Theory of Computation
Problem Set 1

Universidad Politecnica de San Luis Potosi

Please start solving these problems immediately, don't procrastinate, and work in study groups. Please do not simply copy answers that you do not fully understand;

Advice: Please try to solve the easier problems first (where the meta-problem here is to figure out which are the easier ones ☺). Don't spend too long on any single problem without also attempting (in parallel) to solve other problems as well. This way, solutions to the easier problems (at least easier for you) will reveal themselves much sooner (think about this as a "hedging strategy" or "dovetailing strategy").

1. True or false:

   a. $\emptyset \subseteq \emptyset$

   b. $\emptyset \subset \emptyset$

   c. $\emptyset \in \emptyset$

   d. $\{1,2\}\ 2 \in^{\{1,2\}}$

   e. $\{1,2\} \subseteq 2^{\{1,2\}}$

   f. $\{x,y\} \in \{\{x,y\}\}$

2. Write the following set explicitly: $2^{\{1,\ 2\}} \times \{v,w\}$

3. Prove without using induction that for an arbitrary finite set S, the sets 2^S and $\{0,1\}^{|S|}$ have the same number of elements.

4. Which of the following sets are closed under the specified operations?
   a) $\{x \mid x$ is an odd integer$\}$, multiplication
   b) $\{y \mid y=2n$, n some integer$\}$, subtraction
   c) $\{2m+1 \mid m$ some integer$\}$, division
   d) $\{z \mid z=a+bi$ where a and b are real, $|a||b| > 0$, and $i=\sqrt{-1}\ \}$, exponentiation

5. Is the transitive closure of a symmetric closure of a binary relation necessarily reflexive? (Assume that every element of the “universe” set participates in at lease one relation pair.)

6. True or false: a countable union of countable sets is countable.

7. True or false: if T is countable, then the set $\{S \mid S \subseteq T$, S finite$\}$ is also countable.

8. Give a simple bijection for each one of the following pairs of sets:
   a) the integers, and the odd integers.
   b) the integers, and the positive integers.
   c) the naturals, and the rationals crossed with the integers.

9. Is there a bijection between the closed unit interval $\{x \mid x \in \mathbf{R}, 0 \leq x \leq 1\}$ and **R**?

11. Generalize $|S| < |2^S|$ to arbitrary infinite sets (not necessarily countable ones).

12. What is the cardinality of each of the following sets ?
    a. The set of all polynomials with rational coefficients.
    b. The set of all functions mapping reals to reals.
    c. The set of all possible Java programs.
    d. The set of all finite strings over the alphabet $\{0,1,2\}$.
    e. The set of all 5x5 matrices over the rationals.
    f. The set of all points in 3-dimensional Euclidean space.
    g. The set of all valid English words.
    h. $\{\emptyset, \mathbf{N}, \mathbf{Q}, \mathbf{R}\}$
    i. $\mathbf{N} \times \mathbf{Z} \times \mathbf{Q}$
    j. $\mathbf{R} - \mathbf{Q}$

13. Prove without using induction that n^4-4n^2 is divisible by 3 for all $n\geq 0$.

14. How many distinct boolean functions on N variables are there? In other words, what is the cardinality of $|\{f \mid f:\{0,1\}^N \rightarrow \{0,1\}\}|$?

15. How many distinct N-ary functions are there from finite set A to finite set B? Does this generalize the previous question?

16. Show that in any group of people, there are at least two people with the same number of acquaintances within the group. Assume the "acquaintance" relation is symmetric but non-reflexive.

17. Show that in any group of six people, there are either 3 mutual strangers or 3 mutual acquaintances.

18. A clique in a graph is a complete subgraph (i.e., all nodes are connected with edges). Show that every graph with N nodes contains a clique or the complement of a clique, of size at least $\frac{1}{2}\log_2 N$.

19. Show that the set difference of an uncountable set and a countable set is uncountable.

20. Show that the intersection of two uncountable sets can be empty, finite, countably infinite, or uncountably infinite.

21. For an arbitrary language L, prove or disprove each of the following:

    a) $(L^*)^*=L^*$

    b) $L^+=L^*-\{\wedge\}$

22. Characterize completely the cardinalities of all sets of identical test tubes that can be spun simultaneously in a 360-hole centrifuge in a balanced way (i.e. 1 test tube cannot be spun, but 2, 3, 4, and 5 can, etc.)

23. Prove that there are an infinity of prime numbers.

24. Prove or disprove: for any arbitrarily large natural number N, there exists N consecutive composite natural numbers (i.e. argue whether there exists "prime deserts" of arbitrarily large sizes).

25. Compute the infinite sum $(1/16) + (1/16)^2 + (1/16)^3 + (1/16)^4 + \ldots =$? without using induction.

26. Find a formula (as a function of n) for $1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 =$? and prove it using a picture (and without using induction).

27. Prove that the square root of 2 is irrational.

28. Are the complex numbers closed under exponentiation? And if so, what is the value of i^i?

29. Does exponentiation preserve rationality? Does exponentiation preserve irrationality? i.e., are there two irrational numbers x and y such that x^y is rational?