Mathematical Induction

The **principle of mathematical induction** states that if for some property $P(n)$, we have that

If it starts …

$P(0)$ is true

… and it keeps going …

and

For any $n \in \mathbb{N}$, we have $P(n) \rightarrow P(n+1)$

… then it's always true.

Then

For any $n \in \mathbb{N}$, $P(n)$ is true.

Another Example of Induction

Human Dominoes

- Everyone (except that last guy) eventually fell over.
- Why is that?
  - Someone fell over.
  - Once someone fell over, the next person fell over as well.

The **principle of mathematical induction** states that if for some property $P(n)$, we have that

$\boldsymbol{P(0)}$ **is true**

and

For any $\boldsymbol{n \in \mathbb{N}}$**, we have** $\boldsymbol{P(n) \rightarrow P(n+1)}$

Then

For any $\boldsymbol{n \in \mathbb{N}}$**,** $\boldsymbol{P(n)}$ **is true.**

Induction, Intuitively

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- …

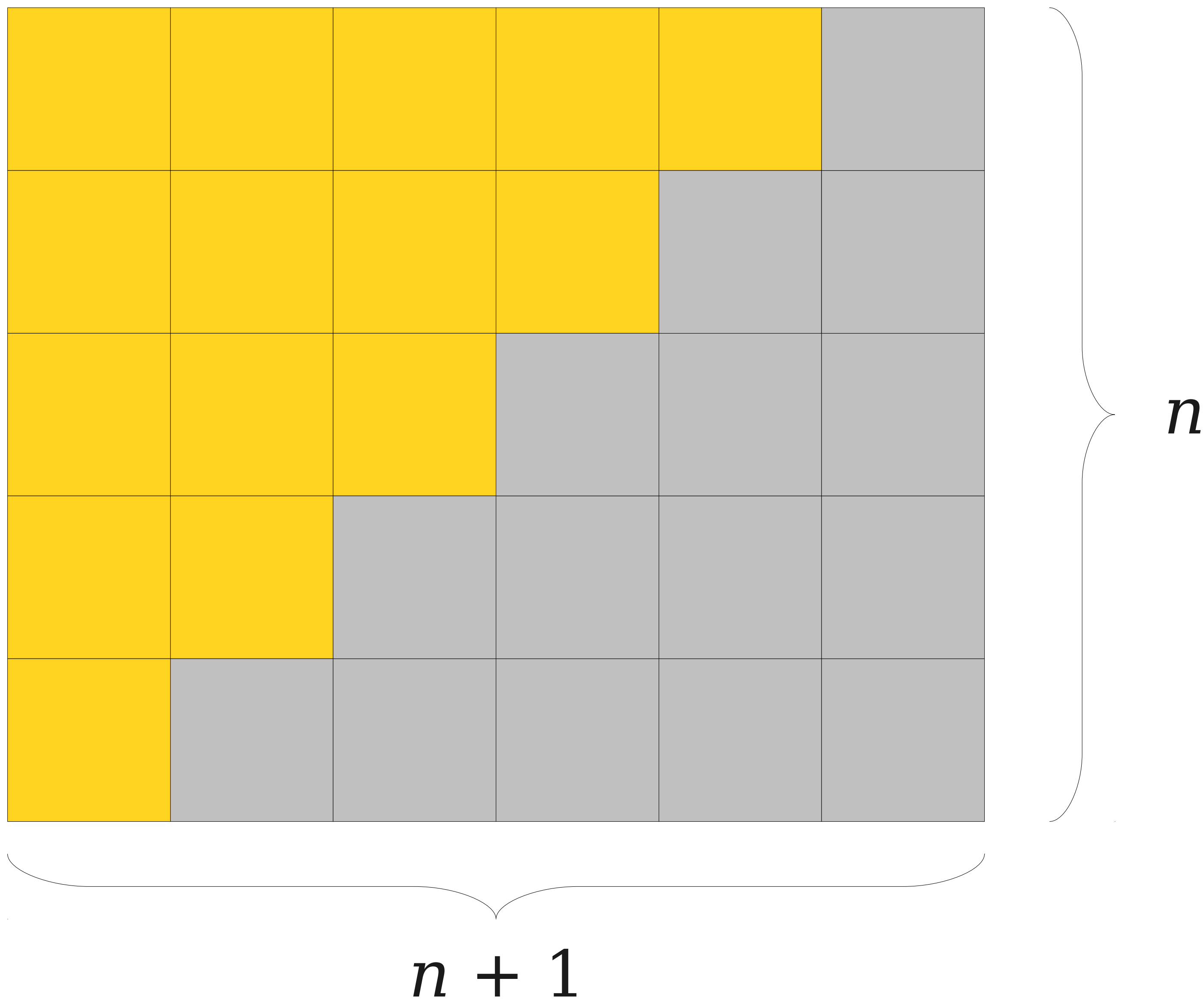
Proof by Induction

- Suppose that you want to prove that some property $P(n)$ holds of all natural numbers. To do so:
  - Prove that $P(0)$ is true.
    - This is called the **basis** or the **base case**.
  - Prove that for all $n \in \mathbb{N}$, that if $P(n)$ is true, then $P(n + 1)$ is true as well.
    - This is called the **inductive step**.
    - $P(n)$ is called the **inductive hypothesis**.
  - Conclude by induction that $P(n)$ holds for all n.

Some Sums

$$1 = 1$$
$$1 + 2 = 3$$
$$1 + 2 + 3 = 6$$
$$1 + 2 + 3 + 4 = 10$$
$$1 + 2 + 3 + 4 + 5 = 15$$

$$1 + 2 + \ldots + (n - 1) + n = n(n + 1) / 2$$



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Some Sums

$1 = 1$ $\mathbf{= 1(1 + 1) / 2}$

$1 + 2 = 3$ $\mathbf{= 2(2 + 1) / 2}$

$1 + 2 + 3 = 6$ $\mathbf{= 3(3 + 1) / 2}$

$1 + 2 + 3 + 4 = 10$ $\mathbf{= 4(4 + 1) / 2}$

$1 + 2 + 3 + 4 + 5 = 15$ $\mathbf{= 5(5 + 1) / 2}$

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Just as in a proof by contradiction or contrapositive, we should mention this proof is by induction.

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Now, we state what property $P(n)$ we are going to prove holds for all $n \in \mathbb{N}$.

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Proof: By induction. Let $P(n)$ be "the sum of the first n positive natural numbers is $n(n + 1) / 2$." We show that $P(n)$ is true for all $n \in \mathbb{N}$.

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For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero positive natural numbers is $0(0 + 1)/2$.

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The first step of an inductive proof is to show $P(0)$. We explicitly state what $P(0)$ is, then try to prove it. We can prove $P(0)$ using any proof technique we'd like.

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For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero positive natural numbers is $0(0 + 1)/2$. Since the sum of the first zero positive natural numbers is $0 = 0(0 + 1)/2$, $P(0)$ is true.

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For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that $1 + 2 + \ldots + n = n(n + 1) / 2$.

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For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that $1 + 2 + \ldots + n = n(n + 1) / 2$.

The goal of this step is to prove

"For any $n \in \mathbb{N}$, if $P(n)$, then $P(n + 1)$"

To do this, we'll choose an arbitrary n, assume that $P(n)$ holds, then try to prove $P(n + 1)$.

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Here, we're explicitly stating $P(n + 1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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Consider the sum of the . This
is the sum of the first n . By
the inductive hypothesis

We're assuming that $P(n)$ is true, so we can replace this sum with the value $\boldsymbol{n(n + 1) / 2}$.

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Thus $P(n + 1)$ is true, completing the induction.

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Structuring a Proof by Induction

- State that your proof works by induction.
- State your choice of $P(n)$.
- Prove the base case:
  - State what $P(0)$ is, then prove it using any technique you'd like.
- Prove the inductive step:
  - State that for some arbitrary $n \in \mathbb{N}$ that you're assuming $P(n)$ and mention what $P(n)$ is.
  - State that you are trying to prove $P(n + 1)$ and what $P(n + 1)$ means.
  - Prove $P(n + 1)$ using any technique you'd like.
- This is very rigorous, so as we gain more familiarity with induction we will start being less formal in our proofs.

Notation: Summations

- Instead of writing 1 + 2 + 3 + … + n, we write

Sum from i = 1 to n

$$\sum_{i=1}^{n} i$$

of i

Summation Examples

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

$$\sum_{i=0}^{2} \left(i^2 - i\right) = \left(0^2 - 0\right) + \left(1^2 - 1\right) + \left(2^2 - 2\right) = 2$$

The Empty Sum

- A sum of no numbers is called the **empty sum** and is defined to be zero.
- Examples:

$$\sum_{i=1}^{0} 2^i = 0 \qquad \sum_{i=137}^{42} i^i = 0 \qquad \sum_{i=0}^{-1} i = 0$$

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Since the empty sum is defined to be 0, this claim is true.

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We need to show that $P(n + 1)$ holds, meaning that

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To see this, note that

$$\sum_{i=1}^{n+1} i$$

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Sums of Powers of Two

(empty sum) $= 0$

$2^0 = 1 = 1$

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$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

A Quick Aside

- This result helps explain the range of numbers that can be stored in an **`int`**.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. We'll see one in a week.

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A Brief Interlude for Announcements

Problem Session Tonight

- Problem Session tonight, 7:00 – 7:50PM in 380-380X
- Purely optional, but should be a lot of fun!
- We'll try to get it recorded and posted online as soon as possible.

Back to our regularly scheduled programming…

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math

How Not To Induct

An Incorrect Proof

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$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + n + 1 = \frac{1}{2}\left(n + \frac{1}{2}\right)^2 + n + 1 = \frac{\left(n + \frac{1}{2}\right)^2}{2} + \frac{2(n+1)}{2} = \frac{\left(n + \frac{1}{2}\right)^2 + 2(n+1)}{2}$$

$$= \frac{n^2 + n + \frac{1}{4} + 2n + 2}{2} = \frac{n^2 + 3n + \frac{9}{4}}{2} = \frac{\left(n + \frac{3}{2}\right)^2}{2}$$

So $P(n + 1)$ holds, completing the induction. ■

An Incorrect Proof

Theorem: For any $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{1}{2}\left(n + \frac{1}{2}\right)^2$

Proof: By induction. Let $P(n)$ be defined as $P(n) \equiv \sum_{i=1}^{n}$

Where did we prove the base case?

Now, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds,

$$\sum_{i=1}^{n} i = \frac{1}{2}\left(n + \frac{1}{2}\right)^2$$

We want to show that $P(n + 1)$ is true, which means that we want to show

$$\sum_{i=1}^{n+1} i = \frac{1}{2}\left(n + 1 + \frac{1}{2}\right)^2 = \frac{1}{2}\left(n + \frac{3}{2}\right)^2$$

To see this, note that

$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + n + 1 = \frac{1}{2}\left(n + \frac{1}{2}\right)^2 + n + 1 = \frac{\left(n + \frac{1}{2}\right)^2}{2} + \frac{2(n+1)}{2} = \frac{\left(n + \frac{1}{2}\right)^2 + 2(n+1)}{2}$$

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An Incorrect Proof

Where did we prove the base case?

$\equiv \sum_{i=1}^{n}$

olds,

means that we want to show

$\frac{3}{2})^2$

$\frac{2(n+1)}{2} = \frac{(n+\frac{1}{2})^2 + 2(n+1)}{2}$

When proving $P(n)$ is true
for all $n \in \mathbb{N}$ by induction,

make sure to show the base case!

Otherwise, your argument is invalid!

The Counterfeit Coin Problem, Take Two

Problem Statement

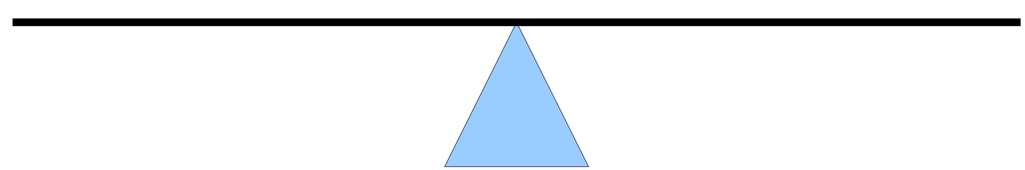
- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

Finding the Counterfeit Coin

1

2

3



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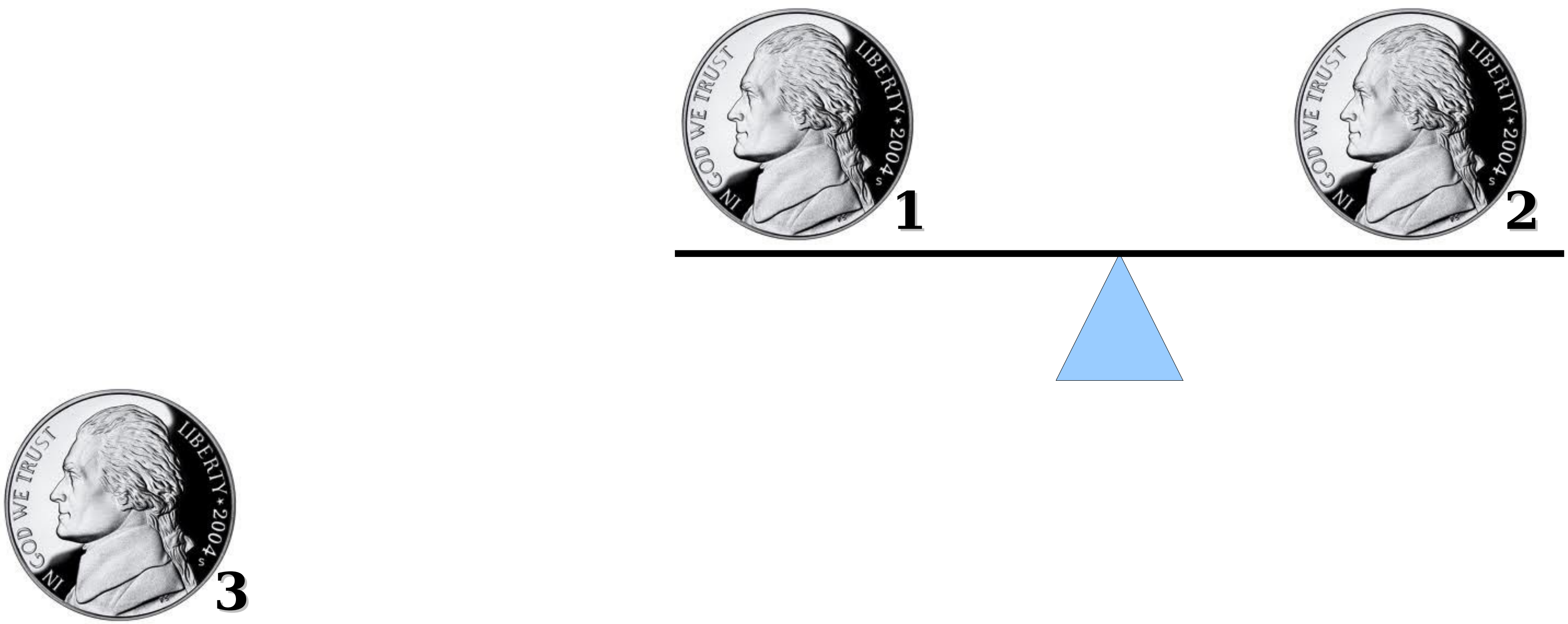

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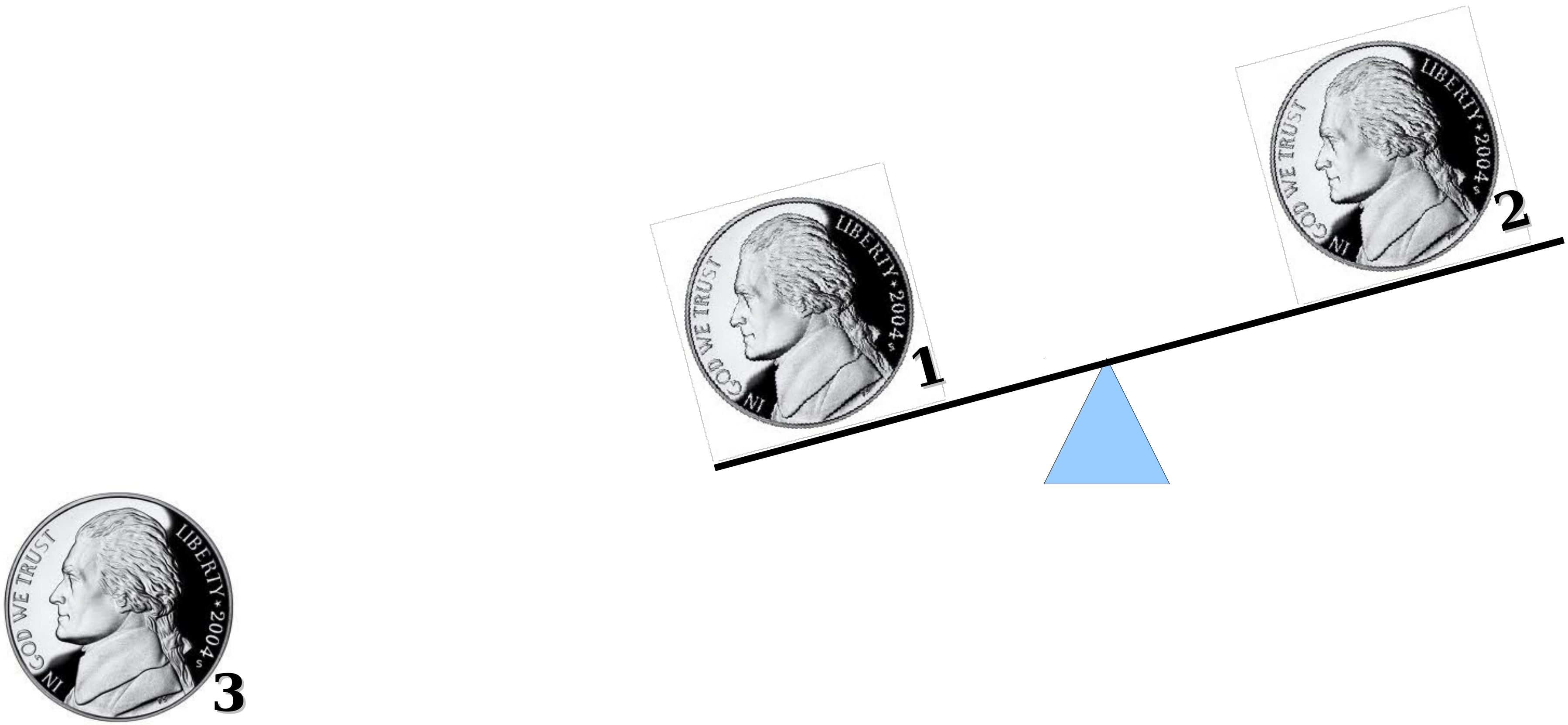

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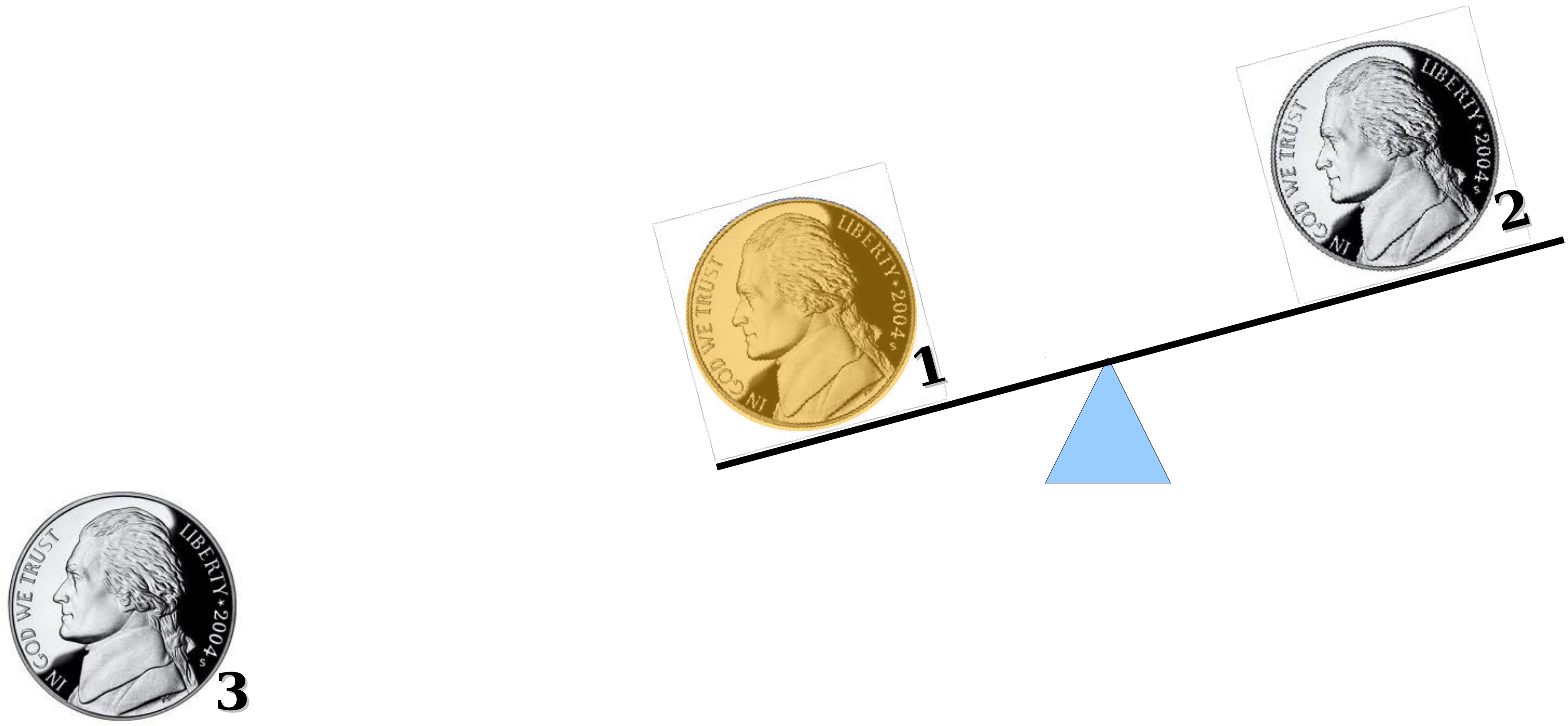

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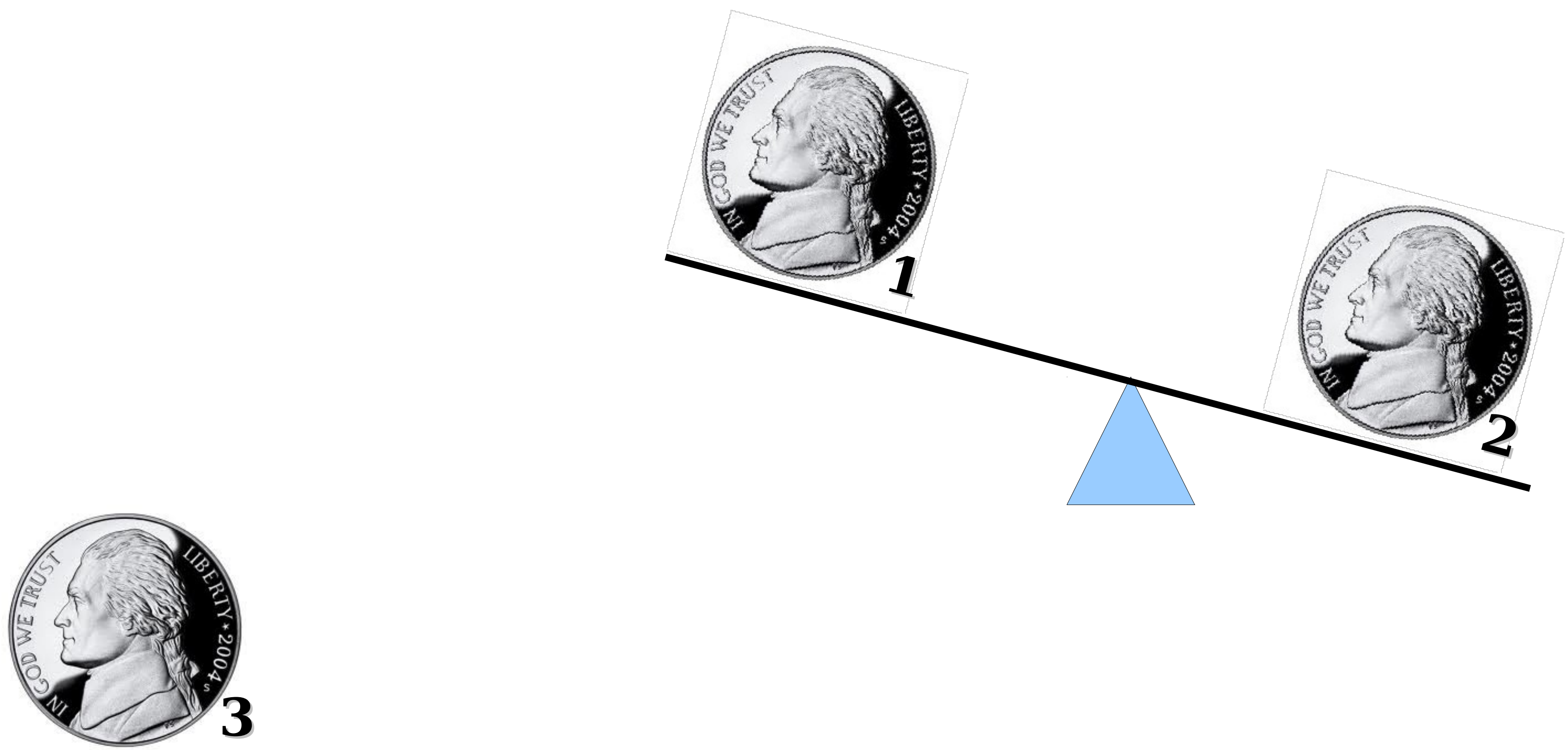

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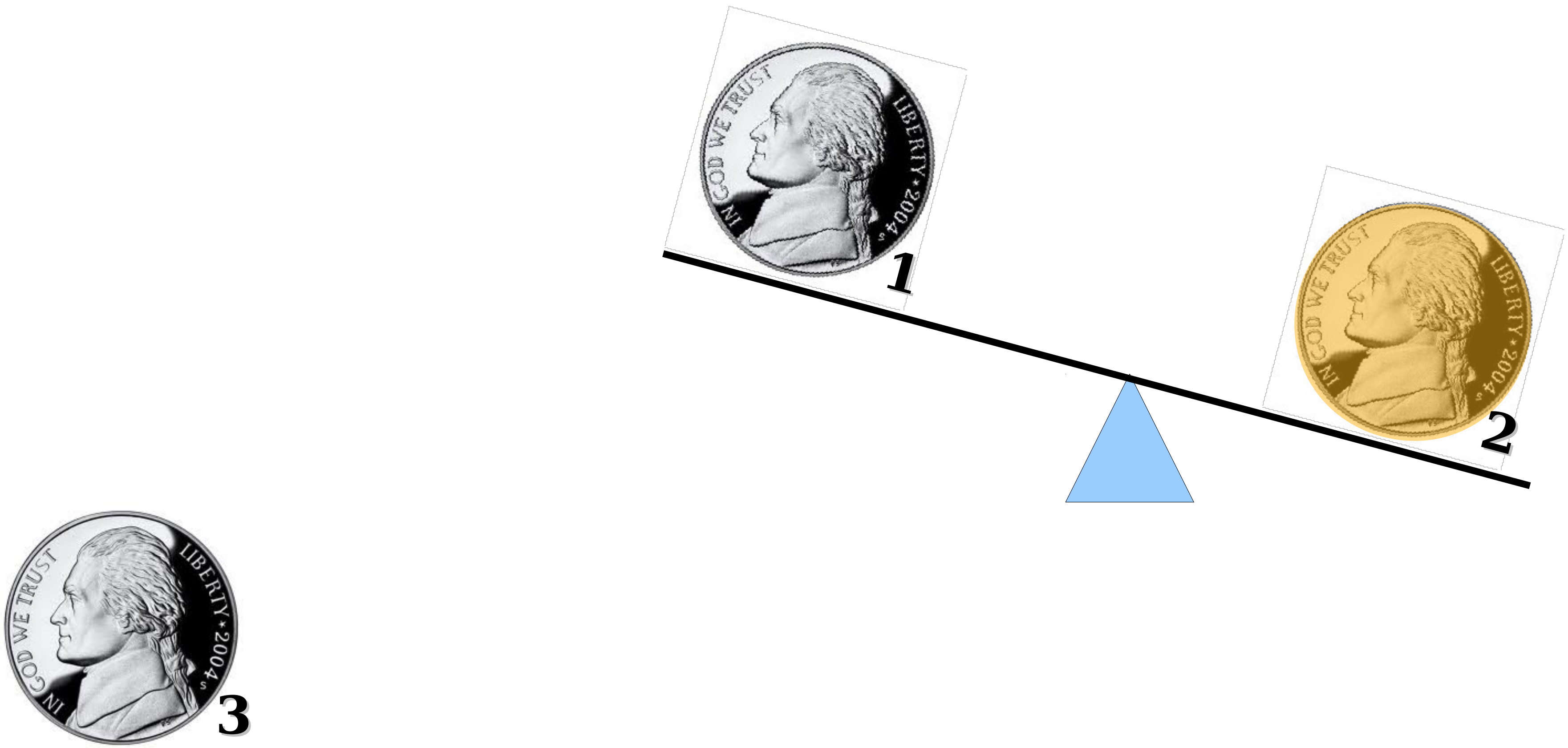

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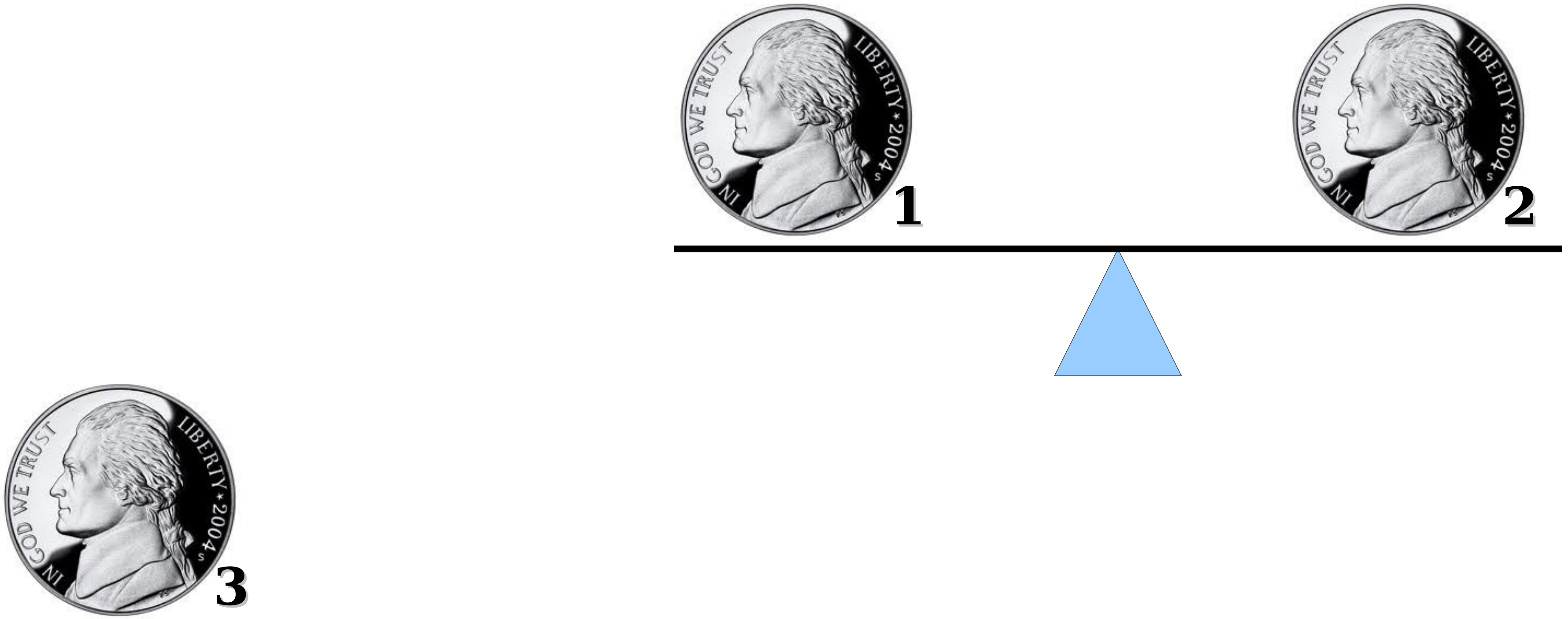

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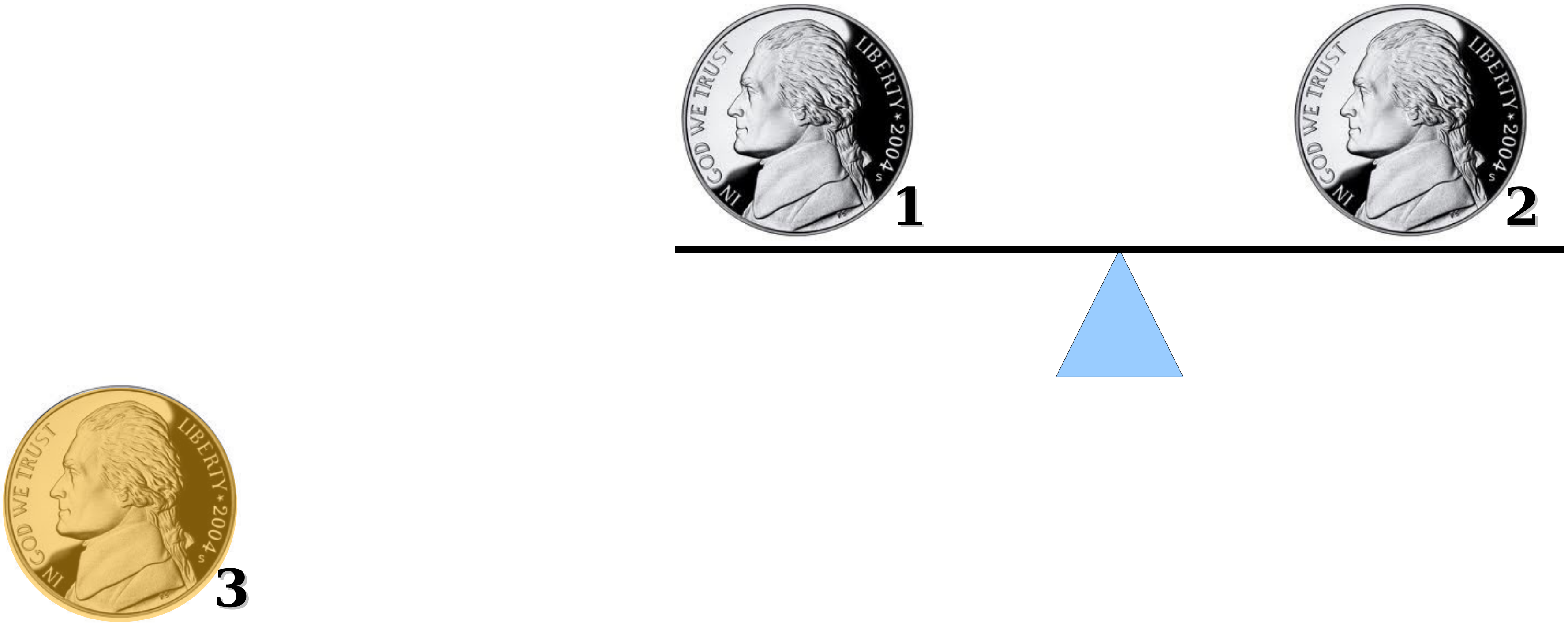
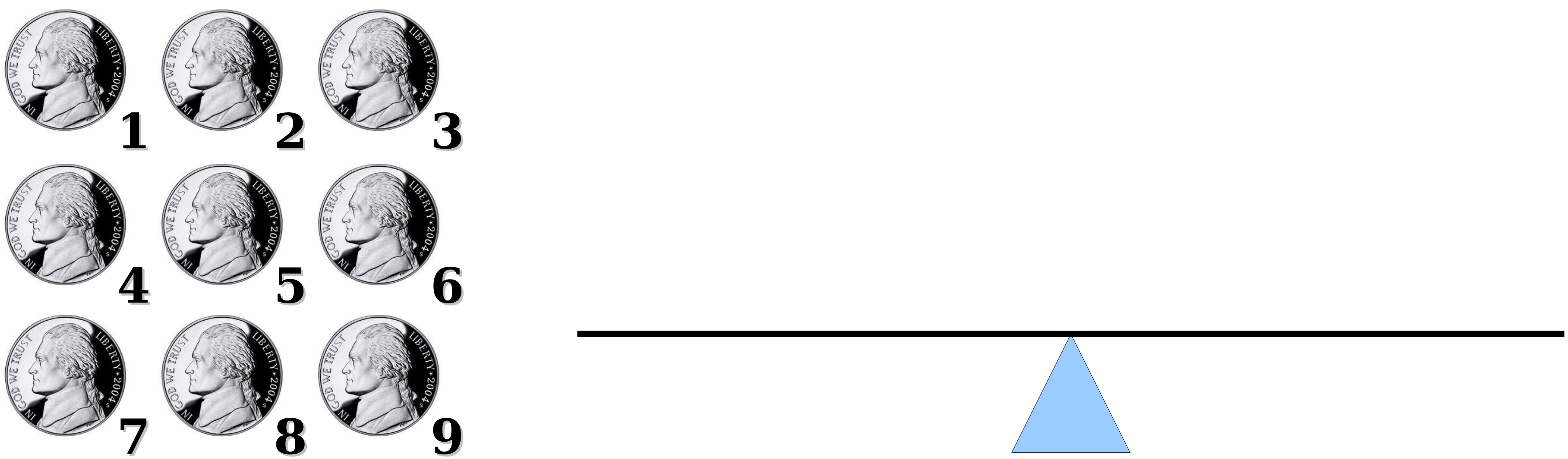


A Harder Problem

- You are given a set of **nine** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only **two** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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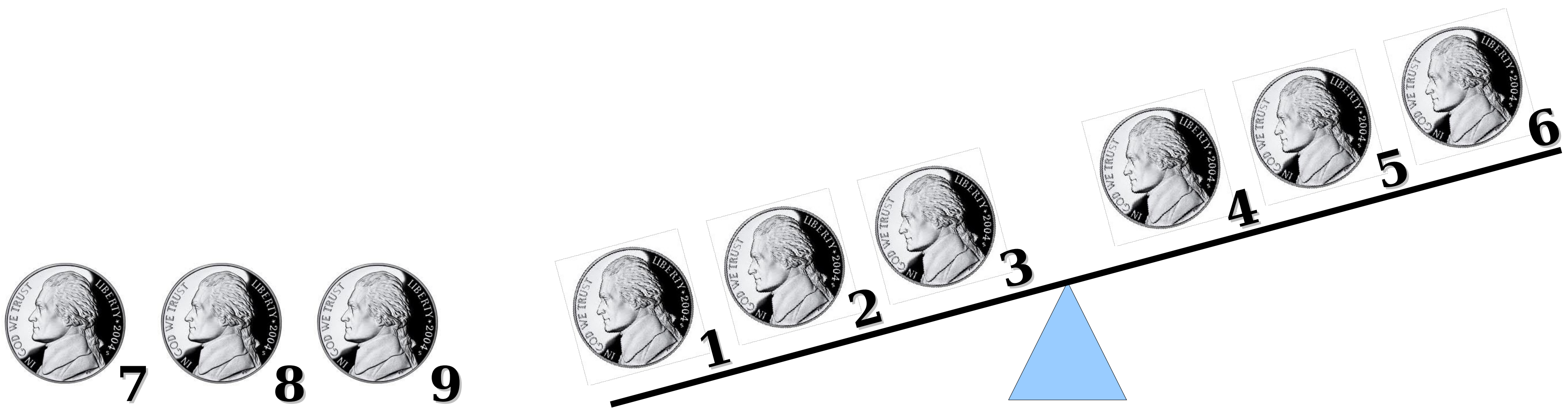

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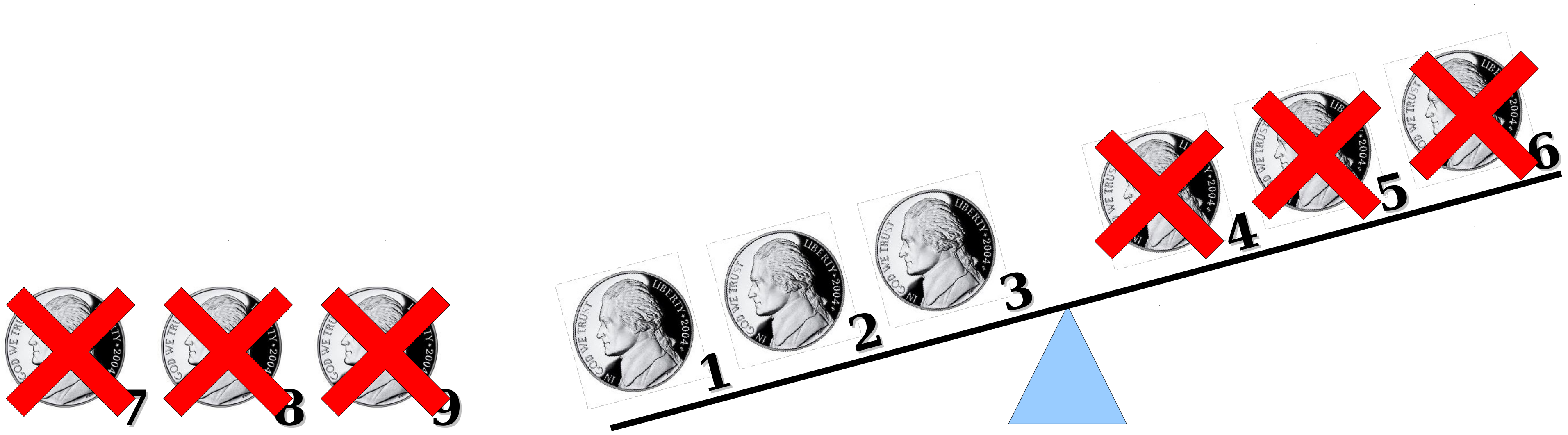

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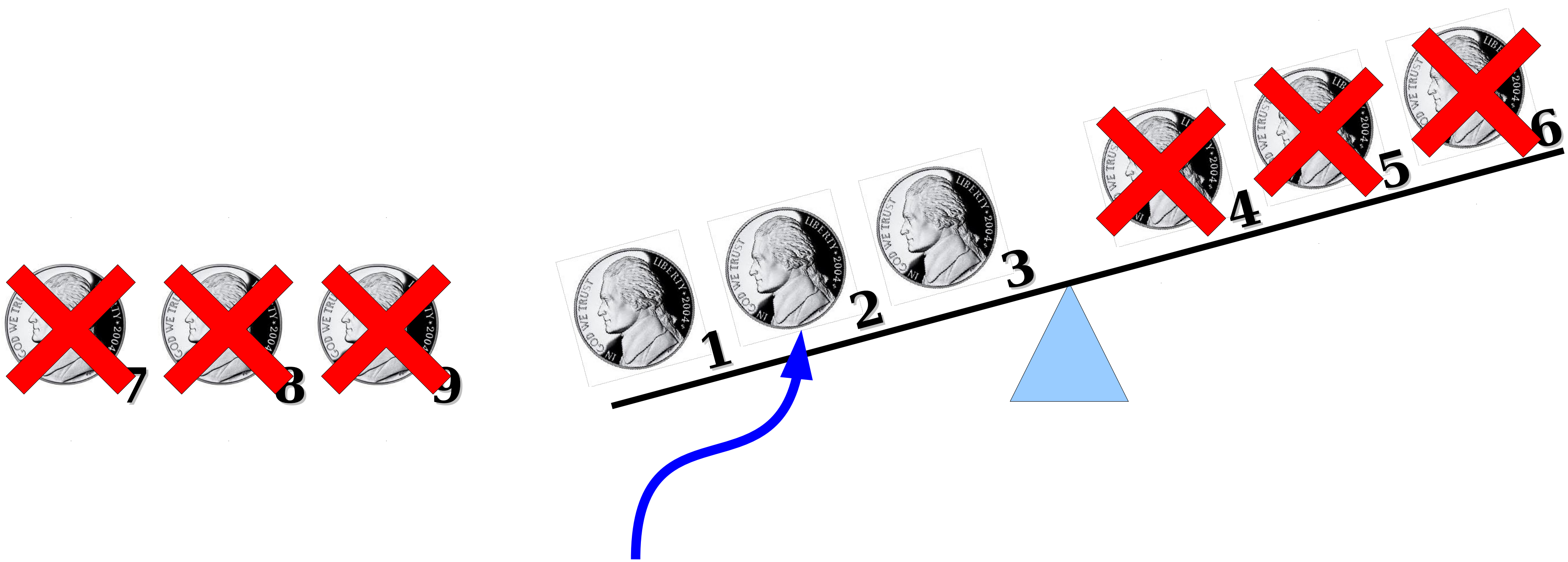
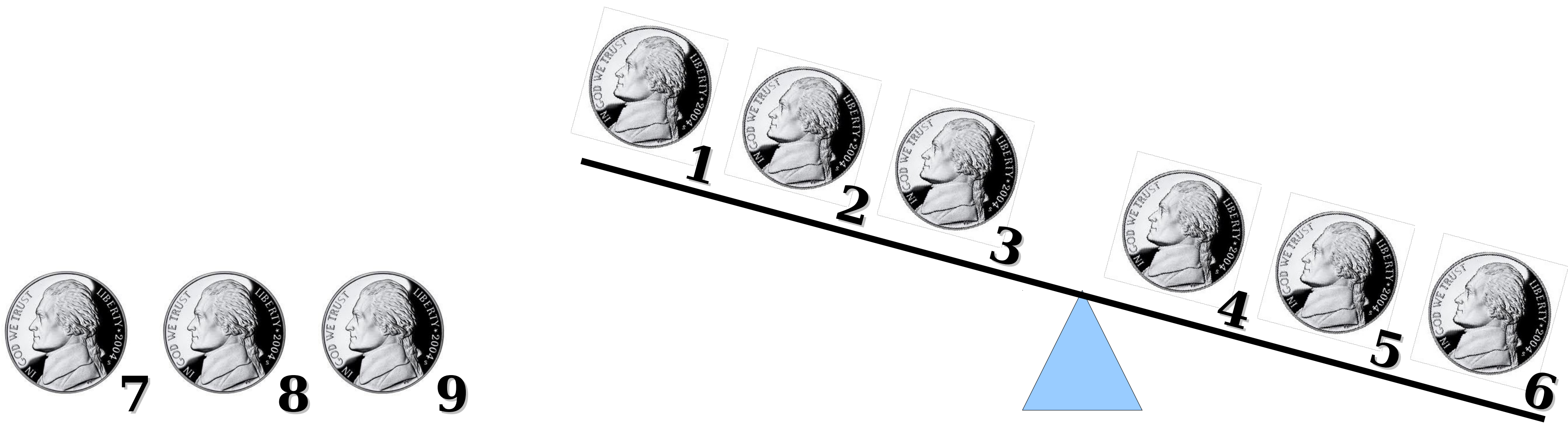


Now we have one weighing to find
the counterfeit out of these three

Finding the Counterfeit Coin



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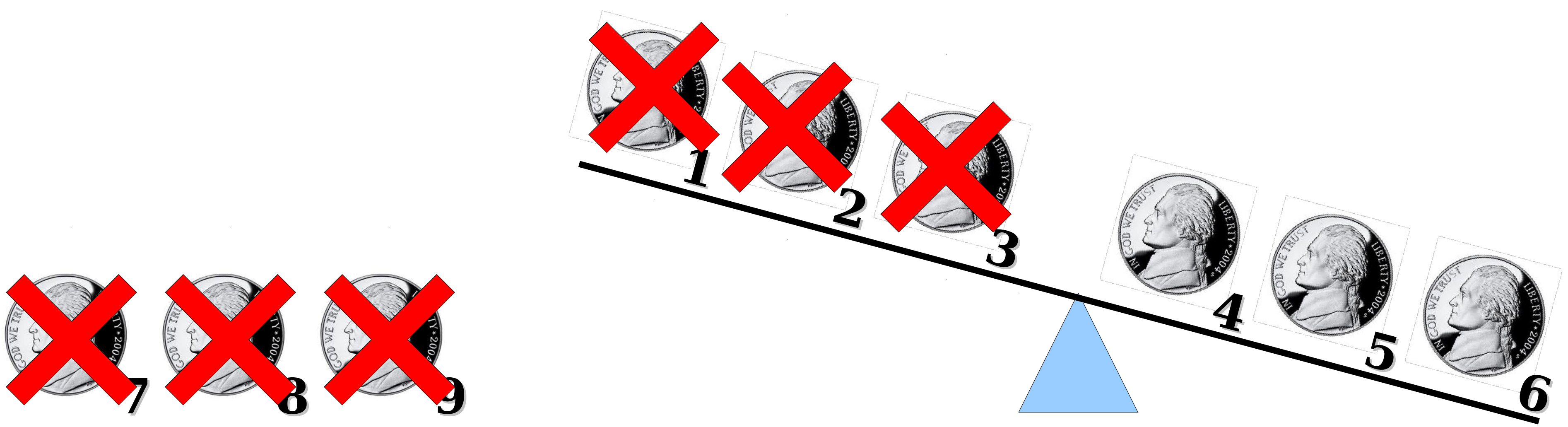

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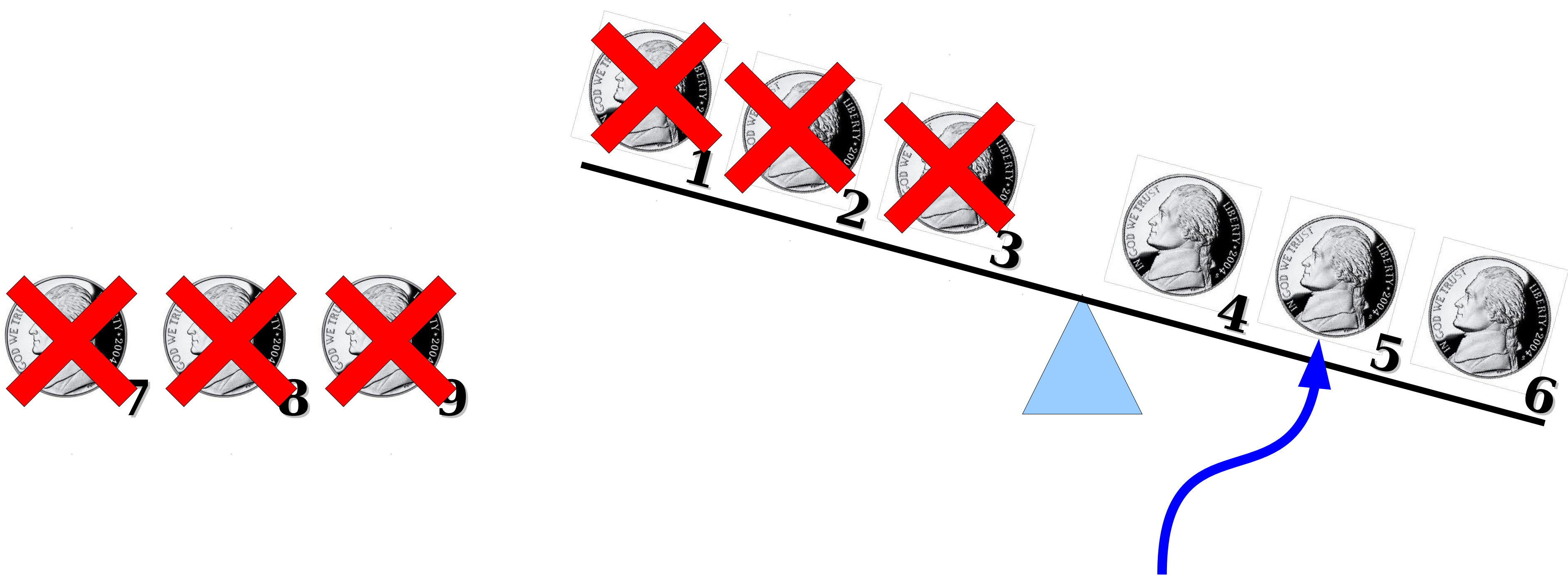

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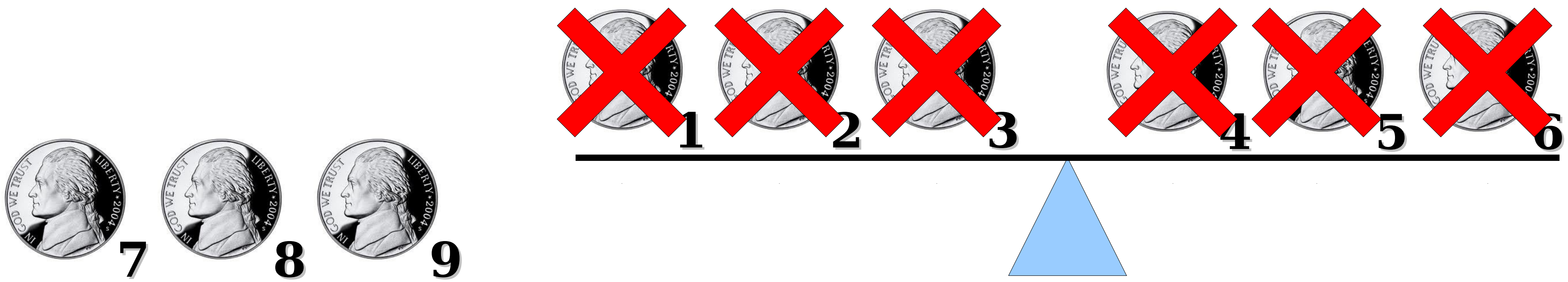

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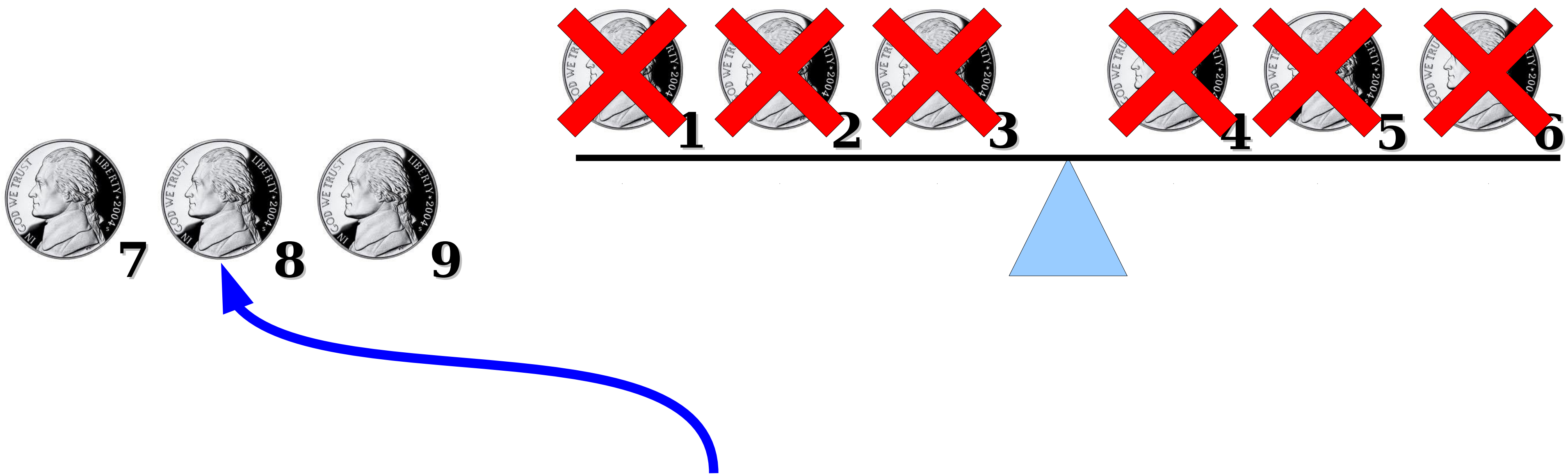

Now we have one weighing to find the counterfeit out of these three

If we have n weighings on the scale, what is the largest number of coins out of which we can find the counterfeit?

A Pattern

- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - **One coin**, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of **three** coins.
- If we have two weighings, we can find the counterfeit out of **nine** coins.

So far, we have

$$\mathbf{1, 3, 9 = 3^0, 3^1, 3^2}$$

Does this pattern continue?

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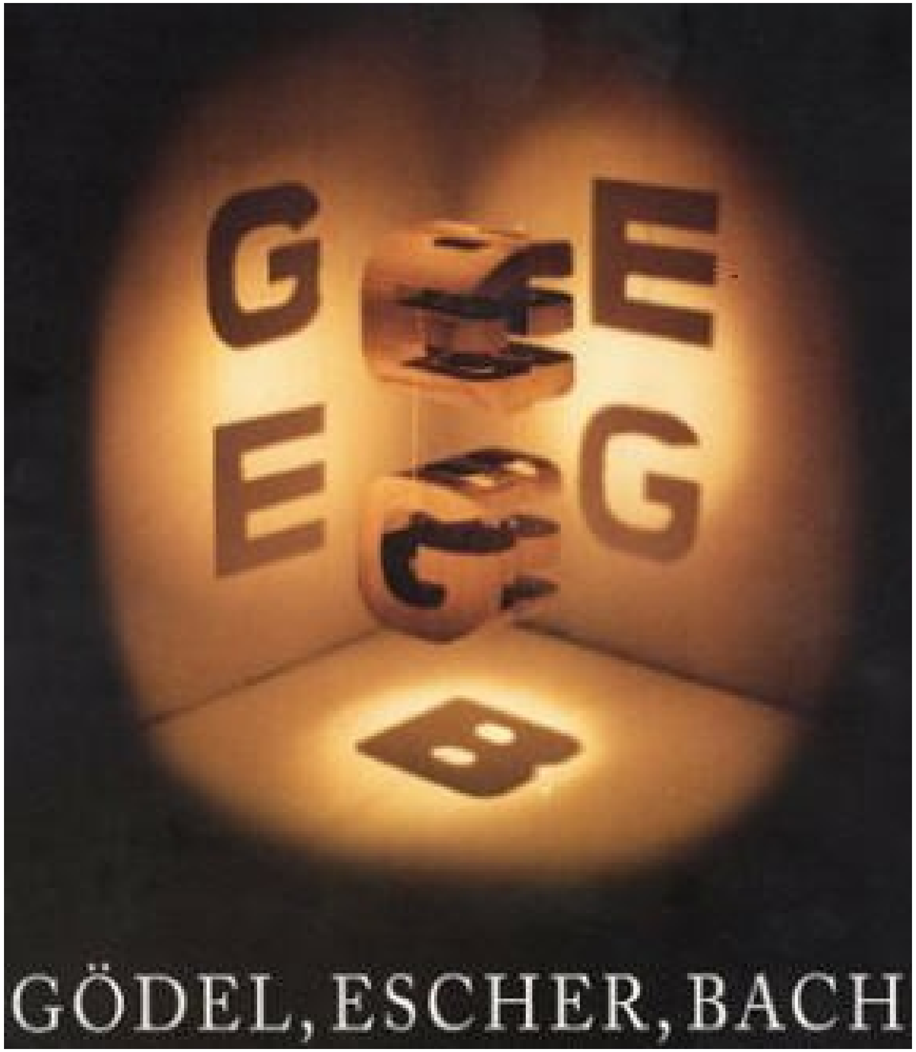
The **MU** Puzzle

Gödel, Escher Bach: An Eternal Golden Braid



- Pulitzer-Prize winning book exploring recursion, computability, and consciousness.
- Written by Douglas Hofstadter, computer scientist at Indiana University.
- A great (but dense!) read.

The **`MU`** Puzzle

- Begin with the string **`MI`**.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the **`M`**: for example, **`MIIU`** becomes **`MIIUIIU`** or **`MI`** becomes **`MII`**.
  - Replace **`III`** with **`U`**: **`MIIII`** becomes **`MUI`** or **`MIU`**
  - Append **`U`** to the string if it ends in **`I`**: **`MI`** becomes **`MIU`**
  - Remove any **`UU`**: **`MUUU`** becomes **`MU`**
- **Question**: How do you transform **`MI`** to **`MU`**?

A) Double the contents of the string after `M`.

B) Replace `III` with `U`.

C) Remove `UU`

D) Append `U` if the string ends in `I`.

`MI`

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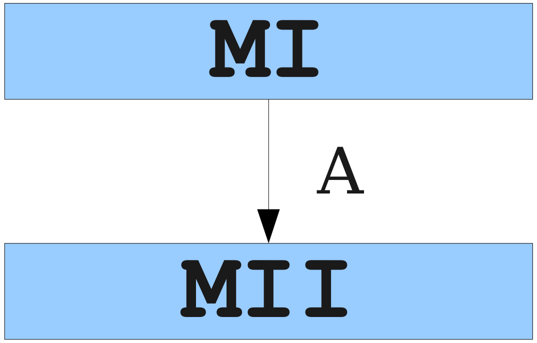

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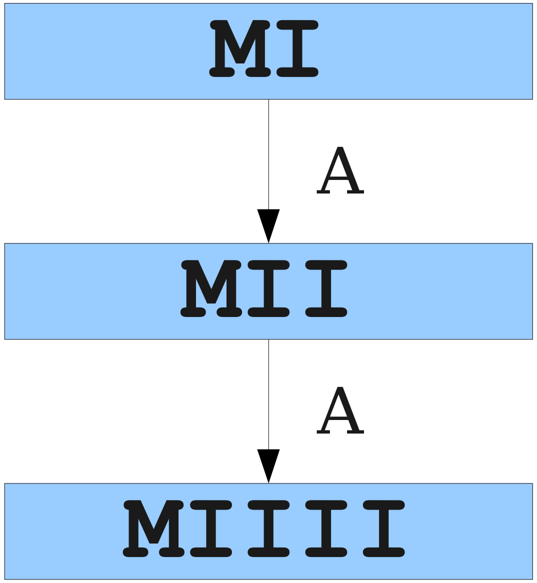

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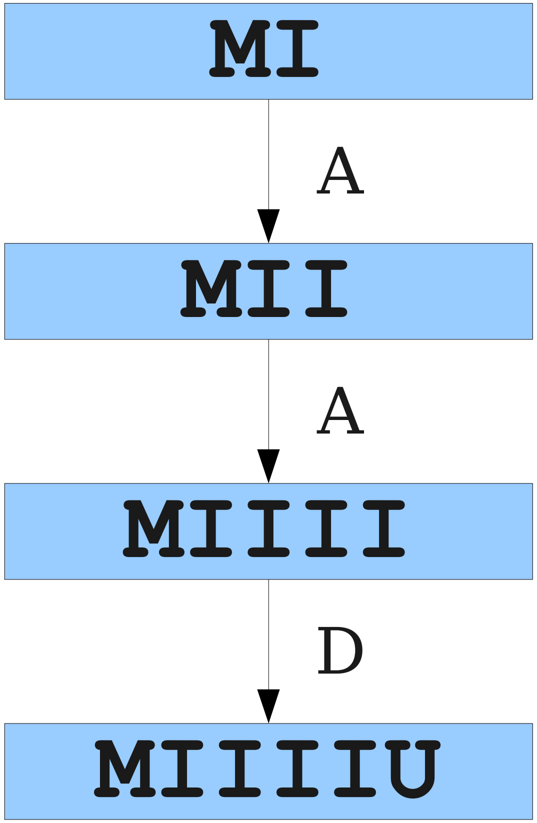

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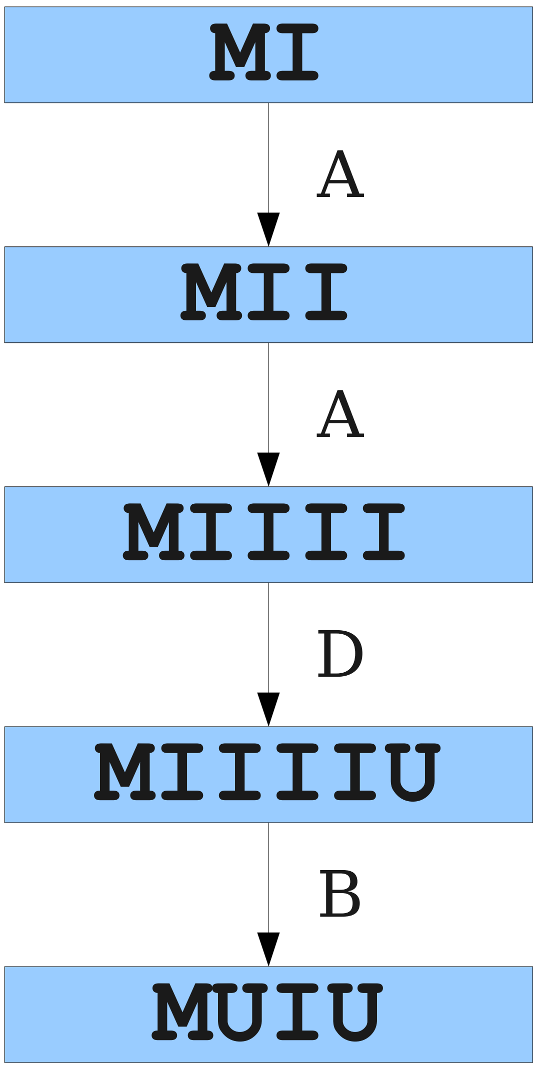

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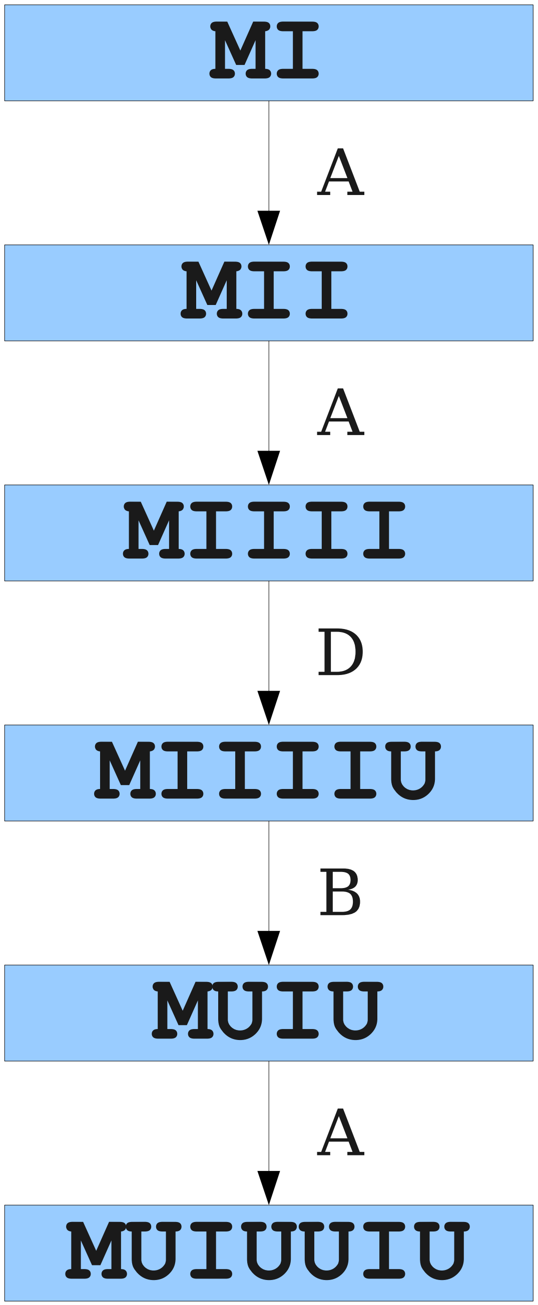

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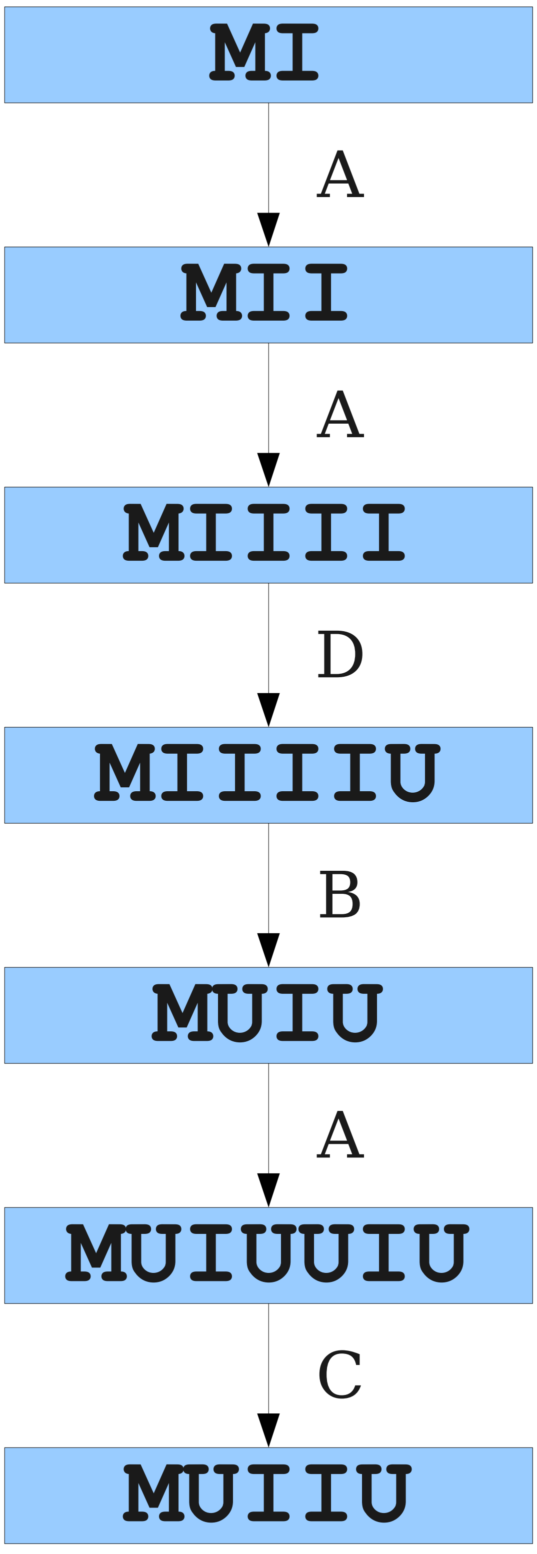

Try It!

Starting with **`MI`**, apply these operations to make **`MU`**:

A) Double the contents of the string after **`M`**.

B) Replace **`III`** with **`U`**.

C) Remove **`UU`**

D) Append **`U`** if the string ends in **`I`**.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting `I`'s

Counting **I**'s

MI

↓

MII

↓

MIIII

↓

MIIIIU

↓

MIIIIUIIIIU

↓

MIIIIUUIU

↓

MIIIIUUIUIIIIUUIU

↓

MUIUUIUIIIIUUIU

Counting **I**'s

MI	1
MII	2
MIIII	4
MIIIIU	4
MIIIIUIIIIU	8
MIIIIUUIU	5
MIIIIUUIUIIIIUUIU	10
MUIUUIUIIIIUUIU	7

Counting **I**'s

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MUIUUIUIIIIUUIU	7

None of these are multiples of three...

The Key Insight

- Initially, the number of `I`'s is **not** a multiple of three.
- To make `MU`, the number of `I`'s must end up as a multiple of three.
- Can we *ever* make the number of `I`'s a multiple of three?

Lemma: Beginning with **`MI`** and applying any legal sequence of moves, the number of **`I`**'s is never a multiple of 3.

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Thus any sequence of $n + 1$ moves starting with **MI** ends with the number of **I**'s not a multiple of three.

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Thus any sequence of $n + 1$ moves starting with **`MI`** ends with the number of **`I`**'s not a multiple of three. Thus $P(n + 1)$ holds, completing the induction.

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Theorem: The **`MU`** puzzle has no solution.

Proof: By contradiction; assume it has a solution. By our lemma, the number of **`I`**'s in the final string must not be a multiple of 3. However, for the solution to be valid, the number of **`I`**'s must be 0, which is a multiple of 3. We have reached a contradiction, so our assumption was wrong and the MU puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
  - "If P is true before we perform an action, it is true after we perform an action."
- We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.
- In algorithmic analysis, this is called a **loop invariant**.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

Mathematical Induction

Part Two

The **principle of mathematical induction** states that if for some property $P(n)$, we have that

If it starts …

$\boldsymbol{P(0)}$ **is true**

… and it keeps going …

and

For any $\boldsymbol{n \in \mathbb{N}}$**, we have** $\boldsymbol{P(n) \rightarrow P(n+1)}$

Then

… then it's always true.

For any $\boldsymbol{n \in \mathbb{N}}$**,** $\boldsymbol{P(n)}$ **is true.**

Theorem: For any natural number n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: By induction. Let $P(n)$ be

$$P(n) \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

For our base case, we need to show $P(0)$ is true, meaning that

$$\sum_{i=1}^{0} i = \frac{0(0+1)}{2}$$

Since the empty sum is defined to be 0, this claim is true.

For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, so

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We need to show that $P(n+1)$ holds, meaning that

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

To see this, note that

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Thus $P(n+1)$ is true, completing the induction. ■

Induction in Practice

- Typically, a proof by induction will not explicitly state $P(n)$.
- Rather, the proof will describe $P(n)$ implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
  - what $P(n)$ is,
  - that $P(0)$ is true, and that
  - whenever $P(n)$ is true, $P(n + 1)$ is true,

  the proof is usually valid.

Theorem: For any natural number n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: By induction on n. For our base case, if $n = 0$, note that

$$\sum_{i=1}^{0} i = \frac{0(0+1)}{2} = 0$$

and the theorem is true for 0.

For the inductive step, assume that for some n the theorem is true. Then we have that

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

so the theorem is true for $n + 1$, completing the induction. ■

A Variant of Induction

n^2 versus 2^n

$0^2 = 0$	$2^0 = 1$
$1^2 = 1$	$2^1 = 2$
$2^2 = 4$	$2^2 = 4$
$3^2 = 9$	$2^3 = 8$
$4^2 = 16$	$2^4 = 16$
$5^2 = 25$	$2^5 = 32$
$6^2 = 36$	$2^6 = 64$
$7^2 = 49$	$2^7 = 128$
$8^2 = 64$	$2^8 = 256$
$9^2 = 81$	$2^9 = 512$
$10^2 = 100$	$2^{10} = 1024$

n^2 versus 2^n

$0^2 = 0$	$<$	$2^0 = 1$
$1^2 = 1$	$<$	$2^1 = 2$
$2^2 = 4$	$=$	$2^2 = 4$
$3^2 = 9$	$>$	$2^3 = 8$
$4^2 = 16$	$=$	$2^4 = 16$
$5^2 = 25$	$<$	$2^5 = 32$
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$3^2 = 9$	$>$	$2^3 = 8$
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2^n is <u>much</u> bigger here. Does the trend continue?

Theorem: For any natural number $n \geq 5$, $n^2 < 2^n$.

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$$(n + 1)^2 = n^2 + 2n + 1$$

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$(n + 1)^2 = n$

$< n$ ... $5 \leq n$)

$= n$ ... $3n$

$$< n^2 + n^2 \qquad \text{(since } 3n < 5n \leq n^2\text{)}$$

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Why is this allowed?

So $(n + 1)^2 < 2n^2$. Now, by our inductive hypothesis, we know that $n^2 < 2^n$. This means that

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Remember: $A \rightarrow B$ means "whenever A is true, B is true." If B is always true, $A \rightarrow B$ is true for any A.

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- We explicitly proved $P(5)$, so $P(4) \to P(5)$

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Again, $A \to B$ is automatically true
if B is always true.

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- $P(3)$ is trivially true, so $P(2) \to P(3)$
- $P(4)$ is trivially true, so $P(3) \to P(4)$
- We explicitly proved $P(5)$, so $P(4) \to P(5)$
- For any $n \geq 5$, we explicitly proved that $P(n) \to P(n + 1)$.

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- $P(4)$ is trivially true, so $P(3) \to P(4)$
- We explicitly proved $P(5)$, so $P(4) \to P(5)$
- For any $n \geq 5$, we explicitly proved that $P(n) \to P(n + 1)$.
- Thus $P(0)$ and for any $n \in \mathbb{N}$, $P(n) \to P(n + 1)$, so by induction $P(n)$ is true for all natural numbers n.

Induction Starting at k

- To prove that $P(n)$ is true for all natural numbers greater than or equal to k:
  - Show that $P(k)$ is true.
  - Show that for any $n \geq k$, that $P(n) \to P(n + 1)$.
  - Conclude $P(k)$ holds for all natural numbers greater than or equal to k.
- You don't need to justify why it's okay to start from k.

An Important Observation

One Major Catch

0	1	2	3	4	5	6	7	8

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One Major Catch

0	1	2	3	4	5	6	7	8

In an inductive proof, to prove P(5), we can only assume P(4). We cannot rely on any of our earlier results!

Strong Induction

The **principle of strong induction** states that if for some property $P(n)$, we have that

$P(0)$ is true

and

For any $n \in \mathbb{N}$ with $n \neq 0$, if $P(n')$ is true for all $n' < n$, then $P(n)$ is true

then

For any $n \in \mathbb{N}$, $P(n)$ is true.

The **principle of strong induction** states that if for some property $P(n)$, we have that

$P(0)$ is true

Assume that P(n) holds for all natural numbers smaller than n.

and

For any $n \in \mathbb{N}$ with $n \neq 0$, if $P(n')$ is true for all $n' < n$, then $P(n)$ is true

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For any $n \in \mathbb{N}$, $P(n)$ is true.

Using Strong Induction

0	1	2	3	4	5	6	7	8

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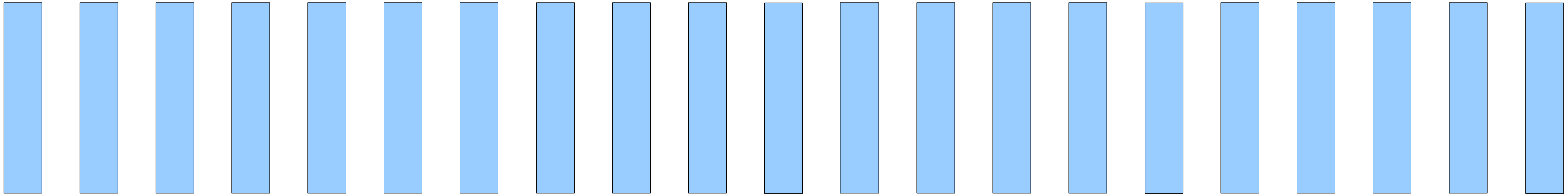
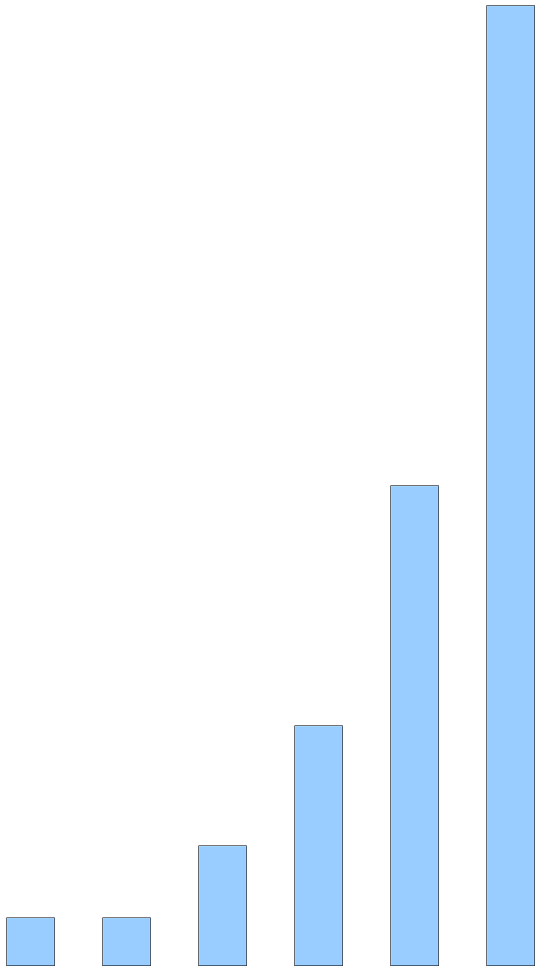
Induction and Dominoes



Strong Induction and Dominoes



Weak and Strong Induction

- **Weak induction** (regular induction) is good for showing that some property holds by incrementally adding in one new piece.
- **Strong induction** is good for showing that some property holds by breaking a large structure down into multiple small pieces.

Proof by Strong Induction

- State that you are attempting to prove something by strong induction.
- State what your choice of $P(n)$ is.
- Prove the base case:
  - State what $P(0)$ is, then prove it.
- Prove the inductive step:
  - State that you assume for all $0 \leq n' < n$, that $P(n')$ is true.
  - State what $P(n)$ is. *(this is what you're trying to prove)*
  - Go prove $P(n)$.

Application: **Binary Numbers**

Binary Numbers

- The **binary number system** is base 2.
- Every number is represented as 1s and 0s encoding various powers of two.
- Examples:
  - $100_2 = 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 4$
  - $11011_2 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 27$
- Enormously useful in computing; almost all computers do computation on binary numbers.
- Question: How do we know that every natural number can be written in binary?

Justifying Binary Numbers

- To justify the binary representation, we will prove the following result:

Every natural number *n*
can be expressed as the sum
of distinct powers of two.

- This says that there's *at least* one way to write a number in binary; we'd need a separate proof to show that there's *exactly* one way to do it.
- So how do we prove this?

One Proof Idea

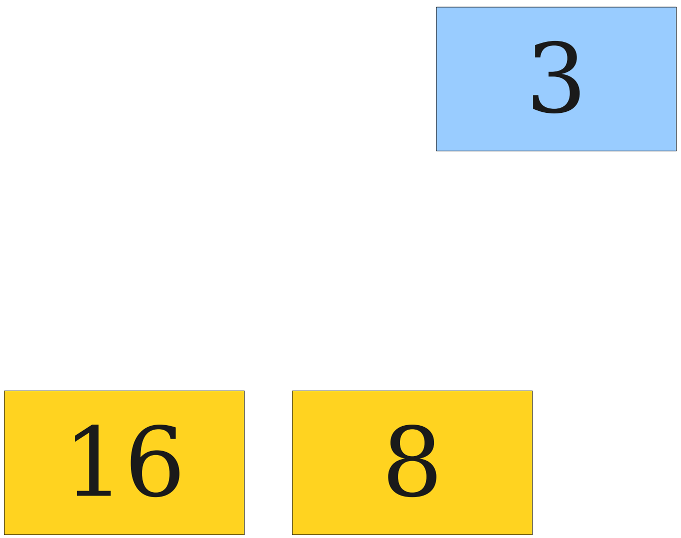
27

One Proof Idea

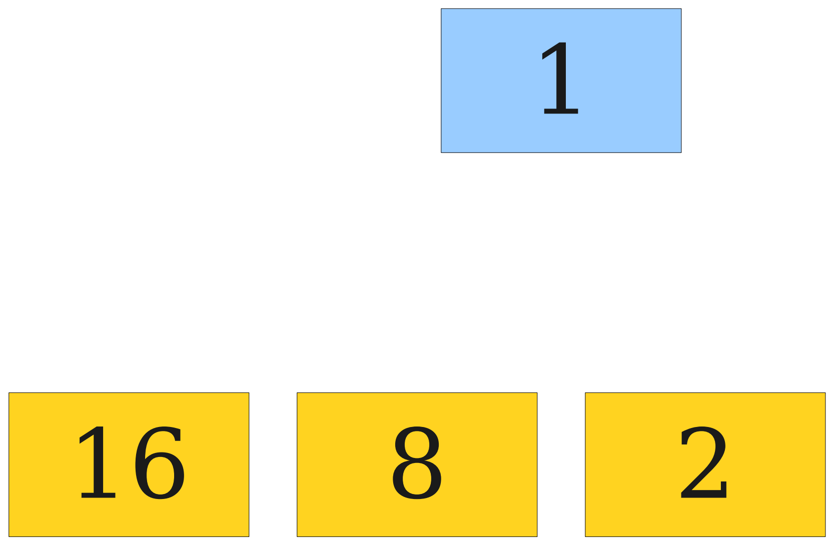
11

16

One Proof Idea



One Proof Idea



One Proof Idea

0

16	8	2	1

General Idea

- Repeatedly subtract out the largest power of two less than the number.
- Can't subtract 2^n twice for any n; otherwise, you could have subtracted 2^{n+1}.
- Eventually, we reach 0; the number is then the sum of the powers of two that we subtracted.
- How do we formalize this as a proof?

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Notice the stronger version of the induction hypothesis. We're now showing that $\boldsymbol{P(n')}$ is true for all natural numbers in the range $\boldsymbol{0 \leq n' < n}$. We'll use this fact later on.

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Let 2^k be the greatest power of two such that $2^k \leq n$. Consider $n - 2^k$.

Here's the key step of the proof.
If we can show that

$$\mathbf{0 \leq n - 2^k < n}$$

then we can use the inductive hypothesis to claim that $\mathbf{n - 2^k}$ is a sum of distinct powers of two.

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Here is where strong induction kicks in. We use the fact that any smaller number can be written as the sum of distinct powers of two to show that $\boldsymbol{n - 2^k}$ can be written as the sum of distinct powers of two.

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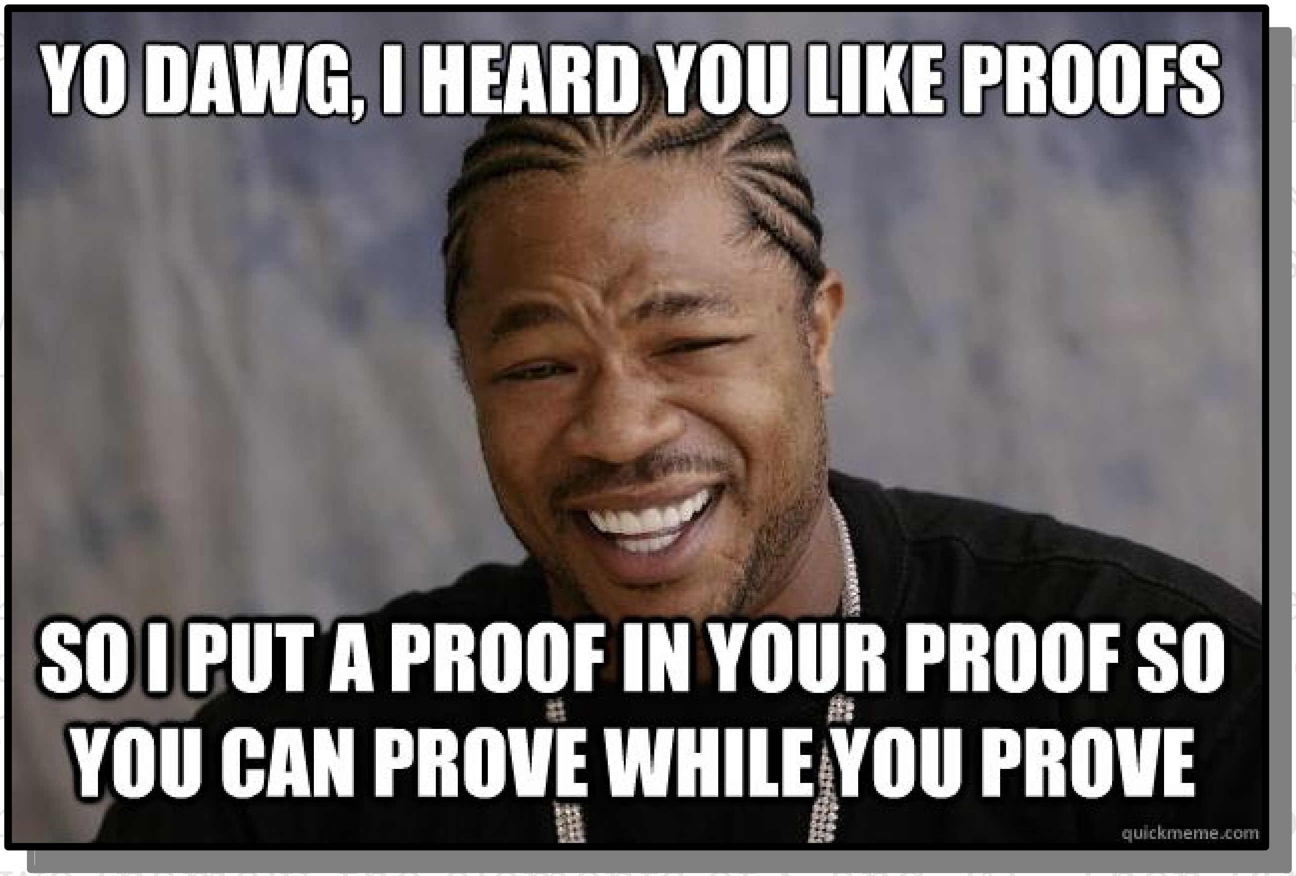

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Application: **Continued Fractions**

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{\boxed{1 + \cfrac{1}{2}}}}$$

Continued Fractions

$$\frac{1}{4 + \cfrac{1}{\cfrac{3}{2}}}$$

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$$\frac{1}{4 + \frac{2}{3}}$$

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$$\cfrac{1}{\boxed{4 + \cfrac{2}{3}}}$$

Continued Fractions

$$\frac{1}{\frac{14}{3}}$$

Continued Fractions

$$\cfrac{1}{\cfrac{14}{3}}$$

Continued Fractions

$$\frac{3}{14}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}$$

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$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\boxed{4 + \cfrac{1}{2}}}}}$$

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$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{9}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{2}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\boxed{1 + \cfrac{2}{9}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{11}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{11}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{9}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{9}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{\cfrac{42}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{\cfrac{42}{11}}$$

Continued Fractions

$$3 + \frac{11}{42}$$

Continued Fractions

$$3 + \frac{11}{42}$$

Continued Fractions

$$\frac{137}{42}$$

Continued Fractions

- A **continued fraction** is an expression of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\dots + \cfrac{1}{a_n}}}}$$

- Formally, a continued fraction is either
  - An integer *n*, or
  - *n* + 1 / *F*, where *n* is an integer and *F* is a continued fraction.
- Continued fractions have numerous applications in number theory and computer science.
- (They're also really fun to write!)

Fun with Continued Fractions

- Every rational number, including negative rational numbers, has a continued fraction representation.
- Harder result: every *irrational* number has an (infinite) continued fraction representation.
- Even harder result: If we truncate an infinite continued fraction for an irrational number, we can get progressively better approximations of that number.

π as a Continued Fraction

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}}}}}}}$$

Approximating π

Approximating π

$\pi = 3$

3 = **3**.0000...

Approximating π

$\pi = 3$

$3 = \mathbf{3}.0000\ldots$

And he made the Sea of cast bronze, ten cubits from one brim to the other; it was completely round. [… A] line of thirty cubits measured its circumference.

1 Kings 7:23, New King James Translation

Approximating π

$$\pi = 3 + \cfrac{1}{7}$$

3 = **3**.0000...

22/7 = **3.14**2857...

Approximating π

$$\pi = 3 + \cfrac{1}{7}$$

3 = **3**.0000...

22/7 = **3.14**2857...

Greek mathematician **Archimedes** knew of this approximation, circa 250 BCE

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

355/113 = **3.141592**92...

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

355/113 = **3.141592**92...

Chinese mathematician 祖沖之 (Zu Chongzhi) discovered this approximation in the early fifth century; this was the best approximation of pi for over a thousand years.

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

3 = **3**.0000...

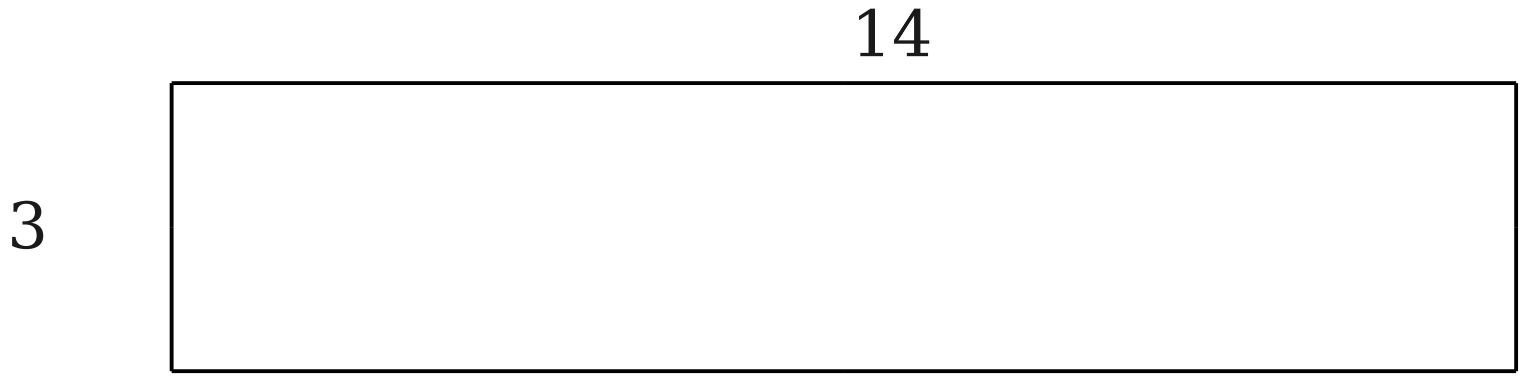
22/7 = **3.14**2857...

336/106 = **3.1415**094...

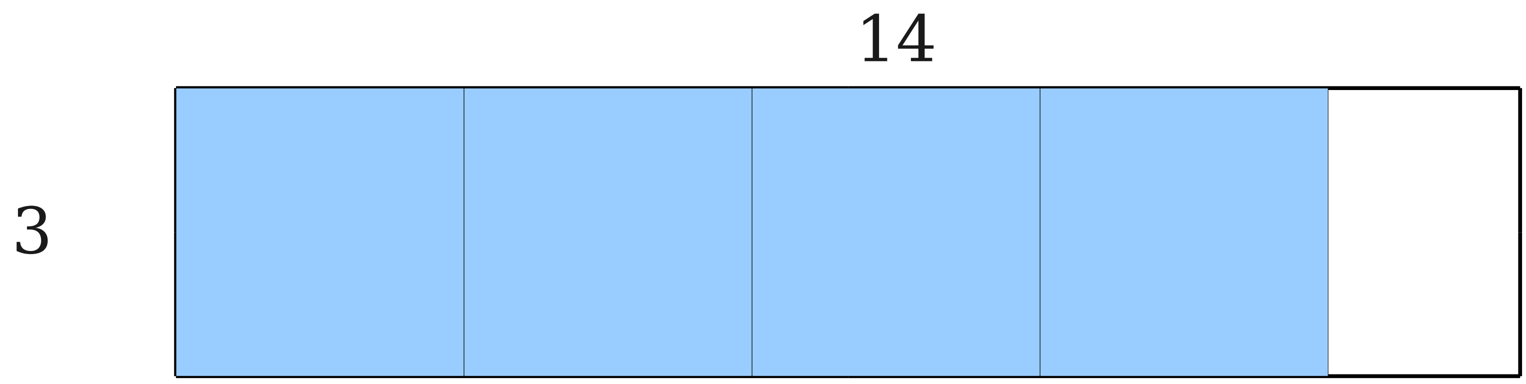
355/113 = **3.141592**92...

103993/33102 = **3.1415926530**...

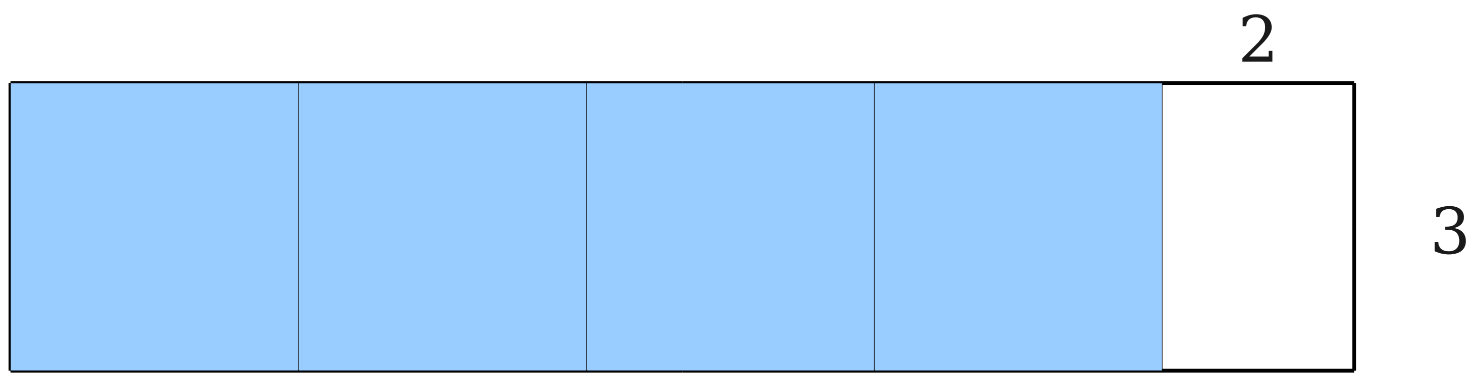
More Continued Fractions



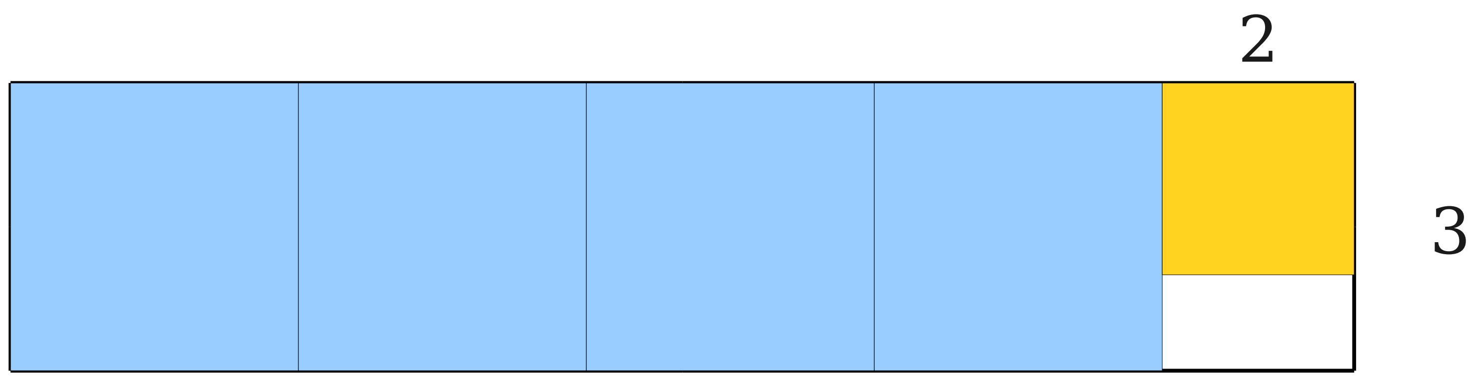
More Continued Fractions



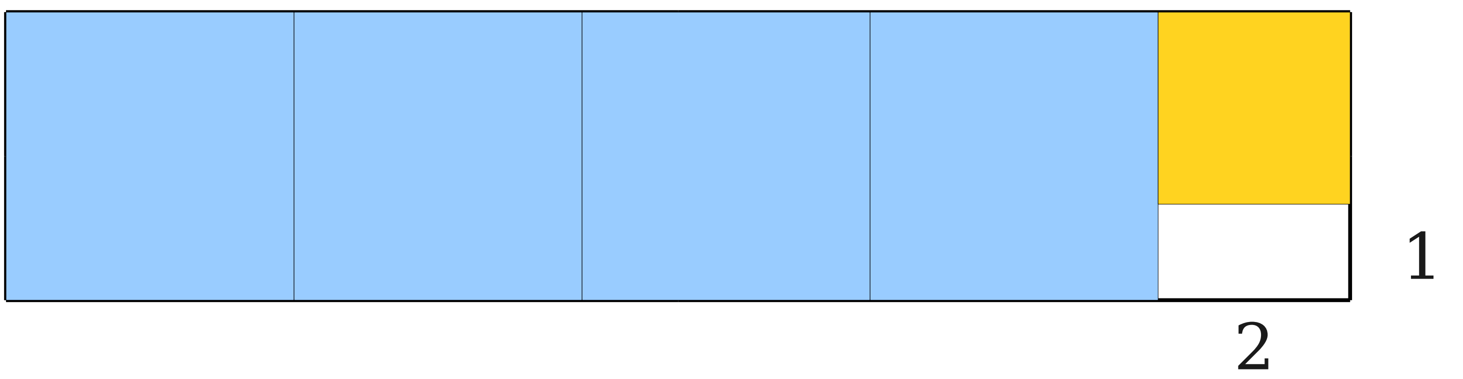
More Continued Fractions



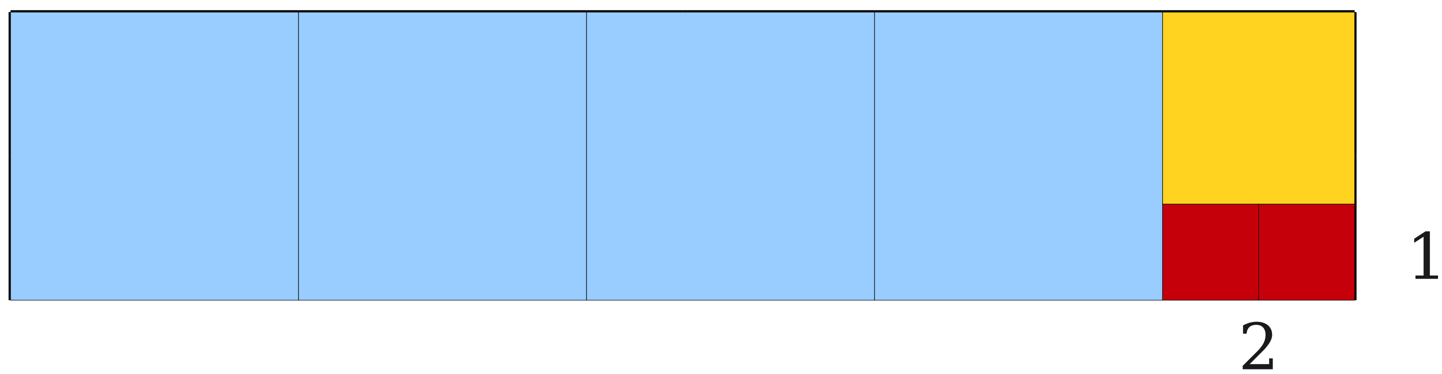
More Continued Fractions



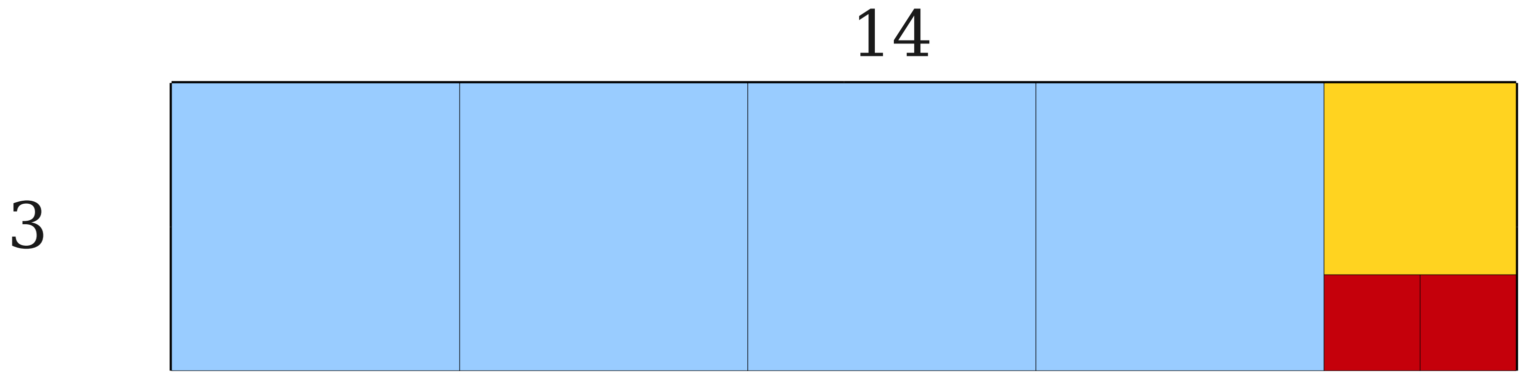
More Continued Fractions



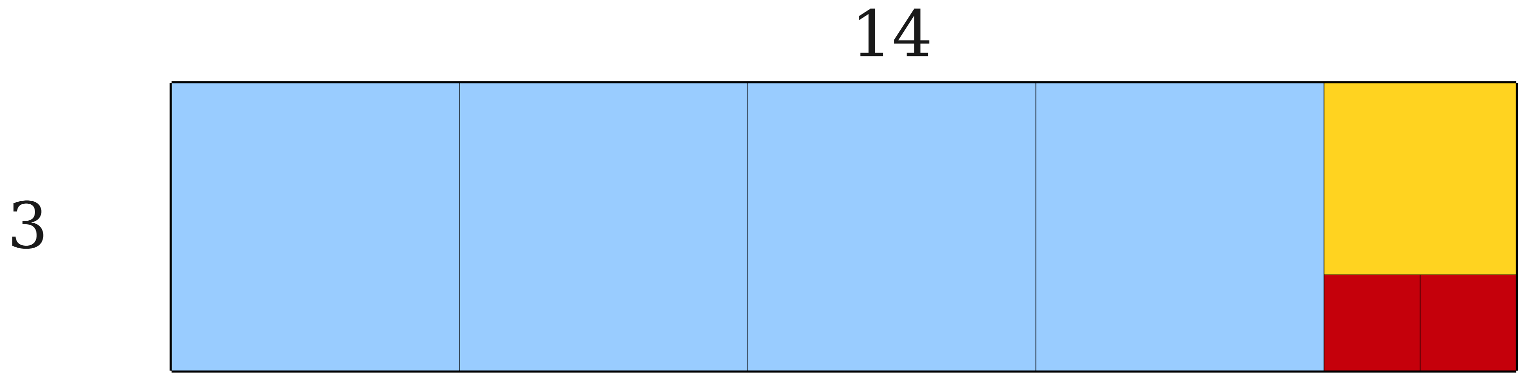
More Continued Fractions



More Continued Fractions

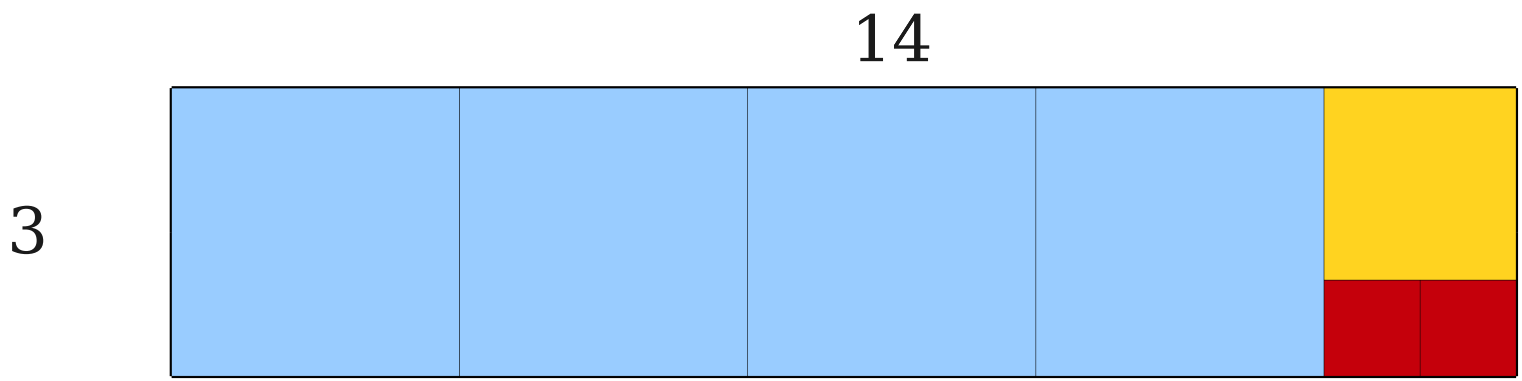


More Continued Fractions



$$\frac{3}{14} = \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

More Continued Fractions

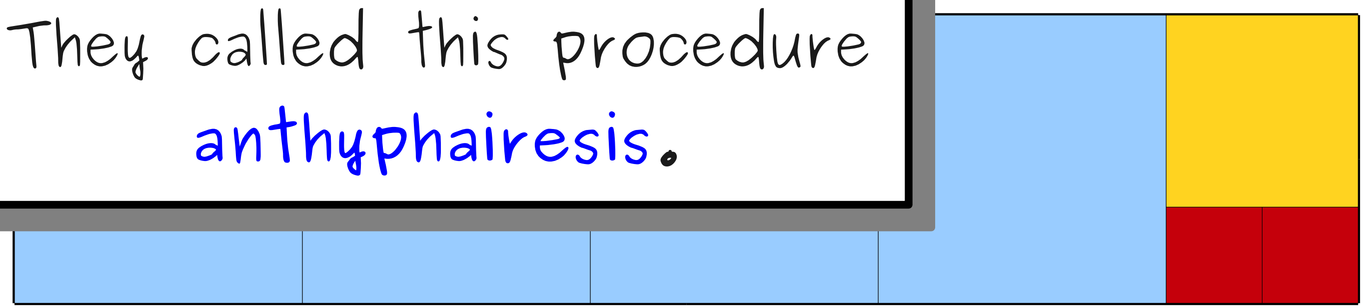


$$\frac{3}{14} = \cfrac{1}{\mathbf{4} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}}}}$$

More Continued Fractions

The Ancient Greeks knew about this connection. They called this procedure anthyphairesis.

3



$$\frac{3}{14} = \cfrac{1}{\mathbf{4} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}}}}$$

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}}}}}$$

An Interesting Continued Fraction

$x = 1$ 1 / 1

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 \cdot}$$

1 / 1

2 / 1

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1}}$$

1 / 1

2 / 1

3 / 2

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

An Interesting Continued Fraction

$$
\begin{array}{ll}
x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}} &
\begin{array}{l}
1/1 \\
2/1 \\
3/2 \\
5/3 \\
8/5
\end{array}
\end{array}
$$

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1.}}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

21 / 13

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}}$$

1 / 1
2 / 1
3 / 2
5 / 3
8 / 5
13 / 8
21 / 13
34 / 21

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}}$$

1 / 1
2 / 1
3 / 2
5 / 3
8 / 5
13 / 8
21 / 13
34 / 21

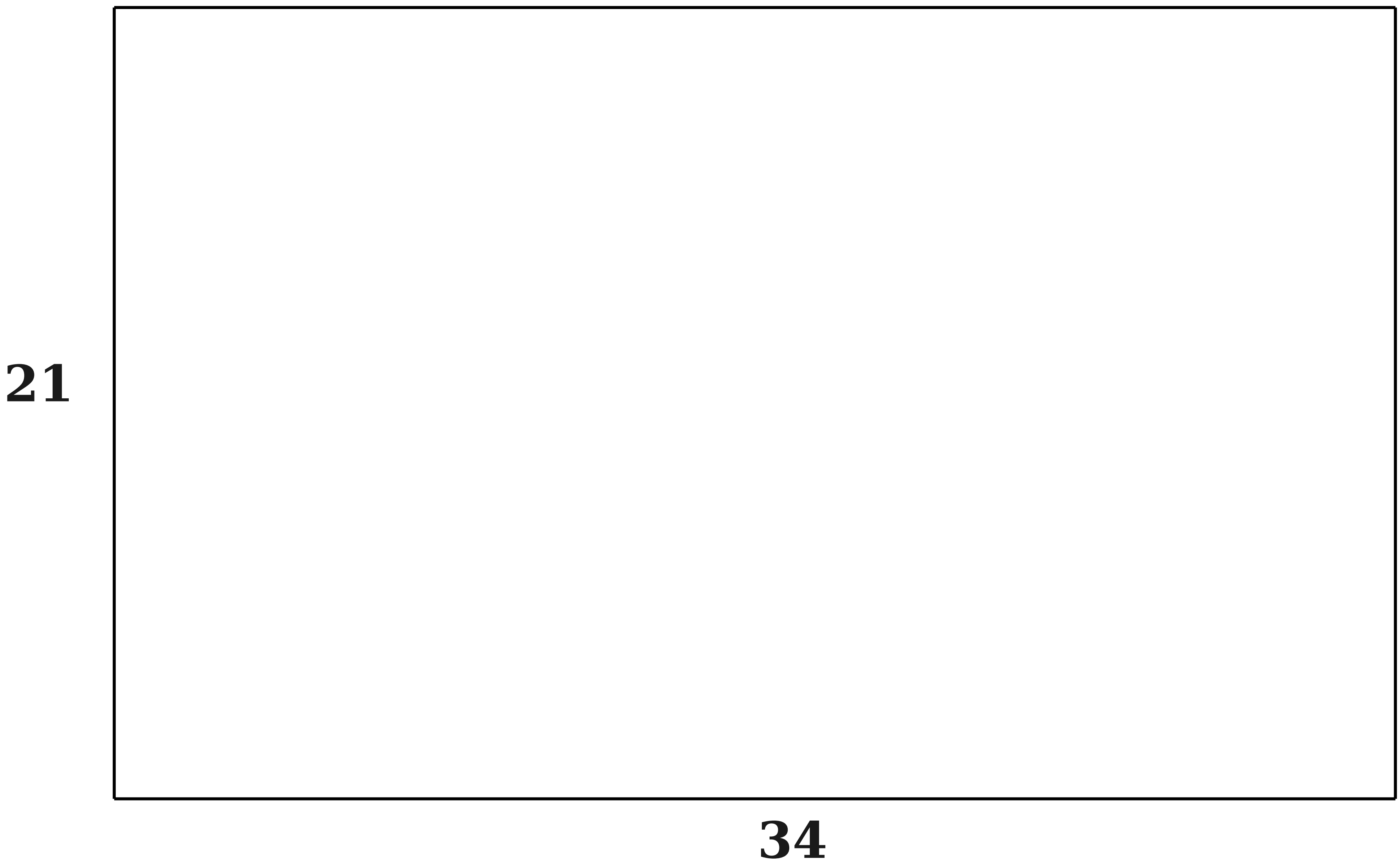
Each fraction is the ratio of consecutive Fibonacci numbers!

The Golden Ratio

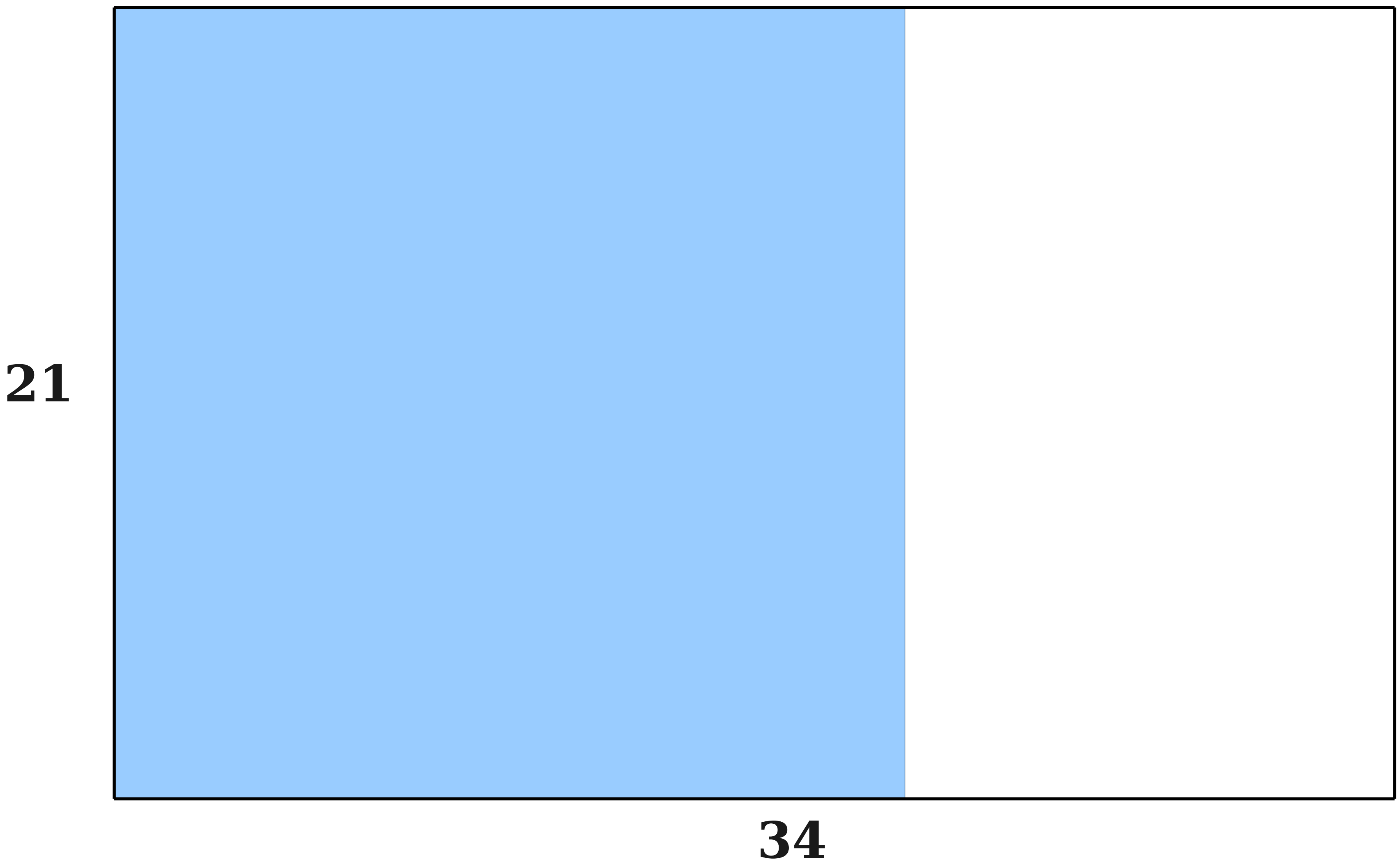
$$\varphi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

$$\varphi \approx 1.61803399$$

The Golden Ratio



The Golden Ratio



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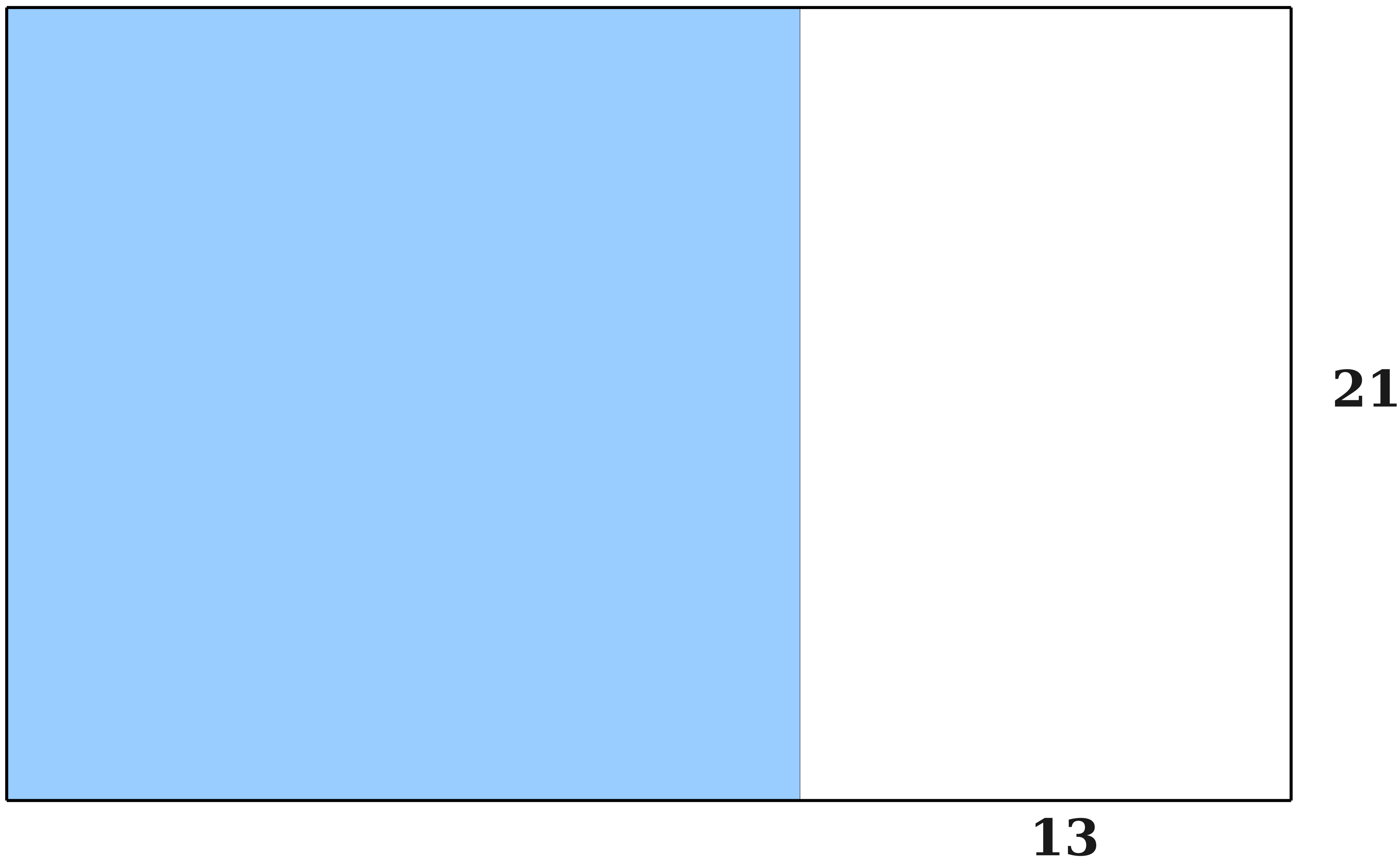

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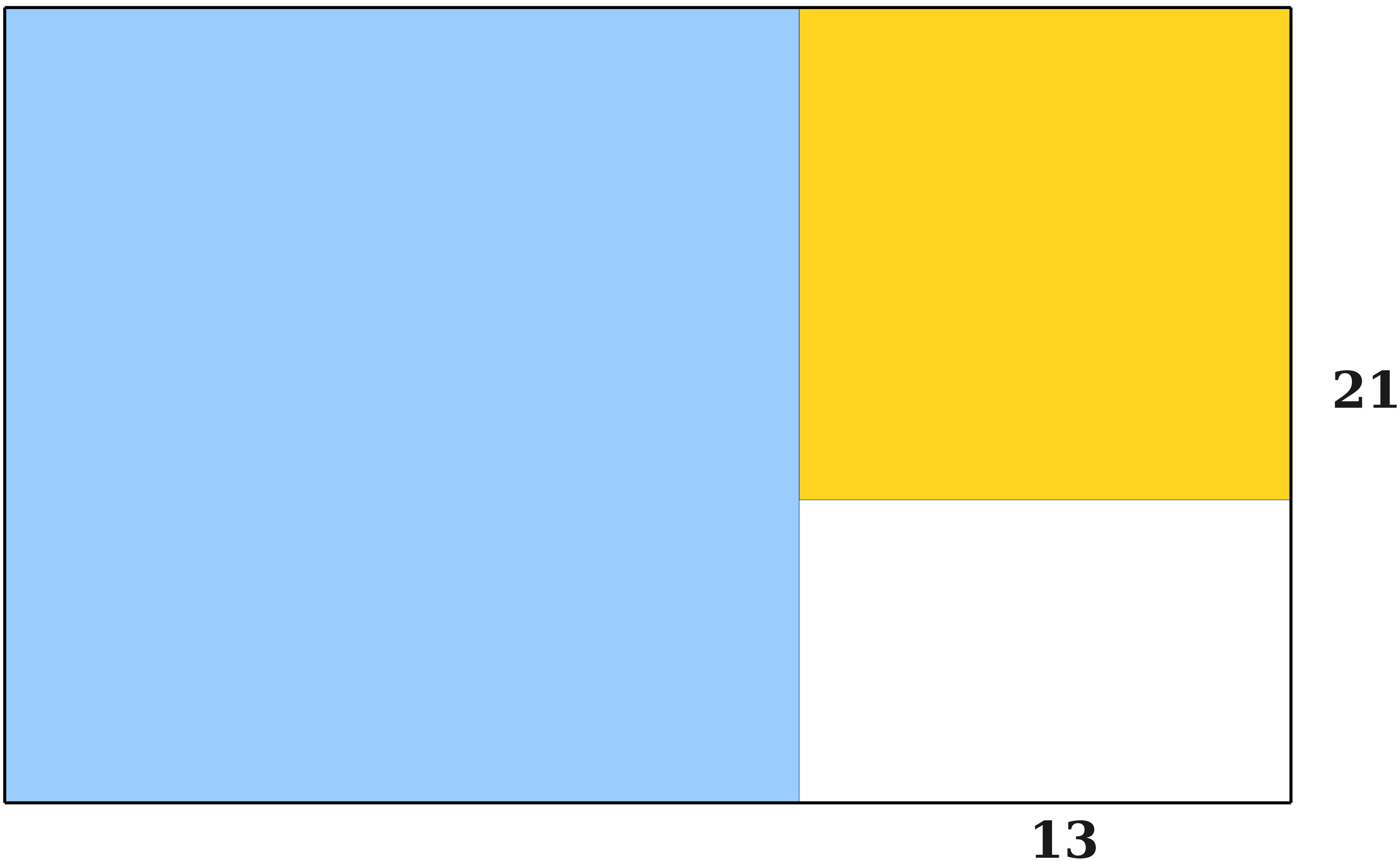

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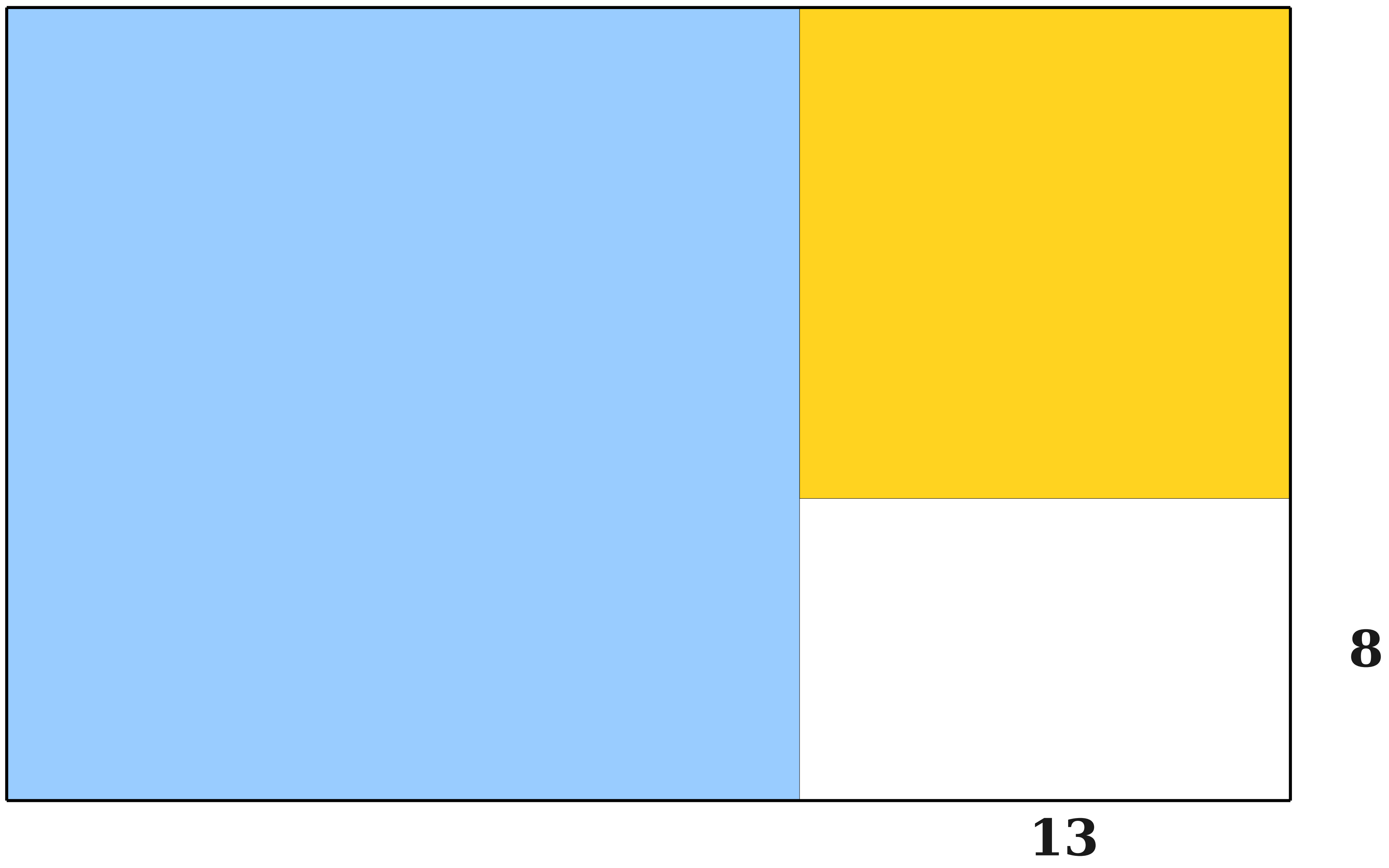

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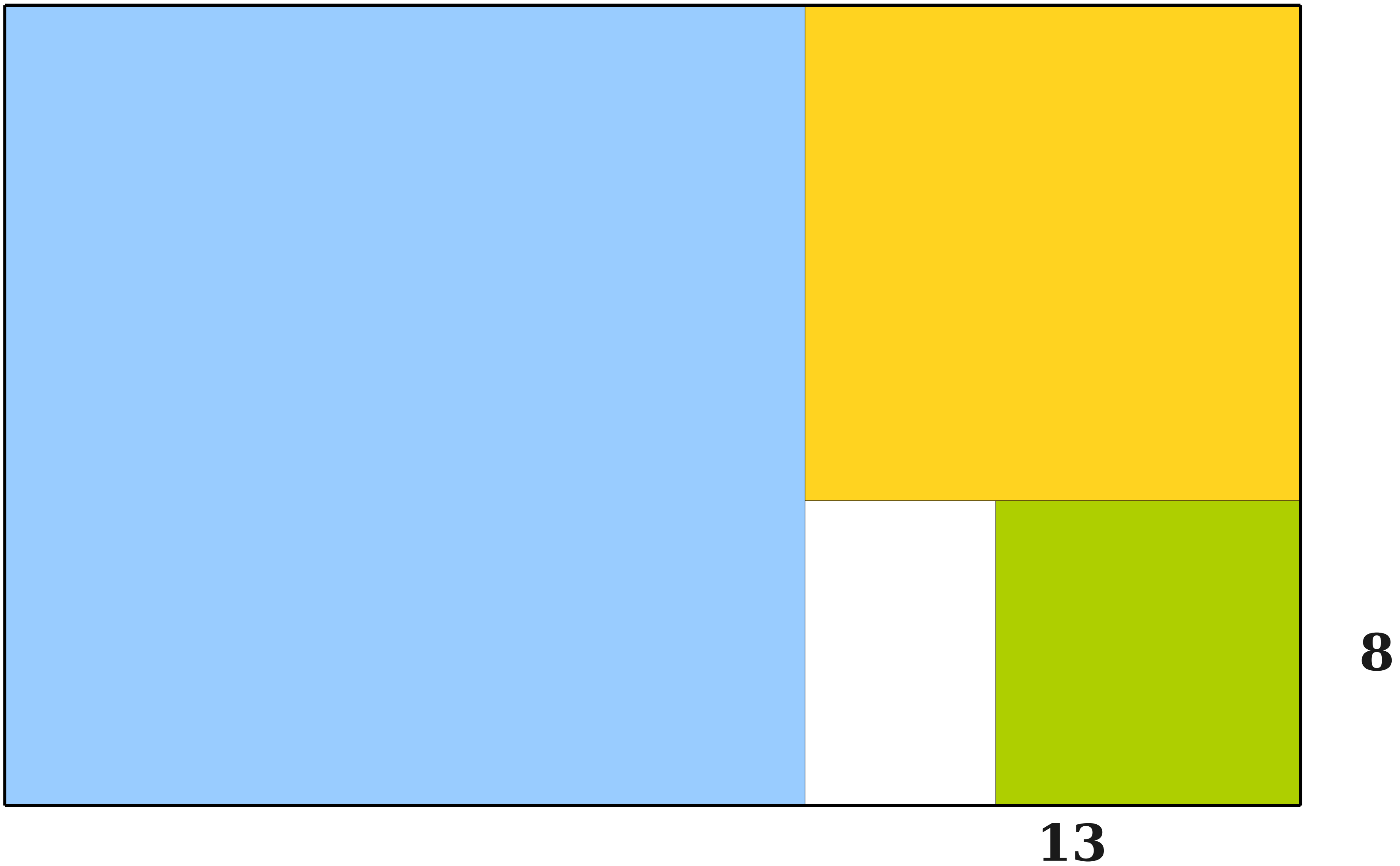

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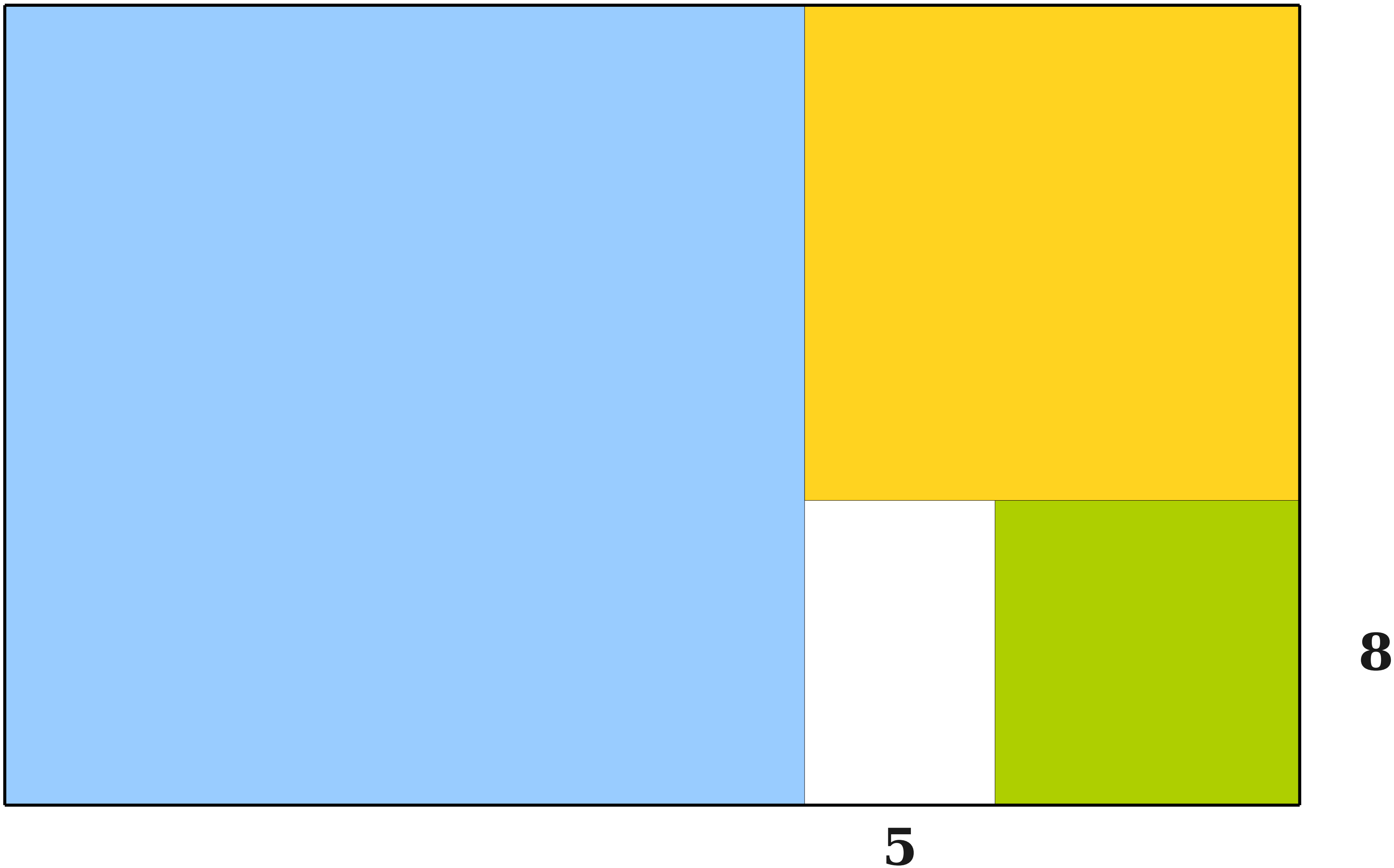
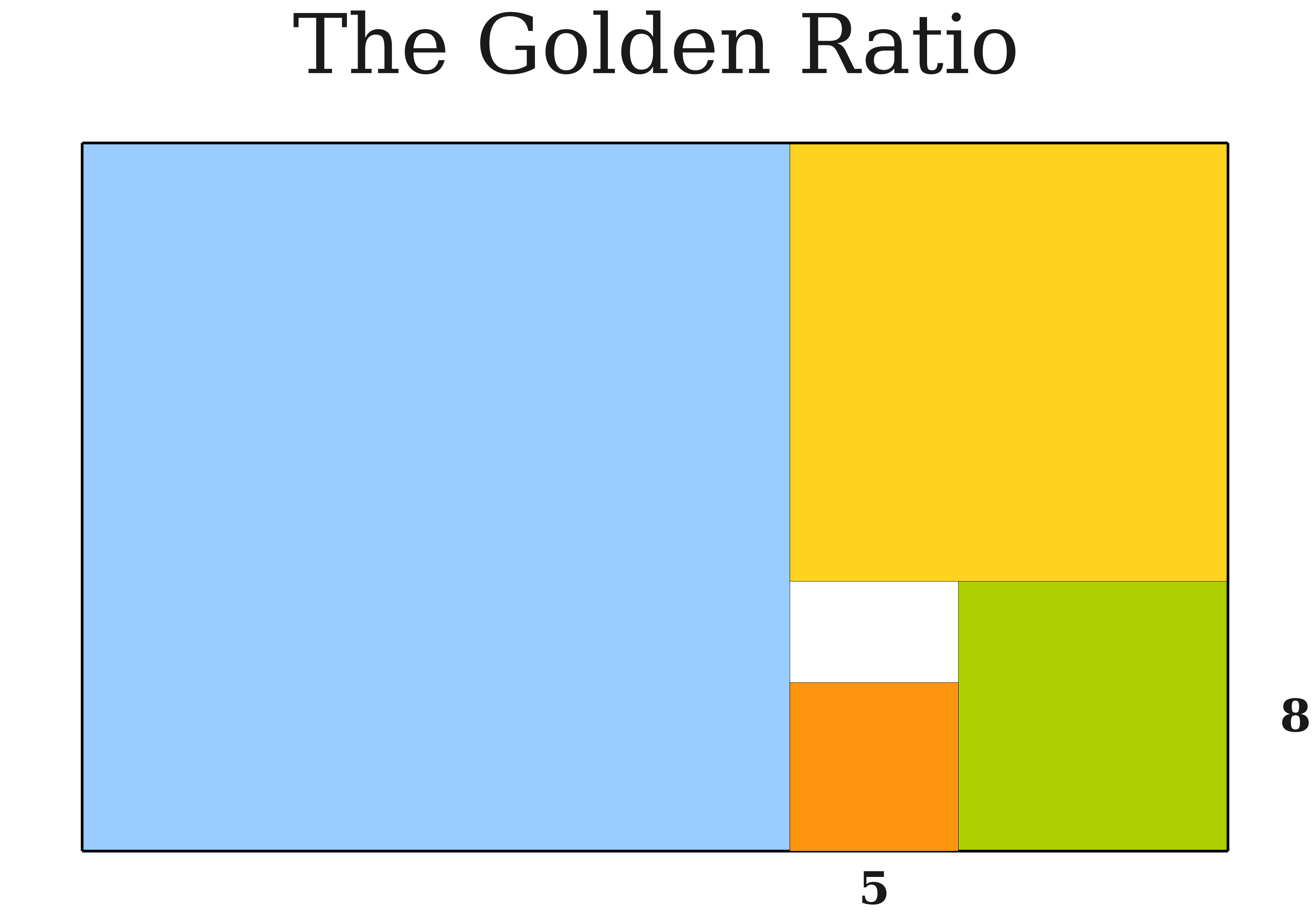
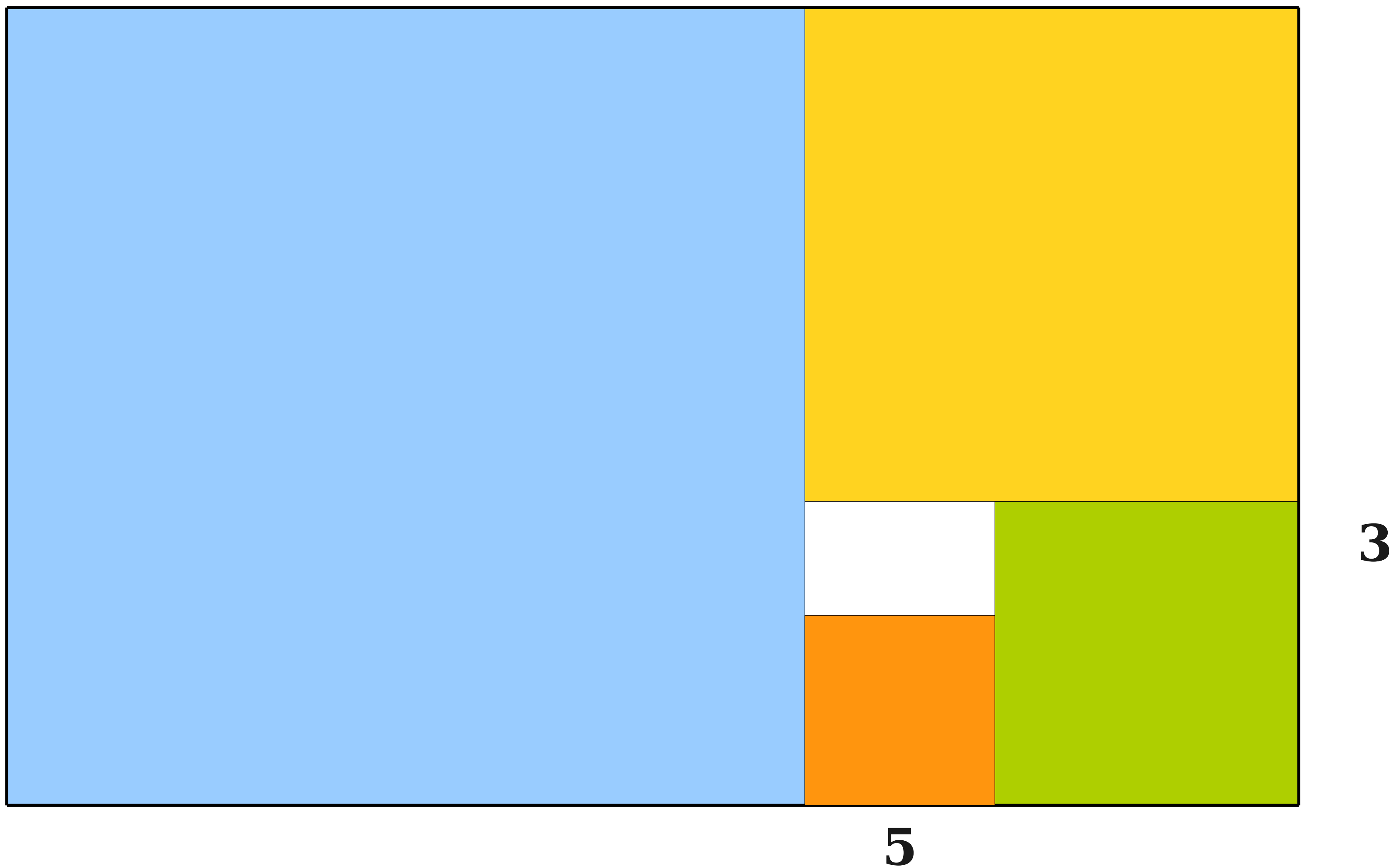



The Golden Ratio
8
5

The Golden Ratio



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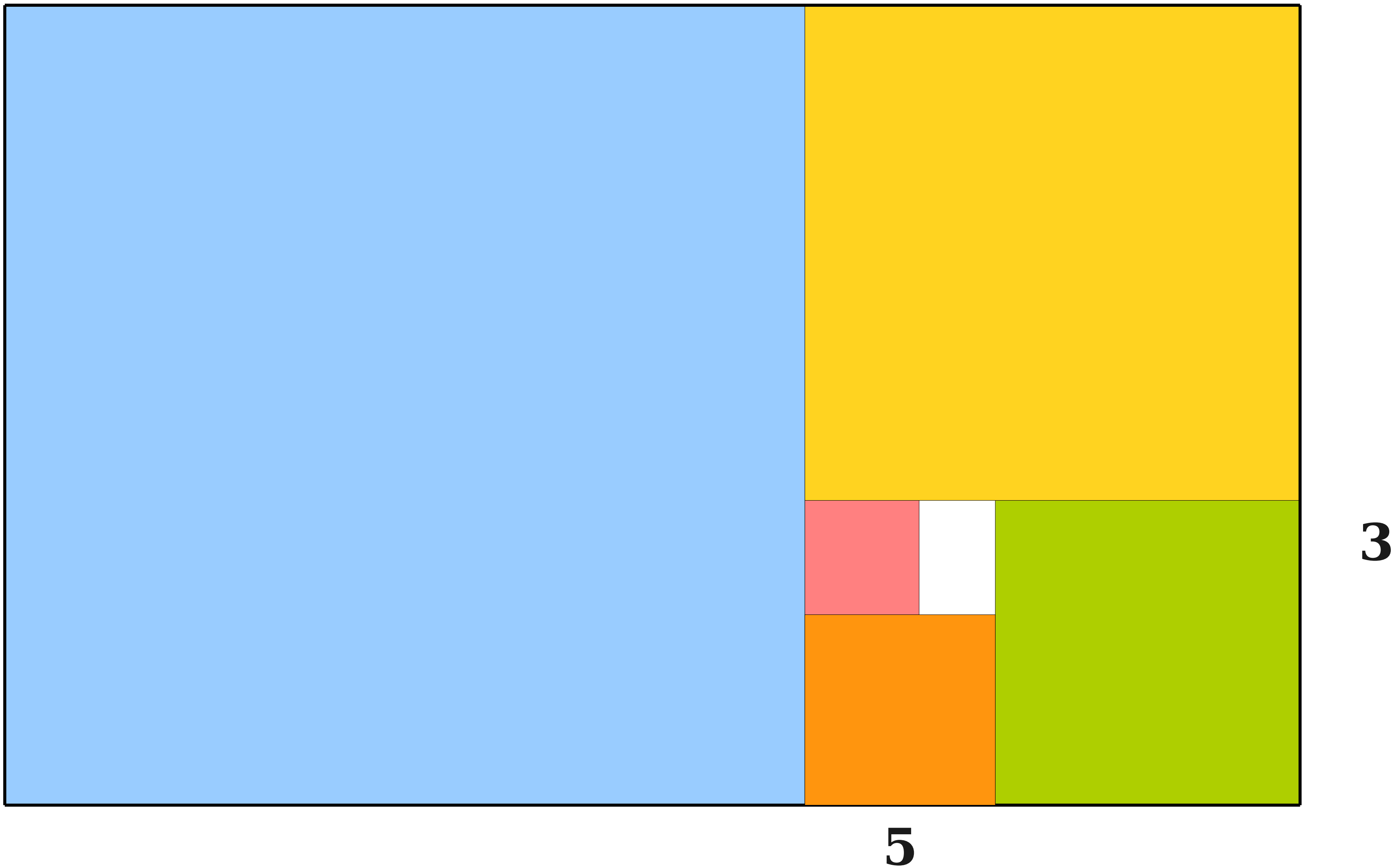

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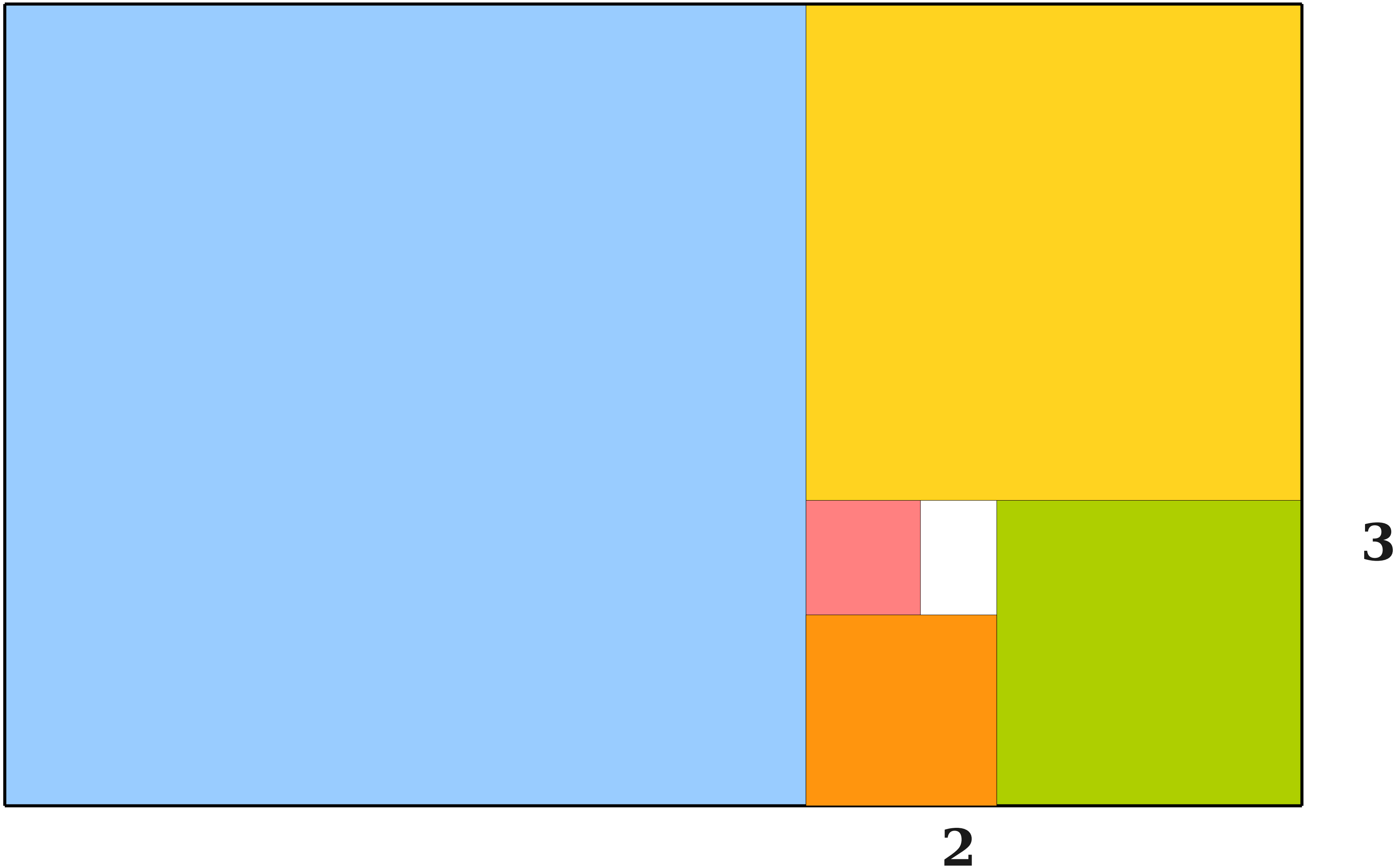

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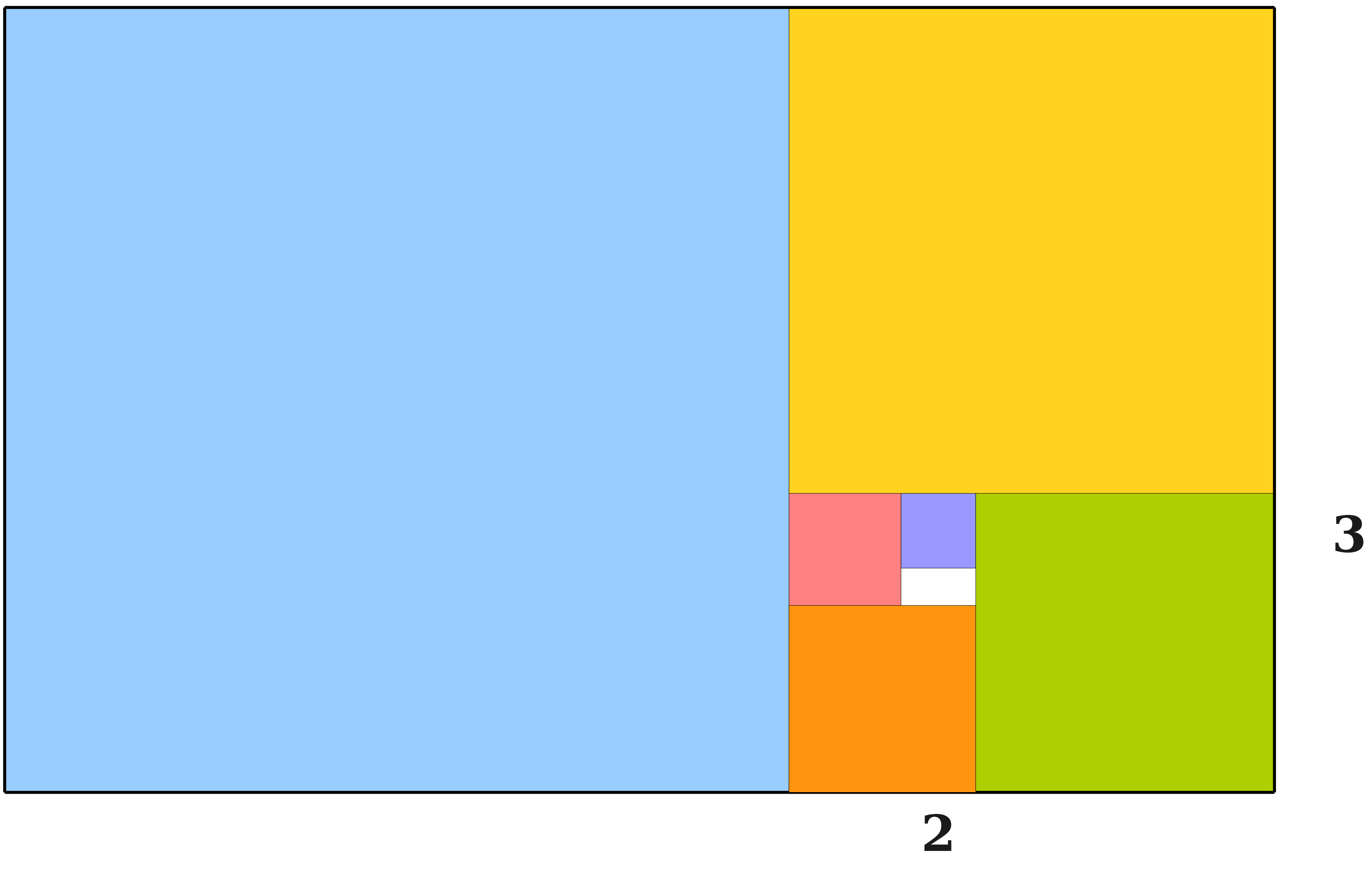

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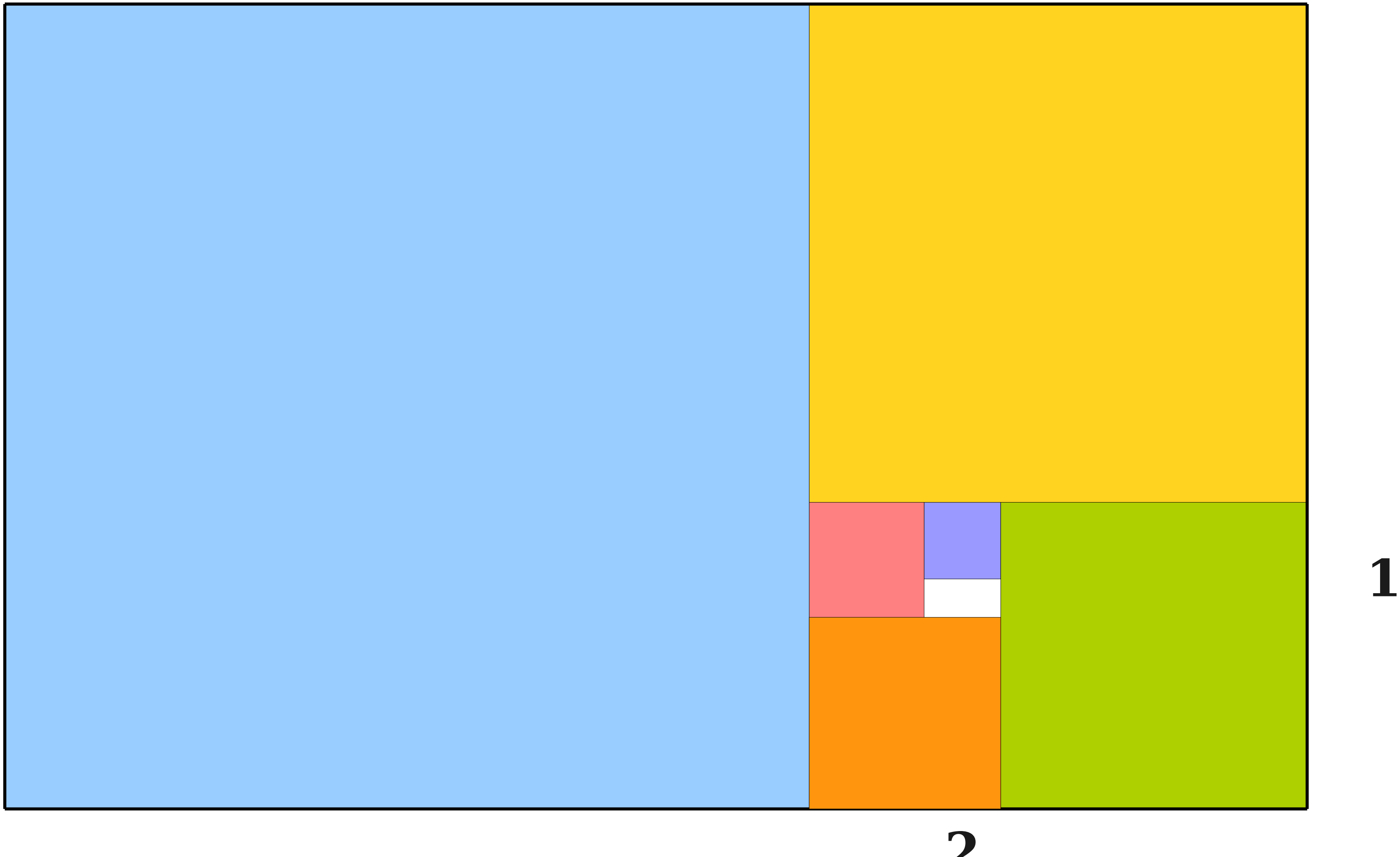

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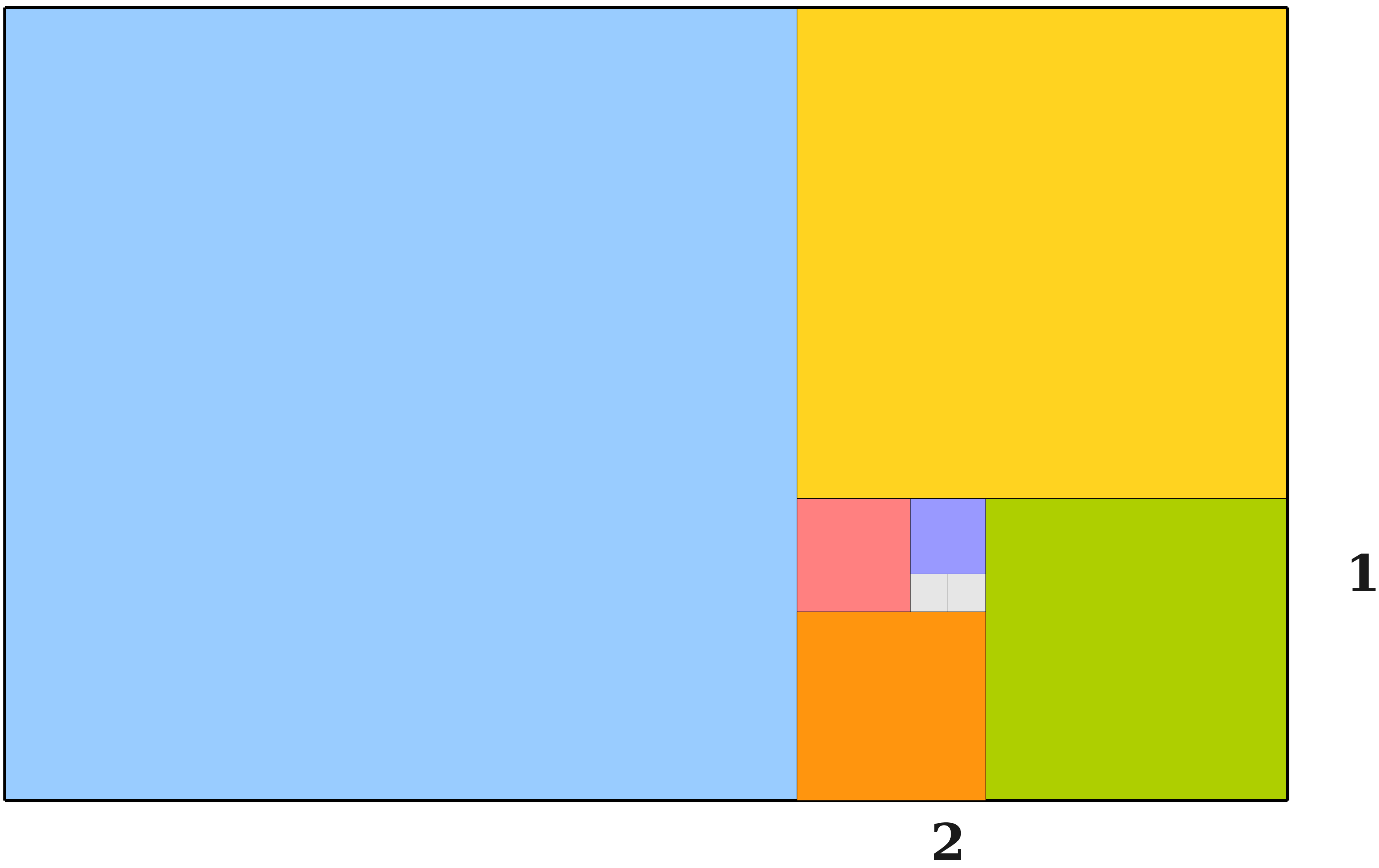
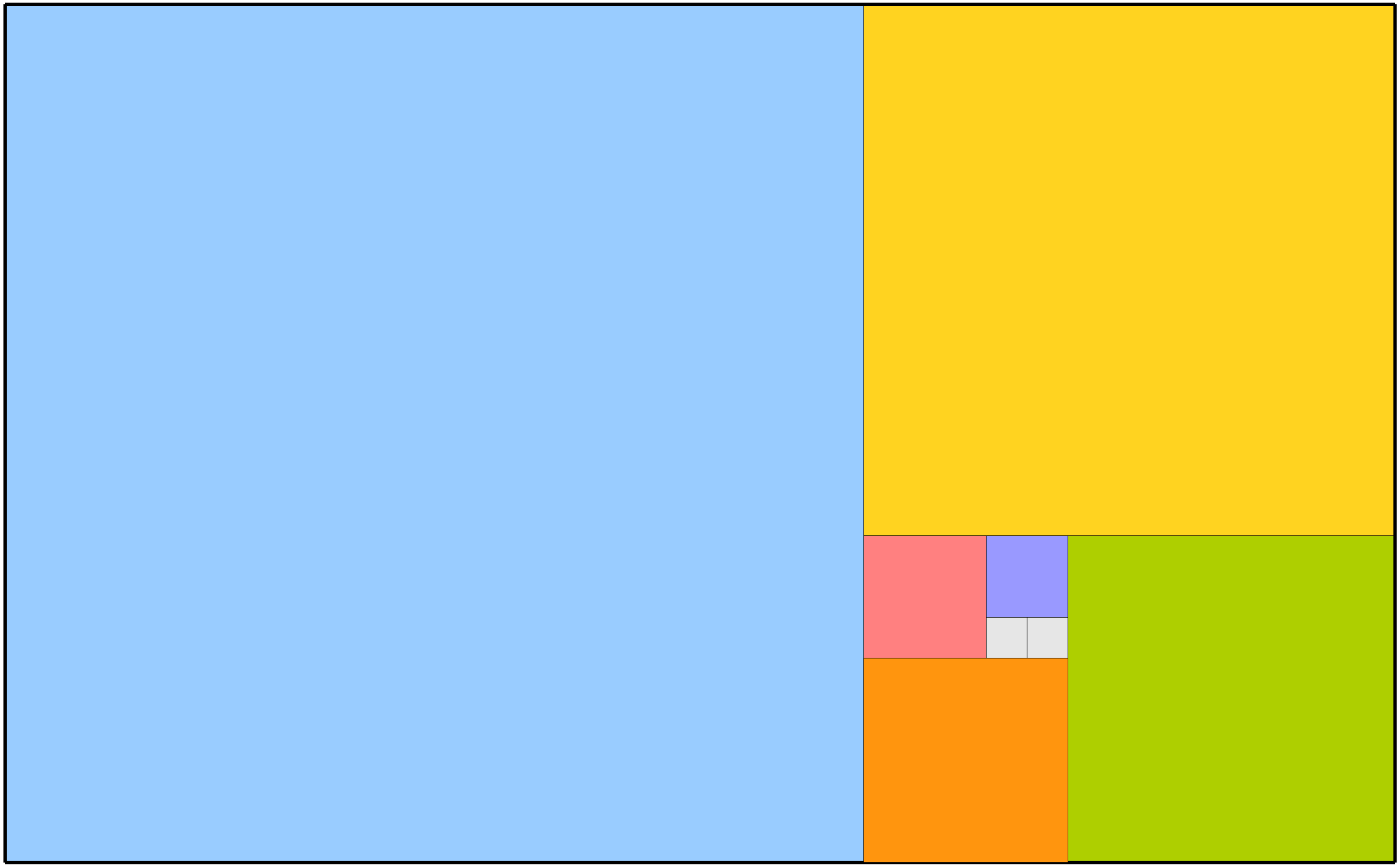
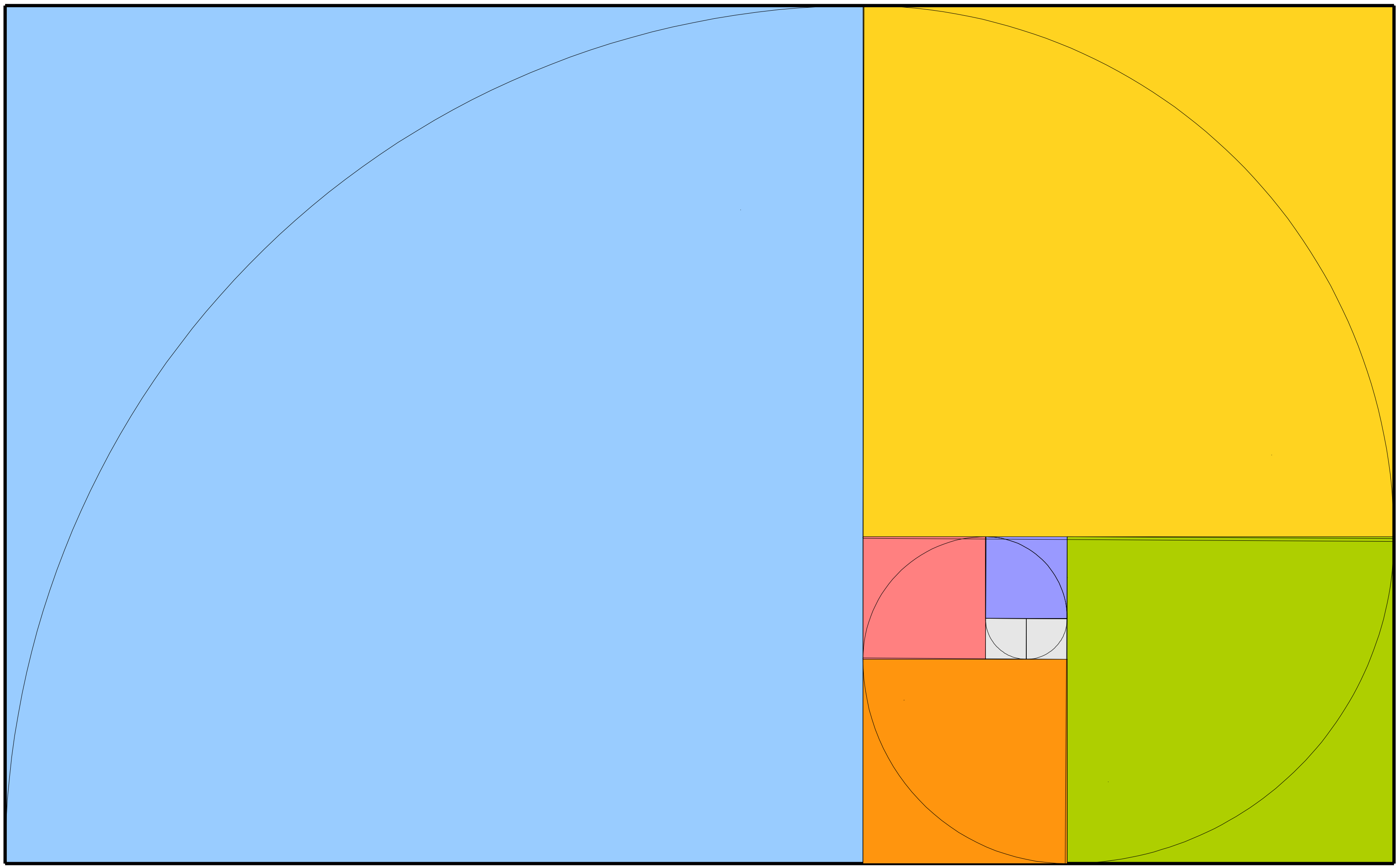


The Golden Ratio



The Golden Spiral



How do we prove all rational numbers have continued fractions?

Constructing a Continued Fraction

$$\frac{7}{9}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \frac{7}{9}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\cfrac{9}{7}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \frac{2}{7}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{\cfrac{7}{2}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{\cfrac{7}{2}}$$

$$\frac{7}{2}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{\cfrac{7}{2}}$$

$$\frac{7}{2} = 3 + \frac{1}{2}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{\boxed{\cfrac{7}{2}}}$$

$$\boxed{\frac{7}{2} = 3 + \frac{1}{2}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{3 + \cfrac{1}{2}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{\boxed{\frac{9}{7}}}$$

$$\boxed{\frac{9}{7} = 1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

Constructing a Continued Fraction

$$\frac{7}{9} = 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

$$\frac{9}{7}$$

$$\frac{7}{2}$$

$$\frac{2}{1}$$

Constructing a Continued Fraction

$$\frac{7}{\mathbf{9}} = 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

$$\frac{9}{\mathbf{7}}$$

$$\frac{7}{\mathbf{2}}$$

$$\frac{2}{\mathbf{1}}$$

Constructing a Continued Fraction

$$\frac{7}{\mathbf{9}} = 0+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2}}}$$

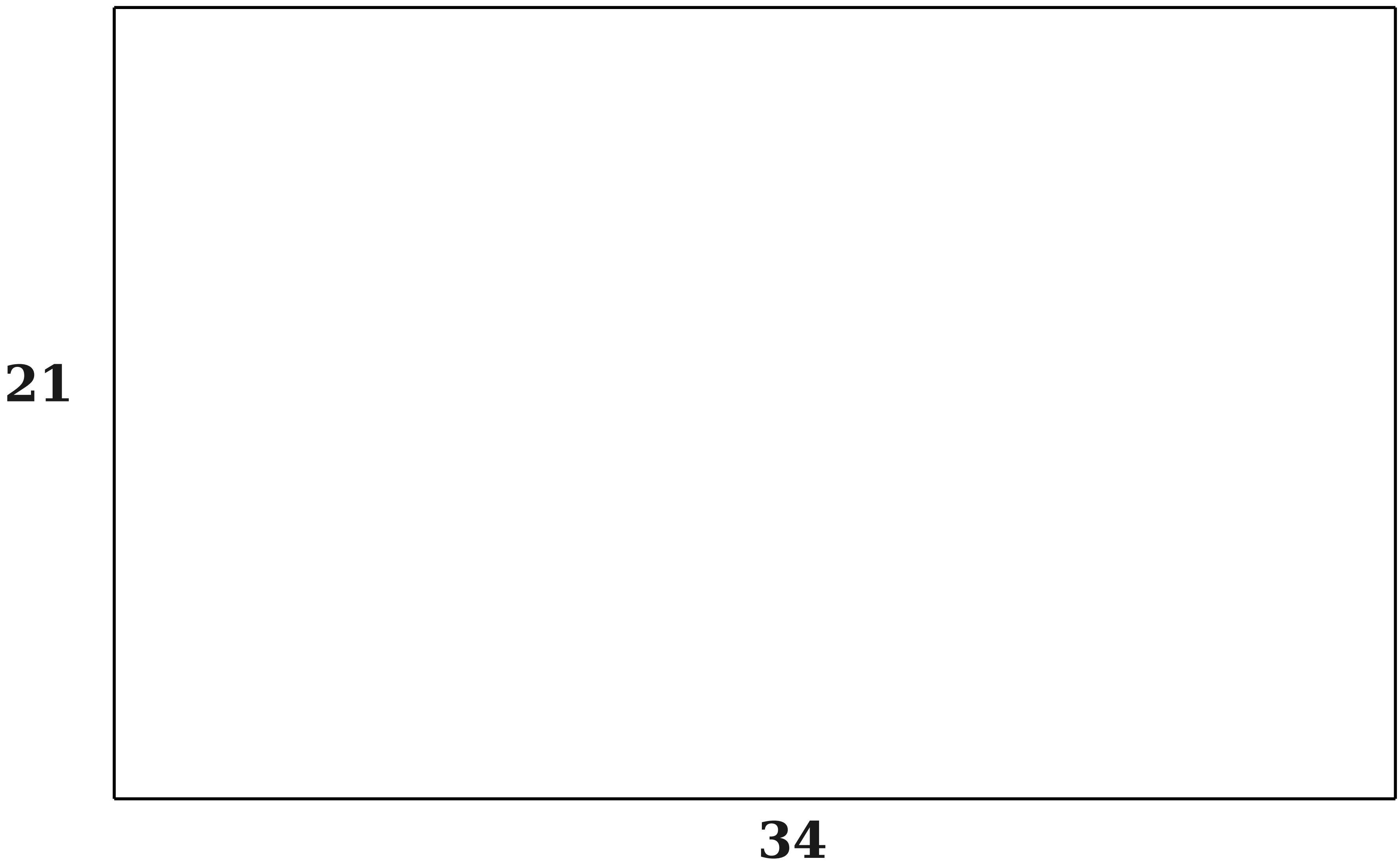
$$\frac{9}{\mathbf{7}}$$

$$\frac{7}{\mathbf{2}}$$

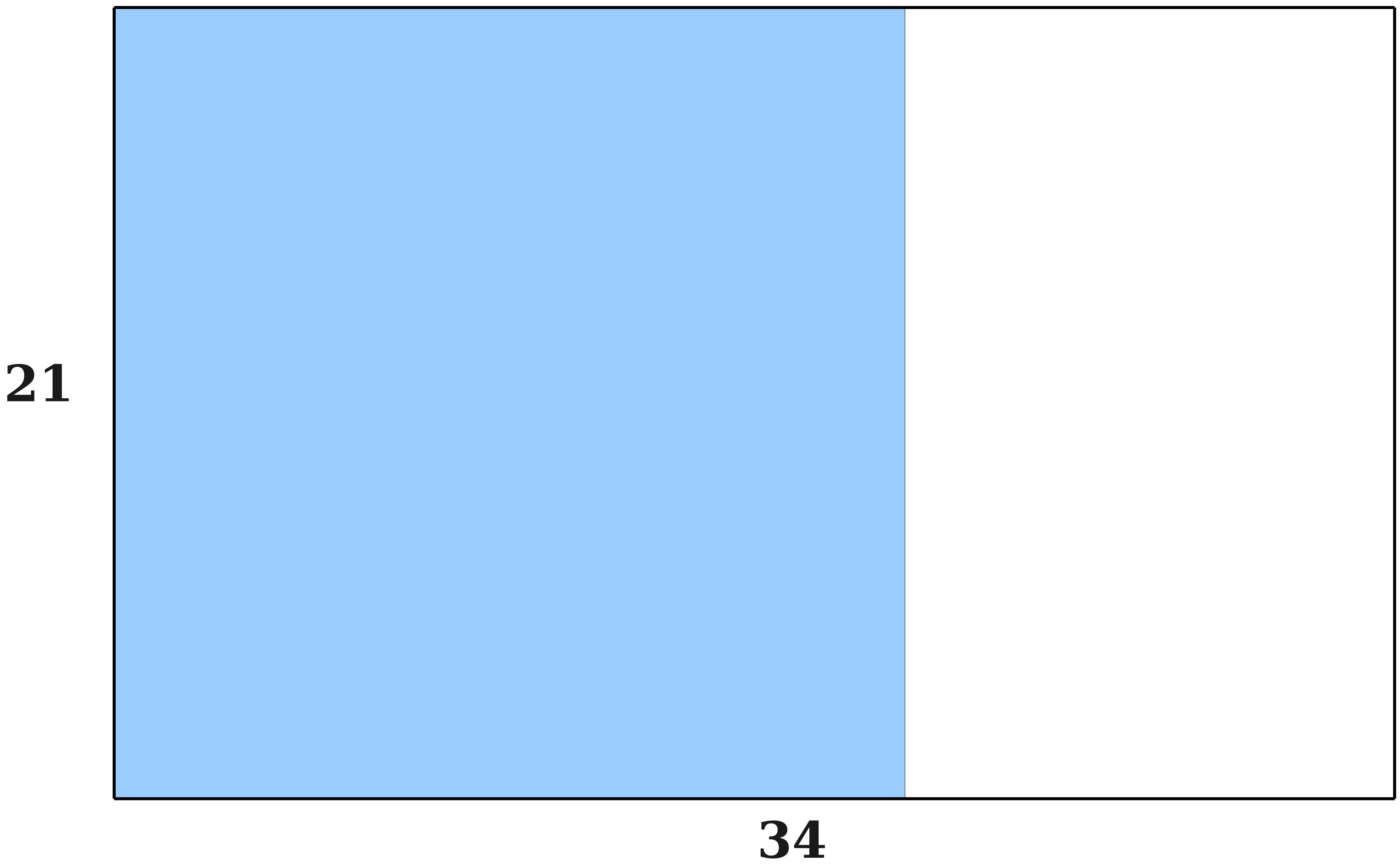
$$\frac{2}{\mathbf{1}}$$

9 > 7 > 2 > 1

The Golden Ratio



The Golden Ratio



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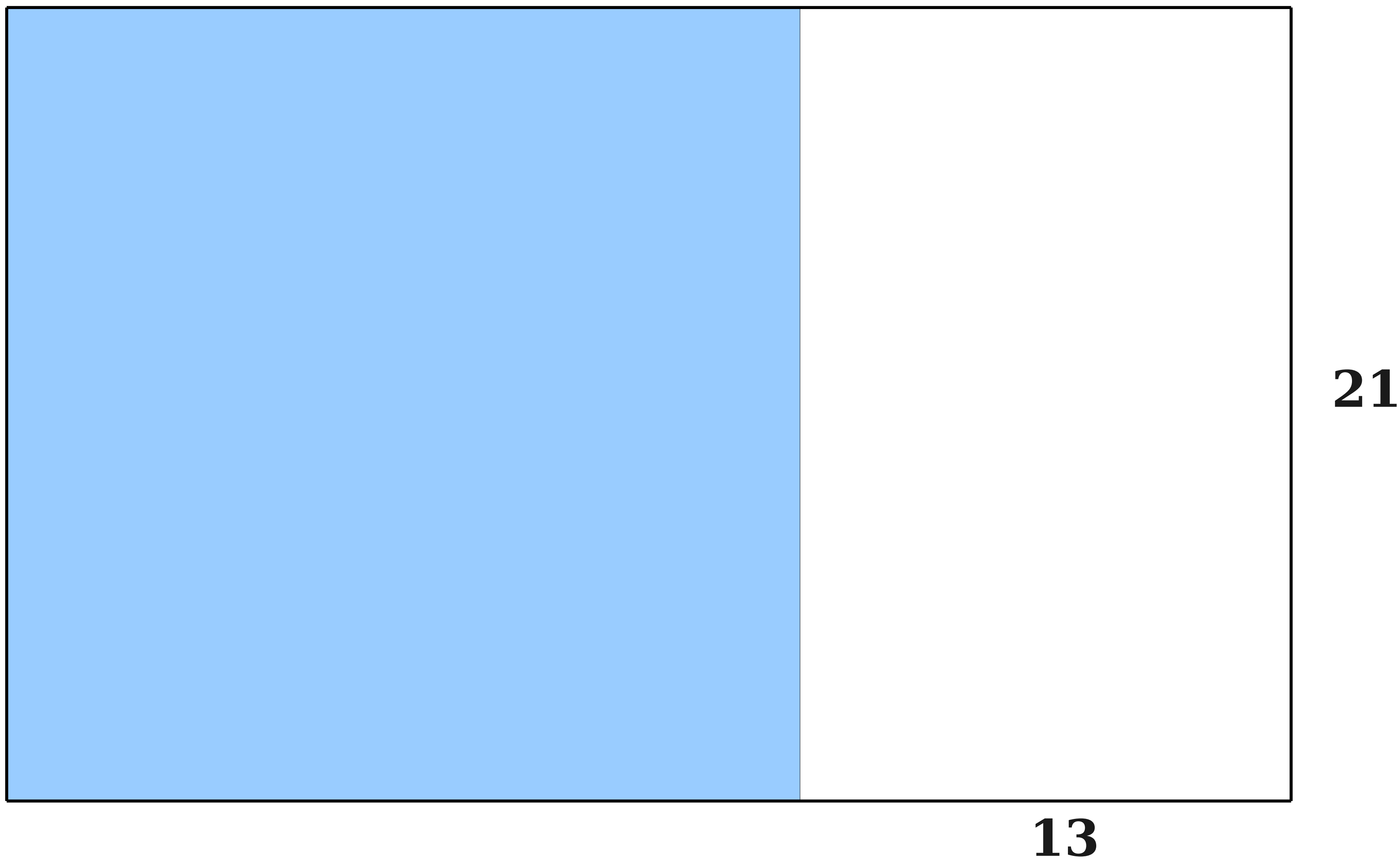
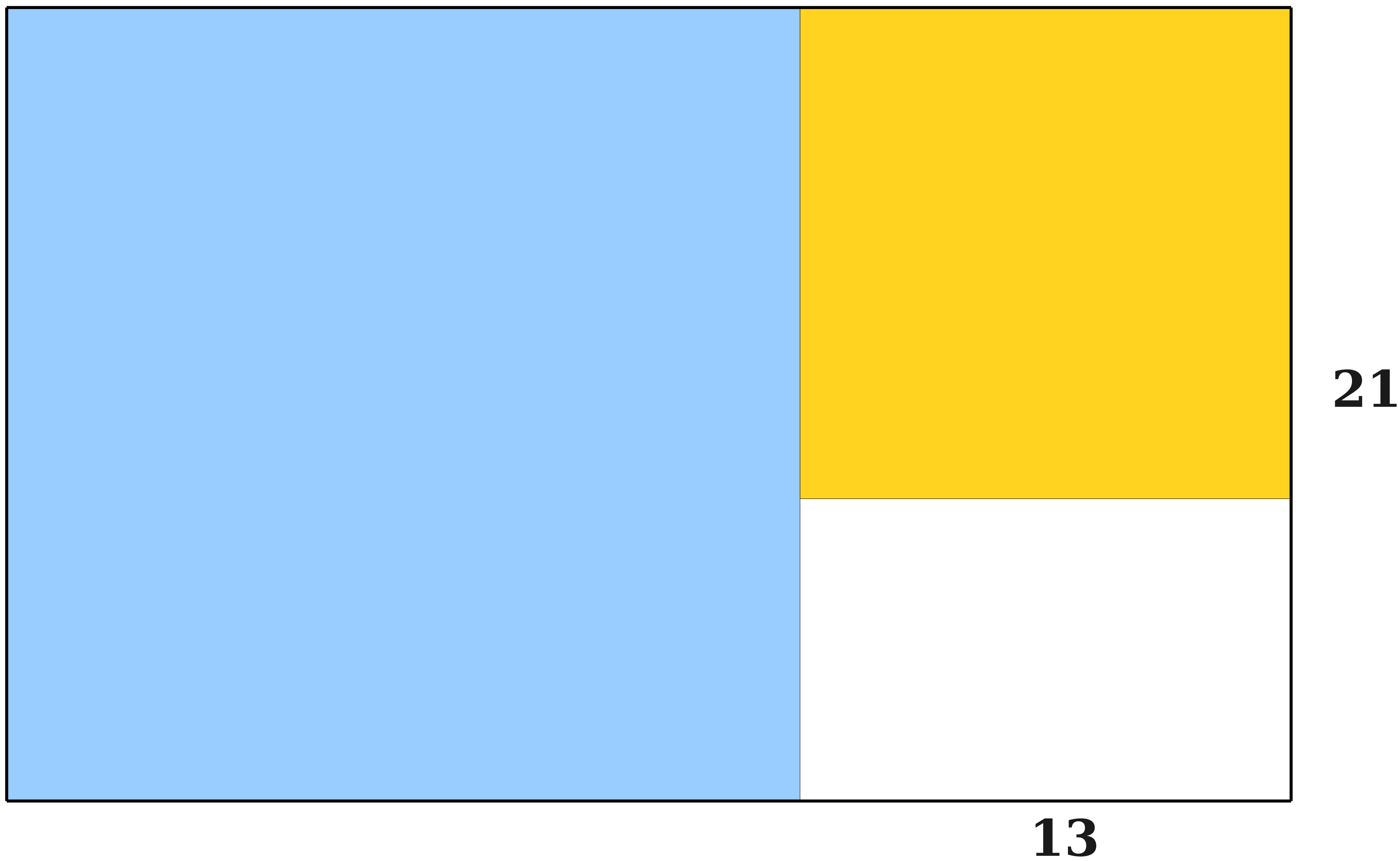


The Golden Ratio



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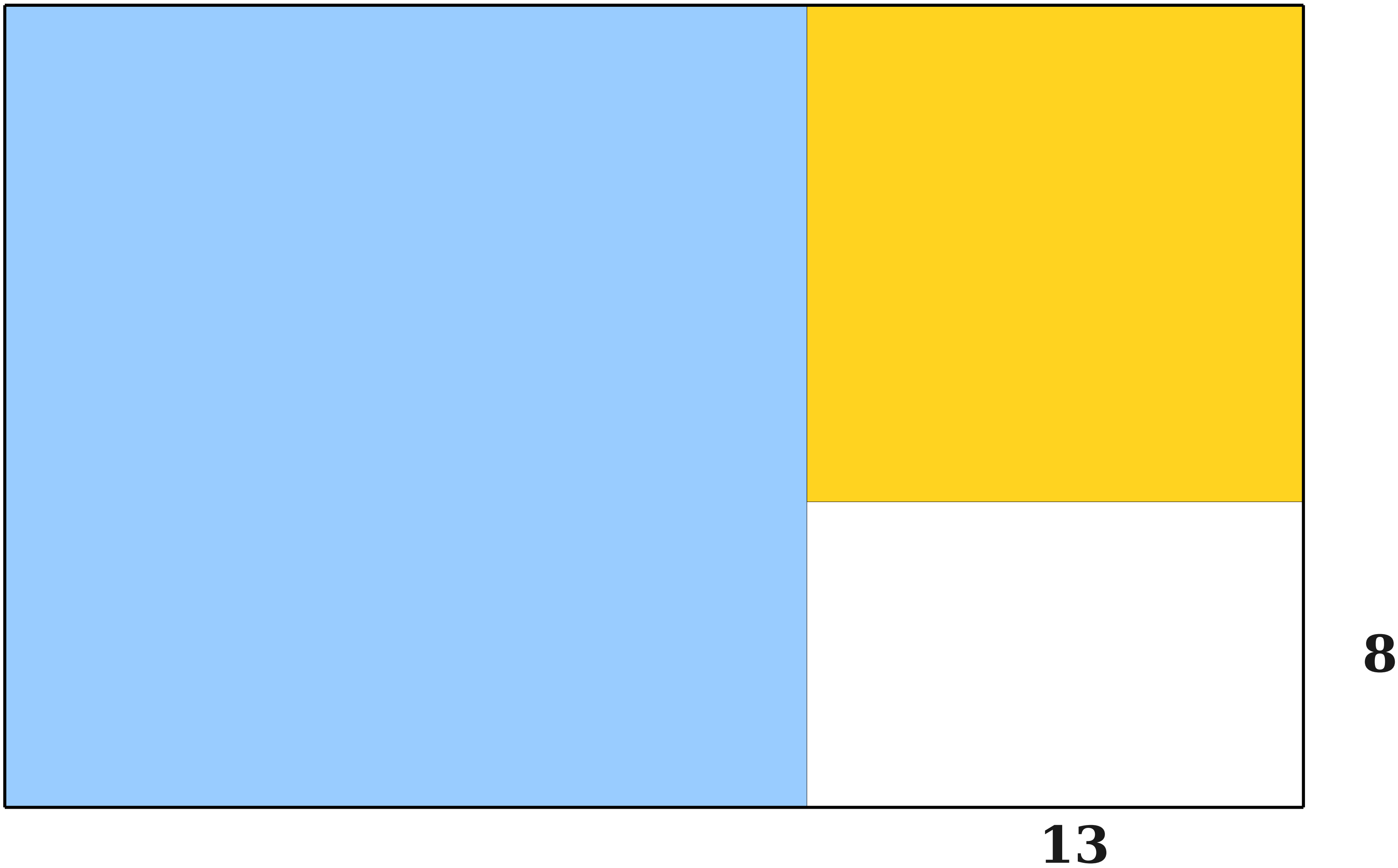

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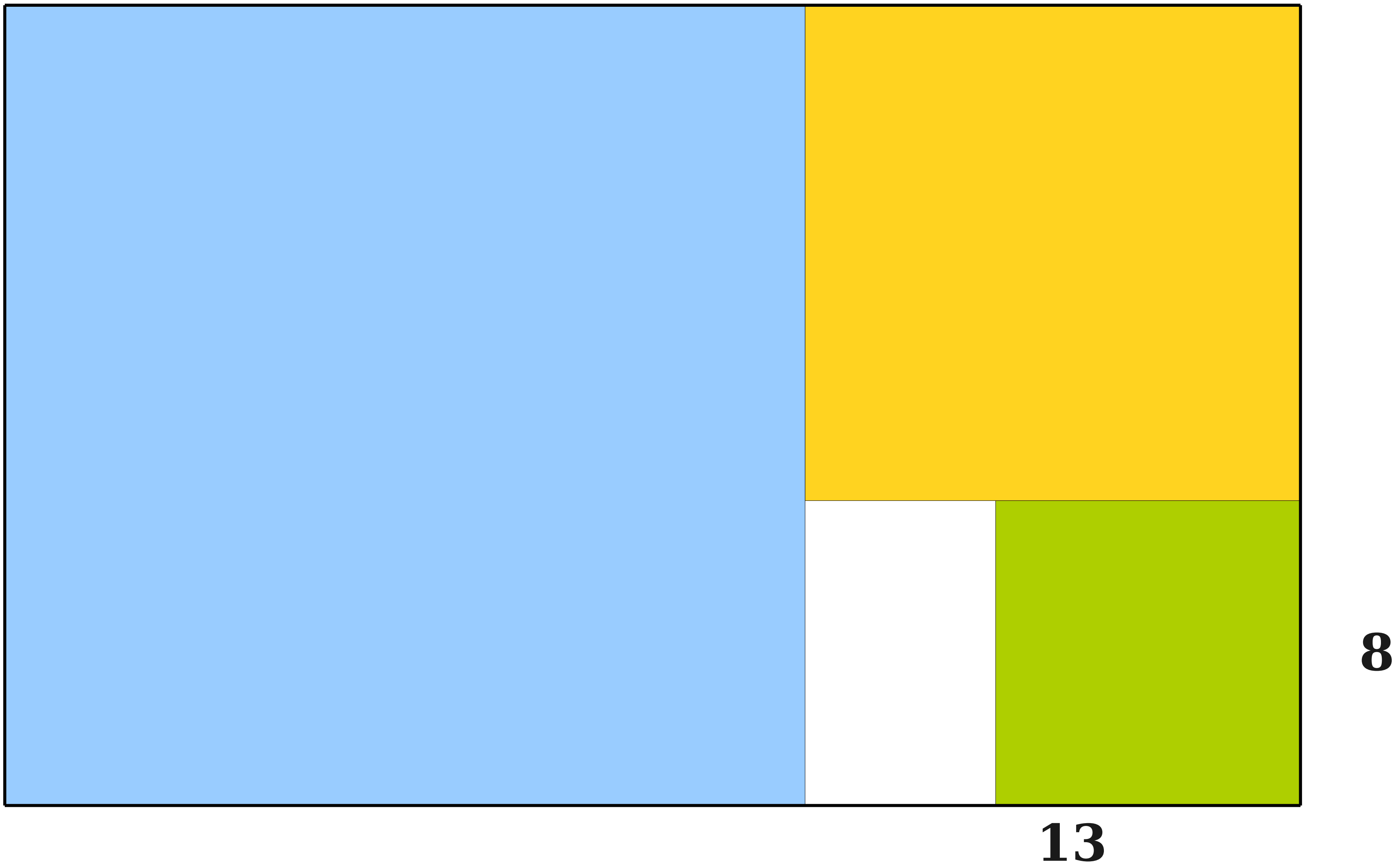

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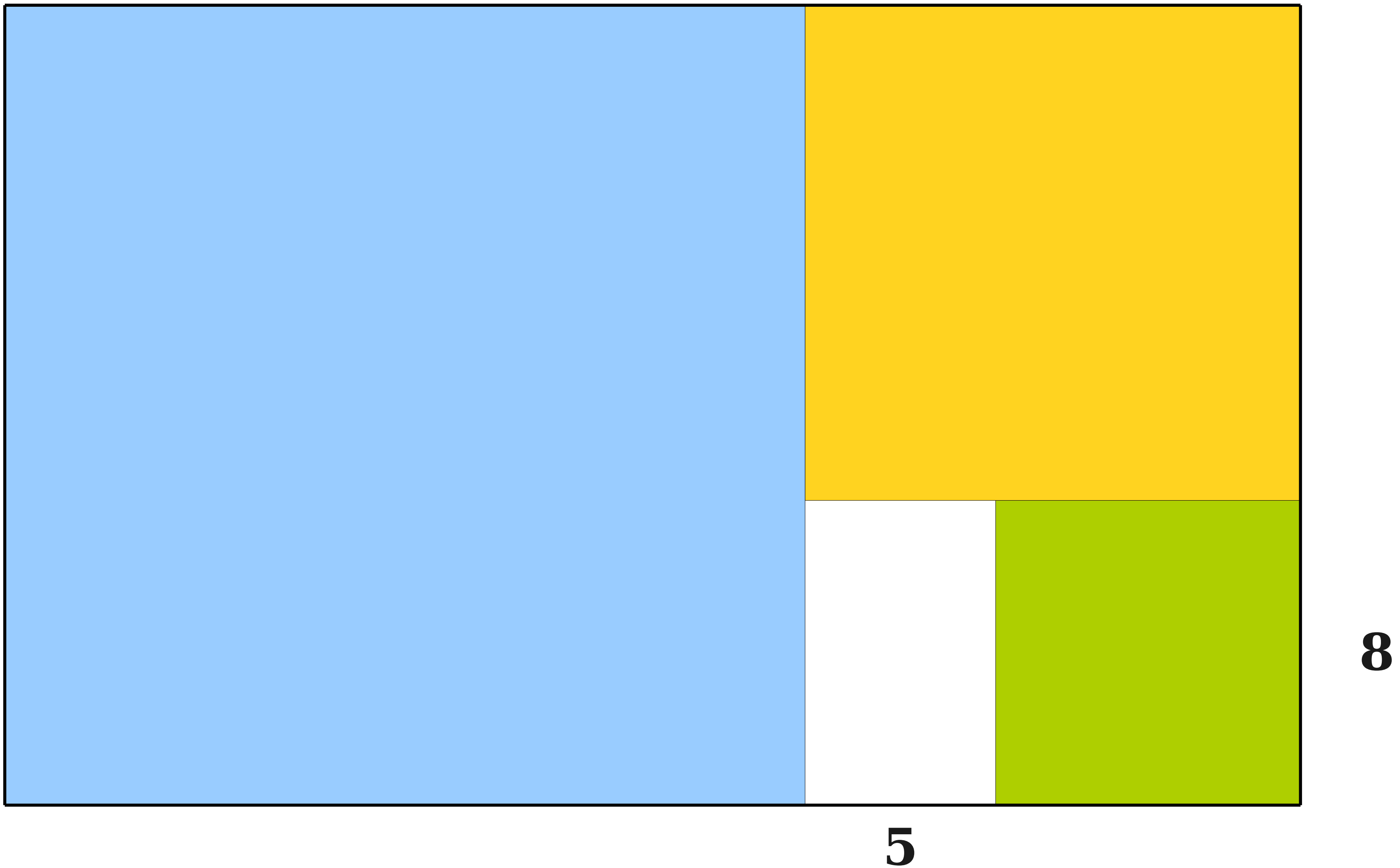
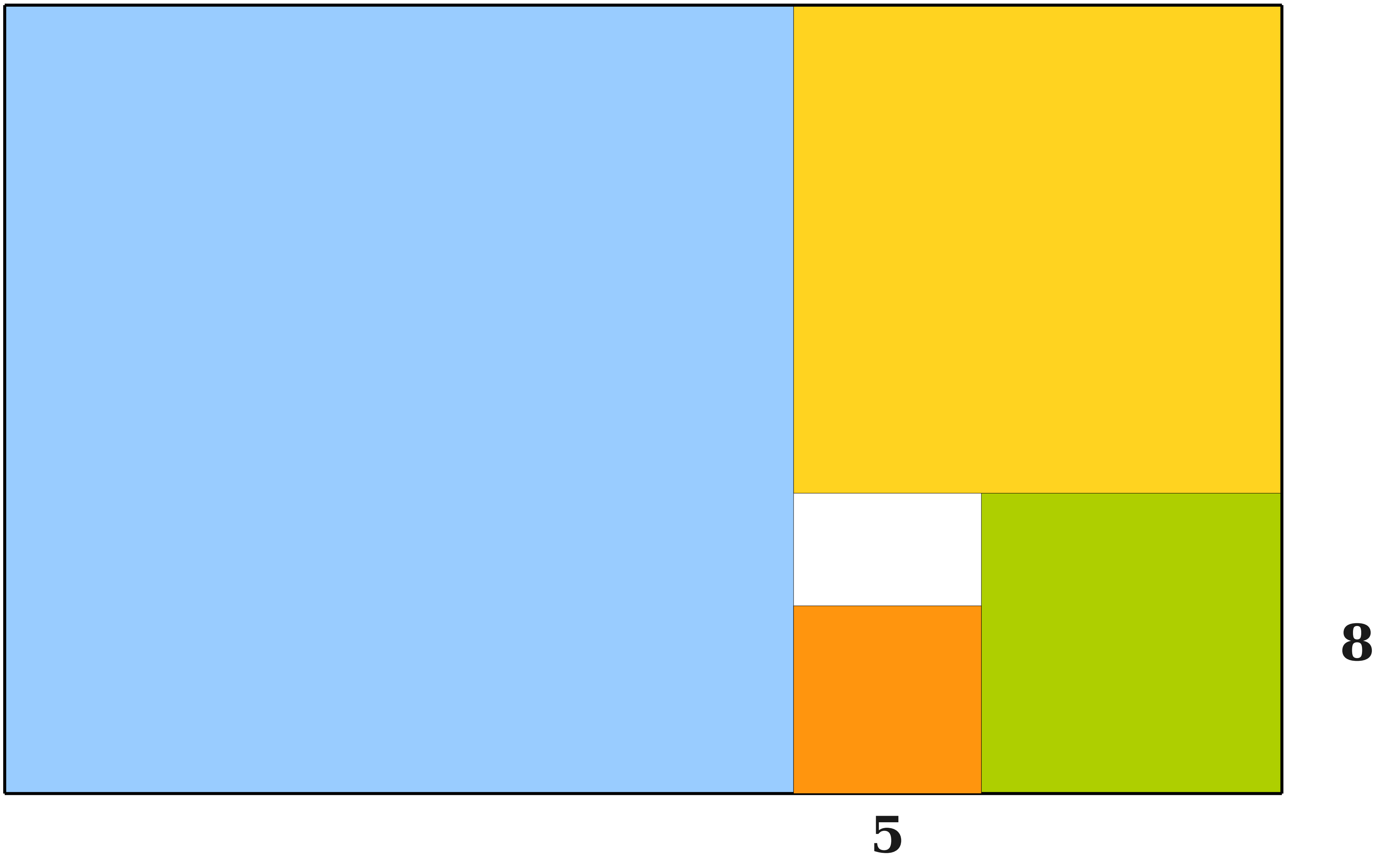


The Golden Ratio



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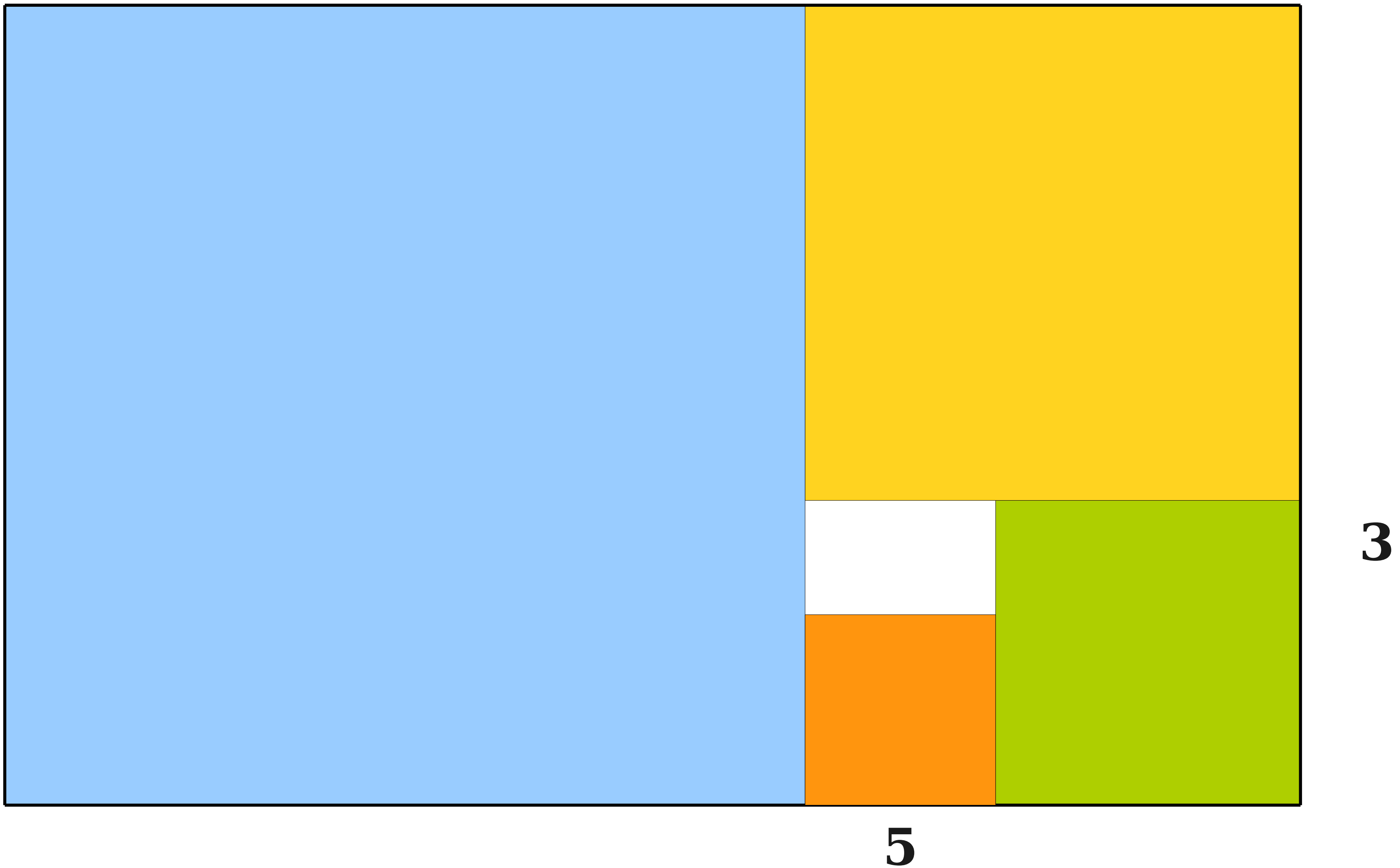

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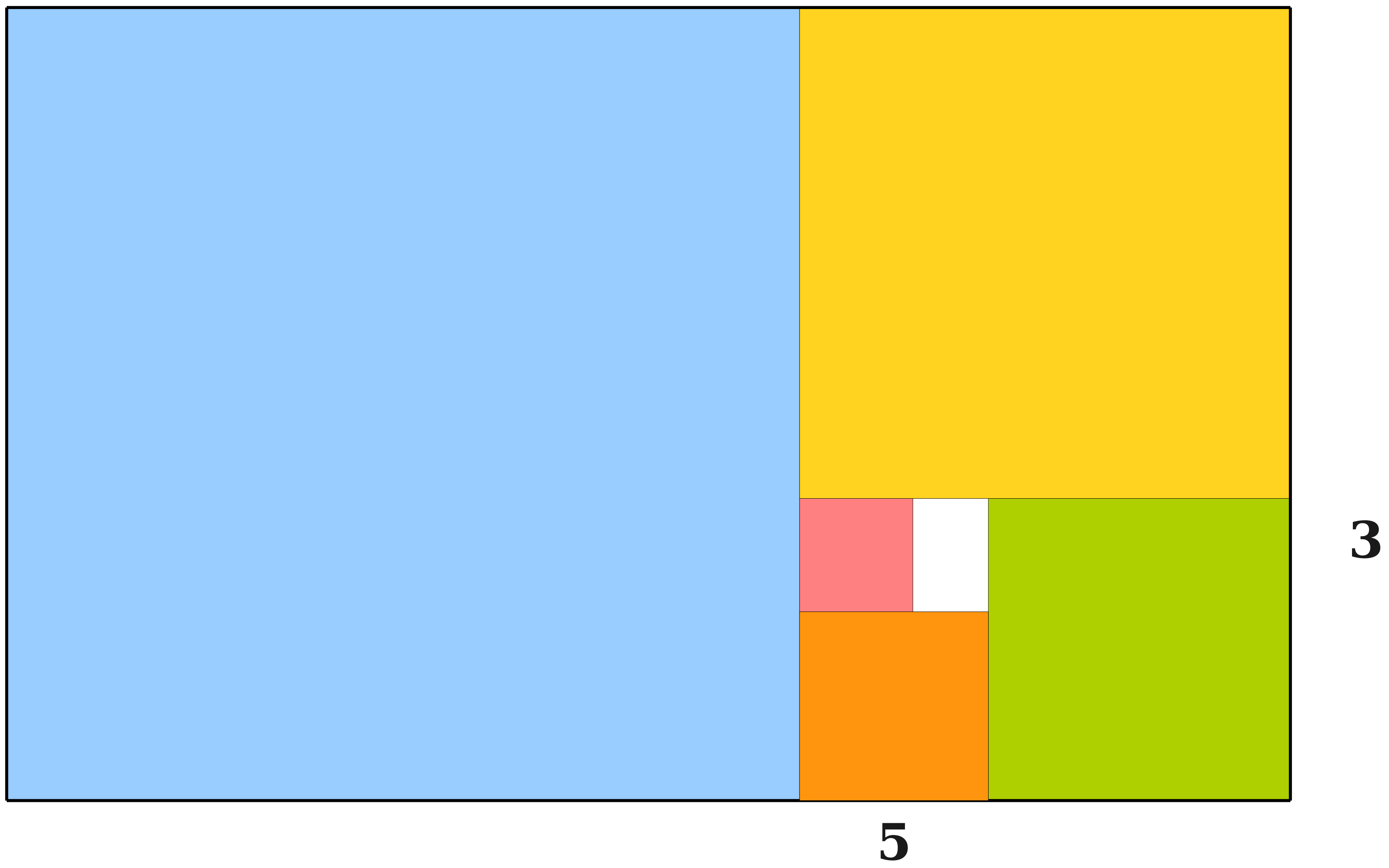
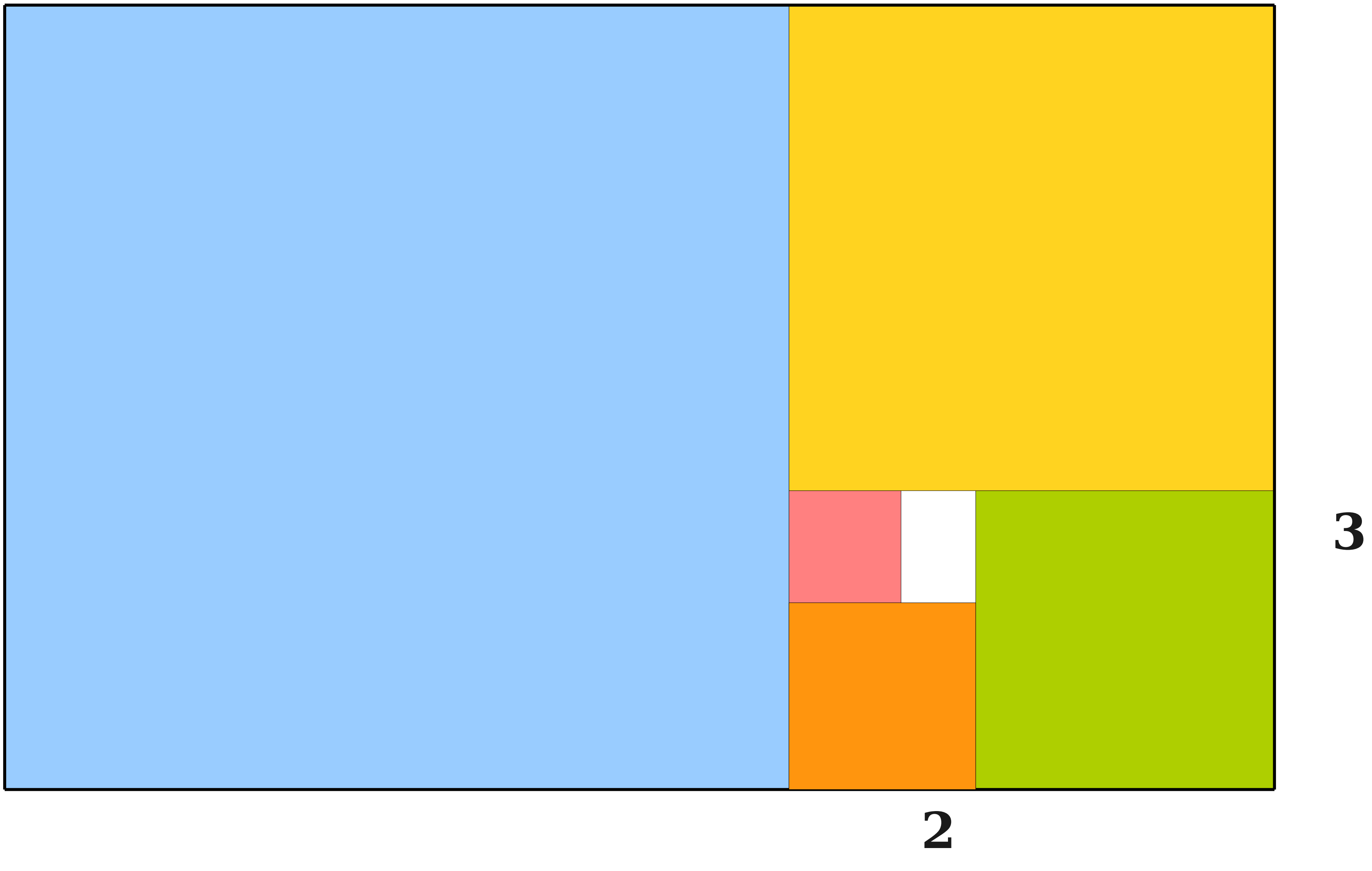


The Golden Ratio



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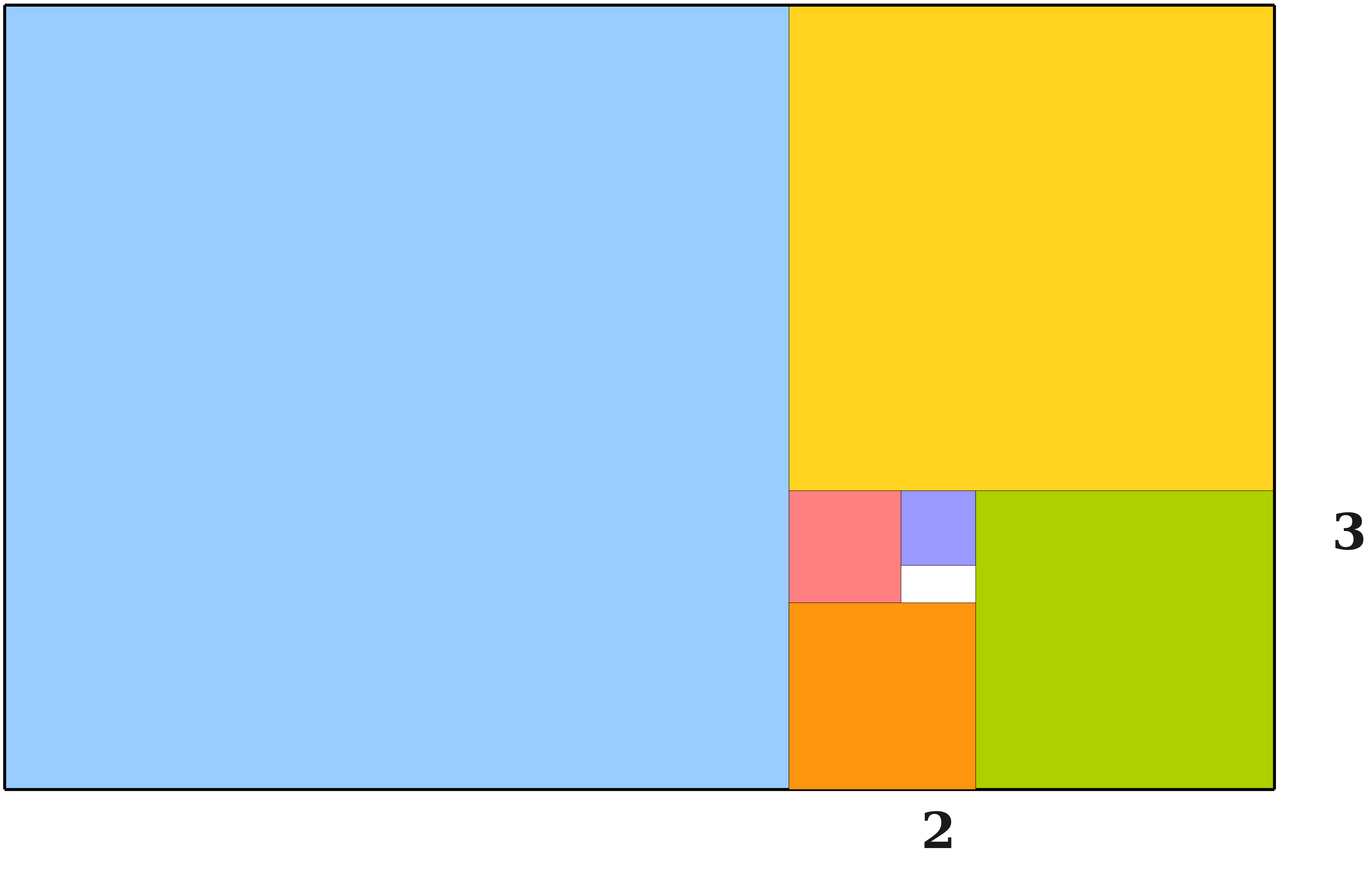

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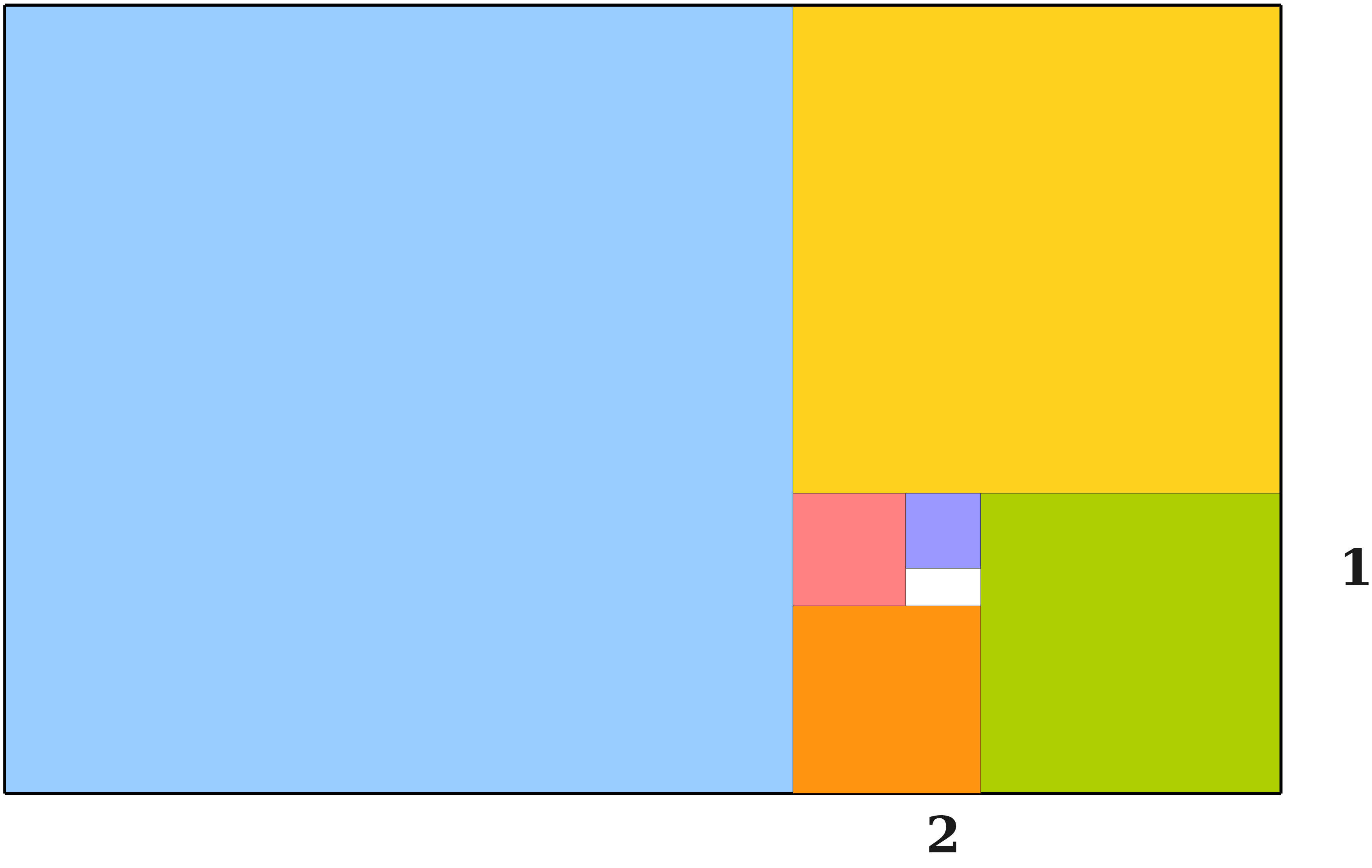

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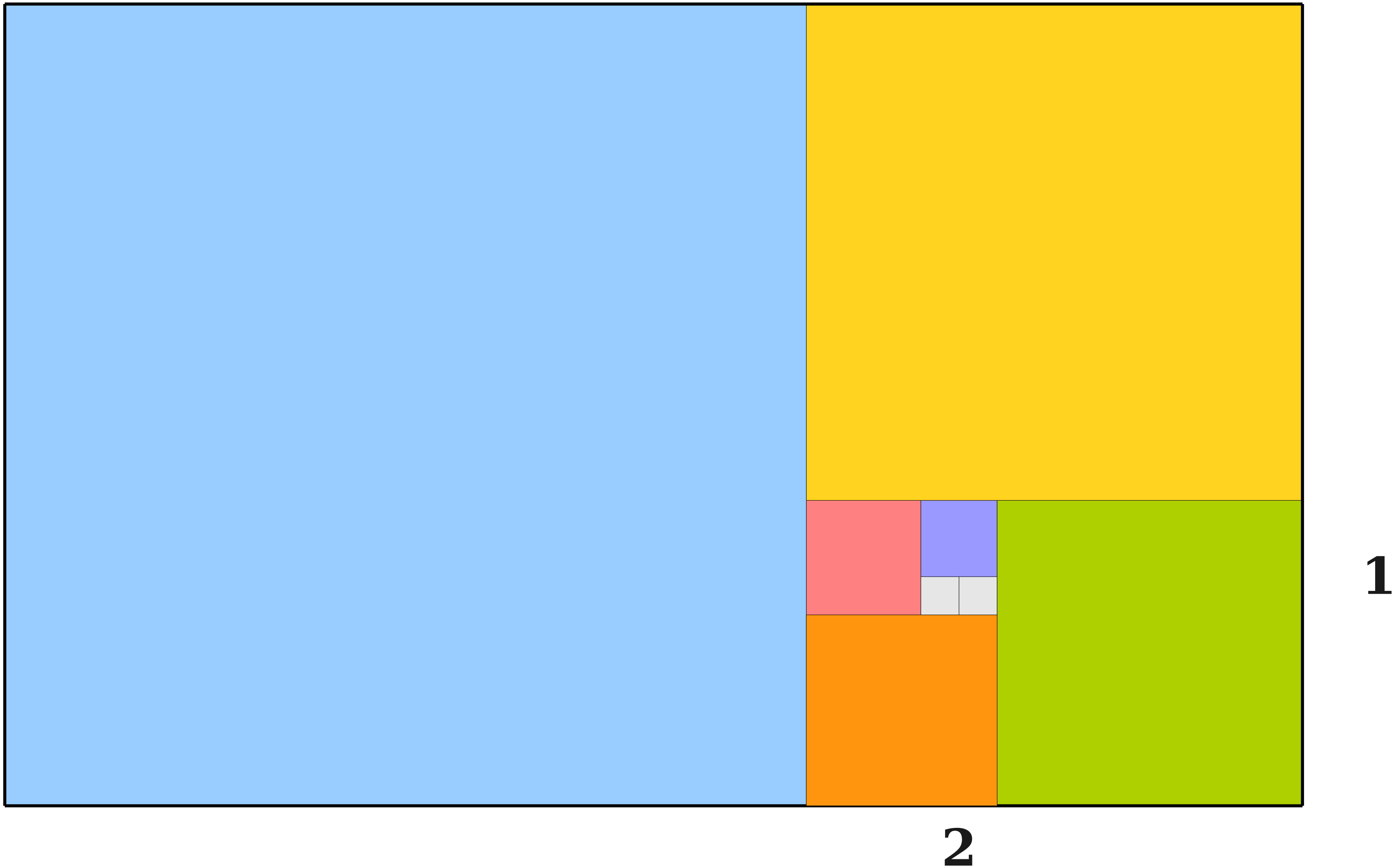

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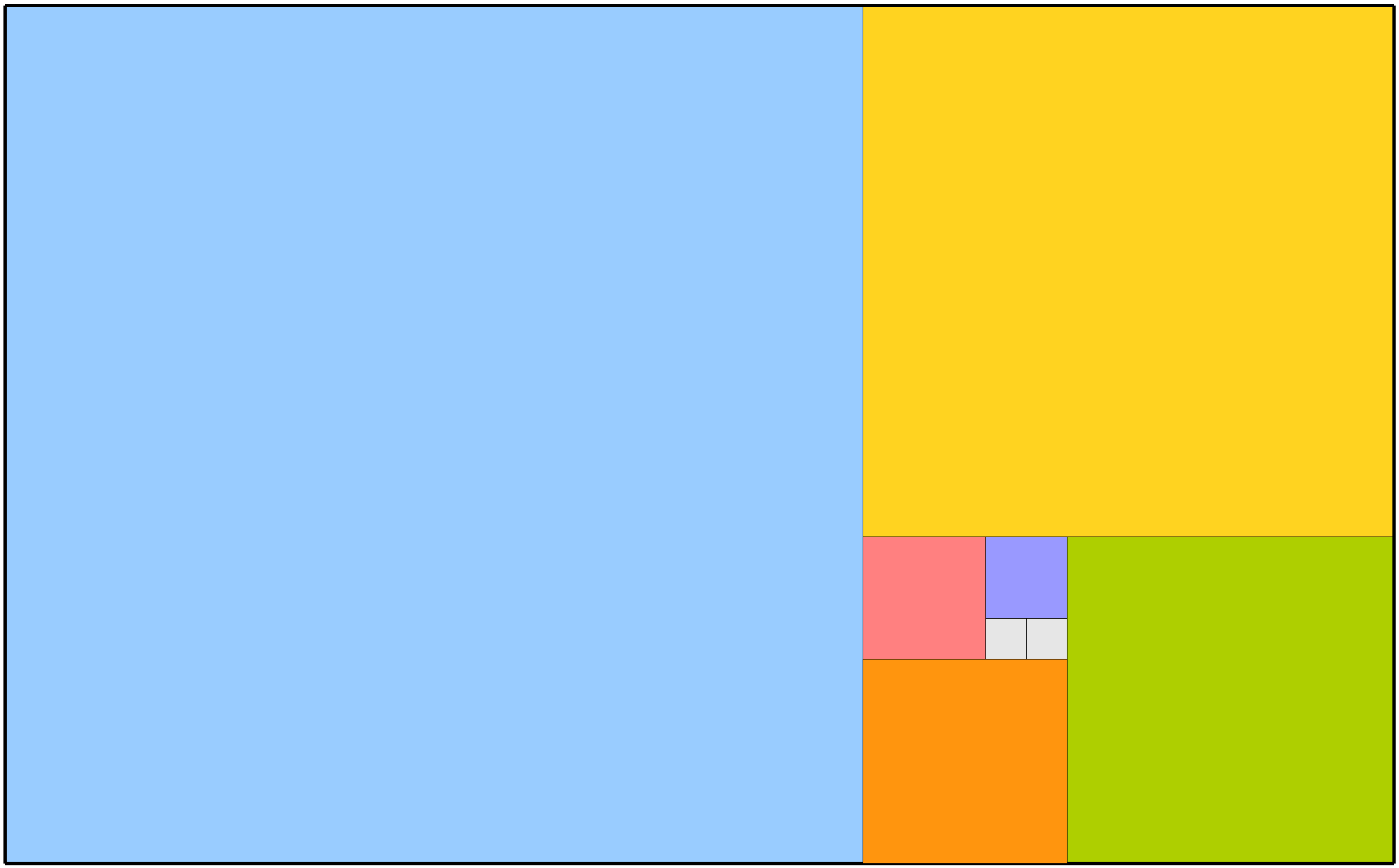

The Division Algorithm

- For any integers a and b, with $b > 0$, there exists **unique** integers $\boldsymbol{q}$ and $\boldsymbol{r}$ such that

$$a = \boldsymbol{q}b + \boldsymbol{r}$$

and

$$0 \le \boldsymbol{r} < b$$

- $\boldsymbol{q}$ is the **quotient** and $\boldsymbol{r}$ is the **remainder**.
- Given $a = 11$ and $b = 4$: $11 = \mathbf{2} \cdot 4 + \mathbf{3}$
- Given $a = -137$ and $b = 42$: $-137 = \mathbf{-4} \cdot 42 + \mathbf{31}$

Theorem: Every rational has a continued fraction.

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Proof: By strong induction.

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Theorem: Every rational has a continued fraction.

Proof: By strong induction. Let $P(d)$ be "any rational with denominator d has a continued fraction." We prove that $P(d)$ is true for all positive natural numbers. Since all rationals can be written with a positive denominator, this proves that all rationals have continued fractions.

For our base case, we prove $P(1)$, that any rational with denominator 1 has a continued fraction.

Theorem: Every rational has a continued fraction.

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Since $1 \leq r < d$, by our inductive hypothesis there is some continued fraction for d / r; call it F.

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We use that $\boldsymbol{r < d}$ to justify using the inductive hypothesis.

Since our induction starts at 1, we also have to show that $\boldsymbol{r \geq 1}$. Otherwise we might be out of the range of where the inductive hypothesis holds.

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For more on continued fractions:

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html