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Mathematical Optimization — Assignment 5

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Representation of Polyhedra

(a) Consider the polyhedron defined by

$$P := \operatorname{conv}\left(\{v_1, v_2, v_3\}\right) + \operatorname{cone}(r) \subset \mathbb{R}^2,$$

where $v_1 = (0,0)^\mathsf{T}, v_2 = (2,1)^\mathsf{T}, v_3 = (0,3)^\mathsf{T}$ and $r = (1,1)^\mathsf{T}$. From the lecture (Theorem of Minkowski-Weyl), we know that P has an *inequality description*. Find A and b such that

$$P = \{x \in \mathbb{R}^2 \mid Ax \leq b\}.$$

(b) Consider the polyhedron defined by

$$P := \{x \in \mathbb{R}^2 \mid x_2 \leq 0,\ x_1 - x_2 \leq 3\}.$$

Find a vertex description of P, namely find $x_1, \ldots, x_k \in P$ and $r_1, \ldots, r_l \in \mathbb{Z}^n$ such that

$$P = \operatorname{conv}(v_1, \ldots, v_k) + \operatorname{cone}(r_1, \ldots, r_l).$$

Exercise 2: Complementary Slackness

Consider the following linear program

$$\begin{array}{rl}
\max & 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5 \\
\text{s.t.} & x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 \leq 4 \\
& 4x_1 + 2x_2 - 2x_3 + x_4 + x_5 \leq 3 \\
& 2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 \leq 5 \\
& 3x_1 + x_2 + 2x_3 - x_4 - 2x_5 \leq 1 \\
& x_j \geq 0 \quad \text{for } j = 1, \ldots, 5
\end{array}$$

of the form $\max\ c^T x$ s.t. $Ax \leq b, x \geq 0$.

(a) Formulate the dual program of the LP.

(b) Consider the primal feasible solution $x^* = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)^T$. Find a $y^* \in \mathbb{R}^4$ such that

$$\sum_{i=1}^{4} a_{ij} y_i^* = c_j \qquad \text{whenever} \quad x_j^* > 0 \quad \text{for } j = 1, \ldots, 5,$$

and

$$y_i^* = 0 \qquad \text{whenever} \quad \sum_{j=1}^{5} a_{ij} x_j^* < b_i \quad \text{for } i = 1, \ldots, 4.$$

Is this y^* uniquely determined?

(c) Use the Optimality Condition from complementary slackness theory to prove that the given x^* is **not** an optimal solution for the above linear program.

Exercise 3: Geometry of LP and Exchange Step
Let us consider the following LP problem given in canonical form:

$$(LP)\qquad \begin{array}{lrrl} \max & x_1 & +2x_2 & \\ \text{s.t.} & x_1 & +x_2 & \leq 8 \\ & -x_1 & +x_2 & \leq 2 \\ & & x_2 & \leq 4 \\ & x_1 & & \geq 0 \\ & & x_2 & \geq 0 \end{array}$$

(a) Draw the feasible set.

(b) Transform (LP) into its corresponding standard form such that the Simplex algorithm is applicable.

(c) For $P_1 = (0,0)$, $P_2 = (0,2)$, $P_3 = (2,4)$, $P_4 = (4,4)$, look at the corresponding point in the standard form P_i', $i = 1,2,3,4$, state which are the basic and non-basic variables and give the corresponding basic solution.

(d) How many exchange steps are necessary to move from point P_1' to P_4'?

(e) Choose to either use the short or long tableau (both are equivalent), construct it corresponding to P_1' and perform the exchange steps to move to P_3':

$$T_{\text{long}} = \left[\begin{array}{c|cc} -{c_B}^\intercal A_B^{-1}b & 0\ldots0 & \bar{c_j},\ j \in N \\ \hline A_B^{-1}b & I & A_B^{-1}A_N \end{array}\right]$$

$$T_{\text{short}} = \left[\begin{array}{c|c} -{c_B}^\intercal A_B^{-1}b & \bar{c_j},\ j \in N \\ \hline A_B^{-1}b & A_B^{-1}A_N \end{array}\right].$$

Have we reached an optimal solution? Why?

Exercise 4: Degeneracy and Cycling of Simplex Method
For this exercise, feel free to use a pivoting tool, such as `http://www.ricoz.net/pivoter.html`.

The goal of this exercise is to show that in case of degeneracy, the finite termination of the Simplex method is not guaranteed by arbitrarily choosing any admissible Simplex pivot.

For the LP

$\max\{x_1 - 2x_2 + x_3 : (x_1, x_2, x_3) \in \mathbb{R}^3_{\geq 0}\ ;\ 2x_1 - x_2 + x_3 \leq 0\ ;\ 3x_1 + x_2 + x_3 \leq 0\ ;\ -5x_1 + 3x_2 - 2x_3 \leq 0\}$,

show that the following sequence of bases corresponds to a legal execution of the Simplex method:

$$\{4,5,6\} \to \{1,4,6\} \to \{1,3,6\} \to \{1,2,3\} \to \{2,3,5\} \to \{2,4,5\} \to \{4,5,6\},$$

where x_4, x_5 and x_6 are the slack variables corresponding to the constraints $2x_1 - x_2 + x_3 \leq 0$, $3x_1 + x_2 + x_3 \leq 0$ and $-5x_1 + 3x_2 - 2x_3 \leq 0$.