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Mathematical Optimization — Assignment 10

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: The pigeonhole principle as LP-formulation
The pigeonhole principle states that we cannot place $n+1$ pigeons in n holes, where we only allow one pigeon per hole. This statement can be expressed by the infeasibility of a binary integer optimization problem. Assume throughout this exercise that $n \geq 2$.

Consider the following two formulations of the Pigeonhole problem, where we define

$$x_{ij} = \begin{cases} 1, & \text{if pigeon } i \text{ is placed in hole } j, \\ 0, & \text{otherwise.} \end{cases}$$

$$(P1) \qquad \begin{array}{ll} \sum_{j=1}^{n} x_{ij} = 1, & i = 1, \ldots, n+1, \\ x_{ij} + x_{kj} \leq 1, & j = 1, \ldots, n, \text{ and } i, k = 1, \ldots, n+1, \text{ where } i \neq k, \\ x_{ij} \in \{0, 1\}, & i = 1, \ldots, n+1, \text{ and } j = 1, \ldots, n. \end{array}$$

$$(P2) \qquad \begin{array}{ll} \sum_{j=1}^{n} x_{ij} = 1, & i = 1, \ldots, n+1, \\ \sum_{i=1}^{n+1} x_{ij} \leq 1, & j = 1, \ldots, n, \\ x_{ij} \in \{0, 1\}, & i = 1, \ldots, n+1, \ j = 1, \ldots, n. \end{array}$$

i) Denote by (P_1^R) and (P_2^R) the LP-relaxation of the polyhedra described by the constraints of $(P1)$ and $(P2)$, respectively. Which of the models above is stronger? In other words, show that either $(P_1^R) \subseteq (P_2^R)$ or $(P_2^R) \subseteq (P_1^R)$.

ii) Show that one of these models has the property that its LP-relaxation is infeasible while the other one isn't.

Exercise 2: Knapsack: Bad approximation of optimal solution
Let $a, c \in \mathbb{Z}_{\geq 0}^n$, $b \in \mathbb{Z}_{\geq 0}$ and assume that $a_i \leq b$, for all $i \in \{1, \ldots, n\}$. Consider the binary knapsack problem

$$\max\{c^{\mathrm{T}}x \mid a^{\mathrm{T}}x \leq b, \ x \in \{0,1\}^n\},$$

and its LP relaxation

$$\max\{c^{\mathrm{T}}x \mid a^{\mathrm{T}}x \leq b, \ x \in [0,1]^n\}.$$

Assume that $\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \ldots \geq \frac{c_n}{a_n}$. We observed in class that the optimal solution for the LP-relaxation x^* is given by

$$x_i^* = \begin{cases} 1, & i \leq i^*, \\ \delta, & i = i^* + 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $i^* \in \{1, \dots, n\}$ is the largest number such that $\sum_{i=1}^{i^*} a_i \leq b$ and $\delta := \frac{b - \sum_{i=1}^{i^*} a_i}{a_{i^*+1}}$.

Define the vectors x^1 and x^2 by

$$x_i^1 = \begin{cases} 1, & i \leq i^* \\ 0, & \text{otherwise,} \end{cases}$$

and

$$x_i^2 = \begin{cases} 1, & i = i^* + 1 \\ 0, & \text{otherwise.} \end{cases}$$

We have seen in class that $\max\{c^{\mathrm{T}} x^1, c^{\mathrm{T}} x^2\}$ is an $\frac{1}{2}$-approximation for the binary knapsack problem. Let x_{BK}^* be the optimal solution to the binary knapsack problem.

i) Construct an instance of the binary knapsack problem where the solution $c^{\mathrm{T}} x^1$ is arbitrarily far away from $c^{\mathrm{T}} x_{BK}^*$.

ii) Construct an instance of the binary knapsack problem where the solution $c^{\mathrm{T}} x^2$ is arbitrarily far away from $c^{\mathrm{T}} x_{BK}^*$.

The term 'arbitrary' means that for a given $N \geq 1$, $c^{\mathrm{T}} x_{BK}^* - c^{\mathrm{T}} x^1 \geq N$, resp., $c^{\mathrm{T}} x_{BK}^* - c^{\mathrm{T}} x^2 \geq N$.

Exercise 3: Integral polyhedra

Let $P \subset \mathbb{R}^n$ be a non-empty polyhedron without lines, i.e. there exists at least one vertex. Prove the following statement:

P is integral if and only if $\max_{x \in P} c^{\mathrm{T}} x$ is integral ($\in \mathbb{Z}$) or unbounded for all $c \underline{\in \mathbb{Z}^n}$.

Hint: Recall that a pointed polyhedron P is called integral if and only if every vertex of P is integral. One direction follows easily. For the other direction, choose a vertex $x^* \notin \mathbb{Z}^n$ and $c \in \mathbb{Q}^n$ such that x^* is an optimal vertex solution w.r.t. c (why does such a c exist?). Rescale c such that $c \in \mathbb{Z}^n$. Let $j \in \{1, \dots, n\}$ such that $x_j^* \notin \mathbb{Z}$. Now, consider both c and $\bar{c} := c + (1/a) \cdot e_j$, where e_j is the j^{th} unit vector and $a \in \mathbb{Z}$ is sufficiently large such that x^* is still optimal w.r.t. $\bar{c}$. Try to conclude that there exists an integral cost vector for which the optimal objective value is fractional.