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Mathematical Optimization — Assignment 2

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Minkowski sum of polyhedra
Let P and Q be polyhedra in $\mathbf{R}^n$. Let $P+Q=\{x+y \mid x\in P, y\in Q\}$ be the Minkowski sum of the two polyhedra.

a) Show that $P+Q$ is a polyhedron.

b) Show that every extreme point of $P+Q$ is the sum of an extreme point of P and an extreme point of Q.

Exercise 2: Facets and extreme points: Canonical examples
a) What are the extreme points of the cube $P=\{x : 0\le x_i\le 1, i=1,\dots,n\}$? How many extreme points are there?

b) Consider the simplex $S=\{x : x_1+\cdots+x_n\le 1, 0\le x_i\le 1, i=1,\dots,n\}$. Prove that $S=\operatorname{conv}(0,e_1,\dots,e_n)$.

Exercise 3: Extreme points of polyhedra
a) Let $P:=\{x\in\mathbf{R}^n \mid Ax=b, x\ge 0\}$ a nonempty polyhedron in standard form. Show that P has an extreme point.

b) We call two vertices v,w of a polyhedron $P=\{x\in\mathbb{R}^n\colon Ax\le b\}$ adjacent if they share an edge, i.e., if there are $n-1$ linearly independent rows $A_{i_1,\cdot},\dots,A_{i_{n-1},\cdot}$ in A such that $\forall k\in\{i_1,\dots,i_{n-1}\}\colon A_{k,\cdot}v=A_{k,\cdot}w=b_k$.

Let $P:=\{x\in\mathbf{R}^n \mid Ax\le b\}$ be a bounded polyhedron, let c be a vector in $\mathbf{R}^n$, and let γ be some scalar. We define
$$Q:=\{x\in P \mid c^Tx=\gamma\}.$$
Assume that Q is nonempty. Show that every extreme point of Q is either an extreme point of P or a convex combination of two adjacent extreme points of P.

Exercise 4: Basic Feasible Solutions
Let $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^m$, $m\le n$, and suppose that A has full row rank. Consider
$$P=\{x\in\mathbb{R}^n | Ax=b, x\ge 0\}.$$
Prove that $x^*\in P$ is a basic feasible solution for P if and only if there exists $B\subseteq\{1,\dots,n\}$, $|B|=m$ such that

i) the submatrix $A_{\cdot B}\in\mathbb{R}^{m\times m}$ is invertible,

ii) $x_i^*=(A_{\cdot B}^{-1}b)_i \ \forall\, i\in B$,

iii) $x_i^*=0 \ \forall\, i\in\{1,\dots,n\}\setminus B$.