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Mathematical Optimization — Assignment 11

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Mixed-integer feasibility with a fixed number of integer variables
The mixed-integer feasibility problem is formulated as follows:

Given $A \in \mathbb{Q}^{m\times d}$, $B \in \mathbb{Q}^{m\times n}$ and $c \in \mathbb{Q}^m$, does there exist $x \in \mathbb{Z}^d$ and $y \in \mathbb{R}^n$ such that $Ax + By \le c$ is satisfied? If yes, find such a pair of x and y.

Under the assumption that d is a constant, solve this problem by using two oracles: One that can solve LP-feasibility in polynomial-time and one that can solve IP-feasibility in polynomial-time if the amount of variables is constant. You may assume that both oracles return feasible solutions if they exist.

Hint: Consider a projection of the LP-relaxation $\{\binom{x}{y} \in \mathbb{R}^{d+n} \mid Ax + By \le c\}$, to which you can apply the IP-feasibility oracle. You may assume that the IP-feasibility oracle can be applied to projections of polyhedra, i.e., does not require an explicit inequality description.

Exercise 2: Matching and Perfect Matching Polytope
Let $G = (V, E)$ be an undirected graph. From the lecture, we know that the matching polytope for a bipartite graph is given by $BM(G) := \left\{x \in \mathbb{R}^{|E|} \;\middle|\; x(\delta(v)) \le 1,\ \forall v \in V,\ x \ge 0\right\}$.

A perfect matching is a matching with the additional property that all vertices are "touched" by the edges in the matching, i.e. a perfect matching is a set $F \subseteq E$ such that $|F \cap \{v\}| = 1$ for all $v \in V$.

For general graphs, it can be shown that an inequality description of the matching polytope is given by

$$\begin{aligned} M(G) := \Big\{x \in \mathbb{R}^{|E|} \;\Big|\; & x(\delta(v)) \le 1,\ \forall v \in V, \\ & x(E[S]) \le \frac{|S|-1}{2},\ \forall S \subseteq V,\ |S| \text{ odd}, \\ & x \ge 0\Big\}, \end{aligned}$$

and an inequality description of the perfect matching polytope is given by

$$\begin{aligned} PM(G) := \Big\{x \in \mathbb{R}^{|E|} \;\Big|\; & x(\delta(v)) = 1,\ \forall v \in V, \\ & x(\delta(S)) \ge 1,\ \forall S \subseteq V,\ |S| \ge 1,\ |S| \text{ odd}, \\ & x \ge 0\Big\}. \end{aligned}$$

Let us define

$$
\begin{aligned}
PM'(G) := \big\{x \in \mathbb{R}^{|E|} \;\big|\; & x(\delta(v)) \leq 1,\ \forall v \in V, \\
& x(E[S]) \leq \frac{|S|-1}{2},\ \forall S \subseteq V,\ |S| \text{ odd}, \\
& x(E) = \frac{|V|}{2}, \\
& x \geq 0\big\}.
\end{aligned}
$$

Show that $PM(G) = PM'(G)$, i.e. that both descriptions are equivalent. You may assume that $PM(G)$ is a valid inequality description of the perfect matching polytope for general graphs.

Hint: Distinguish whether $|V|$ is odd or even.

Exercise 3: LP Solution and the Normal Cone
Let $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a non-empty polyhedron without lines. For a vertex $v \in P$, we define its normal cone to be $\text{cone}\big(\{A_{i,\cdot}^{\mathrm{T}} \mid i \in I\}\big)$, where I is the set of tight inequalities at v.

Prove the following statement:
Let x^* be a vertex where $\max\{c^{\mathrm{T}}x \mid x \in P\}$ is attained, i.e. it exists and is finite. Then, c is contained in the normal cone of x^*.