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Mathematical Optimization — Solution 1

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Knapsack Problem and Branch and Bound

a) The integer program (IP) for the Knapsack Problem can be formulated as follows

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i x_i \leq W \\ & x_i \in \{0,1\} \quad i=1,\dots,n \end{aligned}$$

The variables x_i, $i = 1, \dots, n$, are interpreted as

$$x_i = \begin{cases} 1 & \text{if item } i \text{ is chosen} \\ 0 & \text{otherwise.} \end{cases}$$

b) The relaxed LP problem is

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i x_i \leq W \\ & x_i \leq 1 \quad i=1,\dots,n \\ & x_i \geq 0 \quad i=1,\dots,n \end{aligned} \tag{1}$$

Trivially, if $\sum_{i=1}^{n} w_i \leq W$, we choose all items, and if $\forall i \in \{1,\dots,n\}\colon w_i > W$, we choose no item. So let us assume from now on that we are in neither of these two cases. Then, the following holds:

Lemma 1

Assume that $p_1/w_1 > p_2/w_2 > \dots > p_n/w_n$. Denote by $k \in \{1,\dots,m\}$ the index such that $\sum_{i=1}^{k} w_i \leq W$, $\sum_{i=1}^{k+1} w_i > W$. Then an optimal solution to (1) is given by $x^*(W)$, where

$$x_i^*(W) = \begin{cases} 1, & \text{if } i \leq k, \\ \frac{W - \sum_{i=1}^{k} w_i}{w_{k+1}}, & \text{if } i = k+1, \\ 0, & \text{otherwise} \end{cases} .$$

We give a sketch of the proof for those who are interested.

Proof (Sketch). First, observe that since by assumption $(1,\dots,1)$ is not a solution to (1), there exists an optimal solution y to (1) which fulfills $w^{\mathrm{T}} y = W$. It thus suffices to show that $x^*(W)$ is optimal for

$$
\begin{array}{llll}
\max & \sum_{i=1}^{n} p_i x_i & & \\
\text{s.t.} & \sum_{i=1}^{n} w_i x_i & = & W \\
& x_i & \leq & 1 \quad i = 1, \dots, n \\
& x_i & \geq & 0 \quad i = 1, \dots, n
\end{array} \tag{2}
$$

Let y be optimal for (2). We first show that for all $1 \leq i \leq k : y_i = 1$: Namely, if there was an $l \leq k$ with $y_l < 1$, $w^{\mathrm{T}} y = w^{\mathrm{T}} x^*(W)$ would imply that there is a $o > k$ such that $y_o > 0$. This would imply that there exists $\epsilon > 0$ small enough so that z fulfills $0 \leq z \leq 1$, with z being

$$
z_i = \begin{cases} y_i, & \text{if } i \neq l, o, \\ y_i + \epsilon, & \text{if } i = l, \\ y_i - \frac{w_l}{w_o}\epsilon, & \text{if } i = o \end{cases} .
$$

Note that $w^{\mathrm{T}} z = \sum_{i \neq l,o} w_i y_i + w_l(y_l + \epsilon) + w_o(y_o - w_l \epsilon / w_o) = w^{\mathrm{T}} y = W$ and thus, z is feasible for (2). But then, since $p_l/w_l > p_o/w_o$, $p^{\mathrm{T}} z = p^{\mathrm{T}} y + p_l \epsilon - p_o w_l \epsilon / w_o > p^{\mathrm{T}} y$, contradicting our choice of y.

Thus, both $x^*(W)$ and y are solutions to $\max\{p^{\mathrm{T}} x \colon w^{\mathrm{T}} x = W,\ 0 \leq x \leq 1, x_1 = \cdots = x_k = 1\}$. It is not hard to see that by the choice of our ordering, $y = x^*(W)$. $\square$

By the above Lemma, we can apply the following algorithm:

(i) Sort the variables according to their 'efficiency' $\frac{p_i}{w_i}$.

(ii) Fill the knapsack with items $1, 2, \dots$ consecutively as long as the total weight does not exceed W.

(iii) If the total weight is still strictly less than W, fill the next unused item with the maximum fraction possible.

The variables are already sorted by their 'efficiency' $\frac{p_i}{w_i}$:

$$
\frac{10}{1} > \frac{80}{9} > \frac{40}{5} > \frac{30}{4} > \frac{22}{3}.
$$

The algorithm of the above lemma yields as an optimal solution $x^*(P) = (1, 1, \frac{3}{5}, 0, 0)^T$ with optimal value $z(P) := 114$.

c) The solution to the linear relaxation provides an upper bound on the optimal value of the IP, hence, $UB := 114$ is an upper bound for our IP. We use the (trivial) solution $(0, 0, 0, 0, 0)^T$ to get the (trivial) lower bound $LB := 0$ for our IP.

Denote by $P_{x_J = d}$ the problem $\max\{p^{\mathrm{T}} x \colon w^{\mathrm{T}} x = W,\ 0 \leq x \leq 1, x_J = d\}$, where $J \subseteq \{1, \dots, n\}$ and $d \in \{0, 1\}^{|J|}$. We will gradually set certain variables of x to 0 and 1 to find an optimal integral solution.

We first 'branch' (i.e. decide on a variable value) on x_3 (which is an arbitrary choice), cf. Figure 1. Call $(P_{x_{\{3\}} = 1}) = (P_1)$. Note that (P_1) can be reduced to a Knapsack problem, since it can be written as $\max\{10x_1 + 80x_2 + 40 + 30x_4 + 22x_5 \colon x_1 + 9x_2 + 5 + 4x_4 + 5x_3 \leq 13\}$, which can be reformulated as $\max\{10x_1 + 80x_2 + 30x_4 + 22x_5 \colon x_1 + 9x_2 + 4x_4 + 5x_3 \leq 8\}$. We can thus solve it using the algorithm from part b) of this exercise.

An optimal solution of (P_1) is $x(P_1) = (1, \frac{7}{9}, 1, 0, 0)^T$ with $z(P_1) = 112.2$. We now consider subproblems of (P_1) by branching on the variable x_2 (again an arbitrary choice). Denote by (P_{11}) the problem $(P_{x_{\{2,3\}} = (1,1)})$. As $w_2 + w_3 = 14 > W$, (P_{11}) is infeasible and hence, no subproblem of (P_{11}) needs to be considered.

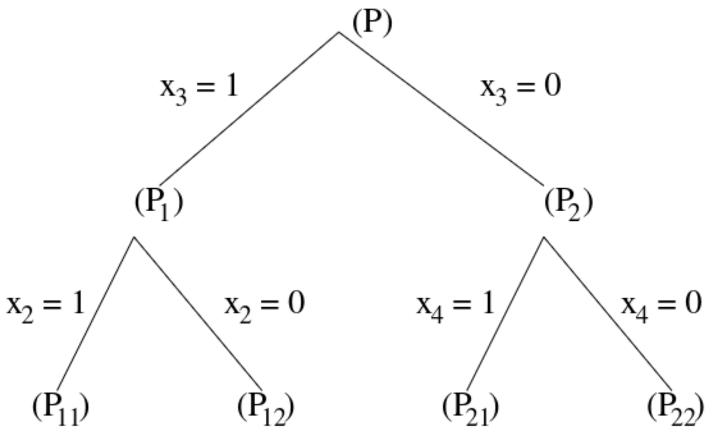

Figure 1: Illustration of our 'branch and bound' approach.

Let (P_{12}) denote the problem $(P_{x_{\{2,3\}}=(0,1)})$. An optimal solution of (P_{12}) (which can again be found by stating the problem as a Knapsack problem) is $x(P_{12}) = (1, 0, 1, 1, 1)^T$ with optimal value $z(P_{12}) = 102$. $x(P_{12})$ is integral, and thus we may update our lower bound for the optimal solution: $LB := 102$. Furthermore, since $x(P_{12})$ is integral, no subproblem of P_{12} needs to be considered.

Let (P_2) denote the problem $(P_{x_{\{3\}}=(0)})$. Our relaxed Knapsack algorithm yields the optimal solution $x(P_2) = (1, 1, 0, \frac{3}{4}, 0)^T$ with $z(P_2) = 112.5$. We now consider subproblems of P_2 by branching on the variable x_4 (yet again, this is an arbitrary choice), and call $(P_{x_{\{3,4\}}=(1,1)}) =: (P_{21})$ and $(P_{x_{\{3,4\}}=(1,0)}) =: (P_{22})$. The optimal solution of (P_{21}) is $x(P_{21}) = (1, \frac{8}{9}, 0, 1, 0)^T$ with $z(P_{21}) = 111.1$. The optimal solution of P_{22} is $x(P_{22}) = (1, 1, 0, 0, 1)^T$ with $z(P_{22}) = 112$. $x(P_{22})$ is integral. We update the lower bound: $LB := 112$. As $x(P_{22})$ is integral, no subproblem of P_{22} needs to be considered. Also, no subproblem of P_{21} needs to be considered because $z(P_{21}) = 111.1 < 112 = LB$. As there is no 'open' branch/subproblem left, the optimal solution is $x(P_{22}) = (1, 1, 0, 0, 1)^T$ and the optimal objective function value is 112.

Exercise 2: Polyhedral Cones

(a) We check condition (i) and (ii) in the definition of a cone. For (i) let $c \in P$, i.e., $Ac \leq 0$. It follows that $A(\lambda c) = \lambda(Ac) \leq 0$ for all $\lambda \geq 0$. For (ii) let $c, d \in P$, i.e., $Ac \leq 0$ and $Ad \leq 0$. This implies $A(c + d) = Ac + Ad \leq 0$ and thus $c + d \in P$. Therefore, P is a cone.

(b) As C is a polyhedron, there are $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ with $C = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. We need to prove that $b = 0$. Each nonempty cone contains the origin $\mathbf{0}$ (choose any $x \in C$,then also $0x \in C$). Thus, it holds $A\mathbf{0} = \mathbf{0} \leq b$.

Next, assume there was an $\bar{x} \in P$ with $(A\bar{x})_i > 0$. Then, were would exist a $\bar{\lambda} \geq 0$ with $(A\bar{\lambda}\bar{x})_i = \bar{\lambda}(A\bar{x})_i = b_i$. For any $\lambda > \bar{\lambda}$ the point $\lambda\bar{x} \notin C$ as $(A\lambda\bar{x})_i > b_i$, which would contradict condition (i) in the definition of a cone. We may thus set $b = \mathbf{0}$.

Exercise 3: Polyhedral Cone with Extreme Point

a)Let $x \in C$ be nonzero. Then $3x/2 \in C$, $x/2 \in C$, and $\frac{3x/2+x/2}{2} = x$. Thus x is not an extreme point.

b)Given $C := \{x \in \mathbb{R}^n : Ax \leq \mathbf{0}\}$, set $c^T := -\mathbf{1}^T A$. Since $x = 0$ is an extreme point, the only solution to the equation system $Ax = \mathbf{0}$ is the trivial one, as otherwise it would contain a whole line and having an extreme point is equivalent to not containing a line. In other words, for every non-zero $x \in C$ there is an index i such that $(Ax)_i < 0$ while $(Ax)_j \leq 0$ for all other $j \neq i$. Thus, we have $c^T x = -\mathbf{1}^T Ax > 0$ for every non-zero $x \in C$.

Exercise 4: The Minkowski Sum of Convex Sets
We have to show that $\lambda x + (1-\lambda)y$ is contained in $A+B$ for all $x, y \in A+B$ and all $\lambda \in [0,1]$. Let $x, y \in A+B$. By definition, there are $a_x, a_y \in A$ and $b_x, b_y \in B$ such that $x = a_x + b_x$ and $y = a_y + b_y$. Now consider

$$\lambda x + (1-\lambda)y = \lambda(a_x + b_x) + (1-\lambda)(a_y + b_y) = \big(\ \lambda a_x + (1-\lambda)a_y\ \big) + \big(\ \lambda b_x + (1-\lambda)b_y\ \big).$$

As A and B are convex, it follows that $\lambda a_x + (1-\lambda)a_y \in A$ and $\lambda b_x + (1-\lambda)b_y \in B$. This shows that $\lambda x + (1-\lambda)y \in A+B$.

The opposite direction is not true, however. Consider for example the following sets on real line

$$A = \{0,1\} \quad B = [0,1].$$

While A is clearly not convex, we have $A+B = [0,2]$, which is a convex set.