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Mathematical Optimization — Solution 10

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: The pigeonhole principle as LP-formulation

i) We want to show that $(P_2^R) \subseteq (P_1^R)$. We therefore need to show that all constraints of the latter are implied by constraints of the former. All constraints apart from $x_{ij} + x_{kj} \leq 1$ for $j = 1, \dots, n$ and $i, k = 1, \dots, n+1$, $i \neq k$ are fulfilled. This set of constraints is fulfilled as well, as $x_{ij} + x_{kj} \leq \sum_{i=1}^{n+1} x_{ij} \leq 1$, using that $x_{ij} \geq 0$.

ii) Model $(P1)$ does not have an empty LP-relaxation, as can be seen by letting $x_{ij} = \frac{1}{n}$, for all $i = 1, \dots, n+1$ and $j = 1, \dots, n$: It holds that $\sum_{j=1}^{n} x_{ij} = n/n = 1$ for all $i = 1, \dots, n+1$ and $x_{ij} + x_{jk} = \frac{2}{n} \leq 1$ as we assumed that $n \geq 2$.

We want to show that the LP-relaxation of the polyhedron in model (P_2) is infeasible. The LP-relaxation is given by:

$$(P_2^R) \qquad \begin{array}{ll} \sum_{j=1}^{n} x_{ij} = 1, & i = 1, \dots, n+1, \\ \sum_{i=1}^{n+1} x_{ij} \leq 1, & j = 1, \dots, n, \\ x_{ij} \geq 0, & i = 1, \dots, n+1,\ j = 1, \dots, n. \end{array}$$

Note that we removed the constraints of the form $x_{ij} \leq 1$ as these are implied by the second constraint already. We want to use Farkas' Lemma. This amounts to calculating the dual of (P_2^R) with $c = \vec{0}$. Define y_i, $i = 1, \dots, n+1$, corresponding to the first type of constraint of (P_2^R), z_j, $j = 1, \dots, n$ for the second type of constraint of (P_2^R) and w_{ij} for the non-negativity constraints. We obtain the following.

$$\min \left\{ \sum_{i=1}^{n+1} y_i + \sum_{j=1}^{n} z_j \;\middle|\; y_i + z_j + w_{ij} \geq 0,\ \forall i, j,\ y_i \text{ free},\ z_j \geq 0,\ w_{ij} \leq 0 \right\}.$$

Choose $y_i = -1$, $z_j = 1$, $w_{ij} = 0$ for all i and j. This forms a feasible solution with objective value $-(n+1) + n = -1$, implying that P_2^R is infeasible.

Exercise 2: Knapsack: Bad approximation of optimal solution

i) Consider the following instance of a binary knapsack problem, where $N \geq 1$, $N \in \mathbb{N}$:

$$\begin{array}{ll} \max & x_1 + N x_2 \\ \text{s.t.} & \frac{1}{N+1} x_1 + x_2 \leq 1, \\ & x \in \{0,1\}^2. \end{array}$$

It holds that $\frac{c_1}{a_1} \geq \frac{c_2}{a_2}$. In this case, $x^1 = (1,0)^{\mathrm{T}}$, while $x^2 = (0,1)^{\mathrm{T}}$. By checking all possible solutions, we can easily see that $x^2 = x^*_{BK}$, where $c^{\mathrm{T}} x^2 = N$, but $c^{\mathrm{T}} x^1 = 1$. For $N \to \infty$, we can get an arbitrarily bad approximation of $c^{\mathrm{T}} x^*_{BK}$.

ii) Consider the following instance of a binary knapsack problem, where $N \geq 1$, $N \in \mathbb{N}$:

$$\begin{array}{ll} \max & N x_1 + x_2 \\ \text{s.t.} & x_1 + 2 x_2 \leq 2, \\ & x \in \{0,1\}^2. \end{array}$$

It holds that $\frac{c_1}{a_1} \geq \frac{c_2}{a_2}$. Again, $x^1 = (1,0)^{\mathrm{T}}$, while $x^2 = (0,1)^{\mathrm{T}}$. By checking all possible solutions, we can easily see that $x^1 = x^*_{BK}$, where $c^{\mathrm{T}}x^1 = N$, but $c^{\mathrm{T}}x^2 = 1$. For $N \to \infty$, we can get an arbitrarily bad approximation of $c^{\mathrm{T}}x^*_{BK}$.

Exercise 3: Integral Polyhedra
Let us start by proving the hint:

Claim:
Let P be a (rational) polyhedron and v a vertex of P. Then, there exists $c \in \mathbb{Q}^n$ such that $c^{\mathrm{T}}v > c^{\mathrm{T}}w$ for $w \neq v$ being a vertex of P.

Proof: Choose $c^{\mathrm{T}} := \sum_{i\in I(v)} A_{i,\cdot} \in \mathbb{Q}^n$ (as P is a rational polyhedron). Let $w \in P$ be a different vertex of P. Then, there exists $i \in I(v)$ such that $A_{i,\cdot}w < b_i$. Therefore, $c^{\mathrm{T}}w < \sum_{i\in I(v)} A_{i,\cdot}v = \sum_{i\in I(v)} b_i$. $\square$

" $\Rightarrow$ ": Suppose that P is a non-empty, pointed, integral polyhedron, i.e. every vertex of P is integral. Assume that for a given $c \in \mathbb{Z}^n$, the problem is bounded. Then, there exists an optimal solution x^* in P which is a vertex and therefore integral.

" $\Leftarrow$ ": Assume by contradiction that P is not integral, i.e. there exists a fractional vertex x^* of P. Let x^*_j be a fractional component of x^*. Since x^* is a vertex of P, we know that there exists $c \in \mathbb{Q}^n$ such that x^* is the unique optimal vertex solution of $\max\{c^{\mathrm{T}}x \mid x \in P\}$.

Now, scale c such that it is in $\mathbb{Z}^n$. Then, for $a \in \mathbb{Z}$ sufficiently large and $\bar{c} = c + (1/a)e_j$, x^* also maximizes $\bar{c}$. Thus, x^* is an optimal solution for the integer cost vectors ac and $a\bar{c}$. Since $a\bar{c}^T x^* - ac^T x^* = x^*_j$, either $a\bar{c}^T x^*$ or $ac^T x^*$ is fractional, implying that there exists an integer vector for which the optimal solution is fractional, a contradiction.