Math 168 Homework 1

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Exercise 1

Dr. Körner thinks that students should treat lectures as a dialogue where the student is following the arguments rather than blindly take notes.

Exercise 2

Exercise 3

1. A polygon is a static simple network. The nodes are the vertices and the edges are the sides of the polygon.
2. Family trees are dynamic (because relationships can form/change) bipartite (because it's a tree) networks. The nodes are the family members and the edges are the relationships between parents and children.
3. Rivers are observable transport networks. The nodes are lakes and oceans, and the edges are rivers.
4. Plumbing is a weighted (weighted by pipe thickness) network with directed edges. The nodes are faucets, and the edges are the pipes.

Exercise 4

(a)

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$L = A - D == \boxed{\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}}$$

(b) There would be $\binom{|N|}{2}$ edges.

(c)

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{pmatrix} \\ &= \lambda^4 - 6\lambda^2 - 8\lambda - 3 \\ &= \lambda^4 - 2\lambda^2 + 1 - 4\lambda^2 - 8\lambda - 4 \\ &= (\lambda^2 - 1)^2 - 4(\lambda + 1)^2 \\ &= (\lambda - 1)^2(\lambda + 1)^2 - 4(\lambda + 1)^2 \\ &= (\lambda + 1)^2((\lambda - 1)^2 - 4) \\ &= (\lambda + 1)^3(\lambda - 3) \end{aligned}$$

The eigenvalues of A are $\lambda = -1, 3$. $\lambda = -1$ has algebraic multiplicity of 3.

(d)

$$\begin{aligned} \det(L - \lambda I) &= \det \begin{pmatrix} -3-\lambda & 1 & 1 & 1 \\ 1 & -3-\lambda & 1 & 1 \\ 1 & 1 & -3-\lambda & 1 \\ 1 & 1 & 1 & -3-\lambda \end{pmatrix} \\ &= \lambda^4 + 12\lambda^3 + 48\lambda^2 + 64\lambda \\ &= \lambda^4 + 8\lambda^3 + 16\lambda^2 + 4\lambda^3 + 32\lambda^2 + 64\lambda \\ &= \lambda^2(\lambda + 4)^2 + 4\lambda(\lambda + 4)^2 \\ &= \lambda(\lambda + 4)^3 \end{aligned}$$

The eigenvalues of L are $\lambda = -4, 0$.

Exercise 5

(a) Adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} A^2 &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \\ A^3 &= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix} \end{aligned}$$

$A^3_{11} = 2$, so there are two paths of length 3 that start and end at node 1.

(b) The entries of A^3 outside the main diagonal are all 3, so between each pair of different nodes there are 3 paths.

Exercise 6

Adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots \\ 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{n\times n}$$

$$\begin{aligned}
&\det(A-\lambda I)\\
&= \det \begin{pmatrix} -\lambda & 1 & 1 & \dots \\ 1 & -\lambda & 0 & \dots \\ 1 & 0 & -\lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{n\times n}\\
&= -1\cdot\det \begin{pmatrix} 1 & 1 & 1 & \dots \\ 0 & -\lambda & 0 & \dots \\ 0 & 0 & -\lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{(n-1)\times(n-1)} \qquad - \lambda \det \begin{pmatrix} -\lambda & 1 & 1 & \dots \\ 1 & -\lambda & 0 & \dots \\ 1 & 0 & -\lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{(n-1)\times(n-1)}
\end{aligned}$$

$$\begin{aligned}
&\det \begin{pmatrix} 1 & 1 & 1 & \dots \\ 0 & -\lambda & 0 & \dots \\ 0 & 0 & -\lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{n\times n}\\
&= -\lambda \det \begin{pmatrix} 1 & 1 & 1 & \dots \\ 0 & -\lambda & 0 & \dots \\ 0 & 0 & -\lambda & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{(n-1)\times(n-1)}
\end{aligned}$$

$$\begin{aligned}
f(n) &= -g(n-1) - \lambda f(n-1)\\
g(n) &= -\lambda g(n-1)\\
g(2) &= \det \begin{pmatrix} 1 & 1 \\ 0 & -\lambda \end{pmatrix}\\
&= -\lambda\\
g(n) &= (-\lambda)^{n-1}\\
f(2) &= \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\\
&= \lambda^2 - 1\\
f(n) &= (-\lambda)^n - (n-1)(-\lambda)^{n-2}
\end{aligned}$$

Verify $f(n)$. Base case:

$$f(2) = (-\lambda)^2 - (2-1)(-\lambda)^0 = \lambda^2 - 1$$

Inductive step:

$$\begin{aligned}
f(n+1) &= -g(n) - \lambda f(n) \\
&= -(-\lambda)^{n-1} - \lambda((-\lambda)^n - (n-1)(-\lambda)^{n-2}) \\
&= -(-\lambda)^{n-1} + (-\lambda)^{n+1} - (n-1)(-\lambda)^{n-1} \\
&= (-\lambda)^{n+1} - n(-\lambda)^{n-1}
\end{aligned}$$

The characteristic polynomial is $f(n) = (-\lambda)^n - (n-1)(-\lambda)^{n-2} = (-\lambda)^{n-2}((-\lambda)^2 - (n-1))$.

The roots are $\lambda = 0, \lambda = \pm\sqrt{n-1}$.

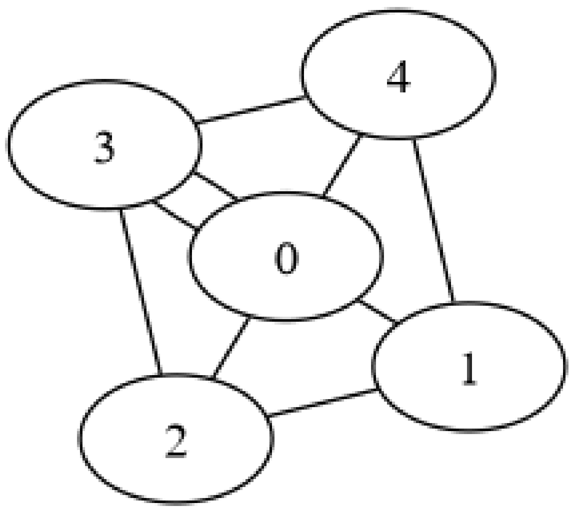
The largest eigenvalue is $\boxed{\sqrt{n-1}}$.

Exercise 7

(a) Eigenvalues:

```
3.749429478214337+0i
-2.3540236479908256+0i
-1.5423248829430551+0i
0.14691905271954572+0i
0.000000000000000021895643135715098+0i
```

(b) Graph of the network:



Exercise 8