Math 168 Homework 3

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Exercise 1

Post: @73.

Exercise 2

Commented on post @70.

Exercise 3

(a) The degree centrality of node 1 is $\boxed{0}$, and the degree centrality of every other node is $\boxed{1}$.

(b) The adjacency matrix looks like this:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is triangular, so we see that the only eigenvalue is 0.

The eigenvector associated with $\lambda = 0$ is $\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^t$

Thus the eigenvalue centrality of every node except node n is $\boxed{0}$, and the eigenvalue centrality of node n is $\boxed{1}$.

(c)

$$(I - \alpha A)x = \vec{1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ 0 & -\alpha & 1 & 0 \\ 0 & 0 & -\alpha & 1 \end{bmatrix} x = \vec{1}$$

$$\begin{bmatrix} x_1 \\ -\alpha x_1 + x_2 \\ -\alpha x_2 + x_3 \\ -\alpha x_4 + x_4 \end{bmatrix} = \vec{1}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 + \alpha \\ 1 + \alpha + \alpha^2 \\ 1 + \alpha + \alpha^2 + \alpha^3 \end{bmatrix}$$

(we must choose $\alpha < 1$ for convergence)

The Katz centrality of node i is

$$\boxed{\frac{1 - \alpha^i}{1 - \alpha}}$$

Exercise 4

$$
\begin{aligned}
C &= \frac{n}{\sum_j d_{ij}} \\
&= \frac{n}{2 \cdot (1 + 2 + \cdots + ((n-1)/2 - 1)) + (n-1)/2} \\
&= \frac{n}{((n-1)/2)((n-1)/2 - 1) + (n-1)/2} \\
&= \frac{n}{((n-1)/2)((n-1)/2)} \\
&= \boxed{\frac{4n}{(n-1)^2}}
\end{aligned}
$$

Exercise 5

Central node

$$
\begin{aligned}
x_1 &= \sum_{st} n_{st}^1 \\
&= \sum_{st} \begin{cases} 1, & s \neq t \lor s = t = 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \boxed{n^2 - n + 1}
\end{aligned}
$$

Other nodes

$$
\begin{aligned}
x_i &= \sum_{st} n_{st}^i \\
&= \sum_{st} \begin{cases} 1, & s = i \lor t = i \\ 0, & \text{otherwise} \end{cases} \\
&= \boxed{2n - 1}
\end{aligned}
$$

Exercise 6

For each node j in the left half, the shortest path to node 2 is $d_{2j} = d_{1j} + 1$ since it must pass through the edge $(1, 2)$. Likewise, for each node k in the right half, the shortest path to node 1 is $d_{1k} = d_{2k} + 1$.

$$\begin{aligned}
C_1 &= \frac{n}{\sum_j d_{1j} + \sum_k d_{1k}} \\
&= \frac{n}{\sum_j d_{1j} + \sum_k (d_{2k} + 1)} \\
&= \frac{n}{\sum_j d_{1j} + \sum_k d_{2k} + n_2} \\
C_2 &= \frac{n}{\sum_j d_{2j} + \sum_k d_{2k}} \\
&= \frac{n}{\sum_j (d_{1j} + 1) + \sum_k d_{2k}} \\
&= \frac{n}{\sum_j d_{1j} + n_1 + \sum_k d_{2k}}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{C_1} + \frac{n_1}{n} &= \frac{\sum_j d_{1j} + \sum_k d_{2k} + n_2}{n} + \frac{n_1}{n} \\
&= \frac{\sum_j d_{1j} + \sum_k d_{2k} + n_1}{n} + \frac{n_2}{n} \\
&= \frac{1}{C_2} + \frac{n_2}{n}
\end{aligned}$$

□

Exercise 7

(a) $(A^3)_{ii}$ is the number of paths of length 3 from node i to itself. A path of length 3 would be a triangle starting and ending at node i. However, we overcount the number of triangles by a factor of 6, because for the same triangle we could count 6 different paths: for each of the 3 vertices, we can count 2 paths in different directions.

(b) Number of 2-paths:

$$\sum_{ij}(A^2)_{ij} - \sum_{i}(A^2)_{ii}$$

(c)

$$\begin{aligned} C &= \frac{\sum_i (A^3)_{ii}}{\sum_{ij}(A^2)_{ij} - \sum_i (A^2)_{ii}} \\ &= \boxed{0.6666666666666666} \end{aligned}$$

Exercise 8

Legend:

x-axis	n
y-axis	C
Blue square	$p = 0.05$
Red triangle	$p = 0.1$
Black circle	$p = 0.2$