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Mathematical Optimization — Assignment 8

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Subgradients
Let $f_1(x,y) = -\sqrt{2-x^2-(y-1)^2}$, $f_2(x,y) = -\sqrt{2-x^2-(y+1)^2}$ be the functions representing the lower half of two balls of radius $\sqrt{2}$ centered in $(0,1,0)$ and $(0,-1,0)$, respectively. We want to consider the function f whose epigraph is the intersection of epif_1 and epif_2, i.e. $f = \max\{f_1, f_2\}$ over $\mathrm{dom} f = \mathrm{dom} f_1 \cap \mathrm{dom} f_2$.

Compute the subdifferential $\partial f(x,y)$ of f for every point of its domain: can you already guess where this set will be empty, where it will be a singleton and where it will contain more than one element? Why?

Exercise 2: Separation of Convex Sets
For this exercise, we will need the following two definitions:

- Two convex sets C and D in $\mathbb{R}^n$ are separated if there is a hyperplane H such that $C \subseteq H^-$ and $C \subseteq H^+$ or vice versa, where H^+ and H^- are the closed halfspaces determined by H.
- Two convex sets C and D in $\mathbb{R}^n$ are strongly separated if there is a $y \in \mathbb{R}^n$, $\alpha < \beta \in \mathbb{R}$ such that $C \subseteq \{x \mid y^\top x \leq \alpha\}$ and $D \subseteq \{x \mid y^\top x \geq \beta\}$, that is, the set $S := \{x \mid \alpha \leq y^\top x \leq \beta\}$ 'separates' C and D.

a) Prove that for two convex sets C and D in $\mathbb{R}^n$ the Minkowski sum $C - D$ is convex and that the following two statements are equivalent:

i) C and D are separated, respectively, strongly separated.

ii) $C - D$ and $\{0\}$ are separated, respectively, strongly separated.

b) Let $C, D \subseteq \mathbb{R}^n$ be convex, C compact, D closed and $C \cap D = \emptyset$. Prove that C and D are strongly separated. You may use the fact that there exist $p \in C, q \in D$ having minimum distance.

Exercise 3: Equality constrained least-squares
Consider the equality constrained least-squares problem

$$\begin{array}{ll} \min & \|Ax - b\|_2^2 \\ s.t. & Gx = h, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ and where $G \in \mathbb{R}^{p \times n}$.Give the KKT conditions.