Math 168 Homework 4

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Exercise 1

(a) Given an E-R graph, we can choose $\binom{n}{3}$ sets of 3 nodes. The probability that each set of 3 nodes is a triangle is p^3. Thus, the expected number of triangles is

$$\binom{n}{3}p^3 = \frac{n(n-1)(n-2)p^3}{6}$$

In the limit of large n, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{n(n-1)(n-2)p^3}{6} &= \lim_{n\to\infty} \frac{(n-1-1)p \cdot (n-1+1)p \cdot (n-1)p}{6} \\
&= \lim_{n\to\infty} \frac{((n-1)p-p)((n-1)p+p)(n-1)p}{6} \\
&= \lim_{n\to\infty} \frac{(c-p)(c+p)c}{6} \\
&= \lim_{n\to\infty} \frac{c^3-cp^2}{6} \\
&= \lim_{n\to\infty} \frac{c^3-0}{6} && \lim_{n\to\infty} cp^2 = 0 \\
&= \frac{c^3}{6}
\end{aligned}$$

(b) We can choose a set of 3 nodes in $\binom{n}{3}$ ways and pick a middle node in 3 ways. For each set, the probability that they are a connected triple is p^2. The expected number of connected triples is

$$\begin{aligned}
3\binom{n}{3}p^2 &= \frac{1}{2}n(n-1)(n-2)p^2 \\
&= \frac{1}{2}(n-1-1)(n-1+1)(n-1)p^2 \\
&= \frac{1}{2}((n-1)^2-1)(n-1)p^2 \\
&= \frac{1}{2}((n-1)^3p^2-(n-1)p^2) \\
&= \frac{1}{2}((n-1)c^2-cp)
\end{aligned}$$

In the limit of large n, this becomes $(n-1)c^2/2$, since cp goes to 0.

(c)

$$\frac{3 \cdot c^3/6}{(n-1)c^2/2} = \frac{c^3}{(n-1)c^2} = \boxed{\frac{c}{n-1}}$$

Exercise 2

(a) Total number of edges

$$
\begin{aligned}
3\binom{n}{3}p &= 3\binom{n}{3}\frac{c}{\binom{n-1}{2}} \\
&= 3 \cdot \frac{n(n-1)(n-2)}{6} \cdot \frac{2c}{(n-1)(n-2)} \\
&= nc
\end{aligned}
$$

Sum of all node degrees would be $2nc$, so the expected degree would be $2nc/n = \boxed{2c}$.

(b) Every time we add a triangle, we add 2 degrees to each vertex of the triangle. Thus all nodes have even degree.

Every node is considered for $\binom{n-1}{2}$ potential triangles independently. Each time it is considered, it has a probability of p to add 2 degrees. For each consideration, let X be the number of added degrees.

$$
\begin{aligned}
p_X(2) &= p \\
p_X(0) &= 1 - p
\end{aligned}
$$

Aggregating the $\binom{n-1}{2}$ events, let Y be the number of degrees of a node.

$$
\begin{aligned}
p_Y(2k) &= \binom{n}{k}p^k(1-p)^{n-k} \\
&= \frac{n!}{k!(n-k)!}\left(\frac{c}{\binom{n-1}{2}}\right)^k\left(1 - \frac{c}{\binom{n-1}{2}}\right)^{n-k} \\
&= \frac{c^k}{k!} \cdot \frac{n!}{(n-k)!} \cdot \left(\frac{2}{(n-1)(n-2)}\right)^k\left(1 - \frac{2c}{(n-1)(n-2)}\right)^{n-k}
\end{aligned}
$$

Exercise 3