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Mathematical Optimization — Assignment 4

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: The Dual Program of an LP

(a) By introducing variables y_1, y_2 and y_3 for the three rows of the primal program, where $y_1 \geq 0$, $y_2 \leq 0$ and $y_3 \in \mathbb{R}$, we can upper-bound the objective function $c^{\mathrm{T}}x$ as follows:

$$\begin{aligned} & x_1 + 2x_2 - x_3 \\ \leq\ & y_1(2x_1 + 3x_2 + 5x_3) \\ & + \ y_2(-x_1 + 2x_2 - x_3) \\ & + \ y_3(x_1 - 3x_3) \\ \leq\ & y_1 + 5y_2 + 2y_3. \end{aligned}$$

As this needs to hold for all x_i ($i \in \{1, 2, 3\}$) separately, we may divide every row by x_i and the third row by $\pm x_3$, leading to

$$\begin{array}{llllllll} \min & y_1 & + & 5y_2 & + & 2y_3 & & \\ & 2y_1 & - & y_2 & + & y_3 & \geq & 1 \\ & 3y_1 & + & 2y_2 & & & \leq & 2 \\ & 5y_1 & - & y_2 & - & 3y_3 & = & -1 \\ & y_1, y_2 & \geq & 0, & & y_3 & \in & \mathbb{R}. \end{array}$$

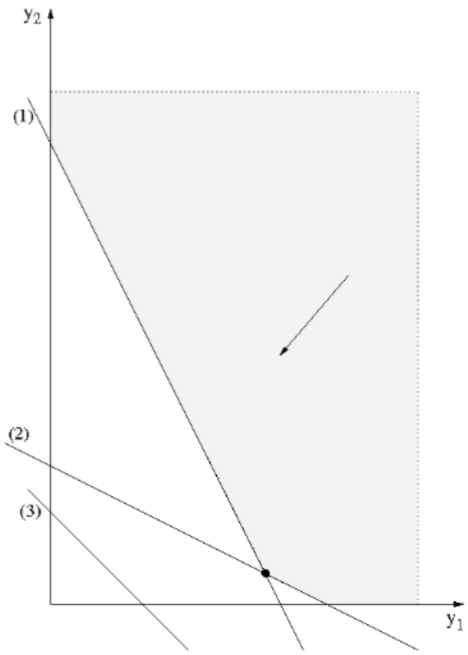
(b) Performing the same operation as in (a) leads to

$$\begin{array}{llllll} \min & y_1^{\mathrm{T}}b & + & y_2^{\mathrm{T}}\gamma & & \\ & A^T y_1 & + & C^T y_2 & = & c \\ & B^T y_1 & + & D^T y_2 & \geq & h \\ & y_1 \geq 0 & & y_2 \text{ free} & & \end{array}$$

Exercise 2: Complementary Slackness Conditions
The dual problem is:

$$
\begin{array}{llcrcll}
\min & 6y_1 & + & 7y_2 & & & \\
\text{s.t.} & 2y_1 & + & y_2 & \geq & 5 & \qquad (1) \\
 & y_1 & + & 2y_2 & \geq & 3 & \qquad (2) \\
 & y_1 & + & y_2 & \geq & 1 & \qquad (3) \\
 & & & y_i & \geq & 0 \quad i = 1,2 &
\end{array}
$$

The feasible reagion of the dual problem is shown in the following graph:



We graphically derive the dual optimal solution: $(y_1^*, y_2^*) = (\frac{7}{3}, \frac{1}{3})$. Restrictions (1) and (2) are satisfied with equality, whereas restriction (3) is not. Using the complementary slackness conditions, we conclude $x_3^* = 0$. Moreover, as $y_1^* \neq 0, y_2^* \neq 0$, the two primal restrictions have to be satisfied with equality by the optimal solution, i.e. (we already use $x_3^* = 0$):

$$2x_1 + x_2 = 6, \qquad x_1 + 2x_2 = 7$$

From the first equation, we get $x_2 = 6 - 2x_1$. Using the second equation, this yields: $-3x_1 = -5$. Hence the primal optimal solution is $(x_1^*, x_2^*, x_3^*) = (\frac{5}{3}, \frac{8}{3}, 0)$ with a (maximum) objective function value of $16\frac{1}{3}$.

Exercise 3: Infeasibility and LP Duality

(a) LPs of this type are exactly all unbounded LPs. Indeed, suppose a feasible LP max $c^T x$ subject to $Ax \leq b$ and $x \geq \mathbf{0}$ is unbounded, that is, there is z such that $z \geq \mathbf{0}$, $Az \leq \mathbf{0}$ and $c^T z > 0$. Direct application of Farkas' Lemma (Theorem 2.4) yields that the system $-A^T x \leq -c$ and $x \geq \mathbf{0}$ is infeasible. Therefore the dual LP min $b^T y$ subject to $A^T y \geq c$ and $y \geq \mathbf{0}$ is infeasible. Under the assumption that the primal LP is feasible, the opposite direction also holds.

Note that if an LP has optimal solution zero, then, by the strong duality theorem, also its dual is feasible and attains the value zero.

(b) The following LP is an example where both the primal and dual problem are infeasible.

Primal problem:
$$\begin{array}{lrcrcrl} \max & x_1 & + & x_2 & & & \\ \text{s.t.} & -x_1 & + & x_2 & \leq & -1 & \\ & x_1 & - & x_2 & \leq & -1 & \\ & & & x_i & \geq & 0 & i=1,2 \end{array}$$

Dual problem:
$$\begin{array}{lrcrcrl} \min & -y_1 & - & y_2 & & & \\ \text{s.t.} & -y_1 & + & y_2 & \geq & 1 & \\ & y_1 & - & y_2 & \geq & 1 & \\ & & & y_i & \geq & 0 & i=1,2 \end{array}$$

Checking infeasiblity is easy. Adding the two nontrivial constraints in each LP gives $0 \leq -2$ and $0 \geq 2$, respectively.

Exercise 4: Recession Cone - Extreme Ray with Positive Cost
"$\Leftarrow$:" Choose r such that $c^{\mathrm{T}}r > 0$ for the given vector c. Now, $Ar \leq 0$, by assumption. Therefore, for $\lambda \geq 0$, $c^{\mathrm{T}}(\lambda \cdot r) = \lambda \cdot c^{\mathrm{T}}r \to \infty$, $\lambda \to \infty$, while $A(\lambda \cdot r) = \lambda \cdot Ar \leq 0$, i.e. $\lambda \cdot r$ remains feasible for all $\lambda \geq 0$.

"$\Rightarrow$:" We present two solutions, the first of which uses the Theorem of Minkowski-Weyl. As this result is part of the proof of Minkowski-Weyl, we present a second solution which is based on an iterative statement.

- **Solution I**: Using Minkowski-Weyl's Theorem. Assume that for all extreme rays r_i of C, $c^{\mathrm{T}}r_i \leq 0$. Choose any $x \in C$ such that $c^{\mathrm{T}}x > 0$. Such an x must exist as $\max\{c^{\mathrm{T}}x \mid Ax \leq 0\} = +\infty$. Write $x = \sum_{i=1}^{k} \lambda_i r_i$ with $\lambda_i \geq 0$ for all $i \in \{1, \ldots, k\}$. Then, $c^{\mathrm{T}}x = \sum_{i=1}^{k} \lambda_i c^{\mathrm{T}} r_i \leq 0$, a contradiction, as $c^{\mathrm{T}}x > 0$.
- **Solution II**: Using an iterative statement. Choose any $x \in C$ such that $c^{\mathrm{T}}x > 0$. Such an x must exist as $\max\{c^{\mathrm{T}}x \mid Ax \leq 0\} = +\infty$. Denote by I the set of row indices of all tight rows of $Ax \leq 0$ at x. Then, either, $\mathrm{rank}(A_{I,\cdot}) = n - 1$, which implies that x is an extreme ray, or $\mathrm{rank}(A_{I,\cdot}) \leq n - 2$.

  In the second case, proceed as follows: Consider $\{x \in \mathbb{R}^n \mid A_{I,\cdot}x = 0,\ c^{\mathrm{T}}x = 0\}$. By assumption, the kernel of the matrix $\begin{bmatrix} A_{I,\cdot} \\ c^{\mathrm{T}} \end{bmatrix}$ has dimension greater or equal than 1, as $\mathrm{rank}(A_{I,\cdot}) \leq n - 2$ (rank–nullity theorem). Therefore, there exists $l \in \mathbb{R}^n$, $l \neq 0$, such that $A_{I,\cdot}l = 0$ and $c^{\mathrm{T}}l = 0$.

  Let $J := [m] \setminus I$. Consider the partition $J = J_1 \cup J_2$ such that $J_2 \neq \emptyset$ and $A_{J_1,\cdot}l \leq 0$, $A_{J_2,\cdot}l > 0$. We can assume without loss of generality that such a partition exists: As C is pointed, A has full rank, which implies that there is at least one index j such that $A_{j,\cdot}l \neq 0$. If $A_{j,\cdot}l > 0$, we are done. Otherwise, setting $l := -l$ and forming the partition again leads to the desired result.

  Now, define $\lambda^* := \min\left\{-\frac{A_{j,\cdot}x}{A_{j,\cdot}l} \mid j \in J_2\right\}$ (note that $A_{j,\cdot}x < 0$ by assumption). Let index $k \in J_2$ attain the minimum. For $x' := x + \lambda^* l$, it holds that

  - $A_{I,\cdot}x' = A_{I,\cdot}x + \lambda^* A_{I,\cdot}l = A_{I,\cdot}x = 0$,
  - $A_{k,\cdot}x' = A_{k,\cdot}x - A_{k,\cdot}\frac{A_{k,\cdot}x}{A_{k,\cdot}l}l = 0$,
  - $A_{j,\cdot}x' \leq A_{j,\cdot}x - A_{j,\cdot}\frac{A_{j,\cdot}x}{A_{j,\cdot}l}l = 0$, $j \in J_2 \setminus \{k\}$,
  - $A_{j,\cdot}x' = A_{j,\cdot}x - A_{j,\cdot}\frac{A_{k,\cdot}x}{A_{k,\cdot}l}l \leq A_{j,\cdot}x \leq 0$, $j \in J_1$.

  Therefore, $\mathrm{rank}(A_{I(x'),\cdot}) > \mathrm{rank}(A_{I,\cdot})$, while $c^{\mathrm{T}}x' = c^{\mathrm{T}}x > 0$. Iterate this statement until we reach an extreme ray r with $c^{\mathrm{T}}r > 0$.