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Mathematical Optimization — Assignment 6

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Modeling linear programs

(a) Consider the LP $\min\{c^\mathsf{T}x \mid x \in \mathbb{R}^n,\ Ax \geq b\}$. How can you find a point in the interior of the feasible set, i.e. in $\{x \in \mathbb{R}^n,\ Ax > b\}$, if one exists?

(b) Assume that X and Y are finite sets of points in $\mathbb{Q}^n$. Write an LP to decide whether there exists a hyperplane $\{x \in \mathbb{R}^n \mid h^\mathsf{T}x = b\}$ with the property that all points in X are lying (strictly) on one side of the hyperplane, while all points in Y are lying (strictly) on the other side. Here, 'strictly' means that we do not want any point to lie directly on the hyperplane.
Hint: The decision whether this is possible is allowed to depend on the resulting objective value.

Exercise 2: Unbounded Simplex Tableau
Consider the following LP:

$$\begin{aligned}
\text{minimize} \quad & \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}^\mathrm{T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
\text{subject to:} \quad & \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

(a) Transform the problem to a LP over a standard-form polyhedron P such that the Simplex algorithm is applicable. Write down a Simplex tableau of your choice corresponding to the transformed problem. Why is it unbounded?

(b) Find a feasible point $x \in P$.

(c) Using the tableau, find $d \in \mathbb{R}^{n+m}$ such that $x + \lambda \cdot d \in P$, for all $\lambda \geq 0$.

Exercise 3: Simplex Phase One
Consider the following LP:

$$\begin{array}{lrcl}
\text{minimize} & \text{x} & & \\
\text{subject to:} & -x & \leq & -3 \\
& x & \leq & 5 \\
& x & \geq & 0
\end{array}$$

Note that $x = 0$ is not a feasible solution to this LP and we therefore cannot use the slack basis, as we usually did. The goal of this exercise is to find an initial basis for the original problem.

(a) Reformulate the problem above such that we are in a setting in which we can apply the Simplex method.

(b) Make sure that $b \geq 0$ by multiplying the rows by (-1), if necessary.

(c) Formulate the auxiliary LP for the problem above corresponding to "Phase One" of the Simplex method. For a minimization problem of the form $\min\{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$, where $b \geq 0$, one way to formulate the auxiliary problem is:

$$\min\left\{\sum_i y_i \;\middle|\; Ax + Iy = b,\ x \geq 0,\ y \geq 0\right\}.$$

(d) Solve the auxiliary LP using the Simplex method with a pivoting method of your choice. Note that we can now start with a slack basis. Is the original problem feasible? If yes, extract an initial basis from the resulting tableau.

Exercise 4: (*) Lexicographic Pivoting Rule

Show that the Simplex algorithm with the lexicographic pivoting rule always terminates after a finite number of steps.

Hint: For the definition of the lexicographic pivoting rule, see the lecture notes. Start by choosing the extended tableau, where the unit matrix corresponding to the basis variables is situated on the very left:

$$T_{\text{long}} = \left[\begin{array}{c|cc} -c_B{}^{\mathsf{T}} A_B^{-1} b & 0 \dots 0 & \bar{c}_j,\ j \in N \\ \hline A_B^{-1} b & I & A_B^{-1} A_N \end{array}\right].$$

Let us enumerate the rows of T by $\{0, 1, \dots, m\}$ and the columns of T by $\{0, 1, \dots, n\}$. Observe that initially, for all $i \geq 1$, $T_{i,\cdot} >_{\text{lex}} 0$. Show that this also holds true after a basis exchange. Use this to show that $T_{0,\cdot}^{\text{new}} >_{\text{lex}} T_0$, where $T_{0,\cdot}^{\text{new}}$ is the first row of T after the basis exchange. Conclude that no basis is visited more than once.