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# The Dynamics of an Elastic Hopping Hoop

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**Abstract**—This paper discusses the motion of a hoop which is loaded with a heavy particle fixed to the rim and which is rolling in a vertical plane. In contrast to previous analyses, the hoop is not rigid; the elasticity in the system produces results that are in agreement with previously reported observations of hopping hula-hoops.

The main result of this analysis is the identification of the conditions that are required for hopping to occur after a rotation through less than 90° after starting with the particle at the highest point.


**Keywords**—Classical mechanics, Elastic, Hopping hoop.

## 1. INTRODUCTION

Hopping hoops have been observed and reported on for some time. This paper is the first to develop a model which gives a reasonable explanation of the phenomenon which occurs when a hoop, loaded with a heavy object fixed to the rim, hops while rolling in a vertical plane.

The hypothetical problem of a rigid, massless hoop, loaded with a particle, which rolls without slipping can be used as an interesting classroom problem for planar motion of rigid bodies, and was first published by Littlewood [1] in 1953 in "A Mathematician's Miscellany". The problem was "revived" by Tokieda [2], whose analysis is based on the geometric aspects of the motion. Both Littlewood and Tokieda conclude that the hoop will hop after rolling through 90° after starting from rest with the particle at the highest point of its cycloidal path.

Variations of this problem have received quite a bit of attention since Tokieda's paper. Butler [3] and Theron [4] attempted to prove that this hop could not occur. However, in both cases, the "proof" was based on an incorrect assumption, as discussed by Theron and du Plessis in [5]. We have been unable to prove conclusively that the truly singular problem of a massless, rigid hoop loaded with a particle does not hop when the normal reaction becomes zero. However, in [5], we include realistic values of the friction coefficient in the model, and find that the rolling phase can be followed by a "skimming" motion in which the massless hoop remains in contact with the surface even though the reaction force between surface and hoop is zero.

Real rigid hoops, where the mass of the hoop as well as the effect of friction is taken into account are analysed in detail by Pritchett [6] and Theron [7]. Pritchett proves formally that such hoops cannot hop from the rolling motion, but have to slide first. An important aspect of his proof is a model where the "particle" is a body with nonzero moment of inertia. Theron uses

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mainly numerical solutions to analyse a number of interesting phenomena in great detail. His results are in agreement with Pritchett in showing that a real hoop will always start sliding before the normal reaction becomes zero. However, the two papers present conflicting results for the motion after sliding starts. Theron maintains that the hoop cannot hop after rotating through approximately 90°, whereas Pritchett shows a graphic result in which it does. This discrepancy is presumably due to the different numerical algorithms used to solve the sliding motion. Theron shows, however, that heavily loaded rigid hoops can hop after rotating through approximately 270°.

Tokieda [2] reports having witnessed an experiment by Almgren [8] in which a hula-hoop was loaded with a battery and hopped as predicted by Littlewood. Pritchett [6] also shows a stroboscopic photograph of the same phenomenon, in which a plastic hula-hoop is loaded with brass rods. It is interesting that in both cases, a hula-hoop, which can surely not be considered to be rigid, was used.

This paper is an attempt to explain this observed phenomenon by taking the elasticity of the hoop into account. An extremely simple elastic model is considered, in which it is assumed inter alia that the deformation of the hoop does not affect the moment of inertia. In spite of its simplicity, this model yields the desired results.

Incidentally, in a recent fascinating paper [9], Fairclough also analyses the rotation of a weighted wheel, albeit in a completely different context.

## 2. THE MATHEMATICAL MODEL

### 2.1. Geometry of the Hoop

Figure 1 shows a hoop with radius $R$ rolling on a horizontal surface. Point O is the centre of the undeformed hoop, which is loaded with a small object (or particle) at point P on the rim, and point C is the point of contact.

Three assumptions regarding the geometry of the elastic hoop are made in order to simplify the model. They are as follows.

1. The moment of inertia is not affected by the deformation of the hoop.
2. The radius to the contact point, OC, deforms elastically.
3. The radius to the particle is constant; OP $= R$.

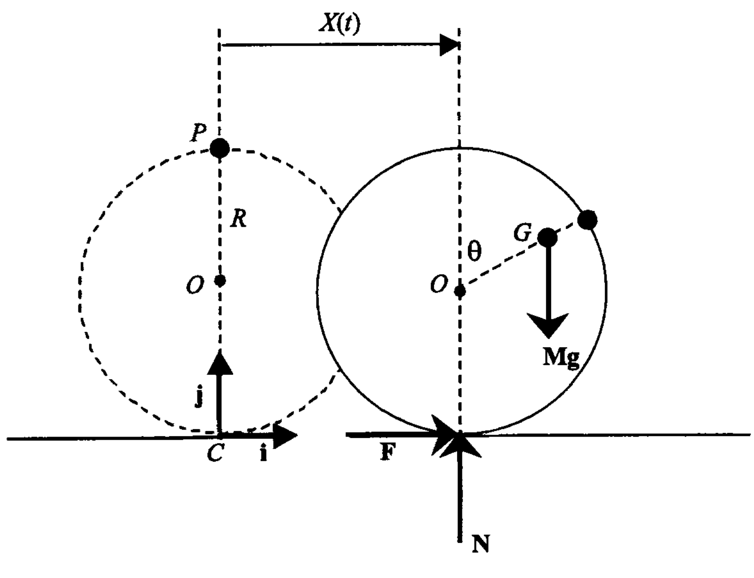


Figure 1. A loaded hoop; coordinates and forces.

By making these assumptions, the complexity of the model is comparable with that of a rigid hoop; any relaxation of these assumptions greatly increases the complexity of the model. These assumptions will be discussed in more detail later.

The position of the hoop is determined by the angular displacement $\theta$, with the initial position $\theta = 0$ being taken as the position when the radii C-O-P form a straight line. This analysis is restricted to $\theta < \pi$.

Let $m_h$ denote the mass of the hoop, $m_p$ the mass of the particle, and $M = m_h + m_p$ the total mass, and define the mass ratio $\gamma$ as $\gamma = m_p/M$.[1] Then the centre of mass is at point G at a radial distance $\gamma R$ from point O. For example, in the *equal mass* case, $m_h = m_p$ and $\gamma = 0.5$; in the case of Littlewood's hoop, $m_h = 0$ and $\gamma = 1$. The cases of interest for the analysis that follows are for (very) heavy particles, approximately in the range $\gamma \in (0.7; 1)$.

Moments of inertia are defined in terms of a factor $\kappa$ such that the moment of inertia is given by $\kappa MR^2$. In all cases, the axis around which the moments are taken is perpendicular to the plane containing the hoop. The hoop is approximated as a rigid wire so that $I_{O(\text{hoop})} = m_h R^2$. In [5] and [7], an infinitely small particle with $I_P = 0$ was used. Here we will follow Pritchett [6] and define $I_{P(\text{particle})} = \epsilon MR^2$. Here $\epsilon$ will always be a very small number in our analysis, with, in most cases, very little influence on the numerical results. Then it follows that for the loaded rigid hoop, the moments of inertia around axes through G and C, respectively, are defined by the dimensionless moment of inertia factors

$$\kappa_G = 1 - \gamma^2 + \epsilon, \qquad \kappa_C = 2 + 2\gamma\cos\theta + \epsilon. \tag{1}$$

Here the significance of $\epsilon$ becomes clear: $\kappa_G$ remains positive even for the massless case.

Assumption 1 implies that (1) holds for elastic as well as for rigid hoops, on the grounds that the small deformation of the elastic hoop will not have a significant influence on the moment of inertia. Later results show that the largest deformation occurs in the initial position and that this value is less than 2% of the radius for moderately soft hoops.

## 2.2. Kinematics

With $t$ denoting time, the coordinates of O are $(X(t), Y(t))$ in a fixed Cartesian coordinate system with axes parallel and perpendicular to the plane as indicated by unit vectors $\mathbf{i}$ and $\mathbf{j}$. The coordinates of the centre of mass G are $(x(t), y(t))$ relative to the same axes. Differentiation with respect to $t$ is denoted by a dot, for example $\dot{X}$. As usual, $g$ denotes the gravitational acceleration.

The subsequent modelling, as well as the numerical solution, is simplified by using nondimensional variables. Defining nondimensional time as $\tau = \sqrt{g/R}t$, and denoting derivatives with respect to $\tau$ by $(\ )'$, it follows that $(\ )' = \sqrt{R/g}(\dot{\ })$. It is convenient to denote the nondimensional angular velocity by $\omega$, where $\omega = \theta' = \sqrt{R/g}\dot{\theta}$. Then $\omega' = \theta'' = (R/g)\ddot{\theta}$ is the nondimensional angular acceleration.

In accordance with Assumption 2, we introduce a new variable $d(t)$ to define the elasticity of radius OC by setting $Y(t) = R + d(t)$, or in nondimensional form as

$$\left(\frac{Y}{R}\right) = 1 + \delta, \tag{2}$$

where the elastic deformation $\delta$ is defined as $\delta = d/R$. The time derivatives of $Y$ and $\delta$ are given by $(Y/R)' = \delta' = \dot{d}/\sqrt{Rg}$ and $(Y/R)'' = \delta'' = \ddot{d}/g$.

The variable $d(t)$ is interpreted as follows. Hopping occurs when $d(t) > 0$; i.e., the hoop is airborne and $Y > R$. This is true for both the rigid and elastic models. In the rigid model,

[1] Here Pritchett's definition for mass ratio is used, as it is slightly simpler than the definition previously used by Theron, which was $m_h/m_p = (1-\gamma)/\gamma$.

$d(t) = 0$ implies contact between the hoop and the surface; $d$ can never become negative. In the elastic model, $d(t) \leq 0$ implies contact between the hoop and the surface and negative values of $d(t)$ denote elastic deformation, which should be recognised as the shortening of the radius OC.

The rigid hoop has three degrees of freedom, which are here selected as $X$, $Y$, and $\theta$. Rigidity of radius OP introduces constraints on $x(t)$ and $y(t)$ in the form

$$x(t) = X(t) + \gamma R \sin\theta(t), \qquad y(t) = Y(t) + \gamma R \cos\theta(t).$$

With simplifying Assumption 3, these constraints are also used in the case of the elastic hoop and can be written in nondimensional form as

$$\frac{x}{R} = \frac{X}{R} + \gamma \sin\theta, \qquad h = \frac{y}{R} = 1 + \delta + \gamma\cos\theta, \tag{3}$$

where it will be found useful to use $h = y/R$, the nondimensional height of G.

Differentiation with respect to time gives the nondimensional acceleration components as

$$\left(\frac{x}{R}\right)'' = \left(\frac{X}{R}\right)'' + \gamma\theta''\cos\theta - \gamma\omega^2\sin\theta, \qquad h'' = \delta'' - \gamma\theta''\sin\theta - \gamma\omega^2\cos\theta. \tag{4}$$

If Assumption 3 was not made, the constant parameter $\gamma$ in the above would be replaced by a new variable, thereby introducing a fourth degree of freedom and greatly increasing the complexity of the problem. This assumption might seem to be unreasonable from a physical point of view, especially in the initial position. However, after some increase in the angular velocity, the centrifugal effect will tend to increase the radius OP once again, making the assumption more realistic.

One of the fundamental motions of the hoop is *rolling*, characterised by the constraints that contact is maintained and that the contact point C is momentarily at rest, i.e., that $\dot{X} = Y\dot{\theta}$. Differentiation with respect to time gives the nondimensional acceleration for rolling motion,

$$\frac{\ddot{X}}{g} = \left(\frac{X}{R}\right)'' = (1+\delta)\theta'' + \delta'\omega. \tag{5}$$

### 2.3. Kinetics

Figure 1 shows the three forces acting on the hoop while moving in contact with the surface. They are the weight $Mg$, the normal reaction $N$, and the friction force $F$. We simplify the elastic model by assuming that $N$ and $F$ act at point C in the direction of the coordinate axes.

Using Newton's second law for components in the positive $x$-direction, $F = M\ddot{x}$, or, in nondimensional form and using (4),

$$\frac{F}{Mg} = \left(\frac{X}{R}\right)'' + \gamma\theta''\cos\theta - \gamma\omega^2\sin\theta. \tag{6}$$

For components perpendicular to the plane, $N - Mg = M\ddot{y}$, or, in nondimensional form and using (2) and (4),

$$\frac{N}{Mg} = 1 + \delta'' - \gamma\theta''\sin\theta - \gamma\omega^2\cos\theta. \tag{7}$$

The maximum value of the friction force places a constraint on the forces. Denoting the coefficient of friction by $\mu$, we use Coulomb's law,

$$|F| \leq \mu N. \tag{8}$$

Pritchett [6] distinguishes between static and dynamic friction coefficients. Although this has very little influence on the results, it does not complicate the model and will be included later where appropriate.

In the elastic model, there is a relationship between the normal force and the elastic deformation. We use the simplest linear relationship $N = -kd$, and define the *static deformation* $d_s = Mg/k$, and the nondimensional *elastic constant* as the ratio $e = R/d_s = Rk/Mg$. Then the nondimensional form of Hooke's law is

$$\frac{N}{Mg} = -e\delta, \qquad \delta \leq 0. \tag{9}$$

For example, a value of $d_s = 0.002R$, or $e = 500$, could be considered reasonably stiff, whereas a plastic hula-hoop with 1 meter diameter might have a static deformation of 10 mm when loaded, in which case, $e = 50$. Clearly, the rigid model is approximated by $e \to \infty$.

Finally, the "torque equation" is obtained by taking clockwise moments about the centre of mass and can be written as $N\gamma R \sin\theta - Fy = I_G\ddot{\theta}$, or, in nondimensional form, as

$$\kappa_G \theta'' = \left(\frac{N}{Mg}\right)\gamma \sin\theta - \left(\frac{F}{Mg}\right)h. \tag{10}$$

### 2.4. An Alternative Elastic Model

A model which is simpler to visualise and more realistic from a physical point of view consists of a rigid hoop rolling on a deformable surface. If the downward displacement of the surface at point C is denoted by $-d$, and the normal reaction is assumed to be proportional to this displacement, equation (9) is again applicable. In this case, rolling motion is defined by $\dot{X} = R\dot{\theta}$ and the torque equation is slightly simpler with $h = 1 + \gamma\cos\theta$. This simplifies the detail of the equations, but not the essential structure of the solution, and the subsequent analysis and results are all based on equation (5) and (10), as they represent a better model of the hula-hoop in Pritchett's photograph.

## 3. MOTIONS OF AN ELASTIC HOOP

### 3.1. Initial Conditions

We will assume that the motion always starts with the particle at the highest point; therefore, $\theta(0) = 0$ and $X(0) = 0$. The initial angular velocity is denoted by $\omega_0 = \sqrt{R/g}\dot{\theta}(0)$. The initial velocity is horizontal, so that $\dot{Y}(0) = 0$ and $\delta'(0) = 0$.

A fourth simplifying assumption is introduced regarding the initial value of $d$. Physically appropriate choices such as $d(0) = 0$ or $d(0) = -d_s$ result in an initial impulse in $N$, thus causing significant initial oscillations. To avoid these, we assume the following.

4. The initial elastic deformation does not cause an impulse.

To achieve this, $\delta''(0)$ is assumed to be zero and the initial normal reaction from (7) is

$$\frac{N(0)}{Mg} = 1 - \gamma\omega^2. \tag{11}$$

Equating this to (9) results in

$$\delta_0 = -\frac{(1 - \gamma\omega_0^2)}{e}. \tag{12}$$

Note that the assumption that this value is small was used in justifying the first simplification.

Finally, we assume that the hoop always starts in rolling mode, so that $\dot{X}(0) = Y(0)\dot{\theta}(0)$ or $X'(0)/R = (1 + \delta_0)\omega_0$. Depending on the value of the different parameters, this rolling phase can be followed by a number of different possibilities, as analysed below.

### 3.2. Rolling Motion

For rolling motion, equations (5)–(7) can be used to eliminate the forces from the torque equation (10) and Hooke's law (9) to obtain the simultaneous equations

$$\begin{bmatrix} \kappa_c + 2(1+\gamma\cos\theta)\delta & -\gamma\sin\theta \\ -\gamma\sin\theta & 1 \end{bmatrix} \begin{bmatrix} \theta'' \\ \delta'' \end{bmatrix} = \begin{bmatrix} \gamma\sin\theta\left(1+\omega^2+\omega^2\delta\right) - \delta'\omega h \\ \omega^2\gamma\cos\theta - e\delta - 1 \end{bmatrix},$$

with solution

$$\theta'' = \frac{\left(\gamma\sin\theta\left(\omega^2 h - e\delta\right) - \delta'\omega h\right)}{D_1}, \tag{13}$$

$$\delta'' = \omega^2\gamma\cos\theta - e\delta - 1 + \theta''\gamma\sin\theta, \tag{14}$$

where the determinant of the coefficient matrix simplifies to

$$D_1 = \kappa_G + (1+\gamma\cos\theta)^2 + 2(1+\gamma\cos\theta)\delta.$$

This pair of second-order equations can be solved numerically in the standard manner by writing them as four first-order equations. From this solution, $X''$ is found from (5), the friction force from (6), and the normal reaction from (7).

This rolling motion continues up to the point where $|F| = \mu_s N$, with $\mu_s$ denoting the static coefficient of friction. The angular displacement at the point where the hoop stops rolling is denoted by $\theta_1$, and the remaining initial values for the subsequent motion by $\omega_1$, $\delta_1$, and $\delta'_1$.

### 3.3. Spinning or Skidding

Depending on the values of the parameters, the rolling motion will end with the friction force being either positive or negative.

The friction force is positive in cases where $Y\dot{\theta} > \dot{X}$; this motion is defined as *spinning*. Then $\mathbf{F} = +\mu_k N\mathbf{i}$, where $\mu_k$ is the kinetic coefficient of friction, and the torque equation (10) becomes

$$\kappa_G\theta'' = \left(\frac{N}{Mg}\right) S(\theta),$$

where it is useful to define the factor

$$S(\theta) = \gamma\sin\theta - \mu_k h.$$

As in the case of rolling, elimination of $N$ by means of (7) from the torque equation and Hooke's law results in a set of simultaneous equations:

$$\begin{bmatrix} \kappa_G + S\gamma\sin\theta & -S \\ -\gamma\sin\theta & 1 \end{bmatrix} \begin{bmatrix} \theta'' \\ \delta'' \end{bmatrix} = \begin{bmatrix} S(1-\omega^2\gamma\cos\theta) \\ \omega^2\gamma\cos\theta - e\delta - 1 \end{bmatrix},$$

with solution

$$\theta'' = -e\left(\frac{S}{\kappa_G}\right)\delta, \tag{15}$$

$$\delta'' = \omega^2\gamma\cos\theta - e\delta - 1 + \theta''\gamma\sin\theta. \tag{16}$$

In this case, the determinant of the coefficient matrix simplifies to $\kappa_G$.

Similarly, the friction force is negative in cases where $Y\dot{\theta} < \dot{X}$; this motion is defined as *skidding*. Then $\mathbf{F} = -\mu_k N\mathbf{i}$, and the factor $S$ in the torque equation is redefined as

$$S(\theta) = \gamma\sin\theta + \mu_k h,$$

whereby equations (15) and (16) are once again obtained.

The numerical solution is once again found in terms of the four nondimensional variables $\theta$, $\omega$, $\delta$, and $\delta'$. From this solution, the normal reaction is found from (7), the friction force from $F = \mu_k N$ or $F = -\mu_k N$ and $X''$ from (6).

In the case of spinning, this motion continues as long as $Y\dot{\theta} > \dot{X}$ and $d \le 0$, with $N > 0$. Similarly, skidding continues as long as $Y\dot{\theta} < \dot{X}$ and $d \le 0$, with $N > 0$. If $N$ remains positive, the hoop will start rolling again at some stage, as the normal reaction increases rapidly when $\theta \to \pi$. This situation was analysed in detail in [7] for rigid hoops, but will not be pursued here. The cases of interest here are situations where the spinning or skidding motion ends when the normal reaction becomes zero at a displacement $\theta \le 90°$, and the hoop hops.

### 3.4. Hopping

The angular displacement at the point where hopping occurs is denoted by $\theta_2$, with $N(\theta_2) = 0$ at time $\tau_2$; this clearly also implies $\delta(\theta_2) = 0$. The remaining initial conditions for the subsequent motion are denoted by $\omega_2$ and $\delta_2'$, the values being known from the solution of the previous phase. Then, for point G, $h_2 = 1 + \gamma\cos\theta_2$ and $h_2' = \delta_2' - \omega_2\gamma\sin\theta_2$; for the hoop, $Y_2 = R$ and $Y_2' = R\delta_2'$. Note that $\delta_2' > 0$ is a sufficient condition for hopping.

With zero reaction force, the weight is the only force and the centre of mass $G$ follows a parabolic path, with the vertical component given by $h(\tau) = h_2 + h_2'(\tau - \tau_2) - 0.5(\tau - \tau_2)^2$. If required, the values of $X_2'$ and $X_2$ can be obtained by numerical integration of the values of $X''$, in order to calculate the horizontal component. The characteristics of this parabolic path were analysed in some detail in [5] for the case of a rigid, massless hoop.

The subsequent motion of the hoop is controlled by the fact that the torque around G is zero, and therefore, provided that $\kappa_G > 0$, the angular acceleration is zero and the hoop rotates with constant angular velocity $\omega_2$, so that

$$\theta(\tau) = \theta_2 + (\tau - \tau_2)\omega_2. \tag{17}$$

The vertical component of the hop is $\delta = Y/R - 1$ which can be written as

$$\delta = \gamma(\cos\theta_2 - \cos\theta) + h_2'(\tau - \tau_2) - 0.5(\tau - \tau_2)^2. \tag{18}$$

This motion continues as long as $\delta > 0$. Denoting the time at the end of the hop by $\tau_3$, i.e., $\delta(\tau_3) = 0$, and defining the *span* of the hop as $\theta(\tau_3) - \theta_2$, the span $= \omega_2(\tau_3 - \tau_2)$. This is one of the measurable characteristics of the hop; another is the maximum height $\delta_{\max}$.

### 3.5. Rigid Model

The model for the rigid hoop, which was derived in [7], can be obtained from the above equations by setting $d = 0$ and ignoring Hooke's law (9). Then the equation for rolling becomes

$$\kappa_c\theta'' = \gamma\left(1 + \omega^2\right)\sin\theta. \tag{19}$$

The analytic solution is easily found as shown in [7]. However, in this paper, all results were obtained using numerical methods. For spinning and skidding, the torque equation for a rigid hoop results in

$$\theta'' = \frac{S(\theta)\left(1 - \gamma\omega^2\cos\theta\right)}{\kappa_G + \gamma\sin\theta S(\theta)}, \tag{20}$$

with $S(\theta) = \gamma\sin\theta - \mu_k(1 + \gamma\cos\theta)$ for spinning, and $S(\theta) = \gamma\sin\theta + \mu_k(1 + \gamma\cos\theta)$ for skidding. No analytic solution could be found for this equation.

Furthermore, the equations for the massless case can be obtained by setting $\gamma = 1$ in the above. In the case of a particle with $\epsilon = 0$, it follows from (1) that $\kappa_G = 0$, and the motion following rolling is given by

$$\theta'' = \frac{\left(1 - \omega^2\cos\theta\right)}{\sin\theta}, \tag{21}$$

which is the equation for skimming as obtained in [5]. This situation where $\kappa_G = 0$ gives rise to a number of subtle issues is discussed in detail in [5].

## 4. RESULTS

The numerical solutions were obtained by using the ODE45 solver in MATLAB 5. This is a fourth-order Runge-Kutta algorithm with automatic choice of time-steps. It also has a facility for finding the zeroes of a specified function; this is called an "event". This facility was extremely useful in determining the point where rolling stops by defining the event as the function $|F| - \mu_s N$.

The behaviour of the loaded elastic hoop essentially depends on the values of four parameters, namely the mass ratio, the initial angular velocity, the friction coefficient, and the elastic constant. In all cases, it will be assumed that the particle moment of inertia factor is $\epsilon = 0.01$ and that $\mu_s = \mu$ and $\mu_k = 0.9\mu$; small variations in these values have very little effect on the results.

### 4.1. Classification of Behavioural Patterns

Without any loss in generality, the initial velocity is kept at $\omega_0 = 0.6$ for this section. Increasing this value reduces $N(0)$ (see (11)) and lowers the graph of the normal reaction.

For a given mass ratio, the behaviour of a rigid hoop is then determined by the value of the friction coefficient, and three essentially different behavioural patterns can be identified. These patterns are best illustrated by graphs of the reaction forces as functions of position. Elastic hoops closely follow these patterns. In Figures 2–4, graphs are drawn for a rigid hoop (solid line), a very stiff hoop with $e = 500$ (dash-dot), and a moderately soft hoop with $e = 50$ (dashed). In all cases, the very stiff hoop closely approximates the behaviour of the rigid hoop.

First, the pattern associated with *heavy friction* is shown in Figure 2, for the case of a reasonably heavy hoop with mass ratio $\gamma = 0.7$. Here the hoop never slips, but continues rolling through a full 360° revolution without any loss of energy. This occurs with (very) large coefficients of friction, or $\mu \geq \mu_2$ where $\mu_2$ will be defined shortly.

Here the upper graphs show the normal reaction, with a positive minimum value in the region of 90°, and a large maximum at 180°, which position is a point of symmetry for the full revolution. The lines for the rigid hoop and stiff hoops are very close together, but the soft hoop deviates quite a bit in the region 100° to 180°; these characteristics are repeated in the lower graph and in Figures 3 and 4.

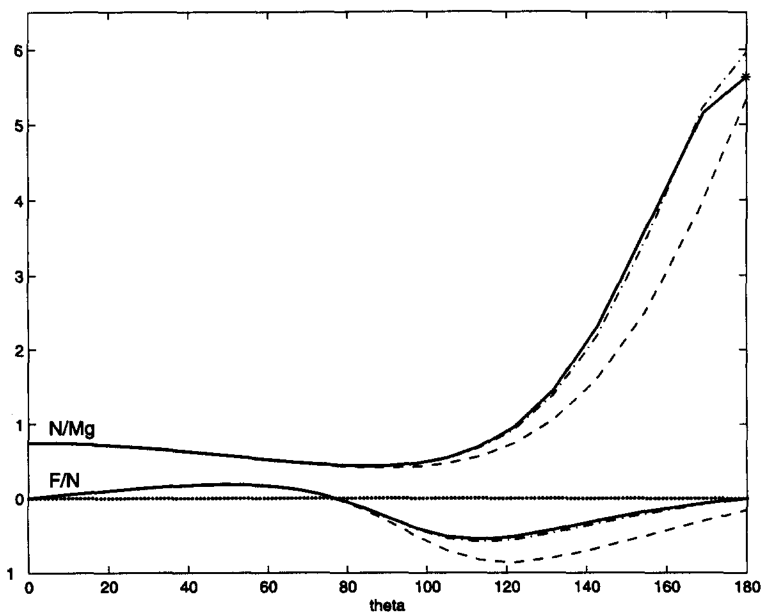


Figure 2. Reaction as functions of $\theta$ for heavy friction.

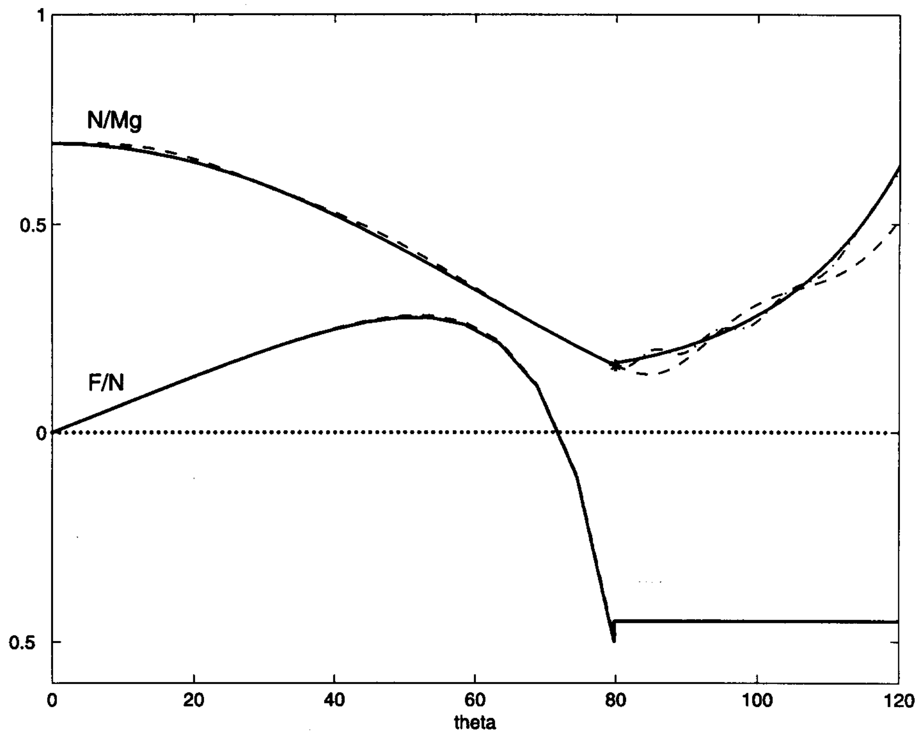


Figure 3. Reactions as functions of $\theta$ for normal friction.

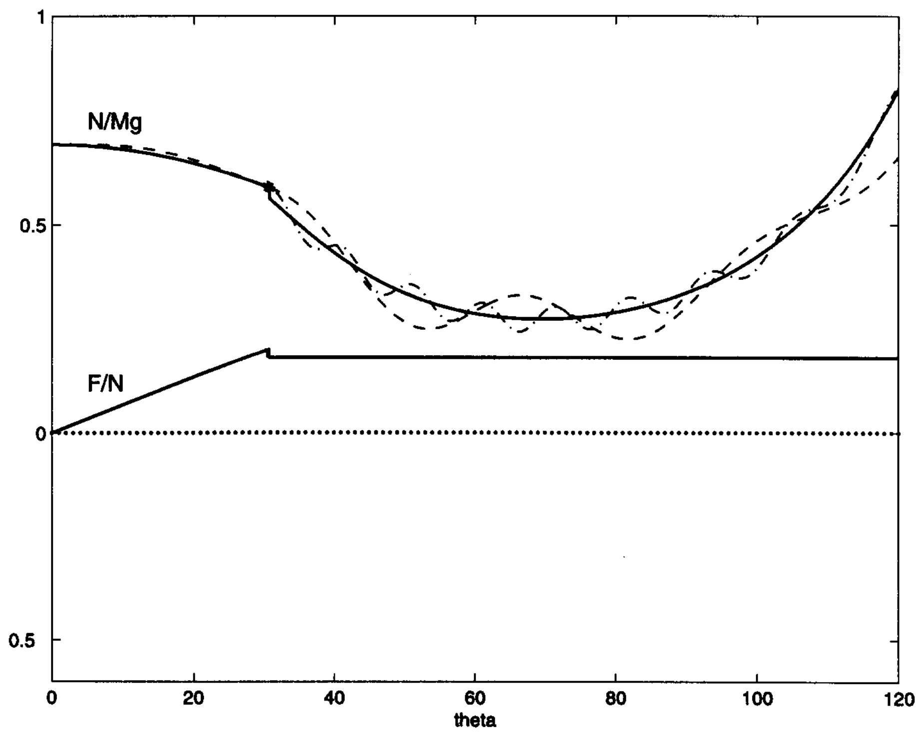


Figure 4. Reactions as functions of $\theta$ for light friction.

The lower graphs in Figure 2 show the $F/N$ ratios for the same hoops. The friction force is zero at the two endpoints, and changes from positive to negative at some intermediate point. We define $\mu_1 = \max(F/N)$ and $\mu_2 = |\min(F/N)|$ as the values for the rigid hoop at the two extreme points, and note that these values are functions of $\omega_0$ and $\gamma$. In this example, $\mu_1 = 0.187$ and $\mu_2 = 0.55$. In all examples, we find $\mu_1 < \mu_2$. Clearly, if $\mu \geq \mu_2$, the hoop will never slip.

The second pattern is associated with *normal friction* which is defined as $\mu_1 < \mu < \mu_2$. An example is shown in Figure 3, for the case of $\gamma = 0.85$. The heavier particle in this system has the effect of decreasing the minimum value of $N$, and increasing $\mu_1$ to 0.275 and $\mu_2$ to 4.85. The value of $N(\pi)$ increases dramatically with very heavy particles, as discussed in [7], but is not shown here.

Taking a value $\mu = 0.5$, it is clear that the hoop will roll past the first critical point in the region of 50°, but will start skidding at approximately 80°. On the upper graph of the normal reaction for a rigid hoop, the end of the rolling phase is indicated by a $*$ and the value at this point is denoted by $N_1 = N(\theta_1)$; in this example, $N_1 = 0.16$. The onset of skidding corresponds with a marked increase in the normal reaction. The corresponding values for the soft hoop show a further decrease in $N$ before oscillating around the values for the rigid hoop. The lower graph of the $F/N$ ratio clearly shows the onset of skidding; the small discontinuity at this point is due to the assumption that $\mu_k < \mu_s$.

The third type of behaviour occurs for *light friction*, defined as $\mu \leq \mu_1$, as shown in Figure 4 for the case $\mu = 0.2$ with the other parameters as before. Here the rolling phase stops at a much earlier position and is followed by spinning with a positive friction force. The normal reaction continues decreasing during the spinning phase, and, for the rigid case, reaches a positive minimum at approximately 70°. In this case, the oscillations of the soft hoop again result in a minimum value of $N$ which is smaller than that for the rigid hoop.

## 4.2. Hopping Hoops

We now come to the primary purpose of this analysis, namely, finding parameters that will cause the hoop to hop in a manner similar to the photograph in Pritchett [6]. It was shown in [7] that this is not possible for rigid hoops. However, the graphs of the normal reaction in Figures 3 and 4 suggest that a lowering of the value of $N_1$, together with a very soft hoop, might create a situation where $N$ becomes zero and the hoop hops.

$N_1$ can be reduced by increasing the mass ratio and/or by increasing the initial velocity, which, in effect, lowers the whole graph and also reduces the value of $\theta_1$. Increasing the friction coefficient also has a small effect, because rolling continues further, and therefore, $N_1$ is decreased. We therefore select a very heavy particle to give a mass-ratio of $\gamma = 0.95$, and use a higher initial velocity of $\omega_0 = 0.8$; then $\mu_1 = 0.242$. With such very heavy particles, the value of $\mu_2$ becomes totally unrealistic and the behaviour associated with heavy friction can never occur.

Figure 5 shows graphs of $N$ for a rigid hoop (solid line) and a very soft hoop ($e = 10$), using the above-mentioned parameter values together with $\mu = 0.7$, representing normal friction. For the soft hoop, the graph of $\delta$, magnified ten times, and labelled as $10 * d/R$ is also shown; clearly, the interval where $\delta > 0$ represents a rather small hop. The values at the start of the hop are found to be $\theta_2 = 64.8°$, $\omega_2 = 1.116$, and $\delta_2' = 0.0812$. Using these values in (17) and (18), we find that the maximum height of the hop is $\delta_{\max} = 0.0059$ and that the span is 17.1°.

With all parameters the same except for $\mu = 0.15$ to give light friction, these values change to $\theta_2 = 47.4°$, span = 40.5°, and $\delta_{\max} = 0.03$, representing a much larger hop. A further reduction in $\mu$ reduces the value of $\theta_2$ and also results in a smaller loss of energy during the spinning phase, and consequently, in a higher velocity $\delta_2'$, resulting in still larger hops.

In Figure 5, we attempt to simulate the hop shown in Pritchett's photograph [6]. Measurements of the photograph indicate that the hoop seems to be airborne for a rotation between approximately 35° and 112° (span = 77°) and reaches a maximum height of close to $R/15$. The last value is approximated very well by using extremely light friction with $\mu = 0.05$ together with $\omega_0 = 0.46$ and $\gamma = 0.917$, keeping $e = 10$, resulting in $\delta_{\max} = 0.0668$; the values $\theta_2 = 40.6°$ and span = 53° are reasonably close to the values in the photo.

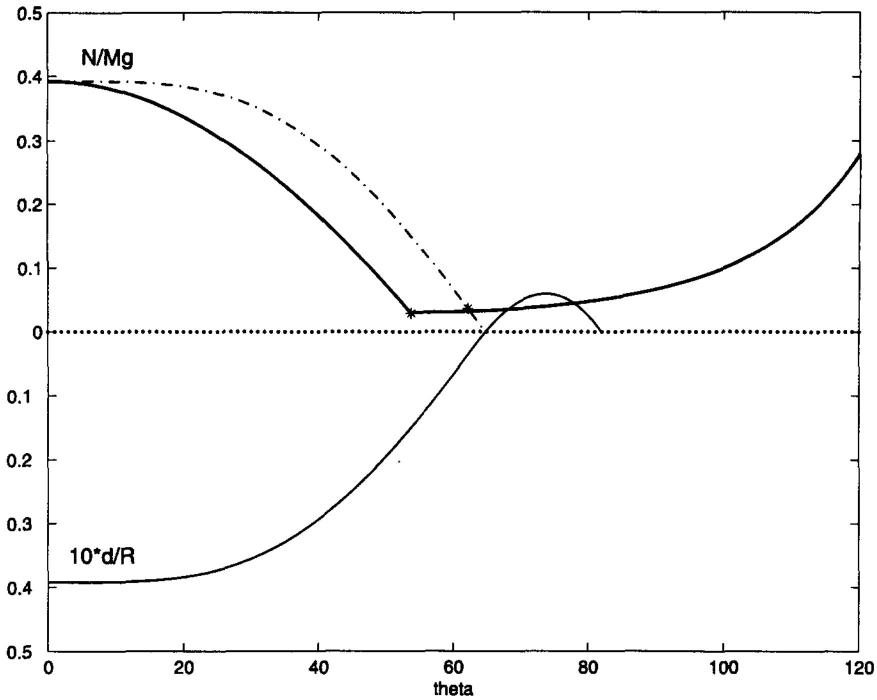


Figure 5. Graphs for a hopping hoop with normal friction.

### 4.3. Parameter Values that Cause Hopping

Finally, it is of some interest to investigate the effect of the parameters that cause hopping. In Figure 7, some results are summarised in plots on the $\omega_0\gamma$-plane space, for given values of $\mu$, keeping $e = 10$. The main results are for very light friction, with $\mu = 0.05$, and are shown by three contour plots for $\delta_{\max}$, labelled as $d/R = 1/15$, 0.03 and zero. Clearly, points below the zero contour indicate parameter values for which the hoop can not hop. Also, as expected, $\delta_{\max}$ increases as the mass ratio is increased.

An interesting feature on these graphs is the existence of an *optimum initial velocity* $\omega_*$, as it is the value of $\omega_0$ which causes the largest hop for a given mass ratio. Equivalently, $\gamma_*$ is the mass ratio of the lightest particle which will cause a hop of a specified magnitude (for given values of $\mu$ and $e$). These points are shown by stars on the zero contour, point $(0.578; 0.807)$, and the 0.03-contour at point $(0.505; 0.882)$. The cross at point $(0.460; 0.917)$ represents the values used for Figure 6.

The heavy dash-dot line connecting these points represents a flat ridge on the contour plot of equal hops on the $\omega_0\gamma$-plane. This phenomenon is presumably due to a resonance between the centripetal acceleration and the elasticity; this aspect has, however, not been investigated.

Increasing the friction coefficient to $\mu = 0.7$ moves the optimum point for the zero contour to $(0.971; 0.849)$. The zero contour and the ridge of optimum values for this case are also shown in Figure 7. The cross at $(0.8; 0.95)$ corresponds with the values used for Figure 5, and clearly indicates that hopping will occur for both light and normal friction.

The heavy, almost vertical line connecting points $(1; 1)$ and $(2/\sqrt{3}; 0.75)$ is drawn by setting $\gamma = 1/\omega_0^2$, and represents the maximum possible values for the initial velocity. It is clear from (11) that $N(0) = 0$ for points on this line; for points to the right of this line, the hoop will leave the surface immediately, at $\theta = 0$. For points to the left of this line and above the zero contour, the hop occurs after larger rotations, with $\theta_2 \to 90°$ as $\omega_0 \to 0$.

Increasing the value of $e$ shifts the contours upwards and to the right; for example, with $e = 50$ and $\mu = 0.05$, the optimum point for the zero contour becomes $(0.666; 0.894)$. For a given set of parameters, hopping only occurs for $e \leq e_{\max}$. For example, using the parameters of Figure 6, we

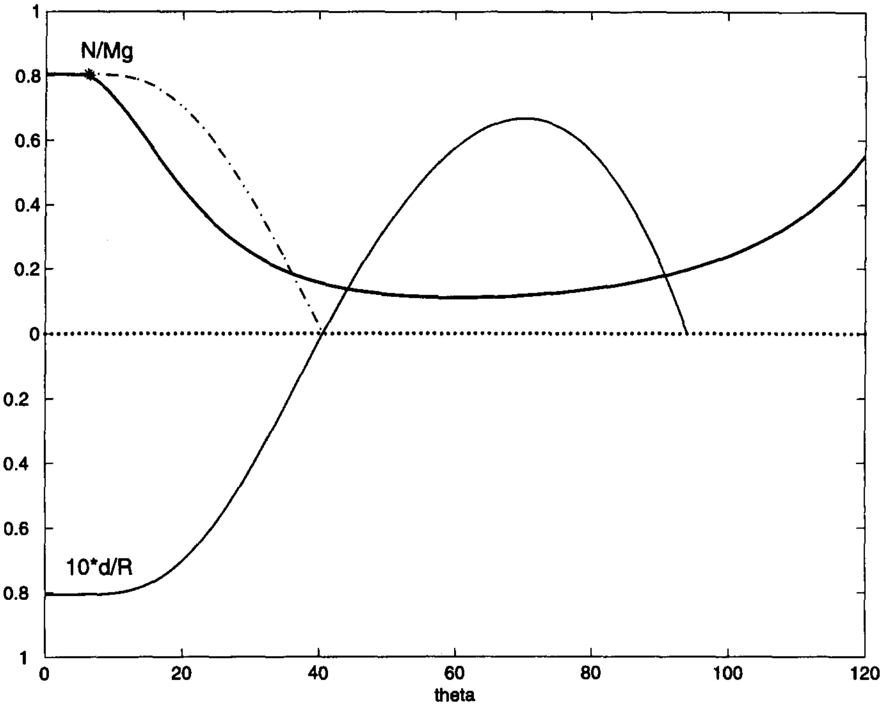


Figure 6. Graphs for a simulation of Pritchett's photograph.

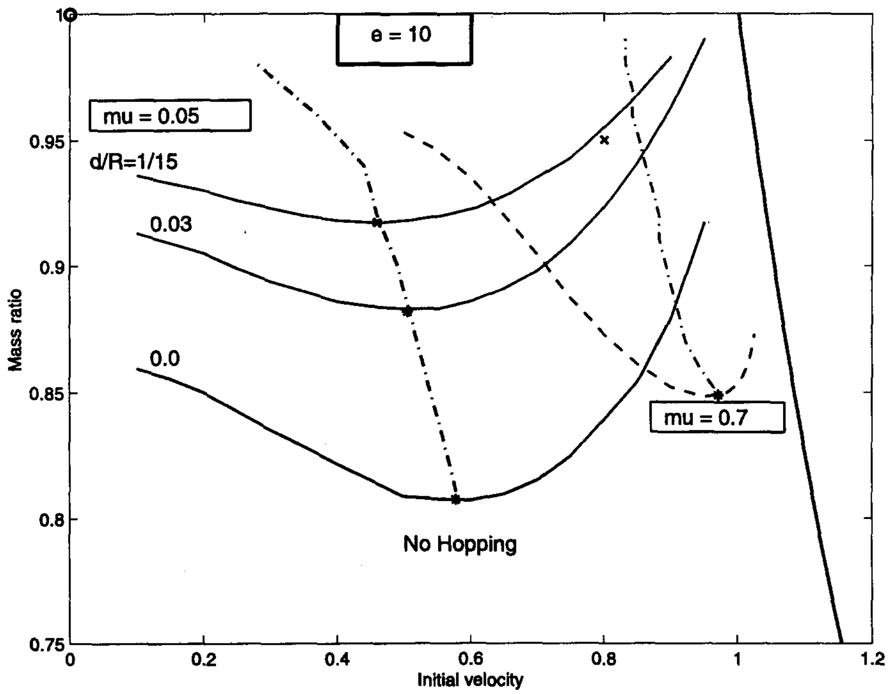


Figure 7. Contours of equal hop size on the $\omega_0\gamma$-plane space.

find that hopping occurs for $e \le 57$. At point (0.78; 0.98), which is near the optimum point for large values of $e$, and keeping $\mu = 0.05$, we find that $e_{\max} \approx 2650$. This is in agreement with the previous conclusion in [7] that rigid hoops will not hop in this position. This is also confirmed by noting that in all cases the results show, not unexpectedly, that $\delta_{\max} \to 0$ as $e$ increases. For example, a value of $\delta_{\max} = 3.6 * 10^{-8}$ was calculated for the example given below. This illustrates the rather amazing robustness of the numerical algorithm in MATLAB, which continues to give sensible values even for such extreme cases.

As a final example, we return to the original problem posed by Littlewood, identified by the circle at point $(0;1)$ in Figure 7. This hypothetical problem can be modelled approximately as follows. First, a small initial velocity is required to start the numerical solution, so we take $\omega_0 = 0.005$. The massless hoop is correctly modelled by $\gamma = 1$, but the particle can only be approximated by taking $\epsilon = 0.001$ in order to prevent a zero moment of inertia. Littlewood's model requires that $\mu \geq 1$; using $\mu = 1$, and iterating for different values of $e$, we find that $e_{\max} = 398$. This suggests that Littlewood's rigid massless hoop cannot hop. Further discussion of the truly singular case with $\kappa_G = 0$ is given in [5].

## 5. CONCLUSIONS

In this analysis of a very simple model of an elastic hoop loaded with a heavy particle and rolling on a rough horizontal surface, it is found that soft hoops can hop given a particle that is heavy enough. An example which is a very good simulation of the photograph of a hopping hoop in [6] is shown.

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