ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Operations Research
ETH Zurich HG G21-22

Stephan Artmann stephan.artmann@ifor.math.ethz.ch
Christoph Glanzer christoph.glanzer@ifor.math.ethz.ch

IFOR
Institute for Operations Research
Department of Mathematics

Mathematical Optimization — Solution 2

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Minkowski sum of polyhedra

We use the following fact: the projection of a polyhedron on a linear space is a polyhedron.

a) Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ and $Q := \{x \in \mathbb{R}^n : Cx \leq d\}$. Define $S = \{(x, y, z) \in \mathbb{R}^{3n} : Ax \leq b, Cy \leq d, x + y - z = 0\}$. Then $P + Q$ is the projection of the polyhedron S on its last n components, and is therefore a polyhedron.

b) Let z be an extreme point of $P + Q$. In particular, there exist $x \in P$ and $y \in Q$ such that $x + y = z$. Since z is an extreme point, there exists $c \in \mathbb{R}^n$ s.t. $c^T w < c^T z$ for all $w \in (P + Q) \setminus \{z\}$. Assume there exists $x' \in P$ with $c^T x' \geq c^T x$. Then we would have $c^T(x' + y) \geq c^T(x + y) = c^T z$ and $x' + y \in (P + Q) \setminus \{z\}$, contradicting the fact that z is a vertex. Hence $c^T x' < c^T x$ for all $x' \in P \setminus \{x\}$, so x is an extreme point of P. The same proof can be repeated to show that $y \in Q$ is an extreme point.

Exercise 2: Facets and extreme points: Canonical examples

a) The extreme points of the cube are the points $\{0, 1\}^n$, so there are 2^n extreme points.

b) The points $\{0, e_1, \dots, e_n\}$ are feasible and basic feasible solutions (though we do not need the latter). Thus $S \supseteq \text{conv}(e_1, \dots, e_n, 0)$.

To prove equality, choose any $x \in S$. Then $1 \geq \sum_{i=1}^n x_i =: 1 - \lambda \geq 0$, and for all $i \in \{1, \dots, n\}$: $0 \leq x_i \leq 1$. Thus, we can write x as

$$x = \left[\mathcal{I} \mid 0\right] \begin{pmatrix} x \\ \lambda \end{pmatrix},$$

and since $0 \leq \binom{x}{\lambda} \leq 1, \sum_{i=1}^n x_i + \lambda = 1$,

$$\left[\mathcal{I} \mid 0\right] \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \text{conv}(e_1, \dots, e_n, 0).$$

Therefore, $S = \text{conv}(e_1, \dots, e_n, 0)$.

Exercise 3: Extreme points of polyhedra

a) $P \subseteq \mathbb{R}^n_{\geq 0}$. Since $\mathbb{R}^n_{\geq 0}$ does not contain a line, the result follows from the theorem covered in the lecture that states that a nonempty polyedron has an extreme point if and only if it does not contain a line.

b) Let x^* be an extreme point of Q. Let $I = I(x^*) := \{i \in \{1, \dots, m\} : A_i x^* = b_i\}$. x^* is basic feasible for Q, and $c^T x^* = \gamma$, so that we have $\text{rank}(< A_i, c >: i \in I) = n$. We distinguish two cases:

   i) if c is linearly dependent with $\{A_i : i \in I\}$, then $\text{rank}(< A_i >: i \in I) = n$ and x^* is an extreme point of P.

   ii) Otherwise, $\dim(< A_i >: i \in I) = n - 1$, and x^* is contained in the line $L := \{x : A_i x = b_i, i \in I\}$. P is bounded, and thus, so is $P \cap L$. Since $P \cap L$ is a 1-dimensional polyhedron, it is the convex hull of two adjacent extreme points (of $P \cap L$) u and v. Being basic feasible, u fulfills n linearly independent constraints of $P \cap L = \{x \in \mathbb{R}^n : Ax \leq b,\ A_I x = b_I\}$, and thus also fulfills n linearly independent constraints of P with equality. Therefore, u is a vertex of P. For the same reason, v is a vertex of P as well. Thus, $x^* \in P \cap L$ is a convex combination of two vertices of P.

Exercise 4: Basic Feasible Solutions

Let us write P as $P = \{x \in \mathbb{R}^n | Cx \leq d\}$, where

$$C = \begin{bmatrix} A \\ -A \\ -\mathcal{I} \end{bmatrix},\ d = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}.$$

Let $x^* \in P$. Note that x^* fulfills the first $2m$ constraints with tightness. Let us denote by $\mathcal{I}_n$ the $(n \times n)$-identity matrix.

We have that

x^* is a basic feasible solution for P
$\Leftrightarrow \operatorname{rank}(\{< C_{i\cdot} > | i \in I(x^*)\}) = n$
$\overset{\text{since } \operatorname{rank}(A)=m}{\Leftrightarrow} \exists I \subseteq \{1, \ldots, 2m+n\} \colon |I| = n,\ \{1, \ldots, m\} \subseteq I,\ \det(C_I) \neq 0,\ C_I x^* = b_I$
$\Leftrightarrow \exists I \subseteq \{1, \ldots, 2m+n\} \colon |I| = n,\ \det(C_I) \neq 0,\ C_I x^* = b_I$ and

$$\{C_{i\cdot} | i \in I\} = \underbrace{\{A_{j\cdot} | j = 1, \ldots, m\}}_{\text{dimension } m} \cup \{-e_{j_1}, \ldots, -e_{j_{n-m}}\}$$

where $\{j_1, \ldots, j_{n-m}\} \subseteq \{1, \ldots, n\}$ and $j_k \neq j_l$ for all $k \neq l$
$\Leftrightarrow \exists I \subseteq \{1, \ldots, 2m+n\}, B \subseteq \{1, \ldots, n\} \colon C_I x^* = b_I,\ |B| = m,\ \det(A_{\cdot,B}) \neq 0$ and using row and column permutations, C_I can be transformed into

$$\begin{bmatrix} -\mathcal{I}_{n-m} & 0 \\ A_{\cdot,\{1,\ldots,m\}\setminus B} & A_{\cdot,B} \end{bmatrix}$$

$\Leftrightarrow \exists B \subseteq \{1, \ldots, m\} \colon |B| = m,\ \det(A_{\cdot,B}) \neq 0,\ x_{\{1,\ldots,m\}\setminus B} = 0,\ x_B = (A_{\cdot,B})^{-1} b$ $\square$