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Mathematical Optimization — Assignment 1

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Knapsack Problem: Continuous and Binary

The **Knapsack Problem** can be formulated as follows

> From n items with given positive values $p_1, p_2, \ldots, p_n$ and positive weights $w_1, w_2, \ldots, w_n$, select items with total weight at most W (a given constant) so as to maximize the total value.

Without loss of generality we assume that the items are sorted such that

$$\frac{p_i}{w_i} \geq \frac{p_{i+1}}{w_{i+1}} \quad i = 1, \ldots, n-1.$$

a) Formulate the Knapsack Problem as a binary integer linear program.

b) Consider the following numerical example with $W = 13$:

j	1	2	3	4	5
p_j	10	80	40	30	22
w_j	1	9	5	4	3

Solve the relaxed linear programming problem, that is, allow all the variables in the problem to be continuous variables in $[0, 1]$, and solve the maximization problem.

c) Suppose now that the variables can only take values in $\{0, 1\}$. Can you solve the maximization problem?

Exercise 2: Polyhedral Cones

A non-empty subset $C \subseteq \mathbb{R}^n$ is called a cone, if (i) $\lambda c \in C$, for all $c \in C$ and for all $\lambda \in \mathbb{R}_{\geq 0}$, and (ii) $c + d \in C$, for all $c, d \in C$.

(a) Let $P = \{x \in \mathbb{R}^n \mid Ax \leq \mathbf{0}\}$ be a polyhedron defined by a matrix $A \in \mathbb{R}^{m \times n}$ and the zero-vector $\mathbf{0} \in \mathbb{R}^m$. Show that P is a cone.

(b) Show that if a non-empty cone $C \subseteq \mathbb{R}^n$ is a polyhedron, then $C = \{x \in \mathbb{R}^n \mid Ax \leq \mathbf{0}\}$ with $A \in \mathbb{R}^{m \times n}$.

Exercise 3: Polyhedral Cone with Extreme Point

A polyhedron of the form $C := \{x \in \mathbb{R}^n : Ax \leq \mathbf{0}\}$ is called a polyhedral cone.

a) Show that the zero vector is the only possible extreme point of C.

b) Assume C has an extreme point. Show that there exists a vector $c \in \mathbb{R}^n$ such that $c^T x > 0$ for all nonzero $x \in C$. Give a construction of such a vector c (depending on A).

Exercise 4: The Minkowski Sum of Convex Sets

For two sets $A, B \subseteq \mathbb{R}^n$, the Minkowski sum is defined as $A + B := \{a + b \in \mathbb{R}^n \mid a \in A, b \in B\}$.

Show that $A + B$ is convex, if both A and B are convex. Is it true that if $A + B$ is convex, then also A and B are convex? Prove the claim, or give a counterexample.