Math 168 Homework 2

Jason Cheng

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Exercise 1

(a)

(b)

Exercise 2

Exercise 3

(a)

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{pmatrix}2-\lambda & -1\\ -1 & 2-\lambda\end{pmatrix}\\ &= (2-\lambda)^2-(-1)^2\\ &= \lambda^2-4\lambda+3\\ &= (\lambda-3)(\lambda-1)\\ &\boxed{\lambda_1=1,\lambda_2=3}\end{aligned}$$

$$A-1I=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\to\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$$
$$\ker(A-1I)=\operatorname{span}\{(1,1)\}$$

The eigenvectors associated with $\lambda_1=1$ are $\boxed{\operatorname{span}\{(1,1)\}}$.

$$A-3I=\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\to\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$$
$$\ker(A-3I)=\operatorname{span}\{(1,-1)\}$$

The eigenvectors associated with $\lambda_2=3$ are $\boxed{\operatorname{span}\{(1,-1)\}}$.

(b) (i)

$$Ax=\lambda x \implies (-A)x=-(Ax)=-\lambda x$$

The eigenvalues of $-A$ are $-\lambda_1,-\lambda_2,\ldots,-\lambda_n$.

(ii)

$$Ax=\lambda x \implies (A+I)x=Ax+Ix=\lambda x+x=(\lambda+1)x$$

The eigenvalues of $A+I$ are $\lambda_1+1,\lambda_2+1,\ldots,\lambda_n+1$.

(c) The eigenvalues of B are its diagonal entries since B is upper triangular. So the eigenvalues are 1, 2, and 3.

Geometric multiplicities

$$\begin{aligned}A-1I &= \begin{bmatrix}0 & 6 & -4 & 5\\ 0 & 1 & 3 & 4\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 2\end{bmatrix}\\ &\to\begin{bmatrix}0 & 0 & -22 & -19\\ 0 & 1 & 3 & 4\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}\end{aligned}$$
$$\dim(\ker(A-1I))=1$$

$$
A - 2I = \begin{bmatrix} -1 & 6 & -4 & 5 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\to \begin{bmatrix} -1 & 6 & -4 & 5 \\ 0 & 0 & 0 & 13 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\to \begin{bmatrix} -1 & 6 & -4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\dim(\ker(A - 2I)) = 1
$$

$$
A - 3I = \begin{bmatrix} -2 & 6 & -4 & 5 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\dim(\ker(A - 3I)) = 1
$$

The geometric multiplicity of all three eigenvalues is 1.

(d) P is symmetric, so it is diagnoalizable.

$$
\begin{aligned}
\det(P - \lambda)I &= \det \begin{pmatrix} 4-\lambda & 0 & 0 & -1 \\ 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ -1 & 0 & 0 & 4-\lambda \end{pmatrix} \\
&= (3-\lambda) \det \begin{pmatrix} 4-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 4-\lambda \end{pmatrix} \\
&= (3-\lambda)(1-\lambda) \det \begin{pmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{pmatrix} \\
&= (3-\lambda)(1-\lambda)((4-\lambda)^2 - (-1)^2) \\
&= (3-\lambda)(1-\lambda)(5-\lambda)(3-\lambda)
\end{aligned}
$$

The eigenvalues are $1, 3, 5$. The geometric multiplicities of $\lambda = 1, 5$ are 1, and the geometric multiplicity of $\lambda = 3$ is 2 since they have to be equal to the arithmetic multiplicities because P is diagnoalizable.

(e)

$$
\begin{gathered}
tr(L) = 2 + 3 + 3 + 2 = 10 \\
(-1) + 1 + \lambda_1 + \lambda_2 = 10 \\
\lambda_1 + \lambda_2 = 10 \\
(-1)(1)\lambda_1\lambda_2 = -9 \\
\lambda_1\lambda_2 = 9 \\
\boxed{\begin{matrix} \lambda_1 = 9 \\ \lambda_2 = 1 \end{matrix}}
\end{gathered}
$$

Exercise 4