ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Operations Research
ETH Zurich HG G21-22

Stephan Artmann stephan.artmann@ifor.math.ethz.ch
Christoph Glanzer christoph.glanzer@ifor.math.ethz.ch

IFOR
Institute for Operations Research
Department of Mathematics

Mathematical Optimization — Solution 6

https://moodle-app2.let.ethz.ch/course/view.php?id=3610

Exercise 1: Modeling linear programs

(a) Consider the following LP:

$$\max \left\{ t \;\middle|\; \begin{array}{c} A_{i,\cdot}x \geq b_i + t,\ i = 1, \ldots, n \\ x \in \mathbb{R}^n,\ t \in \mathbb{R}_+ \end{array} \right\}.$$

If $t > 0$ in an optimal solution, then $x^* \in \operatorname{int}(P)$ exists.

(b) Consider the following LP:

$$\begin{array}{llcll} \text{maximize} & \text{t} & & & \\ \text{subject to:} & h^\mathsf{T}x & \leq & b - t & \forall x \in X, \\ & h^\mathsf{T}y & \geq & b + t & \forall y \in Y, \end{array}$$

where $h \in \mathbb{R}^n$, $b \in \mathbb{R}$, $t \geq 0$. If $t > 0$ in an optimal solution, such a hyperplane is given by $\{x \in \mathbb{R}^n \colon h^\mathrm{T}x = b\}$.

Exercise 2: Unbounded Simplex Tableau

(a) In standard form, the problem is given by

$$\text{minimize} \quad \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^\mathrm{T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$$\text{subject to:} \quad \begin{pmatrix} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & -1 & -2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

The corresponding tableau is:

		x_1	x_2	x_3
	0	0	-1	-1
x_4	0	1	2	-3
x_5	1	2	-1	-2
x_6	0	-1	-1	-1

.

We observe that $\bar{c}_3 < 0$, while the corresponding column admits only non-positive entries. Therefore, the problem is unbounded.

(b) As seen in the lecture, the Simplex algorithm maintains a feasible primal solution when switching bases. Thus, a feasible point is given by the solution corresponding to the current basis, i.e.,

$$x_i = \begin{cases} \left(A_B{}^{-1}b\right)_i & \text{if } i \in B, \\ 0 & \text{else.} \end{cases}$$

Therefore, a feasible solution is given by $x := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{\mathrm{T}}$.

(c) In the lecture, we have seen that in this case, we may consider the feasible direction corresponding to x_3, given by $d := \begin{pmatrix} -A_B^{-1}A_{\cdot,3} \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 3 & 2 & 1 \end{pmatrix}^{\mathrm{T}}$. Indeed, one can check that $c^{\mathrm{T}}d = -1 < 0$, while $Ad = 0$.

Exercise 3: Simplex Phase One

(a) Using slack variables, we can reformulate the problem as:

$$\begin{array}{lllll} \text{minimize} & \text{x} & & & \\ \text{subject to:} & -x & + \; y_1 & = & -3 \\ & x & + \; y_2 & = & 5 \\ & x, y_1, y_2 & & \geq & 0 \end{array}.$$

(b) The result is:

$$\begin{array}{llllll} \text{minimize} & \text{x} & & & & \\ \text{subject to:} & x & - \; y_1 & & = & 3 \\ & x & & + \; y_2 & = & 5 \\ & x, y_1, y_2 & & & \geq & 0 \end{array}.$$

(c) One possibility to formulate the auxiliary LP is:

$$\min \left\{ z_1 + z_2 \;\middle|\; \begin{array}{c} x - y_1 + z_1 = 3 \\ x + y_2 + z_2 = 5 \\ x, y_1, y_2, z_1, z_2 \geq 0 \end{array} \right\}.$$

(d) Using that $c_B = (1,1)^\top$, $c_N = (0,0,0)^\top$, $A_B = I$ and $A_N = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, the corresponding tableau (short) is:

		x	y_1	y_2
	-8	-2	1	-1
z_1	3	1	-1	0
z_2	5	1	0	1

Here, we used that $\overline{c_j} = c_j - c_B^\top A_B^{-1} A_{\cdot,j} = -c_B^\top A_{\cdot,j} \Rightarrow \overline{c} = (-2, 1, -1)^\top$.

By performing the Simplex algorithm, we arrive at the following tableau:

		z_1	y_1	z_2
	0	1	0	1
x	3	1	-1	0
y_2	2	-1	1	1

As the objective value is zero, the original problem is feasible. A feasible basis of the original problem is therefore given by $\{x, y_2\}$.

Exercise 4: Lexicographic Pivoting
We will use the following observations:

i) $u <_{\text{lex}} v \Leftrightarrow u - v <_{\text{lex}} 0$,

ii) For any $v \in \mathbb{R}^n$, $u >_{\text{lex}} 0$ and $\alpha > 0$, it holds that $v + \alpha \cdot u >_{\text{lex}} v$.

As written in the hint, we choose the long tableau where the unit matrix is located to the very left.

$$T_{\text{long}} = \left[\begin{array}{c|cc} -c_B{}^{\mathsf{T}} A_B^{-1} b & 0 \dots 0 & \bar{c}_j,\ j \in N \\ \hline A_B^{-1} b & I & A_B^{-1} A_N \end{array}\right].$$

Let us start by showing that a basis exchange maintains the property that $T_{i,\cdot} >_{\text{lex}} 0$. This clearly holds true for the initial tableau as $A_B^{-1} b \geq 0$ (x is feasible) and due to the unit matrix. Let us change the basis from B to $B \cup \{j\} \setminus \{k\}$. Call the tableau before the basis change T and the new tableau T^{new}. By assumption, $\bar{c}_j < 0$, $\bar{A}_{k,j} > 0$ and by our choice of k, $\frac{T_{k,\cdot}}{A_{k,j}} <_{\text{lex}} \frac{T_{i,\cdot}}{A_{i,j}}$ for all $i \geq 1, i \neq k$ such that $\bar{A}_{i,j} > 0$.

It follows that,

- $T^{\text{new}}_{k,\cdot} = \frac{1}{\bar{A}_{k,j}} T_{k,\cdot} >_{\text{lex}} 0$ as $\bar{A}_{k,j} > 0$,
- $T^{\text{new}}_{i,\cdot} = T_{i,\cdot} - \frac{\bar{A}_{i,j}}{\bar{A}_{k,j}} T_{k,\cdot}$, $\forall i \geq 0,\ i \neq k$. Let us distinguish between two cases:
  - If $\bar{A}_{i,j} \leq 0$, $T^{\text{new}}_{i,\cdot} >_{\text{lex}} 0$ as $T_{i,\cdot} >_{\text{lex}} 0$, where we use that $T_{k,\cdot} >_{\text{lex}} 0$.
  - For $\bar{A}_{i,j} > 0$, it holds that: $\frac{T_{i,\cdot}}{A_{i,j}} >_{\text{lex}} \frac{T_{k,\cdot}}{A_{k,j}} \Leftrightarrow T_{i,\cdot} >_{\text{lex}} \frac{\bar{A}_{i,j}}{\bar{A}_{k,j}} T_{k,\cdot} \Leftrightarrow T_{i,\cdot} - \frac{\bar{A}_{i,j}}{\bar{A}_{k,j}} T_{k,\cdot} >_{\text{lex}} 0 \Leftrightarrow T^{\text{new}}_{i,\cdot} >_{\text{lex}} 0$.

This proves that the desired property still holds true after the basis exchange step.

We will now show that $T^{\text{new}}_{0,\cdot} >_{\text{lex}} T_{0,\cdot}$. As $\bar{c}_j < 0$,

$$T^{\text{new}}_{0,\cdot} = T_{0,\cdot} - \frac{\bar{c}_j}{\bar{A}_{k,j}} T_{k,\cdot} >_{\text{lex}} T_{0,\cdot}.$$

It follows that in every basis exchange step, the topmost row of the tableau will increase strictly (lexicographically). We can therefore never visit the same basis twice, as otherwise, we would end up with the same topmost row of T, a contradiction.