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Mathematical Optimization — Solution 3

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Fourier-Motzkin elimination and optimization

a) Introduce a new variable z corresponding to the objective function value and add the constraint $z \leq c^\intercal x$ to the constraints set. Find $\alpha_1 := \arg\max\{z \in \mathbb{R} \colon z \in \text{proj}_z(P)\}$ - if it does not exist, the original polyhedron is empty, if $\alpha_1 = \infty$, the problem is unbounded.

Otherwise, proceed and find $\binom{\alpha_1}{\alpha_2} \in \{\bar{x} \in \mathbb{R}^2 \colon \bar{x} \in \text{proj}_{x_n,z}(P), \bar{x}_1 = \alpha_1\}$, then find $(\alpha_1, \alpha_2, \alpha_3)^\mathrm{T} \in \{\bar{x} \in \mathbb{R}^3 \colon \bar{x} \in \text{proj}_{x_{n-1},x_n,z}(P), \bar{x}_{1:2} = \binom{\alpha_1}{\alpha_2}\}$ etc. .

b) We perform Fourier-Motzkin-Elimination as follows:

$$
\begin{array}{lrrrrrrrr}
\multicolumn{9}{l}{\textbf{Original System}:}\\
(i) & - & x_1 & - & x_2 & + & z & \leq & 0\\
(ii) & - & 4x_1 & - & x_2 & & & \leq & -8\\
(iii) & - & x_1 & + & x_2 & & & \leq & 3\\
(iv) & & & - & x_2 & & & \leq & -2\\
(v) & & 2x_1 & + & x_2 & & & \leq & 12\\
\multicolumn{9}{l}{\textbf{Projection on } (x_2, z):}\\
2(i)+(v) & & & - & x_2 & + & 2z & \leq & 12\\
(ii)+2(v) & & & & x_2 & & & \leq & 16\\
2(iii)+(v) & & & & 3x_2 & & & \leq & 18\\
(iv) & & & - & x_2 & & & \leq & -2\\
\multicolumn{9}{l}{\textbf{Removing redundant constraints}:}\\
(i) & & & - & x_2 & + & 2z & \leq & 12\\
(ii) & & & & x_2 & & & \leq & 6\\
(iii) & & & - & x_2 & & & \leq & -2\\
\multicolumn{9}{l}{\textbf{Projection on } z:}\\
(i)+(ii) & & & & & & 2z & \leq & 18\\
(ii)+(iii) & & & & 0 & & & \leq & 4
\end{array}
$$

Thus, $z \leq 9$ implies that the maximal objective function value is 9. Setting $z = 9$, the constraint $-x_2 + 2z \leq 12$ leads us to $-x_2 \leq -6$ and together with $x_2 \leq 6$ we obtain $x_2 = 6$. If we plug in $x_2 = 6$ and $z = 9$ into the constraints $-x_1 - x_2 + z \leq 0$ and $2x_1 + x_2 \leq 12$ we obtain $x_1 = 3$. Indeed, $x = \binom{3}{6}$ is a feasible solution with objective function value 9, so we obtained an optimal solution.

Exercise 2: Farkas Lemma for Standard Form Polyhedra

We transform the system $Ax = b,\ x \geq 0$ into the equivalent system

$$
\begin{pmatrix} A \\ -A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}
$$

Applying the Farkas lemma, studied in class, the latter system has a solution if and only if the following system has no solution: $y_1, y_2, y_0 \geq 0$,

$$(y_1^T, y_2^T, y_0^T)\begin{pmatrix} A \\ -A \\ -I \end{pmatrix} = 0$$

and

$$(y_1^T, y_2^T, y_0^T)\begin{pmatrix} b \\ -b \\ 0 \end{pmatrix} < 0.$$

The latter system can be rewritten as $(y_1 - y_2)^T A = y_0$ and $(y_1 - y_2)^T b < 0$. It is solvable if and only if the system $z \in \mathbb{R}^m$, $z^T A \geq 0$ and $z^T b < 0$ is solvable (which can be seen by observing that $y_0 \geq 0$ and by replacing $z = y_1 - y_2$). This finishes the proof.

Exercise 3: Caratheodory's Theorem for Polytopes
Let k be the number of vertices $v_1, \ldots, v_k$. Without loss of generality, $k \geq n+1$ (so that we can talk of subsets of the vertex set of size $n+1$). Let $x \in \operatorname{conv}(v_1, \ldots, v_k)$, then x can be written as $x = \sum_{j=1}^{k} \lambda_i v_i$, with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. Thus,

$$\binom{x}{1} = \sum_i \lambda_i \binom{v_i}{1}.$$

Consequently, $\binom{x}{1} \in \operatorname{cone}(\binom{v_1}{1}, \ldots, \binom{v_k}{1})$. By Caratheodory's Theorem from the lecture, there are indices $\{i_1, \ldots, i_{n+1}\} \subseteq \{1, \ldots, k\}$ and $u_1, \ldots, u_{n+1} \geq 0$ such that

$$\binom{x}{1} = \sum_{j=1}^{n+1} \mu_j \binom{v_{i_j}}{1}.$$

Therefore, $x = \sum_{j=1}^{n+1} \mu_j v_{i_j}$, $\mu \geq 0$ and $\sum_{j=1}^{n+1} \mu_j = 1$, which implies that $x \in \operatorname{conv}(v_{i_1}, \ldots, v_{i_{n+1}})$.

Exercise 4: Iterated Polyhedral Projections
Call $Q := \operatorname{proj}_{(x_1,\ldots,x_{n-2})}(P)$, $S := \operatorname{proj}_{(x_1,\ldots,x_{n-2})}\left(\operatorname{proj}_{(x_1,\ldots,x_{n-1})}(P)\right)$.
Let $\bar{x} \in \mathbb{R}^{n-2}$.

$$\bar{x} \in Q \Leftrightarrow \exists \alpha, \beta \in \mathbb{R}\colon \begin{pmatrix} \bar{x} \\ \alpha \\ \beta \end{pmatrix} \in P \Leftrightarrow \exists \alpha\colon \begin{pmatrix} \bar{x} \\ \alpha \end{pmatrix} \in \operatorname{proj}_{(x_1,\ldots,x_{n-1})}(P) \Leftrightarrow \bar{x} \in S.$$

Exercise 5: Projection
Consider the function $f(x) := ||x - y||_2^2$. In an upcoming lecture you will see that since it is convex and continuously differentiable, we have that $\mathcal{PO}(y) \in Q$ minimizes f over Q if and only if

$$\nabla f(\mathcal{PO}(y))^T (z - \mathcal{PO}(y)) \geq 0 \qquad \text{for all } z \in Q.$$

Therefore

$$2(\mathcal{PO}(y) - y)^T (z - \mathcal{PO}(y)) \geq 0 \qquad \text{for all } z \in Q$$

or

$$(y - \mathcal{PO}(y))^T (z - \mathcal{PO}(y)) \leq 0 \qquad \text{for all } z \in Q.$$

Note that

$$||z - \mathcal{PO}(y)||_2^2 = ||(z-y) + (y - \mathcal{PO}(y))||_2^2 = ||z-y||_2^2 + 2(z-y)^T(y - \mathcal{PO}(y)) + ||y - \mathcal{PO}(y)||_2^2.$$

Thus

$$\begin{aligned}
||z - \mathcal{PO}(y)||_2^2 - ||z-y||_2^2 &= 2(z-y)^T(y - \mathcal{PO}(y)) + ||y - \mathcal{PO}(y)||_2^2 \\
&= 2(z-y)^T(y - \mathcal{PO}(y)) + (y - \mathcal{PO}(y))^T(y - \mathcal{PO}(y)) \\
&= (y - \mathcal{PO}(y))^T(2z - \mathcal{PO}(y) - y) \\
&= 2(y - \mathcal{PO}(y))^T(z - \mathcal{PO}(y)) + (y - \mathcal{PO}(y))^T(\mathcal{PO}(y) - y) \\
&\leq (y - \mathcal{PO}(y))^T(\mathcal{PO}(y) - y) = -||y - \mathcal{PO}(y)||_2^2 \leq 0.
\end{aligned}$$