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Mathematical Optimization — Solution 11

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Mixed-integer feasibility with a fixed number of integer variables
Define $P_{MI} := \{\binom{x}{y} \in \mathbb{Z}^d \times \mathbb{R}^n \mid Ax + By \leq c\}$ and let P be its LP-relaxation. Our goal is to decide whether $P_{MI} \neq \emptyset \Leftrightarrow P \cap (\mathbb{Z}^d \times \mathbb{R}^n) \neq \emptyset$.

Consider $\text{proj}_x P = \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^n \text{ s.t. } Ax + By \leq c\}$. From the lecture, we know that $P = \emptyset \Leftrightarrow \text{proj}_x P = \emptyset$. Additionally, it holds that $P_{MI} = \emptyset \Leftrightarrow (\text{proj}_x P) \cap \mathbb{Z}^d = \emptyset$:

"$\Rightarrow$:" Follows directly as an element in $(\text{proj}_x P) \cap \mathbb{Z}^d$ would be a contradiction.

"$\Leftarrow$:" This implies that $\{x \in \mathbb{Z}^d \mid \exists y \in \mathbb{R}^n \text{ s.t. } Ax + By \leq c\} = \emptyset$, which implies that $P_{MI} = \emptyset$.

By applying the IP-feasibility oracle, we can find an $\bar{x} \in (\text{proj}_x P) \cap \mathbb{Z}^d$ if it exists or decide that no such x exists, which is equivalent to $P_{MI} = \emptyset$. If it exists, the existence of $\bar{x}$ implies that there is a corresponding y such that $\binom{\bar{x}}{y} \in P_{MI}$. In order to find such a y, fix the entries of $\bar{x}$ and apply the LP-feasibility oracle to the following LP: $\{y \in \mathbb{R}^n \mid A\bar{x} + By \leq c\} = \{y \in \mathbb{R}^n \mid By \leq c - A\bar{x}\}$.

Exercise 2: Matching and Perfect Matching Polytope
We claim that $PM'(G) = PM(G)$.

Note that in the case where $|V|$ is odd, $PM'(G) = \emptyset = PM(G)$, as $x(E[V]) = x(E) = \frac{|V|}{2} > \frac{|V|-1}{2}$ for $PM'(G)$ and $x(\delta(V)) = x(\emptyset) = 0 < 1$ for $PM(G)$. Thus, we can assume w.l.o.g. that $|V|$ is even.

"$\subseteq$": Let $x \in$ LHS.

- We know that $2x(E) = \sum_{v \in V} \underbrace{x(\delta(v))}_{\leq 1} \leq |V| = 2x(E) \Rightarrow x(\delta(v)) = 1$.
- Let $S \subseteq V$, $|S|$ odd, $S \neq \emptyset$. Then, $|S^C|$ also has to be odd as $|V| = |S| + |S^C|$ is even. Thus,
$$x(E[S]) + x(E[S^C]) \leq \frac{|S|-1}{2} + \frac{|S^C|-1}{2} = \frac{|V|-2}{2} = \frac{|V|}{2} - 1.$$
As furthermore $\delta(S) = E \setminus \big(E[S] \cup E[S^C]\big)$, it holds that
$$x(\delta(S)) = x(E) - \big(x(E[S]) + x(E[S^C])\big) \geq \frac{|V|}{2} - \frac{|V|}{2} + 1 = 1.$$

"$\supseteq$": Let $x \in$ RHS.

- $x(\delta(v)) \leq 1$ is clearly satisfied for all $v \in V$.
- It holds that $2x(E) = \sum_{v \in V} x(\delta(v)) = |V| \Rightarrow x(E) = \frac{|V|}{2}$.
- Let $S \subseteq V$, $|S|$ odd. Then,
$$|S| = \sum_{v \in S} |\delta(v)| = 2 \sum_{e \in E[S]} x(e) + \sum_{e \in \delta(S)} x(e).$$
Therefore,
$$x(E[S]) = \frac{|S| - x(\delta(S))}{2} \leq \frac{|S|-1}{2},$$
as $x(\delta(S)) \geq 1$.

Exercise 3: LP Solution and the Normal Cone
We assume that $x^* := \max\{c^{\mathrm{T}}x \mid x \in P\}$ exists and is finite, from which follows that also $y^* := \min\{y^{\mathrm{T}}b \mid y^{\mathrm{T}}A = c^{\mathrm{T}},\ y \geq 0\}$ exists and is finite. Using complementary slackness, we know that x^* is optimal for the primal problem and y^* is optimal for the dual problem if and only if $y_i^*\,(A_{i,\cdot}x^* - b_i) = 0$, for all $i \in \{1, \ldots, m\}$. Let I be the constraints where x^* is tight. The previous argument proves that $c \in \operatorname{cone}(\{A_{i,\cdot},\ i \in I\})$ as it gives us a valid combination: We know that $y_i^* = 0$ for $i \notin I$ and therefore, $(y^*)^{\mathrm{T}}A = \sum_{i \in I} y_i^* A_{i,\cdot} = c^{\mathrm{T}}$ and $y^* \geq 0$, from which it follows that $c \in \operatorname{cone}\left(\left\{A_{i,\cdot}^{\mathrm{T}},\ i \in I\right\}\right)$.