168 - Homework 2. Due on Gradescope by January 26th

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Exercise 1. (Getting ready for the final project)

Just as we did in discussion section on Tuesday, think of something that is personally interesting to you from outside of this class. Write a few sentences about a problem that involves this personal interest, and how a network might be involved in it (it is OK to have some uncertainty about what the specific question would be, or have some uncertainty about what the network model might be.

Exercise 2. (Getting ready for the final project, part 2)

Visit the Colorado Index of Complex Networks: `https://icon.colorado.edu/#!/`

This is a website that hosts data from a huge number of networks (some of them huge). Most of the data sets have a corresponding reference (either a scientific paper, or a website). Pick any single network that seems interesting to you, read the associated reference, and write a few sentences describing what the network models and which of the categories we have studied so far it belongs in (e.g. simple, directed, undirected etc.).

(* Optional challenge extension :) Download the data associated with your favorite network, and make a histogram of the distribution of the degrees of the nodes (that is: what proportion of nodes have degree 1, what proportion have degree 2, etc.).

Exercise 3. (Linear algebra practice)

(a) Find the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

(b) Suppose a matrix A has eigenvalues λ_1, λ_2, $\ldots \lambda_n$. Then, what are the eigenvalues of
(i) $-A$
(ii) $A + I$ where I is the $n \times n$ identify matrix?

(c) Consider the matrix

$$B = \begin{pmatrix} 1 & 6 & -4 & 5 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Determine all of B's eigenvalues and their corresponding geometric multiplicities (i.e. the number of linearly independent eigenvectors associated with each eigenvalue). (Hint: you can read off the eigenvalues without needing to factorize the characteristic polynomial, why?)

(d) For the matrix

$$P = \begin{pmatrix} 4 & 0 & 0 & -1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 4 \end{pmatrix},$$

find all of the eigenvalues and their geometric multiplicities.

(e) The matrix

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix},$$

has two eigenvalues $\lambda = -1$ and $\lambda = 1$. Given that $\det(L) = 9$, find the remaining two eigenvalues of B, without finding the characteristic polynomial of L.

Exercise 4. Returning to the star graph of Homework 1, Exercise 6, find the eigenvalue centralities of the nodes in the networks. (NB: in case you attempt this problem before Monday's class the absolute centralities of nodes depend on the precise eigenvalue that you choose, but the ratio of the centralities between the center node and any peripheral node does not). What happens to the ratio of the centralities as the number of nodes, n grows?

Exercise 5. (Based on Newman 6.5a and b)

(a) A $k-$ regular graph is one in which every node has the exact same degree $k_i = k \ , \ \forall i$. Explain why if k is odd, then the graph must have an even number of nodes.

(b) A tree is a connected undirected graph with no loops in it. Read Section 6.8 on trees and the proof that a tree with n nodes always has $n-1$ edges.

(i) What is the average degree of a node in any tree?

(ii) What is the maximum degree a node in a tree graph may have?

Exercise 6. (Computer programming problem) A random network with $|N| = 12$ has the adjacency matrix which is shown in our Piazza post. Make two computer plots of the network (to be included in your answer), showing all of the edges and nodes and with the colors of the nodes showing, respectively:

(a) Eigenvalue centralities.

(b) Degree centralities.