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Mathematical Optimization — Solution 9

https://moodle-app2.let.ethz.ch/course/view.php?id=2180

Exercise 1: Unimodular and Totally Unimodular Matrices
Let $A \in \mathbb{R}^{m\times n}$ and let I be the n-dimensional identity matrix.

a) $\Rightarrow$: Assume that A is totally unimodular. Choose any square submatrix $(A')^{\mathrm{T}}$ of A^{T}. Then, A' is a square submatrix of A and therefore, $\det (A')^{\mathrm{T}} = \det(A') \in \{\pm 1, 0\}$.
$\Leftarrow$: Apply the statement that we proved above to A^{T}.

b) Assume that A is totally unimodular. Choose any square submatrix $B \in \mathbb{R}^{k\times k}$ of $[A \mid I]$. B arises from $[A \mid I]$ by removing some rows and columns. Therefore, $B = [A' \mid I']$, where $A' \in \mathbb{R}^{k\times l}$ and $I' \in \mathbb{R}^{k\times(k-l)}$. Applying Laplace's rule to all columns in I' consecutively erases the corresponding rows in A' (rows corresponding to the unit vector forming the respective column in I') and leads to $\det(B) = \pm\det(B')$, where B' is a sub-matrix of A' and therefore also of A. Thus, $\det(B) \in \{\pm 1, 0\}$.

c) Assume that $[A \mid I]$ is unimodular. Choose any quadratic submatrix $A' \in \mathbb{R}^{k\times k}$ of A. A' arises from A by removing certain columns and rows. We will now choose a $n \times n$ submatrix B from $[A \mid I]$ such that $\det(B) = \pm\det(A')$. First, let B contain all the columns of A which are left in A', which are k in total. Concerning the rows, we removed $m-k$ rows from A to get to A'. Choose the rest of the columns of B to be the columns of I corresponding to the indices of the removed rows. We therefore choose $m-k+k$ columns in total, making B a $m\times m$ matrix and a submatrix of $[A \mid I]$. Therefore, $\det(B) \in \{\pm 1, 0\}$. When calculating the determinant of B, we see when using Laplace on the $m-k$ index vectors (column-wise), that $\det(B) = \det(A')$ as we eliminate the exact same rows. Therefore, $\det(A') \in \{0, \pm 1\}$, proving the statement.

d) " $\Leftarrow$ ": Choose a submatrix A' of A. Then, A' is also a submatrix of $\begin{bmatrix} A \\ -A \end{bmatrix}$, leading to $\det(A') \in \{\pm 1, 0\}$.
" $\Rightarrow$ ": Choose any submatrix B of $\begin{bmatrix} A \\ -A \end{bmatrix}$. Either there are indices i, j such that $B_{i,\cdot} = -B_{j,\cdot}$ in which case the determinant of B is zero or otherwise, B arises from a submatrix of A by multiplying some of its rows by (-1). Therefore, $\det(B) \in \{0, \pm 1\}$.

e) From b), c) and d), we know that
$$A \text{ TU} \Leftrightarrow A^{\mathrm{T}} \text{ TU} \Leftrightarrow [A^{\mathrm{T}} \mid I] \text{ TU} \Leftrightarrow [A^{\mathrm{T}} \mid I \mid -A^{\mathrm{T}} \mid -I] \text{ TU}.$$
Permuting the columns does not change any determinant of any submatrix, therefore
$$[A^{\mathrm{T}} \mid I \mid -A^{\mathrm{T}} - \mid I] \text{ TU} \Leftrightarrow [A^{\mathrm{T}} \mid -A^{\mathrm{T}} \mid I \mid -I] \text{ TU}.$$
Now, as A TU $\Leftrightarrow$ A^{T} TU, $[A^{\mathrm{T}} \mid -A^{\mathrm{T}} \mid I \mid -I]$ TU $\Leftrightarrow [A^{\mathrm{T}} \mid -A^{\mathrm{T}} \mid I \mid -I]^{\mathrm{T}} = \begin{bmatrix} A \\ -A \\ I \\ -I \end{bmatrix}$ is totally unimodular.

f) We use the Theorem by Ghouila-Houri on the columns of A. Let $R \subseteq \{1, \ldots, n\}$ be a subset of the columns. Now, the matrix $A_{\cdot,R}$ is again consecutive-ones, which is why we can w.l.o.g. assume that $R = \{1, \ldots, n\}$. Now, we create our partition $R = R_1 \cup R_2$ such that R_1 contains all even numbers of indices and R_2 contains all odd numbers. Consider now an arbitrary row $i \in \{1, \ldots, m\}$. It holds that $\sum_{j\in R_1} A_{i,j} - \sum_{j\in R_2} A_{i,j} \in \{0, \pm 1\}$ as the matrix is a consecutive-ones matrix and we sum up the elements alternately. This proves the statement by Ghouila-Houri's criterion.

g) The matrix stays TU. This can be seen by Ghouila-Houri's criterion as follows:

Let $R \subseteq \{1, \dots, m+1\}$ be a subset of rows. We do a case by case analysis, depending on whether the $(n+1)^{\text{st}}$ row is part of R or not.

Assume first that $R \subseteq \{1, \dots, m\}$. We assume that G is bipartite, i.e. we can partition $V = V_1 \cup V_2$ such that $V_1 \cap V_2 = \emptyset$ and all edges run between V_1 and V_2. In this case, partition the rows according to whether the vertex corresponding to the row is in V_1 or V_2. Call the partition $R = R_1 \cup R_2$. We know therefore that

$$\sum_{i \in R_1} A_{i,\cdot} - \sum_{i \in R_2} A_{i,\cdot} \in \{0, 1, -1\}^n,$$

in every column, we cannot have more than one entry in R_i, $i = 1, 2$.

In the other case, $\{m+1\} \in R$. Choose now $R_1 := R \setminus \{m+1\}$, $R_2 := \{m+1\}$. Consider any column $j \in \{1, \dots, n\}$. Then,

$$\sum_{i \in R_1} A_{i,j} - \sum_{i \in R_2} A_{i,j} = \underbrace{\sum_{i \in R \setminus \{m+1\}} A_{i,j}}_{\in \{0,1,2\}} - \underbrace{A_{m+1,j}}_{1} \in \{0, 1, -1\}.$$

As this holds for every column, the claim follows by Ghouila-Houri's criterion.

h) Let $J \subseteq \{1, \dots, n_1 + n_2\}$. Define $J_B = J \cap \{n_1 + 1, \dots, n_1 + n_2\}$, $J_A = J \cap \{1, \dots, n_1\}$. Let J_B^+, J_B^- be a partitions for $\begin{bmatrix} b^{\mathrm{T}} \\ B \end{bmatrix}$ according to the Ghouila-Houri Theorem.

- If $\sum_{i \in J_B^+} b_i - \sum_{i \in J_B^-} b_i = 0$, then we can choose a partition of J_A for $[A]$ into J_A^+, J_A^- according to the G.H.-Theorem. Then $J_A^+ \cup J_B^+$ and $J_A^- \cup J_B^-$ give the desired decomposition of J into two sets.
- Otherwise, w.l.o.g. $\sum_{i \in J_B^+} b_i - \sum_{i \in J_B^-} b_i = 1$. Then choose a partition of $J_A \cup \{n_1 + 1\}$ for $[A \mid a]$ into $J_A^+ \cup \{n_1 + 1\}$, J_A^- according to the G.H.-Theorem. Then $J_A^+ \cup J_B^+$ and $J_A^- \cup J_B^-$ give the desired decomposition of J.

Exercise 2: A convex problem in which strong duality fails

(a) $f(x_1, x_2) := e^{-x_1}$ and $g(x_1, x_2) := \frac{x_1^2}{x_2}$ are convex on the domain $D := \{(x_1, x_2) \mid x_2 > 0\}$ since they are smooth and their Hessian is positive semidefinite,

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} e^{-x_1} & 0 \\ 0 & 0 \end{pmatrix} \succeq 0 \qquad \nabla^2 g(x_1, x_2) = \begin{pmatrix} \frac{2}{x_2} & -2\frac{x_1}{x_2^2} \\ -2\frac{x_1}{x_2^2} & 2\frac{x_1^2}{x_2^3} \end{pmatrix} \succeq 0$$

Hence, the problem is a convex optimization problem. The feasible set is $\{(0, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ and the optimal value is $p^* = 1$.

(b) The Lagrangian is $\inf_{x \in D} L(x, u)$, where $L(x, u) = e^{-x_1} + u x_1^2 / x_2$. Then we have

$$\inf_{x \in D} L(x, u) = \inf_{x \in D} \{e^{-x_1} + u x_1^2 / x_2\} = \begin{cases} 0 & u \geq 0 \\ -\infty & u < 0 \end{cases}$$

(note that for $u < 0$ the function $L(x, u)$ is decreasing in x). Thus, we can write the dual problem as

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & u \geq 0 \end{array}$$

with optimal value $d^* = 0$. The optimal duality gap is $p^* - d^* = 1$.

(c) Slater's condition is not satisfied. Namely, no point $(x_1, x_2) \in D$ can satisfy $x_1^2 / x_2 < 0$.