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Mathematical Optimization — Solution 5

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Representation of Polyhedra

(a) We find the solution graphically. The inequality description is

$$P := \{x \in \mathbb{R}^2 \mid -x_1 \leq 0,\ x_1 - 2x_2 \leq 0,\ x_1 - x_2 \leq 1,\ -x + y \leq 3\}.$$

(b) Again, we find the solution graphically. The vertex description is

$$P = \operatorname{conv}\left(\{(3,0)^\mathsf{T}\}\right) + \operatorname{cone}\left(\{(-1,0)^\mathsf{T}, (-1,-1)^\mathsf{T}\}\right).$$

Exercise 2: Complementary Slackness

(a) We denote the given (primal) linear program as (P). Then, its dual program, (D), is the following.

$$\begin{array}{lrcrcrcrcr}
\min & 4y_1 & + & 3y_2 & + & 5y_3 & + & y_4 & & \\
\text{s.t.} & y_1 & + & 4y_2 & + & 2y_3 & + & 3y_4 & \geq & 7 \\
 & 3y_1 & + & 2y_2 & + & 4y_3 & + & y_4 & \geq & 6 \\
 & 5y_1 & - & 2y_2 & + & 4y_3 & + & 2y_4 & \geq & 5 \\
 & -\ 2y_1 & + & y_2 & - & 2y_3 & - & y_4 & \geq & -2 \\
 & 2y_1 & + & y_2 & + & 5y_3 & - & 2y_4 & \geq & 3 \\
 & & & & & & & y_i & \geq & 0 \quad \text{for } i = 1, \ldots, 4
\end{array}$$

(b) For the given $x^* = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)^T$ we want to find $y^* = (y_1^*, y_2^*, y_3^*, y_4^*)^T$ such that the pair x^*, y^* fulfills the complementary slackness conditions.

Since the second, third, and fourth components of x^* are strictly positive, it follows from the complementary slackness conditions that the second, third, and fourth inequality constraints of (D) must be tight at y^*.

Furthermore, plugging in x^* into the set of constraints defining (P), we see that the third inequality is the only one not satisfied with equality at x^*. By the complementary slackness conditions we infer that $y_3^* = 0$.

Taking both facts together we obtain the following system.

$$\begin{aligned}
3y_1^* + 2y_2^* + \ y_4^* &= \ \ 6 \\
5y_1^* - 2y_2^* + 2y_4^* &= \ \ 5 \\
-\ 2y_1^* + \ y_2^* - \ y_4^* &= -2
\end{aligned}$$

The linear system has 3 equations, 3 unknowns, and the 3×3 matrix representing the left-hand side coefficients is nonsingular, thus the linear equation system has a unique solution.

Solving this linear equation system we obtain $y_1^* = y_2^* = y_4^* = 1$, thus the unique dual solution that fulfills the complementary slackness for x^* is $y^* = (1, 1, 0, 1)^T$.

(c) We know that x^* is an optimal solution of (P) if and only if there exists a $y^* = (y_1^*, y_2^*, y_3^*, y_4^*)^T$ such that the pair x^*, y^* fulfills the complementary slackness conditions and y^* is feasible for (D) (i.e., $y^* \geq 0$ and $A^T y^* \geq c$). However, the **unique** y^* such that the pair x^*, y^* fulfills the complementary slackness conditions is not feasible for (D) as it violates the fifth inequality of (D). Therefore, x^* can not be an optimal solution of (P).

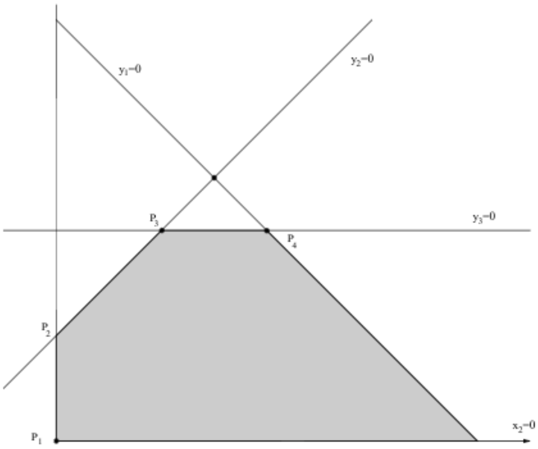

Figure 1: Solution to exercise 3a.

Exercise 3: Geometry of LP and Exchange Step

(a) See Figure 1.

(b)

$$
(LPs) \qquad
\begin{array}{lrrrrrll}
\min & -x_1 & -2\,x_2 & & & & & \\
\text{s.t.} & x_1 & +x_2 & +y_1 & & & = 8 & \\
 & -x_1 & +x_2 & & +y_2 & & = 2 & \\
 & & x_2 & & & +y_3 & = 4 & \\
 & & x_i & & & & \geq 0 & i = 1, 2 \\
 & & & & & y_i & \geq 0 & i = 1, 2, 3.
\end{array}
$$

(c)

- $P_1 = (0, 0)$:
  basic variables $= \{y_1, y_2, y_3\}$
  non-basic variables $= \{x_1, x_2\}$
  basic solution $= (0, 0, 8, 2, 4)$
- $P_2 = (0, 2)$:
  basic variables $= \{x_2, y_1, y_3\}$
  non-basic variables $= \{x_1, y_2\}$
  basic solution $= (0, 2, 6, 0, 2)$
- $P_3 = (2, 4)$:
  basic variables $= \{x_1, x_2, y_1\}$
  non-basic variables $= \{y_2, y_3\}$
  basic solution $= (2, 4, 2, 0, 0)$
- $P_4 = (4, 4)$:
  basic variables $= \{x_1, x_2, y_2\}$
  non-basic variables $= \{y_1, y_3\}$
  basic solution $= (4, 4, 0, 2, 0)$

(d) Since the two sets of basic variables differ by two variables, we need at least two exchange steps. The picture drawn in part (a) shows that this is indeed the right number and it is represented by the path $P_1 \to (8, 0) \to P_4$. Clearly, we can also find longer sequences of exchange steps that start at P_1 and end at P_3.

Remark: In general, the number of basic variables in which two vertices are different determines a lower bound on the number of exchange steps needed to move from one vertex to the other.

(e) Using the short tableau and the route via $P_1 \to P_2 \to P_3 \to P_4$, the initial tableau is (using the slack basis):

	0	x_1	x_2
	0	-1	-2
x_3	8	1	1
x_4	2	-1	1
x_5	4	0	1

Now, when moving to P_4, the short tableaus are:

		x_1	x_4
	4	-3	2
x_3	6	2	-1
x_2	2	-1	1
x_5	2	1	-1

$\rightarrow$

		x_5	x_4
	10	3	-1
x_3	2	-2	1
x_2	4	1	0
x_1	2	1	-1

$\rightarrow$

		x_5	x_3
	12	1	1
x_4	2	-2	1
x_2	4	1	0
x_1	4	-1	1

Thus, the optimal solution is given by -12 and the solution is optimal because the reduced cost vector is ≥ 0. The resulting point corresponds to P_4, which can be verified by setting the non-basic variables to zero.

Remark: This is not the only possible sequence of exchange steps. For instance, we could have moved to $(8, 0)$ and then to P_4, which would have been faster.

Exercise 4: Degeneracy and Cycling of Simplex Method

We transform our given polyhedron to a standard form polyhedron, i.e.

$$
\begin{aligned}
\text{minimize:} \quad & -x_1 + 2x_2 - x_3, \\
\text{subject to:} \quad & 2x_1 - x_2 + x_3 + x_4 = 0 \\
& 3x_1 + x_2 + x_3 + x_5 = 0 \\
& -5x_1 + 3x_2 - 2x_3 + x_6 = 0 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}
$$

Therefore, the initial tableau, using the slack basis, is

		x_1	x_2	x_3
	0	-1	2	-1
x_4	0	2	-1	1
x_5	0	3	1	1
x_6	0	-5	3	-2

Now, we perform the exchange steps.

		x_1	x_2	x_3
	0	-1	2	-1
x_4	0	2	-1	1
x_5	0	3	1	1
x_6	0	-5	3	-2

$\rightarrow$

		x_5	x_2	x_3
	0	0.333	2.333	-0.666
x_4	0	-0.666	-1.666	0.333
x_1	0	0.333	0.333	0.333
x_6	0	1.666	4.666	-0.333

$\rightarrow$

		x_5	x_2	x_3
	0	-1	-1	2
x_3	0	-2	-5	3
x_1	0	1	2	-1
x_6	0	1	3	1

$\rightarrow$

		x_5	x_6	x_4
	0	-0.666	0.333	2.333
x_3	0	-0.333	1.666	4.666
x_1	0	0.333	-0.666	-1.666
x_2	0	0.333	0.333	0.333

$\rightarrow$

		x_1	x_6	x_4
	0	2	-1	-1
x_3	0	1	1	3
x_5	0	3	-2	-5
x_2	0	-1	1	2

$\rightarrow$

		x_1	x_6	x_3
	0	2.333	-0.666	0.333
x_4	0	0.333	0.333	0.333
x_5	0	4.666	-0.333	1.666
x_2	0	-1.666	0.333	-0.666

$\rightarrow$

		x_1	x_2	x_3
	0	-1	2	-1
x_4	0	2	-1	1
x_5	0	3	1	1
x_6	0	-5	3	-2

Thus, we have arrived at the original slack basis and the Simplex algorithm does not terminate with this choice of basis changes.