

Markov Random Fields (MRF)

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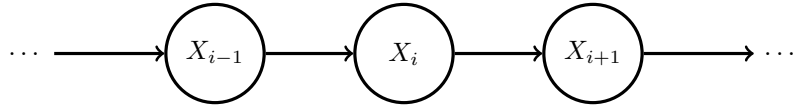
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1 Markov Chains

In a first order Markov chain (MC):

$$\Pr(X_i \mid X_{i-1}, X_{i-2}, \dots, X_1) = \Pr(X_i \mid X_{i-1})$$

For example with:



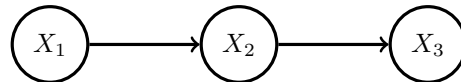
For the sake of argument, let's consider $X_i \in \mathcal{F}_i$ as the event at time i from the possible events \mathcal{F}_i . In all generality, we would expect for a causal process to simplify to:

$$\Pr(X_i \mid \dots, X_{i+1}, X_{i-1}, \dots, X_1) = \Pr(X_i \mid X_{i-1}, X_{i-2}, \dots, X_1)$$

By Markov's property, it adds an additional constraint and says that:

$$\Pr(X_i \mid X_{i-1}, X_{i-2}, \dots, X_1) = \Pr(X_i \mid X_{i-1})$$

In other words, it is the **conditional independence** of X_i w.r.t X_j where $i \neq j$ but not the independence !



As we can see for X_1, X_2, X_3 .

$$\begin{aligned}\Pr(X_3, X_1) &= \Pr(X_3 \mid X_1) \Pr(X_1) \\ &= \Pr(X_3 \mid X_1) \Pr(X_1) \\ &= \Pr(X_1) \sum_{X_2 \in \mathcal{F}_2} \Pr(X_3 \mid X_2) \Pr(X_2 \mid X_1)\end{aligned}$$

Then independence is satisfied only if:

$$\Pr(X_3) = \sum_{X_2 \in \mathcal{F}_2} \Pr(X_3 \mid X_2) \Pr(X_2 \mid X_1)$$

Which is not necessarily the case. The following relation is used:

$$\Pr(X_3 \mid X_1) = \sum_{X_2 \in \mathcal{F}_2} \Pr(X_3 \mid X_2) \Pr(X_2 \mid X_1)$$

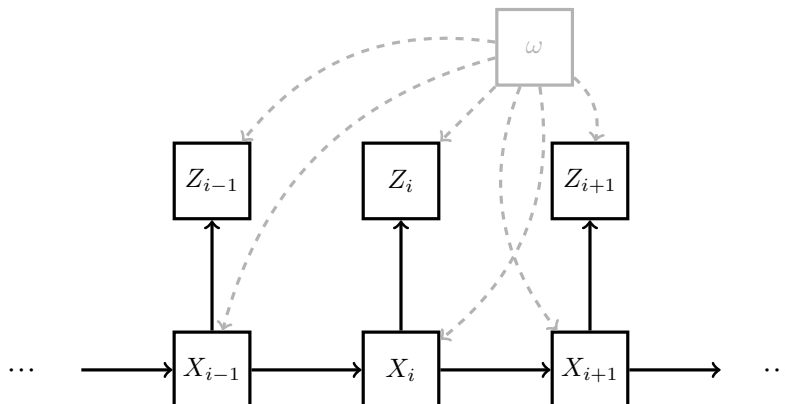
Which is useful to express the dependence events to anterior events. More generally the event at state i is dependent on the event $i - N$ with:

$$\Pr(X_i \mid X_{i-N}) = \sum_{X_{i-2} \in \mathcal{F}_{i-2}, \dots, X_{i-N+1} \in \mathcal{F}_{i-N+1}} \Pr(X_{i-2} \mid X_{i-2}) \dots \Pr(X_{i-N+1} \mid X_{i-N})$$

So relating to algorithms and data structures, the Markov chain formulation may be an instance of memoization to solve a Markov process joint probability !

2 Hidden Markov Model (HMM)

The problem is as follows. We are given a series of observations Z_i where there are hidden events X_i modeled by a Markov chain. We may also have parameters for the model ω . All of these are expressed through a probabilistic graph model such as this one:



Where we can express the likelihood of $\mathbf{Z}_N = \{Z_1, \dots, Z_N\}$ w.r.t $\mathbf{X}_N = \{X_1, \dots, X_N\}$ as:

$$\Pr(\mathbf{Z}_N | \mathbf{X}_N) = \prod_i^N \Pr(Z_i | X_i)$$

The continuous version of HMM is actually a Kalman filter !