Math 168 Homework 5

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Exercise 2

(a) $\sum k^{\text{in}} = \sum k^{\text{out}}$, since every in degree corresponds to an out degree on another node. We also have $\sum k^{\text{in}} = n \langle k^{\text{in}} \rangle$ and $\sum k^{\text{out}} = n \langle k^{\text{out}} \rangle$. Thus $\langle k^{\text{in}} \rangle = \langle k^{\text{out}} \rangle$. Furthermore, $\sum k^{\text{in}} + \sum k^{\text{out}} = 2m$, since the total degree is twice the number of edges. Therefore, $2m = n \langle k^{\text{in}} \rangle + n \langle k^{\text{out}} \rangle = 2n \langle k^{\text{in}} \rangle$.

(b) Assuming $k_i^{\text{out}}, k_j^{\text{in}}$ are nonzero.

The probability that a stub from node i is connected to node j is

$$\frac{k_j^{\text{in}}}{2m-1}$$

The total probability that node i is connected to node j is then

$$\frac{k_i^{\text{out}} k_j^{\text{in}}}{2m-1} \approx \frac{k_i^{\text{out}} k_j^{\text{in}}}{2m}$$

(c) The probability that two nodes i, j are reciprocal is

$$\frac{k_i^{\text{in}} k_i^{\text{out}}}{2m} \cdot \frac{k_j^{\text{in}} k_j^{\text{out}}}{2m}$$

The expected number of reciprocal pairs is

$$\begin{aligned}
&\frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{k_i^{\text{in}} k_i^{\text{out}}}{2m} \cdot \frac{k_j^{\text{in}} k_j^{\text{out}}}{2m} - \sum_{i=1}^{n} \frac{(k_i^{\text{in}} k_i^{\text{out}})^2}{4m^2}\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} \frac{k_i^{\text{in}} k_i^{\text{out}}}{2m} \sum_{j=1}^{n} \frac{k_j^{\text{in}} k_j^{\text{out}}}{2m} - \sum_{i=1}^{n} \frac{(k_i^{\text{in}} k_i^{\text{out}})^2}{4m^2}\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} \frac{k_i^{\text{in}} k_i^{\text{out}}}{2n \langle k^{\text{in}} \rangle} \sum_{j=1}^{n} \frac{k_j^{\text{in}} k_j^{\text{out}}}{2n \langle k^{\text{in}} \rangle} - \sum_{i=1}^{n} \frac{(k_i^{\text{in}} k_i^{\text{out}})^2}{4m^2}\right)
\end{aligned}$$

$$\sum_{j=1}^{n} \frac{k_j^{\text{in}} k_j^{\text{out}}}{2n \langle k^{\text{in}} \rangle} = \frac{\sum_j k_j^{\text{in}} k_j^{\text{out}}}{2 \sum_p k_p^{\text{in}}}$$

TODO

Exercise 3

(a)

$$q_k = \frac{(k+1)p_{k+1}}{\langle k \rangle}$$

$$\begin{aligned}
g_1(z) &= \sum_{k=0}^{\infty} q_k z^k \\
&= \sum_{k=0}^{\infty} \frac{(k+1)p_{k+1}}{\langle k \rangle} z^k \\
&= \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} (k+1)p_{k+1} z^k \\
&= \frac{1}{\langle k \rangle} \sum_{k=0}^{\infty} p_{k+1} \left(\frac{d}{dz} z^{k+1} \right) \\
&= \frac{1}{\langle k \rangle} \frac{d}{dz} \left(\sum_{k=0}^{\infty} p_{k+1} z^{k+1} \right) \\
&= \frac{1}{\langle k \rangle} \frac{d}{dz} (g_0(z) - p_0 z^0) \\
&= \frac{1}{\langle k \rangle} g_0'(z) \\
&= \left(\sum_{k=0}^{\infty} k p_k \right)^{-1} g_0'(z) \\
&= \left(\sum_{k=0}^{\infty} k p_k 1^{k-1} \right)^{-1} g_0'(z) \\
&= (g_0'(1))^{-1} g_0'(z) \\
&= \frac{g_0'(z)}{g_0'(1)}
\end{aligned}$$

(b)

$$\begin{aligned}
g_0(z) &= \sum_{k=0}^{\infty} p_k z^k \\
&= \sum_{k=0}^{\infty} e^{-c} \frac{c^k}{k!} z^k \\
&= e^{-c} \sum_{k=0}^{\infty} \frac{(cz)^k}{k!} \\
&= e^{-c} e^{cz} \\
&= \boxed{e^{c(z-1)}}
\end{aligned}$$

$$
\begin{aligned}
g_1(z) &= \frac{g_0'(z)}{g_0'(1)} \\
&= \frac{e^{c(z-1)}c}{e^{c(1-1)}c} \\
&= \frac{e^{c(z-1)}c}{c} \\
&= e^{c(z-1)}
\end{aligned}
$$

This means the excess degree distribution is equal to the degree distribution. In other words, there is no friendship paradox, since your friends are expected to have the same amount of friends as yourself.

(c) (i)

$$
p_k = \begin{cases} 1, & k = k_0 \\ 0, & k \neq k_0 \end{cases}
$$

This is the unit impulse, whose pgf is $g_0(z) = z^{k_0}$. Using the formula from part a, we get

$$
\begin{aligned}
g_1(z) &= \frac{g_0'(z)}{g_0'(1)} \\
&= \frac{k_0 z^{k_0-1}}{k_0 \cdot 1^{k_0-1}} \\
&= z^{k_0-1}
\end{aligned}
$$

And this pgf corresponds to the probability distribution

$$
q_k = \begin{cases} 1, & k = k_0 - 1 \\ 0, & k \neq k_0 - 1 \end{cases}
$$

which claims that the excess degree of every node is $k_0 - 1$. The formula from part a is correct, and this makes sense, because if every node has k_0 neighbors, one would expect one's neighbors to have $k_0 - 1$ other neighbors.

(ii) If $k_0 \geq 3$, then we have

$$
\begin{aligned}
u &= g_1(u) \\
u &= u^{k_0-1} \\
u &= 0
\end{aligned}
$$

We ignore the trivial solution $u = 1$.

$$
\begin{aligned}
S &= 1 - g_0(u) \\
&= 1 - 0^{k_0} \\
&= 1
\end{aligned}
$$

Exercise 4

(a) Given a random edge, the probability that it leads to node i is

$$p(i) = \frac{k_i}{2m-1} \approx \frac{k_i}{2m}$$

Thus, the expected value of the destination node of a random edge is

$$\begin{aligned}
\langle x \rangle_{\text{edge}} &= \sum_i x_i p(i) \\
&= \frac{1}{2m} \sum_i k_i x_i
\end{aligned}$$

(b) Note that

$$2m = \sum_i k_i$$

Thus

$$\begin{aligned}
\langle x \rangle_{\text{edge}} - \langle x \rangle &= \frac{1}{\sum_i k_i} \sum_i k_i x_i - \langle x \rangle \\
&= \frac{1}{\sum_i k_i} \sum_i k_i x_i - \frac{1}{n} \sum_i x_i \\
&= \frac{1}{1/n \cdot \sum_i k_i} \cdot \frac{1}{n} \sum_i k_i x_i - \frac{1}{n} \sum_i x_i \\
&= \frac{\langle kx \rangle}{\langle k \rangle} - \langle x \rangle \\
&= \frac{\langle kx \rangle - \langle x \rangle \langle k \rangle}{\langle k \rangle} \\
&= \frac{\text{cov}(k, x)}{\langle k \rangle}
\end{aligned}$$

Exercise 5

(a)

$$\begin{aligned}
g_0(z) &= \sum_{k=0}^{\infty}(1-a)a^k z^k \\
&= (1-a)\frac{1}{1-az} \\
&= \frac{1-a}{1-az} \\
g_0'(z) &= (1-a)(-1)(1-az)^{-2}(-a) \\
&= \frac{a(1-a)}{(1-az)^2} \\
g_0'(1) &= \frac{a}{1-a} \\
g_1(z) &= \frac{g_0'(z)}{g_0'(1)} \\
&= \frac{(1-a)^2}{(1-az)^2}
\end{aligned}$$

According to eq. 12.30, $u = g_1(u)$, so

$$\begin{aligned}
u &= g_1(u) \\
u &= \frac{(1-a)^2}{(1-au)^2} \\
u(1-2au+a^2u^2) &= (1-a)^2 \\
a^2u^3 - 2au^2 + u - (1-a)^2 &= 0
\end{aligned}$$

(b) Simply multiply the two factors to get the original equation:

$$\begin{aligned}
(u-1)(a^2u^2 - a(2-a)u + (1-a)^2) &= 0 \\
a^2u^3 - a(2-a)u^2 + (1-a)^2u - a^2u^2 + a(2-a)u - (1-a)^2 &= 0 \\
a^2u^3 + (-2a+a^2)u^2 + (1-2a+a^2)u - a^2u^2 + (2a-a^2)u - (1-a)^2 &= 0 \\
a^2u^3 - 2au^2 + u - (1-a)^2 &= 0
\end{aligned}$$

Thus the nontrivial solution satisfies $a^2u^2 - a(2-a)u + (1-a)^2 = 0$.

(c)

$$\begin{aligned}
u &= \frac{a(2-a) - \sqrt{a^2(2-a)^2 - 4a^2(1-a)^2}}{2a^2} \\
&= \frac{a(2-a) - \sqrt{(a(2-a) - 2a(1-a))(a(2-a) + 2a(1-a))}}{2a^2} \\
&= \frac{a(2-a) - \sqrt{a^2(4a-3a^2)}}{2a^2} \\
&= \frac{2-a-\sqrt{4a-3a^2}}{2a}
\end{aligned}$$

$$
\begin{aligned}
S &= 1 - g_0(u) \\
&= 1 - \frac{1-a}{1-(2-a-\sqrt{4a-3a^2})/2} \\
&= 1 - \frac{1-a}{1-(1-a/2-\sqrt{a-3a^2/4})} \\
&= 1 - \frac{1-a}{a/2+\sqrt{a-3a^2/4}} \\
&= 1 - \frac{(1-a)(a/2-\sqrt{a-3a^2/4})}{a^2/4-a+3a^2/4} \\
&= 1 - \frac{(1-a)(a/2-\sqrt{a-3a^2/4})}{a^2-a} \\
&= 1 + \frac{a/2-\sqrt{a-3a^2/4}}{a} \\
&= 1 + \frac{1}{2} - \sqrt{\frac{1}{a}-\frac{3}{4}} \\
&= \frac{3}{2} - \sqrt{a^{-1}-\frac{3}{4}}
\end{aligned}
$$

(d) If $a \leq 1/3$, then

$$
\begin{aligned}
a^{-1} &\geq 3 \\
\sqrt{a^{-1}-\frac{3}{4}} &\geq \frac{3}{2} \\
S = \frac{3}{2} - \sqrt{a^{-1}-\frac{3}{4}} &\leq 0
\end{aligned}
$$

So the GC would not exist (since the fraction of nodes in the GC is not positive).

Exercise 6

(a) The diameter would be the shortest path from one end of the circle to the opposite end. Since we can jump across $c/2$ neighbors, this distance is $(n/2)/(c/2) = \boxed{n/c}$.

(b)