL},\gamma}\big] = \frac{1}{n}\,\mathrm{D}(P_n\|Q_n) \xrightarrow[n\to\infty]{} \mathsf{d}(P\|Q) \ \geq 0. \tag{5}$$

Proof. Take expectations in (2)–(3) and note $\mathbb{E}_P[-\log_2 P_n(X_{0:n-1})] = H(P_n)$, $\mathbb{E}_P[-\log_2 Q_n(L_n(X_{0:n-1}))] = H(Q_n)$, and $\mathbb{E}_P\big[\log_2 \frac{P_n(X)}{Q_n(X)}\big] = \mathrm{D}(P_n\|Q_n)$. □

Remark 2.3 (Scope)*.* Equation (4) is gauge-invariant and measures net compression or expansion under the loop. Equation (5) is the *irreversibility* functional implemented in our code (KL-rate holonomy): it is observer-transported and non-negative.

3 Canonical Loops and Corollaries

3.1 Gauge invariance for bijective loops

Corollary 3.1 (Gauge invariance)**.** *If each $F_{i,n}$ is a bijection and the loop is the identity on $\mathcal{X}^n$, then*

$$\frac{1}{n}\operatorname{Hol}_n^{\text{out},\gamma}(X_{0:n-1}) \to 0 \quad \textit{and} \quad \frac{1}{n}\operatorname{Hol}_n^{\text{KL},\gamma}(X_{0:n-1}) \to 0$$

in P-probability and in $L^1(P)$.

Proof. Then $Q_n = P_n$ for all n, so both (4) and (5) vanish. □

3.2 Coarse-graining loops via channels

Let K_n be a (possibly many-to-one) Markov kernel on $\mathcal{X}^n$ and R_n any measurable right-inverse (a "lift") so that $L_n := R_n \circ K_n : \mathcal{X}^n \to \mathcal{X}^n$ is a loop. If $Q_n := L_n \# P_n$ arises from a stationary Q, then (5) gives

$$\frac{1}{n}\,\mathbb{E}_P\big[\operatorname{Hol}_n^{\text{KL},\gamma}\big] \to \mathsf{d}(P\|Q) \geq 0,$$

i.e. KL holonomy is non-negative by construction (data-processing monotonicity of KL under channels [8, Ch. 2]).

3.3 Time reversal and entropy production for Markov chains

Let P be a stationary Markov chain on $\mathcal{X} = \{1, \ldots, k\}$ with transition T and stationary π. Its time-reversal P^{rev} has transitions $T^*_{ji} = \frac{\pi_i T_{ij}}{\pi_j}$. For length n, let R_n denote reversal of the string. Consider the canonical loop

$$\textsf{Encode transitions} \;\to\; \textsf{Reverse} \;\to\; \textsf{Decode second},$$

which maps paths back to $\mathcal{X}^n$ (up to a boundary symbol). For Markov P, the pushforward law Q induced by this loop coincides with the path law of P^{rev} on cylinders.

Theorem 3.2 (KL holonomy rate equals entropy production)**.** *For the Markov setting above,*

$$\mathsf{d}(P\|P^{\text{rev}}) \;=\; \sum_{i,j} \pi_i T_{ij}\, \log_2 \frac{\pi_i T_{ij}}{\pi_j T_{ji}} \;=\; \sigma \quad (\textit{bits/step}), \tag{6}$$

and the KL-holonomy satisfies

$$\frac{1}{n}\,\mathbb{E}_P\big[\operatorname{Hol}_n^{\text{KL,time-rev}}\big] \;\to\; \sigma. \tag{7}$$

Proof. The path log-likelihood ratio between P and the reversed path law under P^{rev} is

$$\log \frac{P_n(X_{0:n-1})}{P_n^{\text{rev}}(R_n(X_{0:n-1}))} = \sum_{t=1}^{n-1} \log \frac{\pi_{X_t}}{\pi_{X_{t-1}}} + \sum_{t=1}^{n-1} \log \frac{T_{X_{t-1}X_t}}{T_{X_t X_{t-1}}}.$$

The stationary term telescopes to $O(1)$; divide by n and take expectations. Identity (6) is standard in stochastic thermodynamics [9, 10]. Equation (7) is (5) with $Q = P^{\text{rev}}$. □

Remark 3.3 (Why not merely $h(Q) - h(P)$?). For stationary Markov chains, $h(P) = h(P^{\text{rev}})$, so representation-space holonomy would vanish. The KL version (observer-transported) returns the irreversible production σ.

3.4 General ergodic reversal

Let P^* be any stationary time-reversed process absolutely continuous w.r.t. P on cylinders, with finite $\mathsf{d}(P\|P^*)$. Then, by the same argument,

$$\frac{1}{n}\,\mathbb{E}_P\left[\mathrm{Hol}_n^{\mathrm{KL,time\text{-}rev}}\right] \;\to\; \mathsf{d}(P\|P^*). \tag{8}$$

4 Observer Independence

Theorem 4.1 (Code-robustness of KL holonomy)**.** *Let $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ be universal on $\mathcal{X}$ for the laws appearing in Lemma 2.1. Then, for any fixed loop γ,*

$$\frac{1}{n}\left|\mathrm{Hol}^{\mathrm{KL},\gamma}_{n,\mathcal{E}^{(1)}}(X_{0:n-1}) - \mathrm{Hol}^{\mathrm{KL},\gamma}_{n,\mathcal{E}^{(2)}}(X_{0:n-1})\right| \xrightarrow[n\to\infty]{P\text{-}a.s.} 0,$$

and likewise in $L^1(P)$.

Proof. Apply Lemma 2.1 to both codes and subtract. □

5 Code & Data

All code and data are available at: https://github.com/josh-winters/holonomy

6 Numerical validation (UEC battery)

All experiments were run with the companion script `uec_battery.py`. The suite covers: (i) gauge invariance under bijective recoding; (ii) coarse-graining/refinement loops (non-negativity of KL holonomy); (iii) Markov time-reversal, where KL holonomy matches the analytic entropy production (EP) σ in bits/step; (iv) observer-independence trends (KT vs. LZ code lengths per symbol converge); (v) robustness sweeps (random chains, low/high EP regimes, alignment/segment stability), and (vi) bootstrap confidence intervals for windowed estimates.

AoT demos (audio / sensors / finance). For window-level arrow-of-time classification, two choices align AUC with holonomy and our theory: *loop-negatives* (Encode→Reverse→DecodeSecond) instead of literal reversal, and domain preprocessing (`--aot_diff` for audio/sensors, `--aot_logreturn` for finance). These match the time-reversal loop used by the holonomy and avoid negative-KL pathologies. The script logs per-file AUC and bits/step(/s) and writes a scoreboard CSV.

Artifacts and reproducibility. The script writes

- `results/aot_wav.json`, `results/aot_csv.json` (single-file AoT).
- `results/scoreboard.csv`, `results/scoreboard.json` (folder runs).
- `results/summary.json` (aggregated suite summary for the run).

Representative commands and flags for the AoT demos are documented inline in the repository (e.g., `--aot_bins`, `--aot_win`, `--aot_stride`, `--aot_rate`).

Discussion

We distinguished two operational regimes. If one evaluates evidence *in the representation reached by the loop*, holonomy reduces to the *entropy-rate difference* $h(Q) - h(P)$ (Theorem 2.2); this yields gauge invariance and detects net compression/expansion by the loop. If instead one *transports the observer* and evaluates evidence against the loop's pushforward law on the *original coordinates*, holonomy equals the *relative entropy rate* $\mathsf{d}(P\|Q)$, recovering irreversibility and, for Markov time reversal, the entropy production rate.

Technical extensions. The finite-alphabet assumption can be relaxed via quantization and standard approximation. The Markov time-reversal equality extends to hidden Markov models at the level of path measures; holonomy on observed records gives a certified lower bound by data processing (and in the quantum setting by Lindblad/Uhlmann monotonicity [15, 16]). Absolute continuity requirements ensure finite rates (e.g., $\sigma < \infty$ requires $T_{ij} > 0 \Rightarrow T_{ji} > 0$).

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