Math 168 Homework 6

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Exercise 1

(a)

$$p_q = \frac{1}{2}(qp_{q-1} - (q+1)p_q)$$
$$p_0 = 1 - \frac{p_0}{2}$$

(b)

$$
\begin{aligned}
g_0'(z) &= \sum_{q=0}^{\infty} q p_q z^{q-1} \\
z g_0'(z) &= \sum_{q=0}^{\infty} q p_q z^{q} \\
z g_0'(z) + g_0(z) &= \sum_{q=0}^{\infty} (q+1) p_q z^{q} \\
(z-1)(z g_0'(z) + g_0(z)) &= \sum_{q=0}^{\infty} (q+1) p_q z^{q}(z-1) \\
&= \sum_{q=0}^{\infty} ((q+1) p_q z^{q+1} - (q+1) p_q z^{q}) \\
&= \sum_{q=1}^{\infty} q p_{q-1} z^{q} - \sum_{q=0}^{\infty} (q+1) p_q z^{q} \\
&= \sum_{q=1}^{\infty} q p_{q-1} z^{q} - \sum_{q=1}^{\infty} (q+1) p_q z^{q} - p_0 \\
&= \sum_{q=1}^{\infty} (q p_{q-1} - (1+q) p_q) z^{q} - p_0 \\
&= \sum_{q=1}^{\infty} 2 p_q z^{q} - p_0 \\
&= \sum_{q=0}^{\infty} 2 p_q z^{q} - 2p_0 - p_0 \\
&= \sum_{q=0}^{\infty} 2 p_q z^{q} - 3p_0 \\
&= 2 g_0(z) - 2 \qquad\qquad p_0 = 2/3 \\
&\boxed{g_0(z) = 1 + \frac{1}{2}(z-1)(z g_0'(z) + g_0(z))}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{dh}{dz} &= \left(\frac{d}{dz}z^3 g_0(z)\right)(1-z)^{-2} + z^3 g_0(z)\frac{d}{dz}(1-z)^{-2} \\
&= (3z^2 g_0(z) + z^3 g_0'(z))(1-z)^{-2} + z^3 g_0(z)(-2)(1-z)^{-3}(-1) \\
&= (3z^2 g_0(z) + z^3 g_0'(z))(1-z)^{-2} + 2z^3 g_0(z)(1-z)^{-3} \\
&= \frac{(3z^2 g_0(z) + z^3 g_0'(z))(1-z) + 2z^3 g_0(z)}{(1-z)^3} \\
&= \frac{3z^2 g_0(z) + z^3 g_0'(z) - 3z^3 g_0(z) - z^4 g_0'(z) + 2z^3 g_0(z)}{(1-z)^3} \\
&= \frac{3z^2 g_0(z) + z^3 g_0'(z) - z^3 g_0(z) - z^4 g_0'(z)}{(1-z)^3} \\
&= \frac{z^2(3g_0(z) + z g_0'(z) - z g_0(z) - z^2 g_0'(z)}{(1-z)^3)} \\
&= \frac{z^2(2g_0(z) + g_0(z) + z g_0'(z) - z g_0(z) - z^2 g_0'(z))}{(1-z)^3} \\
&= \frac{z^2(2g_0(z) + (1-z)g_0(z) + (1-z)z g_0'(z))}{(1-z)^3} \\
&= \frac{z^2(2g_0(z) + (1-z)g_0(z) + (1-z)z g_0'(z))}{(1-z)^3} \\
&= \frac{z^2(2g_0(z) - 2(g_0(z) - 1))}{(1-z)^3} \\
&= \frac{2z^2}{(1-z)^3}
\end{aligned}
$$

(d) TODO

Exercise 2

(a)

$$
\begin{aligned}
Q &= \sum_{t\in\{1,2\}} (e_t - a_t^2) \\
&= (e_1 - a_1^2) + (e_2 - a_2^2) \\
&= \left(\frac{r-1}{n-1} - \left(\frac{2(r-1)+1}{2(n-1)}\right)^2\right) + \left(\frac{n-r-1}{n-1} - \left(\frac{2(n-r-1)+1}{2(n-1)}\right)^2\right) \\
&= \frac{n-2}{n-1} - \frac{(2r-1)^2}{4(n-1)^2} - \frac{(2(n-r)-1)^2}{4(n-1)^2} \\
&= \frac{4(n-1)(n-2)}{4(n-1)^2} - \frac{(2r-1)^2}{4(n-1)^2} - \frac{(2(n-r)-1)^2}{4(n-1)^2} \\
&= \frac{4n^2-12n+8}{4(n-1)^2} - \frac{4r^2-4r+1}{4(n-1)^2} - \frac{4(n-r)^2-4(n-r)+1}{4(n-1)^2} \\
&= \frac{4n^2-12n+8}{4(n-1)^2} - \frac{4r^2-4r+1}{4(n-1)^2} - \frac{4n^2-8nr+4r^2-4n+4r+1}{4(n-1)^2} \\
&= \frac{-8n+6}{4(n-1)^2} - \frac{8r^2}{4(n-1)^2} - \frac{-8nr}{4(n-1)^2} \\
&= \frac{-8n+6-8r^2+8nr}{4(n-1)^2} \\
&= \frac{3-4n+4rn-4r^2}{2(n-1)^2}
\end{aligned}
$$

(b)

$$\frac{dQ}{dr} = \frac{4n-8r}{2(n-1)^2}$$

The critical value of this is $r = n/2$, thus it is optimal to divide down the middle.

Exercise 3

Exercise 4

(a) Since there are no self loops, each triangle gives 2 3-paths from a node to itself for each of the 3 vertices, so the total number of triangles is $8 \cdot 6/6 = \boxed{8}$.

(b) The total number of 2-paths is $2 \cdot (2 \cdot 12 + 4 \cdot 3) = 72$. The global clustering coefficient is $8 \cdot 6/72 = \boxed{2/3}$.

Exercise 5

(a)

$$
\begin{aligned}
g_0(z) &= \sum_{k=0}^{\infty} p_k z^k \\
g_0'(z) &= \sum_{k=0}^{\infty} k p_k z^{k-1} \\
g_0'(1) &= \sum_{k=0}^{\infty} k p_k = \langle k \rangle \\
g_0''(z) &= \sum_{k=0}^{\infty} k(k-1) p_k z^{k-2} \\
g_0''(1) &= \sum_{k=0}^{\infty} k(k-1) p_k = \langle k(k-1) \rangle
\end{aligned}
$$

(b) Condition:

$$
\begin{gathered}
\langle k^2 \rangle > 2 \langle k \rangle \\
\sum_{k=0}^{\infty} k^2 p_k > 2 \sum_{k=0}^{\infty} k p_k \\
\sum_{k=0}^{\infty} (k^2 - k) p_k > \sum_{k=0}^{\infty} k p_k \\
\langle k^2 - k \rangle > \langle k \rangle \\
g_0''(1) > g_0'(1)
\end{gathered}
$$

Find g_0', g_0'':

$$
\begin{aligned}
g_0'(z) &= p(-1)(1-qz)^{-2}(-q) \\
&= pq(1-qz)^{-2} \\
&= \frac{pq}{(1-qz)^2} \\
g_0'(1) &= \frac{pq}{(1-q)^2} \\
g_0''(z) &= pq(-2)(1-qz)^{-3}(-q) \\
&= \frac{2pq^2}{(1-qz)^3} \\
g_0''(1) &= \frac{2pq^2}{(1-q)^3}
\end{aligned}
$$

$$
\begin{gathered}
\frac{2pq^2}{(1-q)^3} > \frac{pq}{(1-q)^2} \\
\frac{2p(1-p)^2}{p^3} > \frac{p(1-p)}{p^2} \\
2(1-p) > p \\
\boxed{p < \frac{2}{3}}
\end{gathered}
$$

Exercise 6

(a) Let X be a random variable representing the current degree, n be the total number of nodes. q_k is the probability that a neighbor has k excess degrees.

$$q_k = \frac{n(k+1)p_{k+1}}{n\langle k\rangle}$$

Here, $n(k+1)p_{k+1}$ is the total number of degrees that belong to nodes with degree $k+1$, and $n\langle k\rangle$ is the total degree over all nodes. The fraction represents the probability that an edge from the current node is attached to a degree $k+1$ node. Simplify the expression, and we get

$$q_k = \frac{(k+1)p_{k+1}}{\langle k\rangle}$$

(b)

$$\begin{aligned}
\langle k\rangle &= 1\cdot p + 3(1-p) = 3-2p \\
\sum_{k=0}^{\infty} kq_k &= \sum_{k=0}^{\infty} k\frac{(k+1)p_{k+1}}{\langle k\rangle} \\
&= \frac{1(1+1)p + 3(3+1)(1-p)}{3-2p} \\
&= \frac{2p+12-12p}{3-2p} \\
&= \boxed{\frac{12-10p}{3-2p}}
\end{aligned}$$

Exercise 7

(a) Every node initially has degree $n-1$. Later, each node gains an average of np degrees as it gains edges to the other cluster. Thus, the average degree is $\boxed{n-1+np}$.

(b) Initially, there are $2\binom{n}{2} = n(n-1)$ edges. Later, each of the n^2 candidates is added with uniform probability p, so the expected number of edges is $\boxed{n(n-1)+n^2p}$.

(c)

$$\begin{bmatrix} 0 & 1 & 1 & 1 & p & p & p & p \\ 1 & 0 & 1 & 1 & p & p & p & p \\ 1 & 1 & 0 & 1 & p & p & p & p \\ 1 & 1 & 1 & 0 & p & p & p & p \\ p & p & p & p & 0 & 1 & 1 & 1 \\ p & p & p & p & 1 & 0 & 1 & 1 \\ p & p & p & p & 1 & 1 & 0 & 1 \\ p & p & p & p & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} Q &= \frac{1}{2m}\sum_{i,j}\left(A_{ij} - \frac{k_i k_j}{2m}\right)\delta_{g_i g_j} \\ &= \frac{1}{2n(n-1)+2n^2p}\sum_{i,j}\left(A_{ij} - \frac{n-1+np}{2n(n-1)+2n^2p}\right)\delta_{g_i g_j} \\ &= \frac{1}{2n(n-1)+2n^2p}\sum_{i,j}\left(A_{ij} - \frac{1}{2n}\right)\delta_{g_i g_j} \\ &= \frac{1}{2n(n-1)+2n^2p}\left(\sum_{i,j} A_{ij}\delta_{g_i g_j} - \frac{1}{2n}\sum_{i,j}\delta_{g_i g_j}\right) \\ &= \frac{1}{2n(n-1)+2n^2p}\left(n(n-1) - \frac{n}{2}\right) \\ &= \frac{1}{2(n-1)+2np}\left(n-1-\frac{1}{2}\right) \\ &= \boxed{\frac{n-3/2}{2(n-1)+2np}} \end{aligned}$$