Computational Physics Homework 2

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Problem 1

a) The 4^{th}-order explicit Runge-Kutta code is in the hw2.py file.

b) In problem 1, the orbit of a test particle around a source obeys the following differential equation,

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2 \tag{1}$$

First rewrite Eq.(1) in term of dimensionless variable , using GM/h^2 for u.

$$\frac{d^2\hat{u}}{d\phi^2} + \hat{u} = 1 + \frac{3G^2M^2}{h^2c^2}\hat{u}^2 \tag{2}$$

The exact Newtonian result in term of dimensionless variable is

$$\hat{u} = 1 + e\cos\phi \tag{3}$$

Then, solving Mercury orbiting around the Sun without the relativistic term, and input three different timesteps as 0.01, 0.05 and 0.1 in Eq.(2). Also, compare the results with the exact Newtonian results.

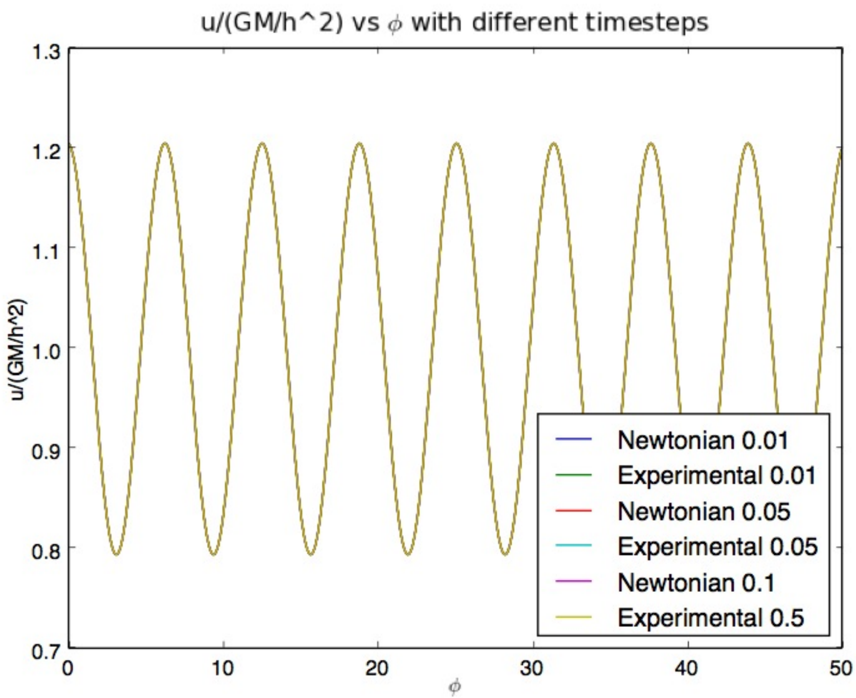

Figure 1: $\hat{u}$ vs ϕ with different timesteps

The relative Error between the results I got and the exact Newtonian results are as below.

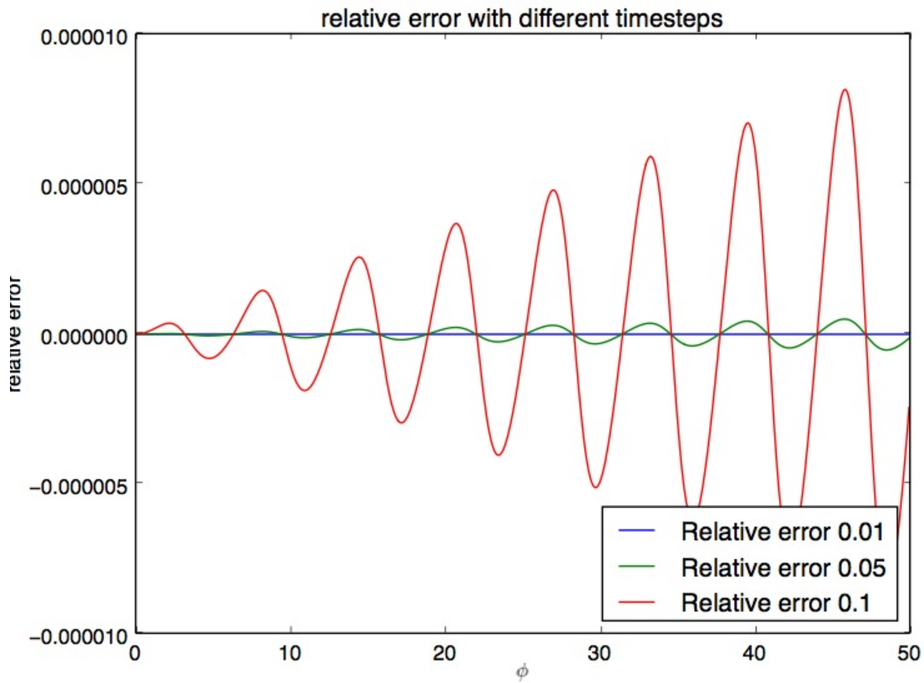

Figure 2: Relative error with different timesteps

It is obvious that a smaller timestep gives a smaller relative error.

c) Next, solving Mercury orbiting around the Sun with the relativistic term in Eq.(2).

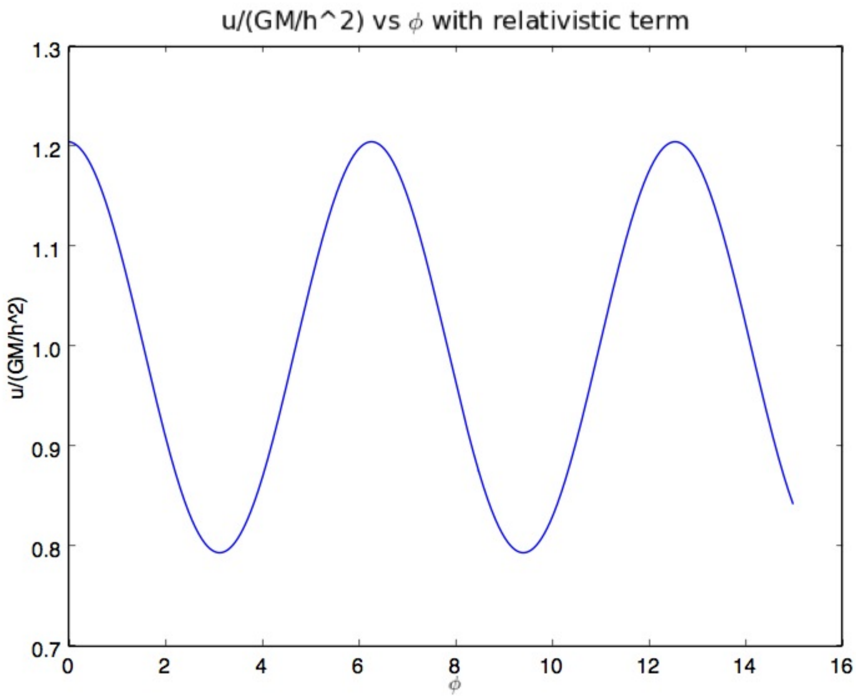

Figure 3: $\hat{u}$ vs ϕ

To compare the result with the famous perturbative result,

$$\triangle\phi^{shift} = \triangle\phi - 2\pi \simeq 6\pi(\frac{GM}{h})^2 \simeq 43\text{"}/century \tag{4}$$

I have to find the angle shift between each perihelion. To get the precise perihelion, I have to fit a quadratic to the three points closest to perihelion to better find the angle at perihelion.

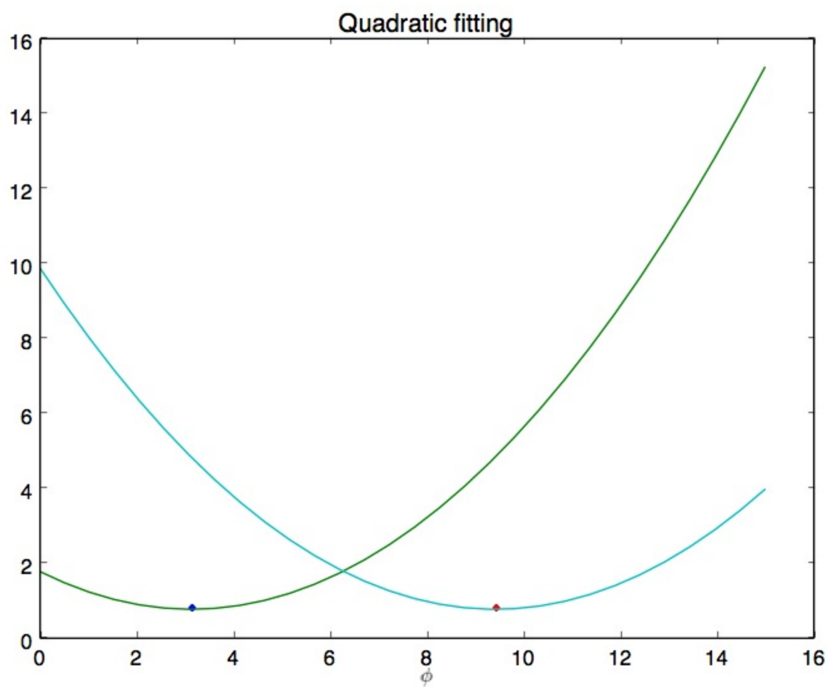

Figure 4: Quadratic fitting

By quadratic fitting, I can get the angle at perihelion and calculate the angle shift is around 43.267"/*century*, which is consistent with the theoretical result.

Problem 2

a) In this problem, the gravitational Newtonian force is changed to

$$F = \frac{GMm}{r^2} \times (\frac{r_0}{r})^\delta$$

Therefore, I have to find the orbit for this force law and write it in a form analogous to Eq.(1)

$$h = r^2\dot{\phi}, \quad u \equiv \frac{1}{r}$$

$$\frac{du}{d\phi} = \frac{-\dot{r}}{h}, \quad \frac{\partial^2 u}{\partial \phi^2} = \frac{-\ddot{r}}{h^2 u^2}$$

$$F = \frac{GMm}{r^2} \times (\frac{r_0}{r})^\delta = ma = m(\ddot{r} - r\dot{\phi}^2) = -m(-h^2u^2\frac{\partial^2 u}{\partial \phi^2} - h^2u^3) = mh^2u^2(\frac{\partial^2 u}{\partial \phi^2} + u)$$

$$\frac{\partial^2 u}{\partial \phi^2} + u = \frac{GM}{h^2}(r_0 u)^\delta$$

Rewrite in term of dimensionless variable , using GM/h^2 for u.

$$\frac{\partial^2 \hat{u}}{\partial \phi^2} + \hat{u} = \hat{u}^\delta \tag{5}$$

b) Solving Eq.(5) by Runge-Kutta, I got

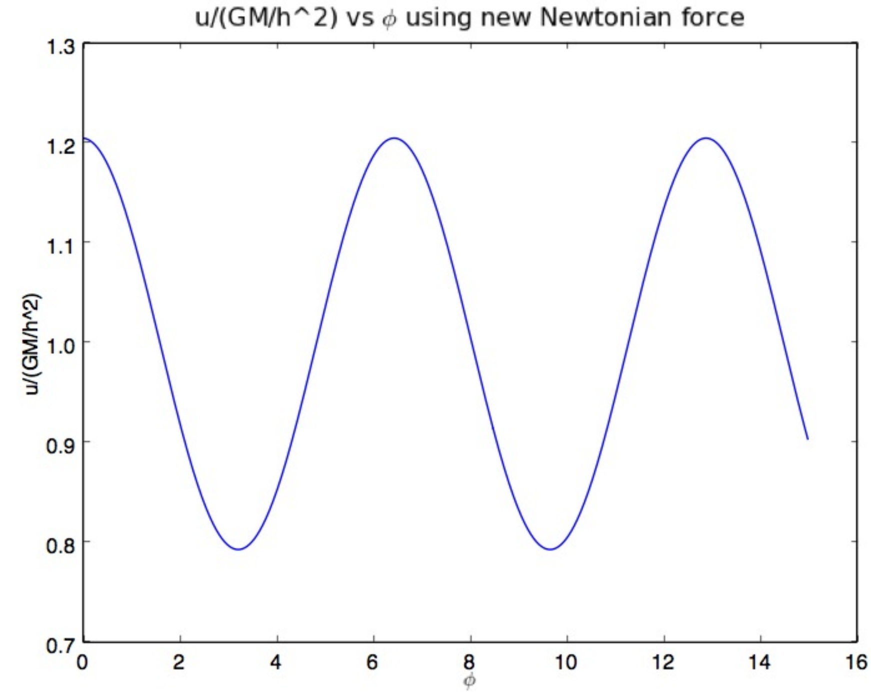

Figure 5: $\hat{u}$ vs ϕ

Similarly, fitting a quadratic to the three points closest to perihelion.

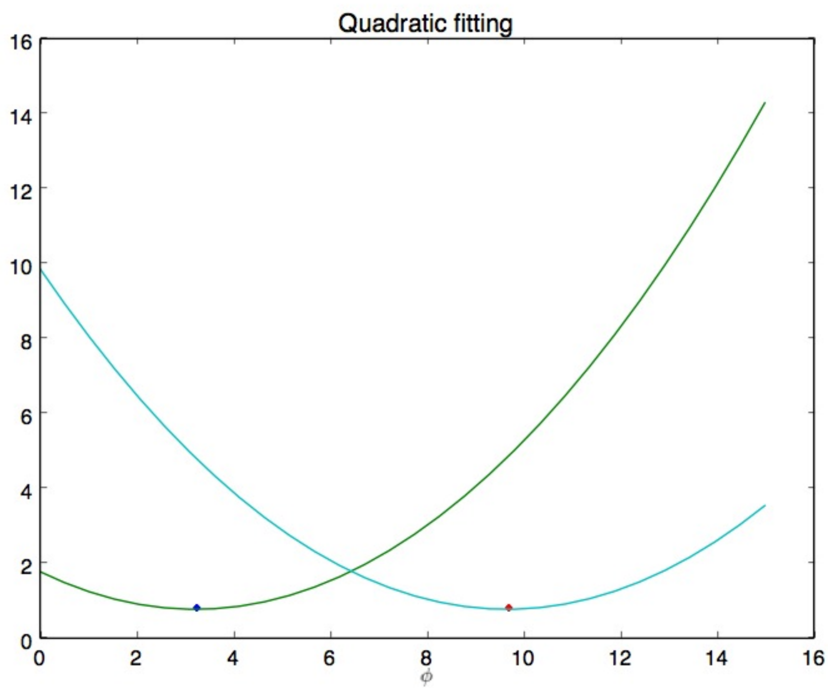

Figure 6: Quadratic fitting

By quadratic fitting, I can get the angle at perihelion and calculate the angle shift is around $0.165 rad/period$.

Problem 3

a) The 4^{th}-order explicit Runge-Kutta code is in the hw2.py file.

b) Rewrite all the equations in terms of dimensionless variables, using ϵ_0 for energy densities and pressure, $M_\odot$ for masses, and km for radii.

$$a = \frac{GM_\odot}{c^2\mathrm{km}}$$

$$\frac{d\hat{p}}{d\hat{r}} = \frac{-a\hat{m}\hat{\epsilon}}{\hat{r}^2} \tag{6}$$

$$\frac{d\hat{m}}{d\hat{r}} = 3\times 10^{-3} \times 4\pi\hat{r}^2\hat{\epsilon} \tag{7}$$

$$\frac{d\hat{p}}{d\hat{r}} = \frac{-a\hat{m}\hat{\epsilon}}{\hat{r}^2}(1+\frac{\hat{p}}{\hat{\epsilon}})(1+\frac{3\times 10^{-3}\times 4\pi\hat{p}\hat{r}^3}{\hat{m}})(\frac{1}{1-\frac{2a\hat{m}}{\hat{r}}}) \tag{8}$$

c) Compare the plots of $\hat{p}$ and $\hat{m}$ as a function of $\hat{r}$ in Newtonian solutions and the relativistic ones with $\hat{p_0}$=0.01.

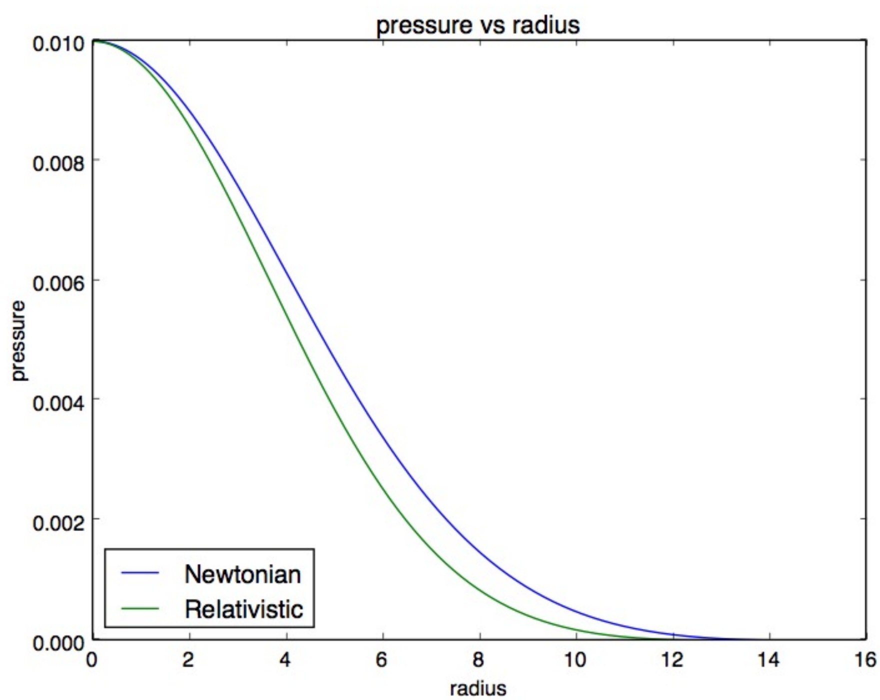

Figure 7: $\hat{p}$ vs $\hat{r}$

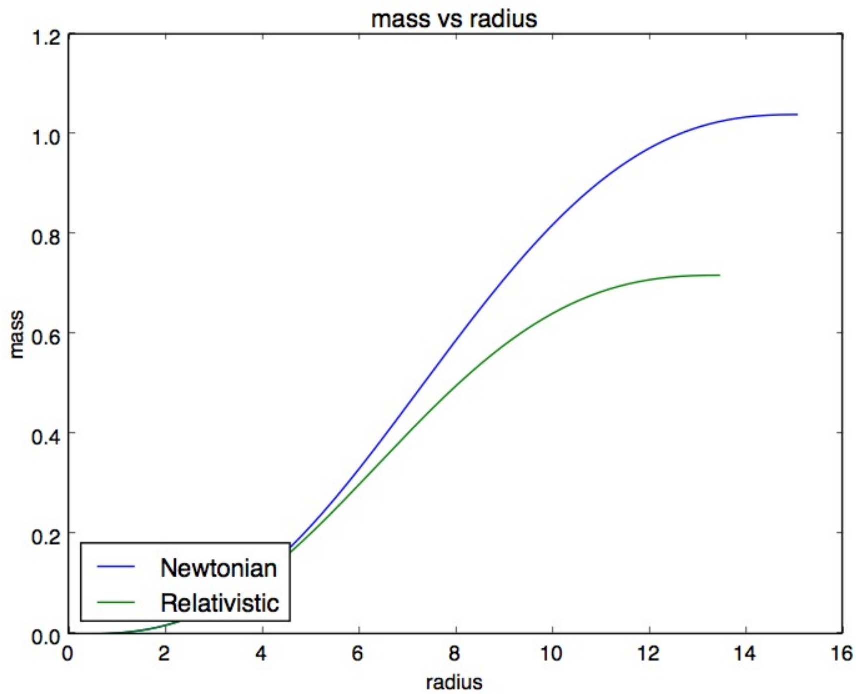

Figure 8: $\hat{m}$ vs $\hat{r}$

The pressure approaches to zero faster in relativistic solutions than in Newtonian ones. The $\hat{r}$ is around 13 and $\hat{m}$ is about 0.75 in relativistic solutions which are both smaller than in Newtonian ones. Actually, I can check this result from Eq.(6) and Eq.(8). The last three terms in Eq.(8) are all larger than 1, so $\hat{p}$ in Eq.(8) must drop faster than in Eq.(6), which agrees with my result.

d) The Plot of $\hat{M}$ and $\hat{R}$ with increasing $\hat{p_0}$ from 0.01 to 0.08 is as below.

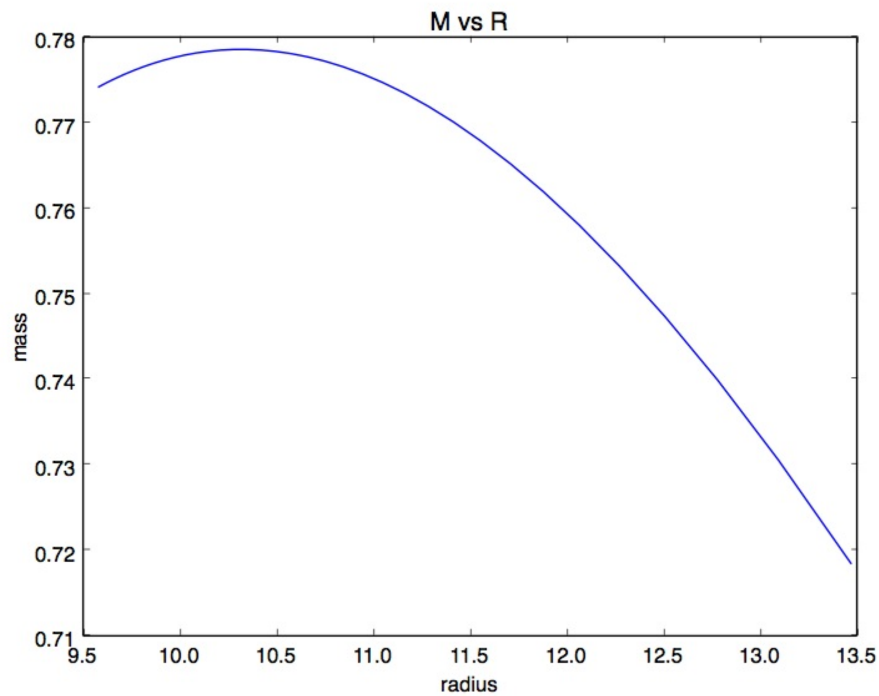

Figure 9: $\hat{M}$ vs $\hat{R}$

I can find that there is a maximum value for the mass, which is around 0.77865, the radius is about 10.35, and the corresponding $\hat{p_0}$ is 0.05. From the paper of Oppenheimer and Volkoff in 1939, I can fiind that the maximum M happends when t_0 is near to 3. According to Table 1, at $t_0 = 3$, $(\frac{N}{V})_{r=0}$ equals to 2.2×10^{39}. Then I can calculate the corresponding $\hat{\epsilon}$ and $\hat{p_0}$ from

$$\frac{(\frac{N}{V}) \times m_n c^2 km^3}{3.0006 \times 10{-}3 \times M_{\odot} c^2 cm^3}, \tag{9}$$

where m_n is the mass of neutron. The result of $\hat{p_0}$ from Eq.(9) is around 0.059, which matches the result I got.