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Mathematical Optimization — Assignment 3

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercises marked with a (*) may require prior knowledge not yet seen in the lecture, but provide statements that are needed later on in the course.

Exercise 1: Fourier-Motzkin elimination and optimization
Consider the following LP:

$$
\text{(LP)} \qquad \begin{array}{rrrrrrr}
\max & & x_1 & + & x_2 & & \\
s.t. & - & 4x_1 & - & x_2 & \leq & -8 \\
 & - & x_1 & + & x_2 & \leq & 3 \\
 & & & - & x_2 & \leq & -2 \\
 & & 2x_1 & + & x_2 & \leq & 12
\end{array}
$$

a) How can the Fourier-Motzkin elimination be modified in order to maximize a linear function over a polyhedron?

b) Apply your idea to the given LP. In particular, derive an optimal solution.

Exercise 2: Farkas Lemma for Standard Form Polyhedra
Prove the following version of the Farkas Lemma: The system $Ax = b$, $x \geq 0$ has a solution if and only if for every y such that $y^T A \geq 0$ it holds that $y^T b \geq 0$.

Exercise 3: Caratheodory's Theorem for Polytopes
Let $P = \text{conv}\{v_1, \cdots v_k\} \subset \mathbb{R}^n$ be a bounded polyhedron (polytope) given as the convex hull of its vertices. Prove that for every point $x \in P$ there exist $n+1$ vertices $v_{i_1}, \cdots, v_{i_{n+1}}$ and nonnegative scalars $\lambda_1, \cdots, \lambda_{n+1}$ with $\sum_{i=1}^{n+1} \lambda_i = 1$ such that

$$x = \sum_{j=1}^{n+1} \lambda_j v_{i_j}.$$

Exercise 4: Iterated Polyhedral Projections
Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \subseteq \mathbb{R}^n$, for $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, be a polyhedron. Show that

$$\text{proj}_{(x_1,\ldots,x_{n-2})}(P) = \text{proj}_{(x_1,\ldots,x_{n-2})}\big(\text{proj}_{(x_1,\ldots,x_{n-1})}(P)\big).$$

Exercise 5: Projection (*)
Let $Q \subset \mathbb{R}^n$ be a non-empty, compact and convex set and let $y \in \mathbb{R}^n$. Consider the projection on Q defined by

$$\mathcal{PO}(y) := \text{argmin}\{||x - y||_2^2 \ : \ x \in Q\}.$$

Show that for all $z \in Q$

$$||z - \mathcal{PO}(y)||_2 \leq ||z - y||_2.$$