ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Operations Research
ETH Zurich HG G21-22

Stephan Artmann stephan.artmann@ifor.math.ethz.ch
Christoph Glanzer christoph.glanzer@ifor.math.ethz.ch

IFOR
Institute for Operations Research
Department of Mathematics

Mathematical Optimization — Solution 12

https://moodle-app2.let.ethz.ch/course/view.php?id=2180

Exercise 1: Submodular Functions

(a) "$\Rightarrow$": Let $S := A \cup \{j\}$, $T := A \cup \{k\}$. Then,

$$f(S \cup T) - f(T) \leq f(S) - f(S \cap T) \quad \Leftrightarrow \quad f(A \cup \{j,k\}) - f(A \cup \{k\}) \leq f(A \cup \{j\}) - f(A).$$

"$\Leftarrow$": Let $A, B \subseteq N$, $S := A \cap B$, $A \setminus B := \{j_1, \dots, j_r\}$, $B \setminus A := \{k_1, \dots, k_s\}$. Then,

$$\begin{aligned} f(B) - f(A \cap B) &= f(S \cup \{k_1, \dots, k_s\}) - f(S) \\ &= \sum_{i=1}^{s} f(S \cup \{k_1, \dots, k_i\}) - f(S \cup \{k_1, \dots, k_{i-1}\}) \\ &\geq \sum_{i=1}^{s} f(S \cup \{k_1, \dots, k_i\} \cup \{j_1\}) - f(S \cup \{k_1, \dots, k_{i-1}\} \cup \{j_1\}) \\ &\geq \dots \geq \sum_{i=1}^{s} f(S \cup \{k_1, \dots, k_i\} \cup \{j_1, \dots, j_r\}) - f(S \cup \{k_1, \dots, k_{i-1}\} \cup \{j_1, \dots, j_r\}) \\ &= \sum_{i=1}^{s} f(A \cup \{k_1, \dots, k_i\}) - f(A \cup \{k_1, \dots, k_{i-1}\}) = f(A \cup B) - f(A). \end{aligned}$$

$\square$

(b) "$\Rightarrow$": Let $S, T \subseteq N$, $T \setminus S := \{j_1, \dots, j_r\}$. Then,

$$\begin{aligned} f(T) &\overset{\text{non-decreasing}}{\leq} f(S \cup T) = f(S) + (f(S \cup T) - f(S)) \\ &= f(S) + \sum_{i=1}^{r} (f(S \cup \{j_1, \dots, j_i\}) - f(S \cup \{j_1, \dots, j_{i-1}\})) \\ &\overset{\text{(a)}}{\leq} f(S) + \sum_{i=1}^{r} (f(S \cup \{j_i\}) - f(S)). \end{aligned}$$

"$\Leftarrow$": Setting $T := S \cup \{j, k\}$ gives

$$\begin{aligned} & f(S \cup \{j,k\}) \leq f(S) + f(S \cup \{j\}) - f(S) + f(S \cup \{k\}) - f(S) \\ \Leftrightarrow\ & f(S \cup \{j\}) - f(S) \geq f(S \cup \{j,k\}) - f(S \cup \{k\}), \end{aligned}$$

which is (a) and therefore implies submodularity.

For monotonicity, let $T \subseteq S$. Now,

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(S \cup \{j\}) - f(S) = f(S).$$

$\square$

Exercise 2: Matroid-union

Let $I, J \in \mathcal{F}$ with $|I| < |J|$. Without loss of generality, assume that $|I \cap E_1| < |J \cap E_1|$ (otherwise, exchange the roles of M_1 and M_2). Using the hint, there exists $e \in (J \cap E_1) \setminus (I \cap E_1)$ such that $(I \cap E_1) \cup e \in \mathcal{F}_1$. Therefore, $I \cup \{e\} \in \mathcal{F}$, as $(I \cap E_1) \cup e = I \cup e \cap E_1 \in \mathcal{F}_1$ and $(I \cup e) \cap E_2 = I \cap E_2 \in \mathcal{F}_2$, where we used that $e \in E_1 \Rightarrow e \notin E_2$.

Exercise 3: Hamiltonian Paths and Matroid Intersection

We consider the following three matroids for intersection:

$\mathcal{M}_1$: **Graphic matroid** $\mathcal{M}_1 = (\mathcal{F}_1, E)$. **Note:** We ignore the direction of the edges.

$\mathcal{M}_2$: **Partition matroid** $\mathcal{M}_2 = (\mathcal{F}_2, E)$ where each node $v \in V$ has at most one outgoing edge: $\mathcal{F}_2 = \{A \subseteq E \quad | \forall v \in V : |\{(v, w) \mid (v, w) \in A\}| \leq 1\}$

$\mathcal{M}_3$: **Partition matroid** $\mathcal{M}_3 = (\mathcal{F}_3, E)$ where each node $v \in V$ has at most one incoming edge: $\mathcal{F}_3 = \{A \subseteq E \quad | \forall v \in V : |\{(w, v) \mid (w, v) \in A\}| \leq 1\}$

Any independent set in $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ represents a disjoint union of directed, simple paths. Therefore, G admits a directed Hamiltonian path if and only if a maximally independent set of cardinality $|V| - 1$ exists in $\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$. Note that in the case where there doesn't exist a directed path between any two vertices of G, an independent set of cardinality $|V| - 1$ cannot exist. Determining whether a Hamiltonian path (or hamiltonian cycle) exists in a given graph is $\mathcal{NP}$-complete. Hence finding the maximal independent set in the intersection of three matroids is $\mathcal{NP}$-hard.