ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Operations Research
ETH Zurich HG G21-22

Stephan Artmann stephan.artmann@ifor.math.ethz.ch
Christoph Glanzer christoph.glanzer@ifor.math.ethz.ch

IFOR
Institute for Operations Research
Department of Mathematics

Mathematical Optimization — Assignment 9

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Unimodular and Totally Unimodular Matrices

Let $A \in \mathbb{R}^{m\times n}$ and let I be the n-dimensional identity matrix. Prove the following statements:

(a) A is totally unimodular $\Leftrightarrow$ A^{T} is totally unimodular.

(b) A is totally unimodular $\Rightarrow$ $[A \mid I]$ is totally unimodular.

(c) $[A \mid I]$ unimodular $\Rightarrow$ A is totally unimodular.

(d) A is totally unimodular $\Leftrightarrow$ $\begin{bmatrix} A \\ -A \end{bmatrix}$ is totally unimodular.

(e) A is totally unimodular $\Leftrightarrow$ $\begin{bmatrix} A \\ -A \\ I \\ -I \end{bmatrix}$ is totally unimodular.

(f) Assume that $A \in \{0,1\}^{m\times n}$ is a matrix with the *consecutive-ones property*, i.e. for all $i \in \{1,\ldots,m\}$, there is $j_1 \leq j_2$, $j_1, j_2 \in \{1,\ldots,n\}$ such that

$$\begin{aligned} A_{i,k} &= 0,\ k \leq j_1 - 1, \\ A_{i,k} &= 1,\ k \in \{j_1,\ldots,j_2\}, \\ A_{i,k} &= 0,\ k \geq j_2 + 1. \end{aligned}$$

Prove that A is TU.

(g) Let $G = (V,E)$ be a bipartite graph and $A \in \{0,1,-1\}^{|V|\times|E|}$ be its incidence matrix, i.e. $A_{v,e} = 1 \Leftrightarrow v \in e$ and 0 otherwise. Add an additional row $A_{n+1,\cdot} = (1,1,\ldots,1)$ to A. Does the matrix stay TU? Give a proof or counterexample.

(h) Assume that the matrices $[A \mid a] \in \mathbb{Z}^{m_1,n_1+1}$ and $\begin{bmatrix} b^{\mathrm{T}} \\ B \end{bmatrix} \in \mathbb{Z}^{m_2+1\times n_2}$ are totally unimodular, where a and b are column vectors of appropriate dimension. Show that then, the so-called 2-sum is totally unimodular as well:

$$T := \begin{bmatrix} A & ab^{\mathrm{T}} \\ 0 & B \end{bmatrix}.$$

Note: From statements b) and c) it follows easily that A TU $\Leftrightarrow$ $[A \mid I]$ TU and A TU $\Leftrightarrow$ $[A \mid I]$ unimodular.

Exercise 2: A convex problem in which strong duality fails

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & e^{-x_1} \\ \text{subject to} & x_1^2/x_2 \leq 0 \\ & x \in D \end{array}$$

where $D := \{(x_1,x_2) \in \mathbb{R}^2 | x_2 > 0\}$.

(a) Verify that this is a convex optimization problem and find the optimal value.

(b) Give the convex optimization dual maximization problem and find the optimal dual solution. What is the optimal duality gap?

(c) Why does the duality theorem for convex optimization not apply here?