Hopping Hoops Don't Hop

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The dynamical behavior of a massless hoop with an attached point mass under the influence of gravity is an old chestnut, with some surprising features, including the question of whether or not it can "hop". It appears in Littlewood's *Miscellany* **[1]** and most recently in this MONTHLY [**2**]. That there are many interesting aspects of this problem was suggested to me many years ago by the late Prof. J.C. Miller of Pomona College, who may have gotten a hint of the trickier parts directly from Littlewood himself. The purpose of this Note is to show that Littlewood's and Tokeida's asserted solution is wrong, even with a more realistic hoop and more realistic friction, and to suggest some approaches to a self consistent investigation of the rolling hoop.

In its simplest form, the problem asks for the behavior of a rough massless hoop of radius R with a point mass M attached to its rim, rolling in a vertical plane on a level floor under gravity. Implicitly in Littlewood's problem, and explicitly in this Note, we understand the concept of massless to be the limit of a positive hoop mass tending to 0; otherwise the rotational behavior of the hoop in free fall is undefined. The idea of "rough" is central to the problem; it is (roughly) defined as a no-slip constraint at the point of contact of the hoop and the floor. Let $\vartheta \in [0,\pi]$ be the angle from the radius vector to the mass, measured from vertical; other ranges of ϑ are not considered in this Note. If the total energy is equal to the gravitational potential energy of the mass one diameter above the floor, the "solution" to the problem is the assertion by Littlewood that the hoop "lifts off the ground" or by Tokeida that the mass "pulls the hoop up" at $\vartheta = \pi/2$; this particular value of ϑ depends on the total energy, but is especially simple in the case cited. Denote the normal force conferred by the floor on the hoop by n. The hopping conclusion is alleged to follow from the observation that, for zero kinetic energy at $\vartheta = 0$ and for the rim constrained to be in no-slip contact with the floor, $n > 0$ for $\vartheta < \pi/2$ and $n < 0$ for $\vartheta > \pi/2$. The evaluation of $n(\vartheta)$ is elementary using Newton's second law and conservation of energy (the rough condition is conservative). An equivalent argument [**2**] is that the motion of the point mass either follows a cycloid for $0 \le \vartheta \le \pi/2$, or its free fall parabolic preference for $\pi/2 \le \vartheta$; here the upper limit was not specified.

This solution is wrong on both mathematical and physical grounds. To show this, we begin with the equations of motion, with the following notational conventions. Nondimensionalize the problem by measuring distance in units of radius R, mass in units of M, and time in units of $\sqrt{R/g}$, where g is gravitational acceleration. In these units, gravitational acceleration is -1. The horizontal and vertical coordinates of the point mass are $x = \vartheta + \sin\vartheta$ and $y = 1 + \cos\vartheta$, respectively. The kinetic energy is given by $(1/2)(\dot{x}^2 + \dot{y}^2) = \dot{\vartheta}^2(1 + \cos\vartheta)$, where the overdot denotes the time derivative, and the potential energy by $y = 1 + \cos\vartheta$. With this notation, Newton's law takes the form

$$\ddot{y} = n - 1 = -\cos(\theta)\dot{\theta}^2 - \sin(\theta)\ddot{\theta}.$$

If we take the total energy to be 2 (the potential energy at $\vartheta = 0$), the conservation of energy yields

$$\dot{\vartheta}^2(1 + \cos\vartheta) = 1 - \cos\vartheta.$$

From these two equations, it is easily verified that n is positive for $\vartheta < \pi/2$, zero for $\vartheta = \pi/2$, and negative for $\vartheta > \pi/2$; since the floor cannot exert a negative normal force on the hoop, one concludes (incorrectly) that the hoop hops past $\vartheta = \pi/2$.

Theorem. *Littlewood's hoop doesn't hop.*

Proof: Let the coordinates of the hoop's center be x_c, y_c. For the hoop to hop, y_c must be greater than one (radius) above the floor (as long as the hoop is in contact with the floor, $y_c = y - \cos\vartheta = 1$). The proof that this cannot happen is by contradiction. Assume that past $\vartheta = \vartheta_0 = \pi/2$ the hoop hops, implying that the mass falls freely, then solve for y_c, and show that it cannot be greater than one. Let t_0 be the time at which $\vartheta = \vartheta_0$ and $n = 0$. Assume that for $t_0 < t < t_1$, the hoop hops, meaning $y_c > 1$. In this interval, the point mass is in free fall; the equations of motion are different from those with rough contact and a positive normal force. In particular, the vertical coordinate of the mass is given by $y = y_0 + \dot{y}_0(t - t_0) - (1/2)(t - t_0)^2$ and ϑ is given by uniform angular rotation $\vartheta = \vartheta_0 + \dot{\vartheta}_0(t - t_0)$, where the subscript 0 indicates the quantity evaluated at $t = t_0^-$. During the hop, therefore,

$$\dot{y}_c = \dot{y} + \sin(\vartheta)\dot{\vartheta} = \dot{y}_0 - (t - t_0) + \sin(\vartheta)\dot{\vartheta}_0,$$

$$\ddot{y}_c = \ddot{y} + \cos(\vartheta)\dot{\vartheta}^2 + \sin(\vartheta)\ddot{\vartheta} = -1 + \cos(\vartheta)\dot{\vartheta}_0^2.$$

At the beginning of the hop, $y_c(t_0) = 1$, $\dot{y}_c(t_0) = 0$, and importantly, $\ddot{y}_c(t_0) = -1$. It follows that for the open time interval $t_0 < t < t_1$, $\ddot{y}_c < 0$. Over this interval therefore, the velocity of the center of the hoop is negative, i.e., it must anti-hop, or push through the floor. This is inconsistent with the assumption of a hop. ■

What this theorem shows is that Littlewood's problem is singular in the sense that there is no solution past t_0 consistent with Newton's laws, an impenetrable floor ($y_c \geq 1$), and no-slip ($x_c = \vartheta$) when $n = 0$. What then does the real hoop do? If we retain Newton's laws and an impenetrable floor, then the no-slip condition must be violated when $n = 0$, implying that the hoop skids. One must check that the skidding solution is self consistent; that this is true is sketched in the following arguments.

There are several issues and questions raised by Littlewood's problem. First, with respect to the hopping conclusion, there is the explicit error in not using the different equations of motion for the different periods. Equivalently, the idea of simultaneously using a constraint such that $y_c = 1$ together with $n < 0$ to argue that $y_c > 1$ is clearly inconsistent. Second, the concept of "rough" contact is not well defined. As we have argued, we may retain a no-slip condition for two objects in contact with $n > 0$, but not for $n = 0$; in what follows we define no-slip conditions only for $n > 0$. Third, what then happens to such a real hoop during the skid phase? Can it subsequently hop? Fourth, if we impose realistic frictional conditions rather than rough ones, can such a hoop hop without skidding, or can it hop following a skid phase?

It is not the purpose of this Note to give complete answers to these questions, but a few points can be made. The idea of a point mass and a massless hoop suggests a second type of singular character to the equations of motion. This singularity manifests itself as a reduction of the number of equations of motion from three to two when the moment of inertia about the center of mass is zero, and as a concomitant reduction in the number of degrees of freedom for the frictional force from two to one (the vector friction force must point in the direction of the point mass). In approximating the behavior of real hoops, it is thus appropriate to consider distributing the mass on a real hoop so that its center of mass is at a radial distance $\lambda < 1$ and its moment of inertia about that point is $I > 0$; we must investigate the hoop's behavior for λ near 1 and for I near zero. With coordinates x, y generalized to the center of mass ($x = x_c + \lambda \sin \vartheta$, $y = y_c + \lambda \cos \vartheta$) and denoting the tangential frictional force of the floor on the hoop by f, the equations of motion, valid irrespective of no-slip, skidding, or free-fall conditions, are

$$\begin{aligned} \ddot{x} &= f, \\ \ddot{y} &= n - 1, \\ I\ddot{\vartheta} &= n(x - x_c) - fy = n\lambda \sin \vartheta - f(1 + \lambda \cos \vartheta). \end{aligned}$$

The no-slip condition remains $x_c = \vartheta$. It turns out that in this case as well, our Theorem remains true. The proof by contradiction is similar, and may be sketched as follows. When the hoop hops, we have $n(t_0) = 0$, $\ddot{y}(t_0) = -1$, $\ddot{y}_c(t_0^-) = -1 + \lambda \cos(\vartheta_0)\dot{\vartheta}_0^2 + \lambda \sin(\vartheta_0)\ddot{\vartheta}_0 = 0$, and $\ddot{y}_c(t_0^+) = -1 + \lambda \cos(\vartheta_0)\dot{\vartheta}_0^2$. Eliminating $\ddot{\vartheta}$ from the equations of motion (written in terms of ϑ) easily shows that $f < 0$ and $\ddot{\vartheta} > 0$ when $n = 0$. Comparison of $\ddot{y}_c(t_0^-)$ with $\ddot{y}_c(t_0^+)$ then shows $\ddot{y}_c(t_0^+) < 0$, which as before implies a contradictory anti-hop and therefore a skid. That the skid phase is well defined is shown by the existence of a finite f consistent with keeping the hoop on the floor.

Having established that (at least for $\vartheta \in [0, \pi]$) an ideal hoop and its real cousin must skid if the normal force goes to zero, we now ask for a little more reality in the friction law. This might be Coulomb friction, with skidding occurring at some critical ratio of the tangential and normal force. One might even include a distinction between the ratio at which skidding begins (static friction), and the ratio during the skid (dynamic friction). In any case, the physics is no longer conservative, and the absence of an energy integral means the equations are more difficult to analyze. Nevertheless, some general conclusions may still be drawn. (A) Is the skid forward ($\dot{\vartheta} > \dot{x}_c$: hoop is rotating faster than necessary for no-slip (an accelerating train with slipping wheels)), or backwards ($\dot{\vartheta} < \dot{x}_c$: hoop is rotating slower than required for no-slip (a decelerating train with slipping wheels))? The answer is that both behaviors are possible. Starting from ϑ near zero, the initial tangential force is clearly positive, and if the hoop is sufficiently greased, f/n can match the static friction; the hoop skids forward. If it's not, there is necessarily a backwards skid with $f < 0$. The sign of f at the beginning of the skid must therefore be considered. (B) While skidding, is a hop possible at some t_1? There are several possibilities here. (B1) Call a hop a "smooth take-off" if it occurs when $n = 0$. During the skid $|f/n|$ is fixed for Coulomb friction, and thus a smooth take-off requires $f = 0$ also. In this case (by analogy with the sketched proof), $\ddot{\vartheta}(t_1^-) = 0$, $\ddot{y}_c(t_1^-) = \ddot{y}_c(t_1^+) = 0$, but $\dddot{y}_c(t_1^+) = -\lambda \sin(\vartheta_1)\dot{\vartheta}_1^3 < 0$, again implying an anti-hop. (B2) Call a hop a "semi-smooth take-off" if it occurs when the skidding hoop catches up to the floor, and dynamic friction switches over to static friction. That this additional complication doesn't change matters is easily seen by the observation that a change in force to a different, but finite, value for an infinitesimal amount of time cannot change the subsequent dynamics. (B3) What remains is the possibility of a "jump hop", associated with impulsive forces, approximated by delta functions in "stick-slip" dynamical problems such as chattering chalk on a blackboard. This phenomenon appears to be the most likely candidate for the origin of the real hop of real hoops, but raises more complicated questions about the physics, and makes the analysis commensurately more difficult.

This analysis of the hopping hoop leads to several conclusions. First, and most important, is that Newton's laws and the kinematical constraint for "rough" contact are in general inconsistent when the normal force is zero. Second, real hoops that hop must skid first, and the subsequent hop cannot be smooth nor semi-smooth. Third, there is a rich structure in the behavior of real hoops: vary λ and I, vary the initial conditions, let ϑ be unbounded, follow the bounce(s) after the hop. Finally, with respect to this isolated singularity in Mr. Littlewood's *Miscellany*, he did say that in practice, "the hoop skids", but seemed to imply this to be due to a realistic friction law rather than a necessary consequence even with an unbounded coefficient of static friction. The answer to his query whether the behavior of the hoop is intuitive is given by the following

Theorem. *The behavior of hopping hoops is not intuitive.*

Proof: By inspection.

REFERENCES

1. J. E. Littlewood, *Littlewood's Miscellany*, Cambridge University Press., Cambridge, 1986.
2. T. F. Tokieda, The Hopping Hoop, *Amer. Math. Monthly* **104** (1997) 152–154.

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Approximation of Hölder Continuous Functions by Bernstein Polynomials

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In a recent MONTHLY [**5**], a special instance of the Weierstraß approximation theorem attracted attention: approximation of real Lipschitz functions on $[0,1]$ by Bernstein polynomials

$$B_n(f,x) := \sum_{j=0}^{n} \binom{n}{j} x^j (1-x)^{n-j} f\left(\frac{j}{n}\right).$$

The authors of [**5**] provided a rate of uniform convergence of $B_n(f,\cdot)$ to f using large deviations techniques. It is the aim of this note to discuss the optimal rate of approximation with some historical remarks. More generally we consider the class $\mathrm{Lip}_\alpha(L)$ of Hölder continuous functions with exponent α for some $0<\alpha\le 1$ and constant L, i.e., functions that obey

$$|f(x)-f(y)| \le L|x-y|^\alpha \quad \text{for all } x,y \in [0,1].$$