In [ ]:
```
import Pkg
Pkg.activate(@__DIR__)
Pkg.instantiate()
using LinearAlgebra, Plots
import ForwardDiff as FD
import MeshCat as mc
using Test
```

```
[ Info: Precompiling IJuliaExt [2f4121a4-3b3a-5ce6-9c5e-1f2673ce168a]
[ Info: Precompiling ForwardDiff [f6369f11-7733-5829-9624-2563aa707210]
[ Info: Precompiling SpecialFunctionsExt [997ecda8-951a-5f50-90ea-61382e9770
4b]
[ Info: Precompiling MeshCat [283c5d60-a78f-5afe-a0af-af636b173e11]
[ Info: Precompiling ForwardDiffStaticArraysExt [b74fd6d0-9da7-541f-a07d-1b6
af30a262f]
[ Info: Precompiling GeometryBasicsExt [b238bd29-021f-5edc-8b0e-16b9cda5f63
a]
```

Julia Warmup

Just like Python, Julia lets you do the following:

In [ ]:
```
let
    x = [1, 2, 3]
    @show x
    y = x # NEVER DO THIS, EDITING ONE WILL NOW EDIT BOTH

    y[3] = 100 # this will now modify both y and x
    x[1] = 300 # this will now modify both y and x

    @show y
    @show x
end
```

```
x = [1, 2, 3]
y = [300, 2, 100]
x = [300, 2, 100]
```

Out[ ]:
```
3-element Vector{Int64}:
 300
   2
 100
```

In [ ]:
```
# to avoid this, here are two alternatives
let
    x = [1,2,3]
    @show x

    y1 = 1*x         # this is fine
    y2 = deepcopy(x) # this is also fine
```

```
    x[2] = 200 # only edits x
    y1[1] = 400 # only edits y1
    y2[3] = 100 # only edits y2

    @show x
    @show y1
    @show y2
end
```

```
x = [1, 2, 3]
x = [1, 200, 3]
y1 = [400, 2, 3]
y2 = [1, 2, 100]
```

Out[ ]:
```
3-element Vector{Int64}:
   1
   2
 100
```

Optional function arguments

We can have optional keyword arguments for functions in Julia, like the following:

In [ ]:
```
## optional arguments in functions

# we can have functions with optional arguments after a ; that have default
let
    function f1(a, b; c=4, d=5)
        @show a,b,c,d
    end

    f1(1,2)               # this means c and d will take on default value
    f1(1,2;c = 100,d = 2) # specify c and d
    f1(1,2;d = -30)       # or we can only specify one of them
end
```

```
(a, b, c, d) = (1, 2, 4, 5)
(a, b, c, d) = (1, 2, 100, 2)
(a, b, c, d) = (1, 2, 4, -30)
```

Out[ ]:
```
(1, 2, 4, -30)
```

Q1: Integration (25 pts)

In this question we are going to integrate the equations of motion for a double pendulum using multiple explicit and implicit integrators. We will write a generic simulation function for each of the two categories (explicit and implicit), and compare 6 different integrators.

The continuous time dynamics of the cartpole are written as a function:

$$\dot{x} = f(x)$$

In the code you will see `xdot = dynamics(params, x)`.

Part A (10 pts): Explicit Integration

Here we are going to implement the following explicit integrators:

- Forward Euler (explicit)
- Midpoint (explicit)
- RK4 (explicit)

In [ ]:
```julia
# these two functions are given, no TODO's here
function double_pendulum_dynamics(params::NamedTuple, x::Vector)
    # continuous time dynamics for a double pendulum given state x,
    # also known as the "equations of motion".
    # returns the time derivative of the state, ẋ (dx/dt)

    # the state is the following:
    θ1, θ̇1, θ2, θ̇2 = x

    # system parameters
    m1, m2, L1, L2, g = params.m1, params.m2, params.L1, params.L2, params.g

    # dynamics
    c = cos(θ1 - θ2)
    s = sin(θ1 - θ2)

    ẋ = [
        θ̇1
        (
            m2 * g * sin(θ2) * c - m2 * s * (L1 * c * θ̇1^2 + L2 * θ̇2^2) -
            (m1 + m2) * g * sin(θ1)
        ) / (L1 * (m1 + m2 * s^2))
        θ̇2
        (
            (m1 + m2) * (L1 * θ̇1^2 * s - g * sin(θ2) + g * sin(θ1) * c) +
            m2 * L2 * θ̇2^2 * s * c
        ) / (L2 * (m1 + m2 * s^2))
    ]

    return ẋ
end
function double_pendulum_energy(params::NamedTuple, x::Vector)::Real
    # calculate the total energy (kinetic + potential) of a double pendulum


    # the state is the following:
    θ1, θ̇1, θ2, θ̇2 = x

    # system parameters
    m1, m2, L1, L2, g = params.m1, params.m2, params.L1, params.L2, params.g
```

```
    # cartesian positions/velocities of the masses
    r1 = [L1 * sin(θ1), 0, -params.L1 * cos(θ1) + 2]
    r2 = r1 + [params.L2 * sin(θ2), 0, -params.L2 * cos(θ2)]
    v1 = [L1 * θ̇1 * cos(θ1), 0, L1 * θ̇1 * sin(θ1)]
    v2 = v1 + [L2 * θ̇2 * cos(θ2), 0, L2 * θ̇2 * sin(θ2)]

    # energy calculation
    kinetic = 0.5 * (m1 * v1' * v1 + m2 * v2' * v2)
    potential = m1 * g * r1[3] + m2 * g * r2[3]
    return kinetic + potential
end
```

Out[ ]: double_pendulum_energy (generic function with 1 method)

Now we are going to simulate this double pendulum by integrating the equations of motion with the simplest explicit integrator, the Forward Euler method:

$$x_{k+1} = x_k + \Delta t \cdot f(x_k) \qquad \text{Forward Euler (explicit)}$$

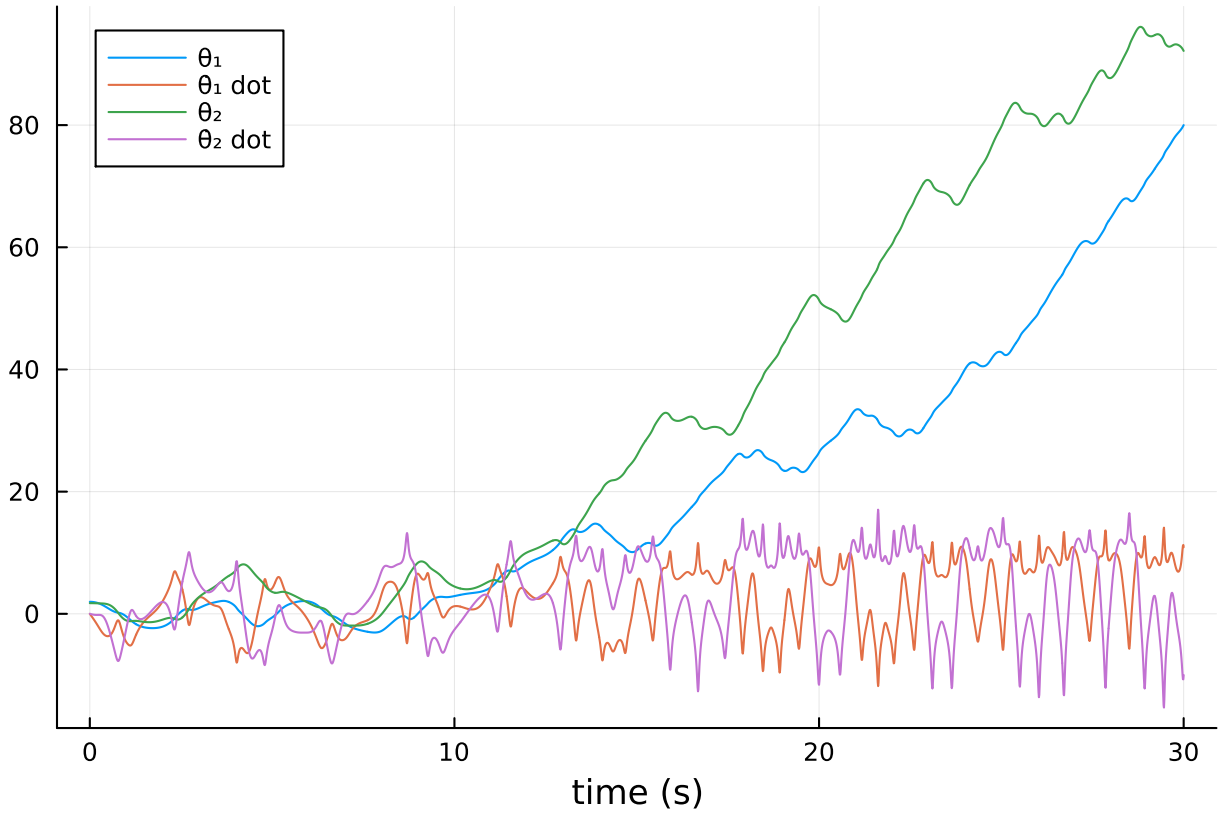
In [ ]:
```
"""
    x_{k+1} = forward_euler(params, dynamics, x_k, dt)

Given `ẋ = dynamics(params, x)`, take in the current state `x` and integrate
using Forward Euler method.
"""
function forward_euler(
    params::NamedTuple,
    dynamics::Function,
    x::Vector,
    dt::Real,
)::Vector
    # ẋ = dynamics(params, x)
    # TODO: implement forward euler
    return x + dt * dynamics(params, x)
end
```

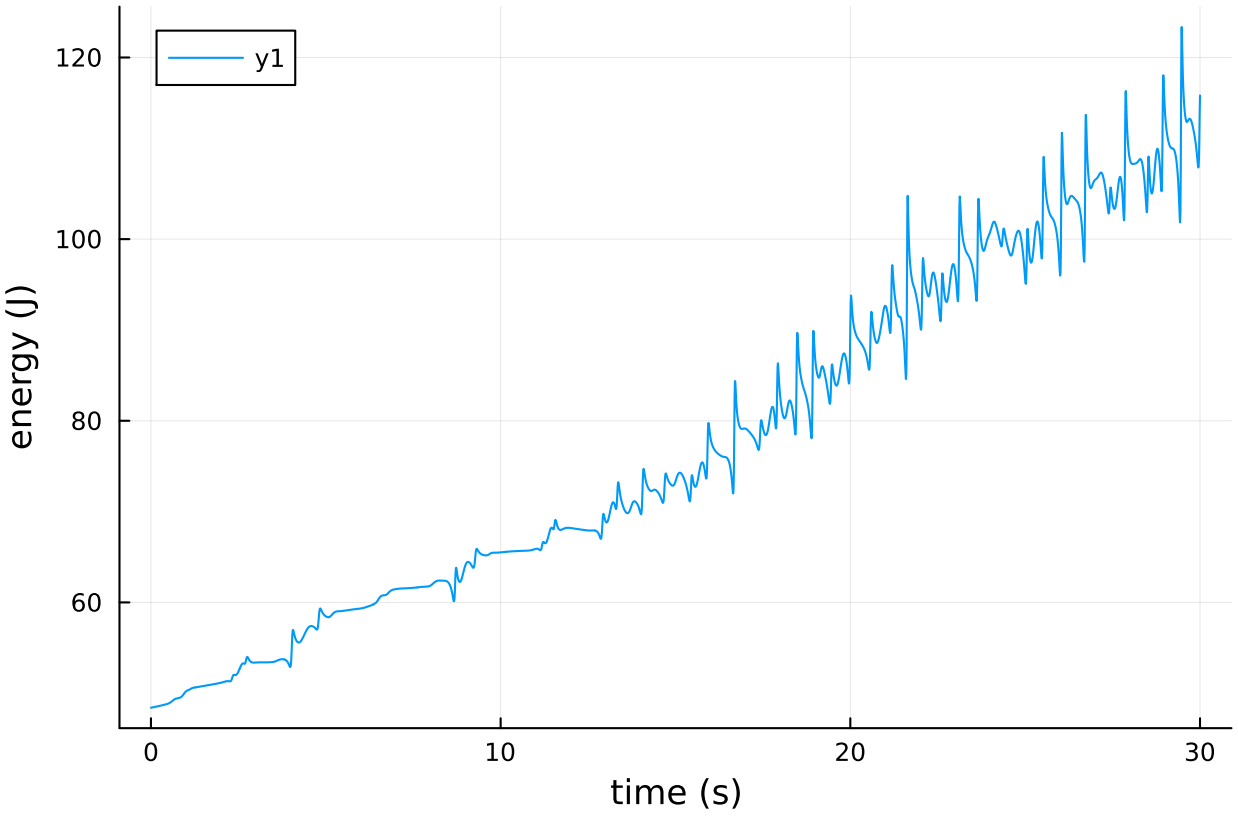
Out[ ]: forward_euler

In [ ]:
```
include(joinpath(@__DIR__, "animation.jl"))

let

    # parameters for the simulation
    params = (m1 = 1.0, m2 = 1.0, L1 = 1.0, L2 = 1.0, g = 9.8)

    # initial condition
    x0 = [pi / 1.6; 0; pi / 1.8; 0]

    # time step size (s)
    dt = 0.01
    tf = 30.0
    t_vec = 0:dt:tf
```

```
    N = length(t_vec)

    # store the trajectory in a vector of vectors
    X = [zeros(4) for i = 1:N]
    X[1] = 1 * x0

    # TODO: simulate the double pendulum with `forward_euler`
    # X[k] = `x_k`, so X[k+1] = forward_euler(params, double_pendulum_dynami
    for k = 1:N-1
        X[k+1] = forward_euler(params, double_pendulum_dynamics, X[k], dt)
    end

    # calculate energy
    E = [double_pendulum_energy(params, x) for x in X]

    @show @test norm(X[end]) > 1e-10   # make sure all X's were updated
    @show @test 2 < (E[end] / E[1]) < 3  # energy should be increasing

    # plot state history, energy history, and animate it
    display(
        plot(
            t_vec,
            hcat(X...)',
            xlabel = "time (s)",
            label = ["θ₁" "θ̇₁ dot" "θ₂" "θ̇₂ dot"],
        ),
    )
    display(plot(t_vec, E, xlabel = "time (s)", ylabel = "energy (J)"))
    meshcat_animate(params, X, dt, N)


end
```

```
#= In[8]:36 =# @test(norm(X[end]) > 1.0e-10) = Test Passed
#= In[8]:37 =# @test(2 < E[end] / E[1] < 3) = Test Passed
```






```
[ Info: Listening on: 127.0.0.1:8700, thread id: 1
┌ Info: MeshCat server started. You can open the visualizer by visiting the
following URL in your browser:
└ http://127.0.0.1:8700
```

Out[ ]:

Now let's implement the next two integrators:

Midpoint:

$$x_m = x_k + \frac{\Delta t}{2} \cdot f(x_k) \tag{1}$$
$$x_{k+1} = x_k + \Delta t \cdot f(x_m) \tag{2}$$

RK4:

$$k_1 = \Delta t \cdot f(x_k) \tag{3}$$
$$k_2 = \Delta t \cdot f(x_k + k_1/2) \tag{4}$$
$$k_3 = \Delta t \cdot f(x_k + k_2/2) \tag{5}$$
$$k_4 = \Delta t \cdot f(x_k + k_3) \tag{6}$$
$$x_{k+1} = x_k + (1/6) \cdot (k_1 + 2k_2 + 2k_3 + k_4) \tag{7}$$

In [ ]:
```julia
function midpoint(params::NamedTuple, dynamics::Function, x::Vector, dt::Rea
    # TODO: implement explicit midpoint
    x_m = x + 0.5*dt*dynamics(params,x)
    return x + dt*dynamics(params,x_m)
```

```
end
function rk4(params::NamedTuple, dynamics::Function, x::Vector, dt::Real)::V
    # TODO: implement RK4
    k1 = dynamics(params,x) * dt
    k2 = dynamics(params,x + 0.5*k1) * dt
    k3 = dynamics(params,x + 0.5*k2) * dt
    k4 = dynamics(params,x + k3) * dt
    return x + (k1 + 2*k2 + 2*k3 + k4)/6
end
```

Out[ ]:
```
rk4 (generic function with 1 method)
```

In [ ]:
```
function simulate_explicit(params::NamedTuple,dynamics::Function,integrator:
    # TOOD: update this function to simulate dynamics forward
    # with the given explicit integrator


    # take in
    t_vec = 0:dt:tf
    N = length(t_vec)
    X = [zeros(length(x0)) for i = 1:N]
    X[1] = x0

    # TODO: simulate X forward
    for k = 1:N-1
        X[k+1] = integrator(params, dynamics, X[k], dt)
    end

    # return state history X and energy E
    E = [double_pendulum_energy(params,x) for x in X]
    return X, E
end
```

Out[ ]:
```
simulate_explicit (generic function with 1 method)
```

In [ ]:
```
# initial condition
const x0 = [pi/1.6; 0; pi/1.8; 0]

const params = (
    m1 = 1.0,
    m2 = 1.0,
    L1 = 1.0,
    L2 = 1.0,
    g = 9.8
)
```

Out[ ]:
```
(m1 = 1.0, m2 = 1.0, L1 = 1.0, L2 = 1.0, g = 9.8)
```

```
WARNING: both Plots and LinearAlgebra export "rotate!"; uses of it in module
Main must be qualified
```

Part B (10 pts): Implicit Integrators

Explicit integrators work by calling a function with x_k and Δt as arguments, and returning x_{k+1} like this:

$$x_{k+1} = f_{explicit}(x_k, \Delta t)$$

Implicit integrators on the other hand have the following relationship between the state at x_k and x_{k+1}:

$$f_{implicit}(x_k, x_{k+1}, \Delta t) = 0$$

This means that if we want to get x_{k+1} from x_k, we have to solve for a x_{k+1} that satisfies the above equation. This is a rootfinding problem in x_{k+1} (our unknown), so we juse have to use Newton's method.

Here are the three implicit integrators we are looking at, the first being Backward Euler (1st order):

$$f(x_k, x_{k+1}, \Delta t) = x_k + \Delta t \cdot \dot{x}_{k+1} - x_{k+1} = 0 \qquad \text{Backward Euler}$$

Implicit Midpoint (2nd order)

$$x_{k+1/2} = \frac{1}{2}(x_k + x_{k+1}) \tag{8}$$

$$f(x_k, x_{k+1}, \Delta t) = x_k + \Delta t \cdot \dot{x}_{k+1/2} - x_{k+1} = 0 \qquad \text{Implicit Midpoint} \tag{9}$$

Hermite Simpson (3rd order)

$$x_{k+1/2} = \frac{1}{2}(x_k + x_{k+1}) + \frac{\Delta t}{8}(\dot{x}_k - \dot{x}_{k+1})$$

$$f(x_k, x_{k+1}, \Delta t) = x_k + \frac{\Delta t}{6} \cdot (\dot{x}_k + 4\dot{x}_{k+1/2} + \dot{x}_{k+1}) - x_{k+1} = 0 \qquad \text{Hermite-Si}$$

When you implement these integrators, you will update the functions such that they take in a dynamics function, x_k and x_{k+1}, and return the residuals described above. We are NOT solving these yet, we are simply returning the residuals for each implicit integrator that we want to be 0.

In [ ]:
```julia
# since these are explicit integrators, these function will return the resid
# NOTE: we are NOT solving anything here, simply return the residuals
function backward_euler(
    params::NamedTuple,
    dynamics::Function,
    x1::Vector,
    x2::Vector,
    dt::Real,
)::Vector
    return x1 + dt * dynamics(params, x2) - x2
end
```

```
function implicit_midpoint(
    params::NamedTuple,
    dynamics::Function,
    x1::Vector,
    x2::Vector,
    dt::Real,
)::Vector
    return x1 + dt * dynamics(params, 0.5 * (x1 + x2)) - x2
end
function hermite_simpson(
    params::NamedTuple,
    dynamics::Function,
    x1::Vector,
    x2::Vector,
    dt::Real,
)::Vector
    x_mid =
        0.5 * (x1 + x2) +
        0.125 * dt * (dynamics(params, x1) - dynamics(params, x2))
    return x1 +
           1 / 6 *
           dt *
           (
               dynamics(params, x1) +
               4 * dynamics(params, x_mid) +
               dynamics(params, x2)
           ) - x2
end
```

Out[ ]:
```
hermite_simpson (generic function with 1 method)
```

In [ ]:
```
# TODO
# this function takes in a dynamics function, implicit integrator function,
# and uses Newton's method to solve for an x2 that satsifies the implicit ir
# that we wrote about in the functions above
function implicit_integrator_solve(
    params::NamedTuple,
    dynamics::Function,
    implicit_integrator::Function,
    x1::Vector,
    dt::Real;
    tol = 1e-13,
    max_iters = 10,
)::Vector

    # initialize guess
    x2 = 1 * x1

    # TODO: use Newton's method to solve for x2 such that residual for the i
    # DO NOT USE A WHILE LOOP
    for i = 1:max_iters
        J = FD.jacobian(
            x -> implicit_integrator(params, dynamics, x1, x, dt),
```

```
            x2,
        )
        x2 = x2 - inv(J) * (implicit_integrator(params, dynamics, x1, x2, dt

        # TODO: return x2 when the norm of the residual is below tol
        if norm(implicit_integrator(params, dynamics, x1, x2, dt)) < tol
            return x2
        end


    end
    error("implicit integrator solve failed")
end
```

Out[ ]: implicit_integrator_solve (generic function with 1 method)

In [ ]:
```
@testset "implicit integrator check" begin

    dt = 1e-1
    x1 = [0.1, 0.2, 0.3, 0.4]

    for integrator in [backward_euler, implicit_midpoint, hermite_simpson]
        println("-----testing $integrator ------")
        x2 = implicit_integrator_solve(
            params,
            double_pendulum_dynamics,
            integrator,
            x1,
            dt,
        )
        @test norm(integrator(params, double_pendulum_dynamics, x1, x2, dt))
              1e-10
    end

end
```

```
-----testing backward_euler ------
-----testing implicit_midpoint ------
-----testing hermite_simpson ------
Test Summary:                | Pass  Total  Time
implicit integrator check |    3      3  1.6s
```

Out[ ]: Test.DefaultTestSet("implicit integrator check", Any[], 3, false, false, tr
ue, 1.706280499859626e9, 1.706280501501644e9, false)

In [ ]:
```
function simulate_implicit(
    params::NamedTuple,
    dynamics::Function,
    implicit_integrator::Function,
    x0::Vector,
    dt::Real,
    tf::Real;
    tol = 1e-13,
)
```

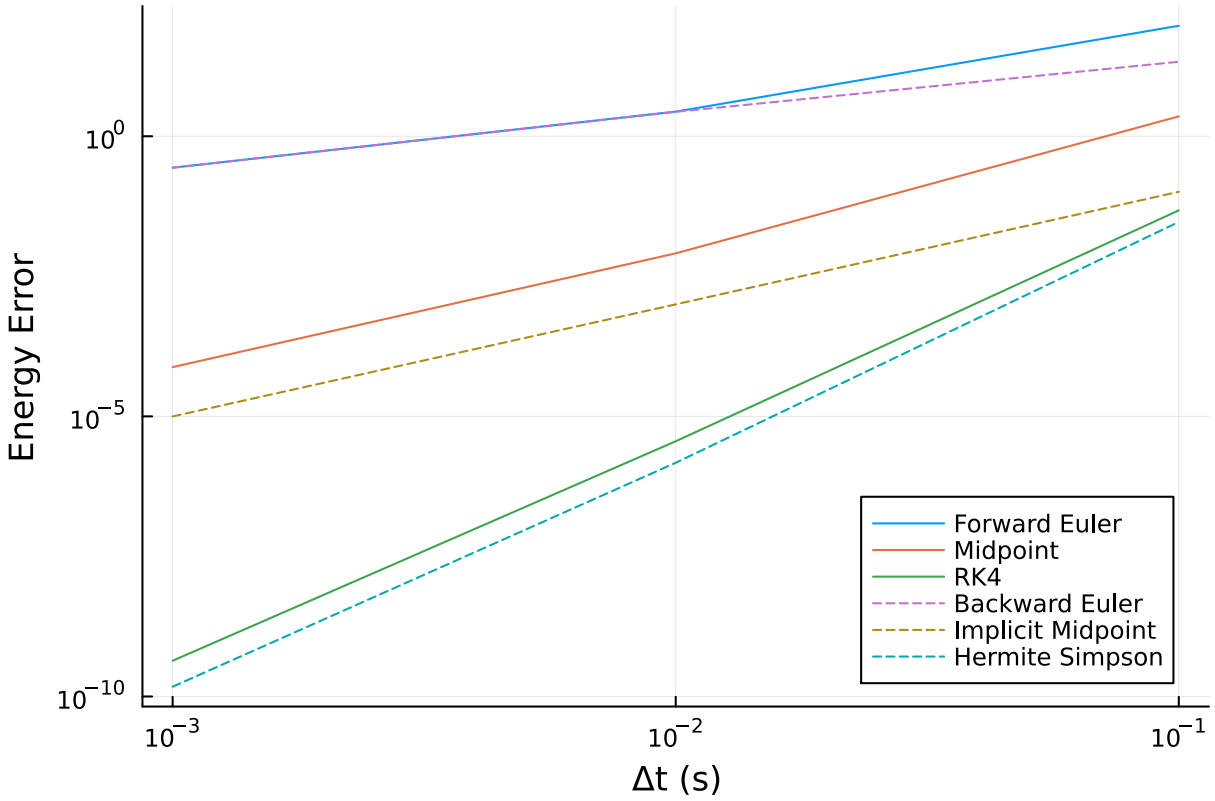
```
    t_vec = 0:dt:tf
    N = length(t_vec)
    X = [zeros(length(x0)) for i = 1:N]
    X[1] = x0

    # TODO: do a forward simulation with the selected implicit integrator
    # hint: use your `implicit_integrator_solve` function
    for k = 1:N-1
        X[k+1] = implicit_integrator_solve(
            params,
            dynamics,
            implicit_integrator,
            X[k],
            dt,
        )
    end

    E = [double_pendulum_energy(params, x) for x in X]
    @assert length(X) == N
    @assert length(E) == N
    return X, E
end
```

Out[ ]: simulate_implicit (generic function with 1 method)

In [ ]:
```
function max_err_E(E)
    E0 = E[1]
    err = abs.(E .- E0)
    return maximum(err)
end
function get_explicit_energy_error(integrator::Function, dts::Vector)
    [
        max_err_E(
            simulate_explicit(
                params,
                double_pendulum_dynamics,
                integrator,
                x0,
                dt,
                tf,
            )[2],
        ) for dt in dts
    ]
end
function get_implicit_energy_error(integrator::Function, dts::Vector)
    [
        max_err_E(
            simulate_implicit(
                params,
                double_pendulum_dynamics,
                integrator,
                x0,
                dt,
```

```
                tf,
            )[2],
        ) for dt in dts
    ]
end


const tf = 2.0
let
    # here we compare everything
    dts = [1e-3, 1e-2, 1e-1]
    explicit_integrators = [forward_euler, midpoint, rk4]
    implicit_integrators = [backward_euler, implicit_midpoint, hermite_simps

    explicit_data = [
        get_explicit_energy_error(integrator, dts) for
        integrator in explicit_integrators
    ]
    implicit_data = [
        get_implicit_energy_error(integrator, dts) for
        integrator in implicit_integrators
    ]

    plot(
        dts,
        hcat(explicit_data...),
        label = ["Forward Euler" "Midpoint" "RK4"],
        xaxis = :log10,
        yaxis = :log10,
        xlabel = "Δt (s)",
        ylabel = "Energy Error",
    )
    plot!(
        dts,
        hcat(implicit_data...),
        ls = :dash,
        label = ["Backward Euler" "Implicit Midpoint" "Hermite Simpson"],
    )
    plot!(legend = :bottomright)
end
```

Out[ ]:

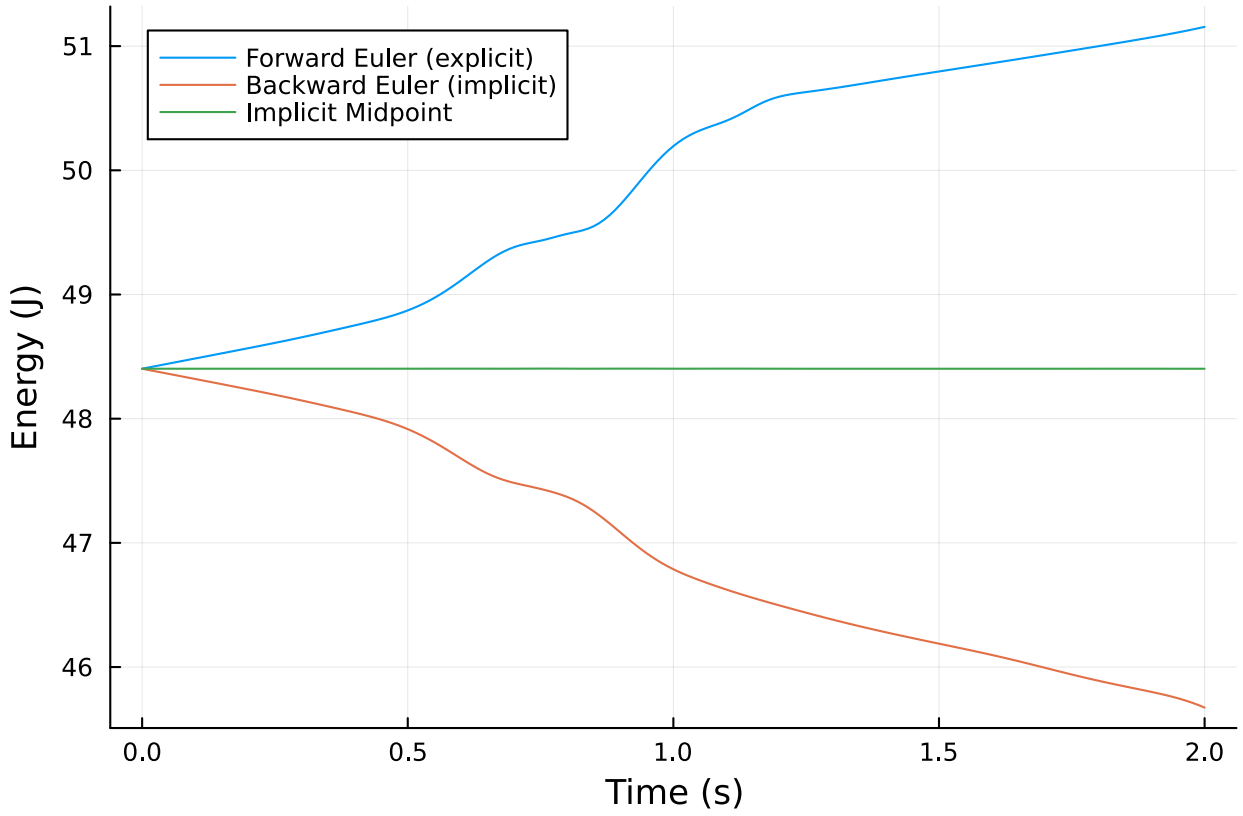


What we can see above is the maximum energy error for each of the integration methods. In general, the implicit methods of the same order are slightly better than the explicit ones.

In [ ]:

```
@testset "energy behavior" begin

    # simulate with all integrators
    dt = 0.01
    t_vec = 0:dt:tf
    E1 = simulate_explicit(
        params,
        double_pendulum_dynamics,
        forward_euler,
        x0,
        dt,
        tf,
    )[2]
    E2 = simulate_implicit(
        params,
        double_pendulum_dynamics,
        backward_euler,
        x0,
        dt,
        tf,
    )[2]
    E3 = simulate_implicit(
        params,
        double_pendulum_dynamics,
```

```
        implicit_midpoint,
        x0,
        dt,
        tf,
    )[2]
    E4 = simulate_implicit(
        params,
        double_pendulum_dynamics,
        hermite_simpson,
        x0,
        dt,
        tf,
    )[2]
    E5 = simulate_explicit(
        params,
        double_pendulum_dynamics,
        midpoint,
        x0,
        dt,
        tf,
    )[2]
    E6 = simulate_explicit(params, double_pendulum_dynamics, rk4, x0, dt, tf

    # plot forward/backward euler and implicit midpoint
    plot(t_vec, E1, label = "Forward Euler (explicit)")
    plot!(t_vec, E2, label = "Backward Euler (implicit)")
    display(
        plot!(
            t_vec,
            E3,
            label = "Implicit Midpoint",
            xlabel = "Time (s)",
            ylabel = "Energy (J)",
        ),
    )

    # test energy behavior
    E0 = E1[1]

    @test 2.5 < (E1[end] - E0) < 3.0
    @test -3.0 < (E2[end] - E0) < -2.5
    @test abs(E3[end] - E0) < 1e-2
    @test abs(E0 - E4[end]) < 1e-4
    @test abs(E0 - E5[end]) < 1e-1
    @test abs(E0 - E6[end]) < 1e-4
end
```




```
Test Summary:   | Pass  Total  Time
energy behavior |    6      6  0.1s
```

Out[ ]:
```
Test.DefaultTestSet("energy behavior", Any[], 6, false, false, true, 1.7062
80537683998e9, 1.706280537763158e9, false)
```

Another important takeaway from these integrators is that explicit Euler results in unstable behavior (as shown here by the growing energy), and implicit Euler results in artificial damping (losing energy). Implicit midpoint however maintains the correct energy. Even though the solution from implicit midpoint will vary from the initial energy, it does not move secularly one way or the other.

Part C (5 pts): One sentence short answer

1. Describe the energy behavior of each integrator. Are there any that are clearly unstable?

Put ONE SENTENCE answer here

- Forward Euler: Energy grows unbounded, which implies that it is unstable.
- Backward Euler: Energy decreases unbounded. The system is artificially damped.
- Implicit Midpoint: Energy is bounded and stays close to the initial energy, which implies that it is stable.