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Mathematical Optimization — Assignment 12

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Submodular Functions
We are given a finite set $N = \{1, \dots, n\}$ and a function $f : 2^N \mapsto \mathbb{R}_+$. Prove the following two statements.

(a) The function f is submodular if and only if for all j, k, $j \neq k$ and $S \subseteq N \setminus \{j, k\}$:

$$f(S \cup \{j\}) - f(S) \geq f(S \cup \{j, k\}) - f(S \cup \{k\}).$$

Hint: Telescoping sum.

(b) The function f is submodular and non-decreasing (i.e., for all $S \subseteq T \subseteq N$ we have $f(S) \leq f(T)$) if and only if for all $S, T \subseteq N$:

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} (f(S \cup \{j\}) - f(S)).$$

Hint: Use (a) iteratively.

Exercise 2: Matroid-union
Let $M_1 := (E_1, \mathcal{F}_1)$ and $M_2 := (E_2, \mathcal{F}_2)$ be two matroids with $E_1 \cap E_2 = \emptyset$.
Let $M := (E_1 \cup E_2, \mathcal{F})$ with

$$\mathcal{F} = \{F \subset E_1 \cup E_2 \mid F \cap E_i \in \mathcal{F}_i \;\; \forall\, i \in \{1, 2\}\}$$

Show that M is a matroid.

Hint: Use the following equivalent definition of a matroid: An independence system $(E, \mathcal{F})$ is a matroid if for every $I, J \in \mathcal{F}$ with $|I| < |J|$ there exists $a \in J \setminus I$ such that $I \cup \{a\} \in \mathcal{F}$.

Exercise 3: Hamiltonian Paths and Matroid Intersection
A Hamiltonian Path in a directed graph is a directed path that visits every vertex exactly once. The Hamiltonian path problem is defined as follows: Given a directed graph $G = (V, E)$, decide whether it contains a Hamiltonian path or not. This problem is $\mathcal{NP}$-hard.

Show that the Hamiltonian path problem in directed graphs can be modeled as the problem of finding a maximum-cardinality independent set in the intersection of three matroids defined on the arc set of the graph. In other words, prove that finding a maximum-cardinality independent set in the intersection of three matroids is NP-hard.

Hint: The partition matroid is defined as follows: Given a ground set N, disjoint sets $B_1, \dots, B_k$ and $d_1, \dots, d_k \in \mathbb{Z}$ such that $0 \leq d_i \leq |B_i|$ for all i, define $\mathcal{I} := \{I \subseteq N \text{ s.t. } \forall i\colon |I \cap B_i| \leq d_i\}$.