NOTES

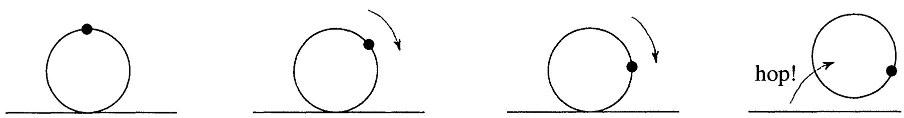
Edited by **Jimmie D. Lawson**

The Hopping Hoop

Tadashi F. Tokieda

'A weight is attached to a point of a rough weightless hoop, which then rolls in a vertical plane, starting near the position of unstable equilibrium. What happens, and is it intuitive?'

The problem just quoted is from Littlewood's delightful *Miscellany* [**1**, p.37]. As is often the case in phenomena involving a no-slip constraint ('rough' indicates that the hoop is to roll *without slipping*), what happens is rather unintuitive. Declares Littlewood: 'The hoop lifts off the ground when the radius vector to the weight becomes horizontal.'



Perhaps the most ingenuous approach to proving that the hoop indeed hops is to calculate the force that the hoop exerts against the floor at the point of contact, and to check that it changes to negative after the hoop has rolled $\pi/2$. The approach works, but hardly explains why the hoop should hop at all.

It is more pleasant to reason as follows. If the hoop is always kept in contact with the floor, then the weight (call it m) travels along a cycloid. Now imagine that, when m comes to a certain point P on the cycloid, the hoop suddenly disappears. Then m would continue to free-fall along a parabola tangent to the cycloid at P. If, however, the hoop fails to disappear (as it usually does), then

(1) m presses the hoop *down* as long as the imagined parabola at P departs *below* the cycloid;

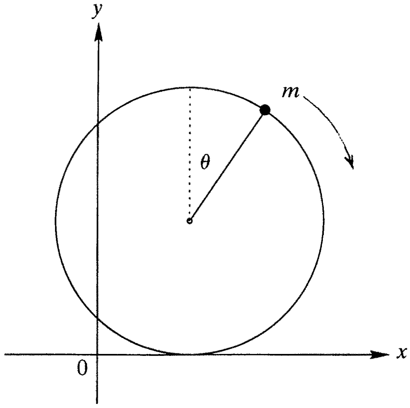
(2) m pulls the hoop *up*, and so the hoop hops, as soon as the imagined parabola at P departs *above* the cycloid.

By construction the parabola and the cycloid have the same zeroth and first terms in their Taylor series around P. Therefore departure below or above will be decided by an inequality between their *second* derivatives.

Let us determine the earliest P for which (2) occurs. We generalize the problem somewhat by taking the liberty of shoving m off the point of unstable equilibrium with initial velocity v_0 (in Littlewood's original formulation, $v_0 = 0$).

In coordinates as shown, with g and R denoting the gravitational acceleration and the radius of the hoop, the conservation of energy dictates

$$\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + mgy = \frac{m}{2}v_0^2 + mg2R.$$



Along the cycloid

$$x(t) = R\theta(t) + R\sin\theta(t)$$
$$y(t) = R + R\cos\theta(t)$$

we have

$$\frac{m}{2}\left[\left(R\dot{\theta} + R\cos\theta\cdot\dot{\theta}\right)^2 + \left(-R\sin\theta\cdot\dot{\theta}\right)^2\right] + mg(R + R\cos\theta) = \frac{m}{2}v_0^2 + mg2R,$$

which unravels to

$$\dot{\theta}^2 = \frac{4gR\sin^2(\theta/2) + v_0^2}{4R^2\cos^2(\theta/2)}.$$

This relation enables us to express the derivatives of $y(t)$ in terms of θ:

$$\dot{y} = -R\sin\theta\cdot\dot{\theta}$$
$$= -\sin(\theta/2)\sqrt{4gR\sin^2(\theta/2) + v_0^2},$$
$$\ddot{y} = -2g\sin^2(\theta/2) - \frac{v_0^2}{4R}.$$

As remarked earlier, m pulls the hoop up, thereby making it hop, as soon as the second derivative of the parabola exceeds that of the cycloid; i.e. the hop occurs at minimal θ such that $-g \geq \ddot{y}(\theta(t))$, or

$$\sin(\theta/2) \geq \frac{1}{\sqrt{2}}\left(1 - \frac{v_0^2}{4gR}\right)^{1/2}.$$

In particular, for $v_0 = 0$ the hoop hops at $\theta = \pi/2$, as claimed. We also observe that for $v_0 = \sqrt{4gR}$ the hoop 'glides' immediately without rolling. Naturally, this escape velocity ought to be larger than the escape velocity $\sqrt{2gR}$ for a circle: since the cycloid has smaller curvature than the circle does at the peak $\theta = 0$, it is harder to escape from the cycloid than from the circle.

The author thanks F. Almgren for collaborating on an experiment. We taped a battery on a hula-hoop and rolled it down the twelfth-floor hallway in Fine Hall; it actually hopped.

REFERENCES

1. J. E. Littlewood, *Littlewood's Miscellany*, Cambridge University Press, 1986.

Department of Mathematics
Princeton University
Princeton NJ 08544-1000
tokieda@math.princeton.edu

Principal Ideal Domains Are Almost Euclidean

John Greene

In most undergraduate level books on abstract algebra, it is shown that every Euclidean domain (ED) is a principal ideal domain (PID) and every principal ideal domain is a unique factorization domain (UFD). We thus have a set of implications: ED $\Rightarrow$ PID $\Rightarrow$ UFD. Most (but not all!) books mention that neither converse is true. But while it is very easy to show that $Z[x]$ is an example of a UFD that is not a PID, an example of a PID that is not a ED is harder to come by. In [**2**], Campoli gives an easy proof that $Z[\zeta]$ has the desired properties, where $\zeta = (-1 + \sqrt{-19})/2$, by showing that, in his words, $Z[\zeta]$ is "almost Euclidean." In this note, we show that Campoli's "almost Euclidean" condition is, in fact, equivalent to the PID condition.

Definition. An integral domain D is said to be **almost Euclidean** if there is a function $d: D \to Z^+ \cup \{0\}$ (called an almost Euclidean function) such that

1) $d(0) = 0$, $d(a) > 0$ if $a \neq 0$,
2) If $b \neq 0$, then $d(ab) \geq d(a)$ for all $a \in D$,
3) for any $a, b \in D$, if $b \neq 0$ then either
   i) $a = bq$ for some $q \in D$ or
   ii) $0 < d(ax + by) < d(b)$ for some $x, y \in D$.

Our functions d in this paper will satisfy the stronger condition (2′) that for all a, b in D, $d(ab) = d(a)d(b)$, from which (2) follows trivially.

Our main result is the following:

Theorem 1. *An integral domain D is a principal ideal domain if and only if it is almost Euclidean.*

Proof: Campoli [**2**] proved that if a ring is almost Euclidean, it is a PID. For completeness, we repeat the proof here. Let D be almost Euclidean, and let I be a nonzero ideal in D. Among the elements $x \in I$, let b be an element with a minimal positive value for $d(x)$. Given $a \in I$, for any $x, y \in D$, $ax + by$ is in I. By definition of b, it cannot be that $0 < d(ax + by) < d(b)$, so the second condition, $a = bq$ must hold for some $q \in D$. Thus, $I = (b)$.