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Andrew Taylor and Mary Fehrs

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The dynamics of an eccentrically loaded hoop

Andrew Taylor[a)] and Mary Fehrs
Department of Physics, Pacific University, Forest Grove, Oregon 97116

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Several papers have modeled the hopping behavior of a rolling massless hoop with a point mass on the rim and predicted the angle of the point mass with respect to the vertical when the hoop first hopped. The predictions vary from no hop at all to a hop at $\pi/2$ or $3\pi/2$. Theron more recently modeled the hoop, assuming that it has mass. For a mass ratio γ of the rim-loaded mass to the total hoop mass, he predicted hopping angles between $3\pi/2$ and 2π. We tested these predictions experimentally for hoops with γ between 0.5 and 0.8 and found that Theron's model correctly predicts the angles at which the hoop hops. There is a slight disagreement between our results and Theron's for mass ratios near the crossover between hopping and nonhopping.

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I. INTRODUCTION

An eccentrically loaded wheel is a hoop with its center of mass displaced from the geometric center. A loaded hoop will hop if its velocity is sufficiently high. We achieved a consistent hopping velocity by rolling the eccentrically loaded hoop down a ramp. The question investigated in our experiment is the angle with respect to the vertical of the attached mass when the hoop first hops.

Littlewood proposed this question in 1953 (Ref. 1) and predicted the hoop will hop at $\pi/2$ or $3\pi/2$, but he presented no justification for these predictions. Three papers in the 1990s reconsidered this question. Each modeled the eccentrically loaded hoop as a massless hoop with a point mass on the rim. Two papers concluded that the hoop hops at an angle of $\pi/2$,[2,3] and the third concluded that the hoop will not hop at all.[4]

In 2000 Theron did an extensive analysis of the hoop with a more realistic model in which the mass of the hoop is taken into account.[5] He concluded the hoop would hop at an angle ranging from $\pi/2$ to 2π. We created a quick demonstration of a rolling eccentrically loaded hoop to test the claims in Refs. 1–5. Because visual observations showed the hoop hopped at an angle clearly greater than $\pi/2$, we conducted an experiment to test Theron's analysis.

II. THERON'S MODEL

Theron modeled a perfectly rigid hoop with an attached point mass.[5] Figure 1 shows the hoop with radius R rolling down a ramp, which makes an angle ϕ with the horizontal. Point P is the attached mass, θ is the angle that the mass has rotated from its starting position, and G represents the center of mass and is located a distance γR from the geometric center, where γ is a measure of the eccentricity,

$$\gamma = \frac{m_{\text{attached}}}{m_{\text{total}}}. \tag{1}$$

Point O is the contact point between the hoop and the ramp where F and N act. F is the frictional force between the ramp and the hoop, and N is the normal force exerted by the ramp on the hoop.

The coordinates of the geometric center are denoted by $(X(t),Y(t))$ with the origin O as the initial contact point and the y'-axis perpendicular to the plane and the x'-axis along the plane. The coordinates of G are $(x(t),y(t))$ relative to the same axes. The wheel rotates in the $(x'(t),y'(t))$ plane and has three degrees of freedom denoted by X, Y, and θ.

The constraints implied by the rigidity of the wheel are

$$y(t) = Y(t) + R\gamma \cos\theta(t), \tag{2}$$

$$Y(t) \geq R. \tag{3}$$

Equation (3) is a sufficient criterion for hopping because $Y(t)=R$ describes the hoop maintaining contact with the surface and $Y(t)>R$ describes leaving the surface. The dimensionless centripetal acceleration and angular acceleration (respectively) are

$$\eta = (R/g)\dot{\theta}^2, \tag{4}$$

$$\zeta = (R/g)\ddot{\theta}. \tag{5}$$

A dot denotes differentiation with respect to time. If we differentiate Eq. (2) twice with respect to time and substitute the dimensionless variables where appropriate, we obtain

$$\ddot{y}/g = \ddot{Y}/g - \gamma\zeta \sin\theta - \gamma\eta \cos\theta. \tag{6}$$

While in contact with the ramp, there are three forces acting on the hoop: The weight of the hoop and attached mass, the normal force N, and the frictional force F. A necessary condition for hopping is that $N=0$. Because the frictional force between two surfaces is $F=\mu N$, F is also zero when N is zero. The only force acting on the hoop as it hops is the weight of the hoop and attached mass. The components of acceleration when hopping are

$$\ddot{y} = -g\cos\phi, \tag{7}$$

$$\ddot{\theta} = 0. \tag{8}$$

If we use Eqs. (7) and (8) in Eq. (6) and solve for $\ddot{Y}$, we find

$$\ddot{Y} = g\gamma(\eta\cos\theta - \eta_0), \tag{9}$$

where

$$\eta_0 = (1/\gamma)\cos\phi. \tag{10}$$

Immediately following the instant the normal force becomes zero, the position of the center of the hoop C must be

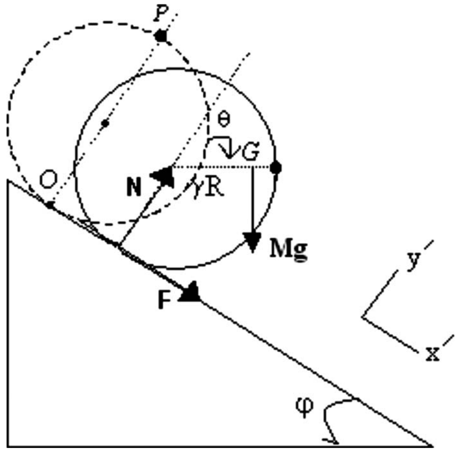


Fig. 1. Diagram describing the loaded hoop on an incline. The hoop rolls in the $(x'(t), y'(t))$ frame and the mass P starts at $\theta=0$ when released. Different γ/ϕ combinations were measured to test under what conditions the loaded hoop would hop.

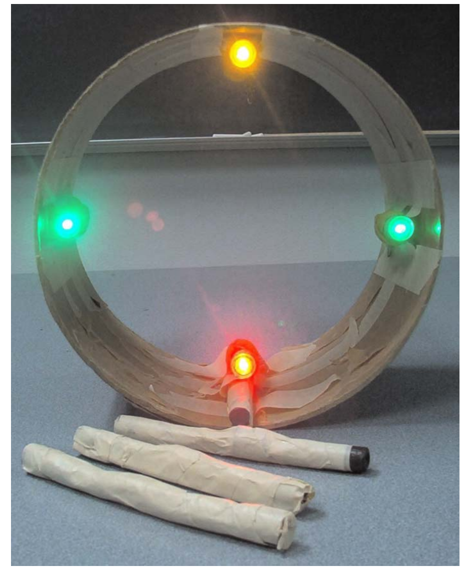


Fig. 2. Hoop with LED lights and lead weights. Lights used for tracking could not be placed on the radius but were secured to the inside of the hoop. The weights ran the width of the hoop. Extra weights for larger gamma were taped around the bottom light.

at a distance greater than R from the horizontal axis. Before the hop $\dot{Y}(t)$ is zero, and $Y(t)=R$. For $Y(t)>R$ the hoop must accelerate in the y'-direction. Therefore, hopping will occur when Eq. (9) is positive, which will occur when the quantity in parentheses is positive,[5]

$$\eta \cos \theta - \eta_0 > 0. \tag{11}$$

Equation (11) is the critical equation for hopping: η_0 is determined by both γ and the angle ϕ of the incline of the ramp; η is determined by the radius and the angular velocity of the hoop. The variables γ, ϕ, and R are determined by the experimental setup; only $\dot{\theta}$ and θ are measured during a trial. If the inequality is positive definite, then Ref. 5 predicts a hop; otherwise, it does not.

III. EXPERIMENTAL ARRANGEMENT

The eccentrically loaded hoop was constructed out of a cardboard tube 20.5 cm in diameter and cut to 12.7 cm in width. It weighed 143 g. The weights were small lead slugs taped together to be as long as the hoop's width; these weights were attached as the point mass. Four small LED lights were taped to the inside of the hoop at the four cardinal points to more accurately determine the position of the hoop as it rolled, as shown in Fig. 2. A wooden plank on top of a metal track resting against a small stepladder constituted the ramp. The hoop was placed at the top of the ramp with $\theta=0$. A black marker was used to draw two small lines from the inside of the hoop onto the wood to ensure that the hoop started in the same position for every trial.

This setup was videotaped by a camcorder, which was connected to a computer; the videos were converted into QUICKTIME movie files using IMOVIE. These movies were then opened in VIDEOPOINT[6] for analysis. The angle θ was measured visually by watching the videos frame by frame. The angular velocity $\dot{\theta}$ was measured using the VIDEOPOINT software. VIDEOPOINT marked the coordinates of the four LED lights plotted in the x' and y' coordinates and determined the angular velocity. From the frame for which $\theta=3\pi/2$, the four angular velocities of each point were averaged together to obtain the hoop angular velocity.

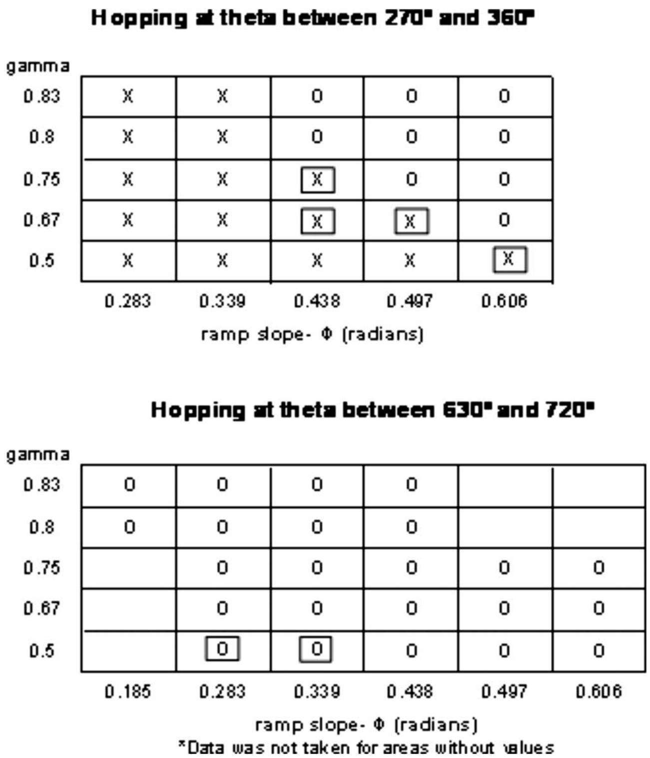
Six ramp angles were considered: $\phi=0.185$, 0.283, 0.339, 0.438, 0.497, and 0.606 rad. The attached mass could be easily varied from one hoop mass to five hoop masses. Thus, there were 30 combinations of mass and ramp angle to test the predictions in Ref. 5, and each combination was videotaped twice.

IV. RESULTS

The length of the ramp allowed the hoop to rotate twice and hence two chances to hop. Some γ/ϕ combinations yielded a hop on the first rotation, and the smaller eccentricities yielded a hop on the second. Some hopped as soon as $\theta=3\pi/2$, while most hopped at $\theta=2\pi$. When modeling Theron's equations, we overestimated hopping by assuming $\theta=2\pi$ for the greatest spread in the inequality in Eq. (11). The modeled inequality was then checked against the visual observations.

Our experiment generally supports the analysis in Ref. 5. This result is at odds with earlier analyses.[2,4] Most of the trials hopped when Theron's model predicts hopping and did not hop when it did not predict hopping. Five of the 48 combinations (25 first rotations and 23 second rotations) showed visual disagreement, and three other combinations were not accurate enough to predict due to the resolution of our QUICKTIME movies.

To determine whether these represented real disagreements with Ref. 5, more videos were taken. The eight combinations that did not fully agree were recorded using a higher frame rate camera with progressive scan at 240 frames per second (fps), again with two trials each. Unfortunately, the only video analysis program we had access to was unable to analyze this data properly. The program split the frames into four quadrants. We chose the frame where $\theta=3\pi/2$. Each quadrant was analyzed separately for the angular velocity and then averaged to obtain the average angular velocity per frame. This procedure effectively cut the fps



Hopping at theta between 270° and 360°

gamma	0.283	0.339	0.438	0.497	0.606
0.83	X	X	0	0	0
0.8	X	X	0	0	0
0.75	X	X	$\boxed{\text{X}}$	0	0
0.67	X	X	$\boxed{\text{X}}$	$\boxed{\text{X}}$	0
0.5	X	X	X	X	$\boxed{\text{X}}$

ramp slope- Φ (radians)

Hopping at theta between 630° and 720°

gamma	0.185	0.283	0.339	0.438	0.497	0.606
0.83	0	0	0	0		
0.8	0	0	0	0		
0.75		0	0	0	0	0
0.67		0	0	0	0	0
0.5		$\boxed{0}$	$\boxed{0}$	0	0	0

ramp slope- Φ (radians)
*Data was not taken for areas without values

Fig. 3. The measured combinations of γ/ϕ. An *H* represents that the hoop was observed to hop, and an *O* represents nonhopping. A bracket is placed when theory differs with the results.

to 60, only twice that of our original camera. Experimentally, only one of the eight combinations hopped, but there was no longer ambiguity between trials.

Of the eight γ/ϕ combinations we remeasured, two agreed with Ref. 5 (γ=0.75, ϕ=0.339 rad; γ=0.75, ϕ=0.497 rad). Six combinations disagreed still. Figure 3 shows the results taken with the high speed camera. An *N* is placed if we did not see the hoop hop and an *H* if we did see it hop. The brackets represent disagreement between the model of Ref. 5 and our observations.

V. CONCLUSION

An eccentrically loaded hoop was observed for 30 combinations of the hoop eccentricity γ and ramp angle ϕ to determine the angle at which the hoop hops. Our results support the predictions in Ref. 5. Ambiguities in the observations near the transition between hopping and nonhopping regions are most likely due to the difficulty in accurately determining the position of the lights on the hoop using the VIDEOPOINT program because of the low resolution of our videos. Also, we used a real mass, while in Ref. 5 a point mass was modeled.

The experimental confirmation that the hoop hops at $3\pi/2 < \theta < 2\pi$ supports Eq. (11) and contradicts Ref. 3. Further experiments could study the effects of slipping or the use a more rigid hoop. Equation (11) could also be modified to more accurately model a more realistic hoop.

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[a]Electronic mail: tayl8485@pacificu.edu

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