Design of Tableau-Based Automated Theorem Provers and Production of Machine-Checkable Proofs

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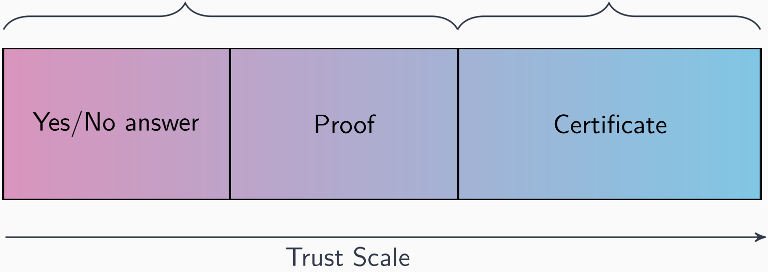
Proofs ♥

Automated Theorem Proving

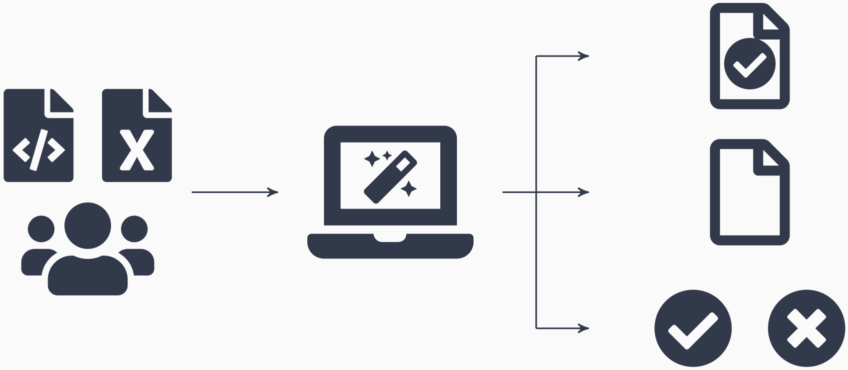
- Click-and-prove software
- Searching for a proof all by themselves
- Output a statement or a proof-like trace

Interactive Theorem Proving

- Proof assistants
- Guide humans towards proofs
- Proofs are machine-checkable and certified



Big Picture



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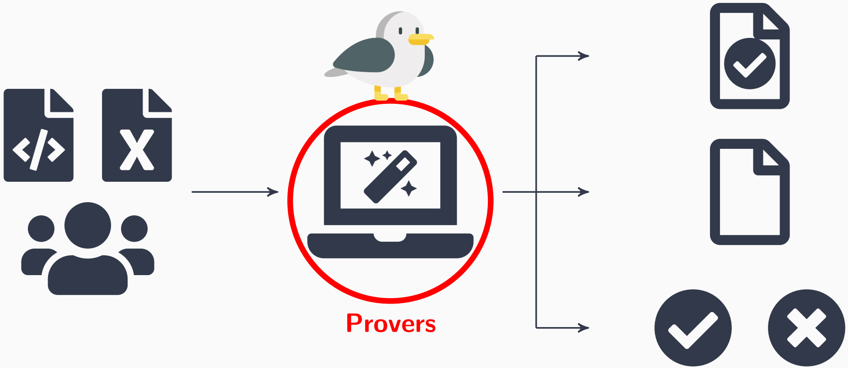

1. Preliminary Notions

Logic

What is Logic?

- Study of correct reasoning
- Mathematical representation of the world
- Truth value of a statement

First-Order Logic (FOL)

- Expressivity: elements and properties about them
- Efficient reasoning methods
- Semi-decidable

Method of Analytic Tableaux

Principle

- A set of axioms and one conjecture
- Refutation
- Syntactic rules: $\odot, \alpha, \delta, \beta, \gamma$
- Close all the branches

Origin & Strengths

- Beth and Hintikka
- Extended by Smullyan and Fitting
- Unaltered original formula
- Output a proof

$$\dfrac{\neg(\exists x.\ P(x) \Rightarrow (P(a) \land P(b)))}{\dfrac{\neg(P(a) \Rightarrow (P(a) \land P(b)))}{\dfrac{P(a), \neg(P(a) \land P(b))}{\dfrac{\neg P(a)}{\odot}\ \odot \qquad \dfrac{\neg P(b)}{\dfrac{\neg(P(b) \Rightarrow (P(a) \land P(b)))}{\dfrac{P(b), \neg(P(a) \land P(b))}{\odot}\ \odot}\ \alpha_{\neg\Rightarrow}}\ \gamma_{\neg\exists}}\ \beta_{\neg\land}}\ \alpha_{\neg\Rightarrow}}\ \gamma_{\neg\exists}$$

Tableau-Based Proof-Search Procedure

Rules

- $\odot$: Closure rule
- α, β: Expands the tree
- γ: Free variables
- δ: Skolemization

Tableaux in AR

- Free variables
- Substitutions (local & global)

$$\frac{Human(Socrates), \neg Human(Socrates)}{\odot}\ \odot$$

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$$\neg(\exists x.\, P(x) \Rightarrow (P(a) \wedge P(b)))$$

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$$\dfrac{\dfrac{\dfrac{\neg(\exists x.\ P(x) \Rightarrow (P(a) \wedge P(b)))}{\neg(P(\color{red}{a}) \Rightarrow (P(a) \wedge P(b)))}\ \gamma_{\forall}}{P(\color{red}{a}), \neg(P(a) \wedge P(b))}\ \alpha_{\neg\Rightarrow}}{\dfrac{\neg P(a)}{\sigma = \{\color{red}{X} \mapsto a\}}\ \odot_{\sigma} \qquad \neg P(b)}\ \beta_{\neg\wedge}$$

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$$
\begin{array}{c}
\neg(\exists x.\, P(x) \Rightarrow (P(a) \wedge P(b))) \\
\hline
\neg(P(\color{red}{a}) \Rightarrow (P(a) \wedge P(b))) \\
\hline
P(\color{red}{a}), \neg(P(a) \wedge P(b)) \\
\hline
\end{array}
\begin{array}{l}
\gamma_\forall \\
\alpha_{\neg\Rightarrow} \\
\beta_{\neg\wedge}
\end{array}
$$

$$
\begin{array}{c}
\neg P(a) \\
\hline
\sigma = \{\color{red}{X} \mapsto a\}
\end{array}
\odot_\sigma
\qquad
\begin{array}{c}
\boxed{\neg P(b)} \\
\hline
\neg(P(X_2) \Rightarrow (P(a) \wedge P(b))) \\
\hline
\boxed{P(X_2)}, \neg(P(a) \wedge P(b)) \\
\hline
\end{array}
\begin{array}{l}
\gamma_\forall \\
\alpha_{\neg\Rightarrow}
\end{array}
$$

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2. Fairness Management in Tableau Proof-Search Procedure: a Concurrent Approach

Fairness in Tableaux

Fairness

A proof-search procedure is *fair* if and only if each formula on which a non-γ-rule can be applied occurs in a subsequent step, and every γ-rules will be computed an arbitrary number of times.

"At the present time, no strongly complete, destructive tableau proof procedure is known that works well in practice."
— Reiner Hähnle, *Handbook of Automated Reasoning Vol.1*, 2001

Fariness Management

Unfairness Sources

- The selection of a branch B (*select branch*)
- Determining whether B should be closed or expanded (*select mode*)
- If B is to be closed, the choice of a pair of complementary literals and thus a closing substitution (*select pair*)
- If B is to be expanded, the selection of a formula to which an expansion rule is applied (*select formula*)

State-of-the-Art Answers & Heuristics

- Limit the number of application of γ-rules
- Iterative deepening
- Rules ordering ($\odot \prec \alpha \prec \delta \prec \beta \prec \gamma$)

Select Pair and Select Mode

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)$$

Select Pair and Select Mode

$$\dfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)\end{array}}{P(X) \vee Q(X), \forall y.\ S(X)}\ \gamma\forall$$

Select Pair and Select Mode

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(X) \lor Q(X), \forall y.\ S(X)}\ \gamma_{\forall}}{P(X), \forall y.\ S(X) \qquad\qquad Q(X), \forall y.\ S(X)}\ \beta_{\Leftrightarrow}$$

Select Pair and Select Mode

$$\neg P(a)$$
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$$\dfrac{\dfrac{\forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)}{P(X) \vee Q(X), \forall y.\ S(X)}\ \gamma_\forall}{P(X), \forall y.\ S(X) \qquad\qquad Q(X), \forall y.\ S(X)}\ \beta_\Leftrightarrow$$

Select Pair and Select Mode

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$$\dfrac{\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(X) \lor Q(X), \forall y.\ S(X)}\ \gamma_{\forall}}{\dfrac{P(X), \forall y.\ S(X)}{\sigma = \{X \mapsto a\}}\ \odot_{\sigma} \qquad\qquad Q(X), \forall y.\ S(X)}\ \beta_{\Leftrightarrow}$$

Select Pair and Select Mode

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Select Pair and Select Mode

$$\begin{array}{c} \neg P(a) \\ \neg Q(b) \\ \neg S(c) \\ \dfrac{\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(a) \lor Q(a), \forall y.\ S(a)}\ \gamma_\forall}{\dfrac{P(a), \forall y.\ S(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma \qquad\qquad \dfrac{Q(a), \forall y.\ S(a)}{S(a)}\ \gamma_\forall}\ \beta_\Leftrightarrow \end{array}$$

Select Pair and Select Mode

$$\neg P(a)$$
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$$\dfrac{P(a) \lor Q(a), \forall y.\ S(a)}{\dfrac{P(a), \forall y.\ S(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma \qquad \dfrac{\dfrac{Q(a), \forall y.\ S(a)}{S(a)}\ \gamma_\forall}{P(X_2) \lor Q(X_2), \forall y.\ S(X_2)}\ \gamma_\forall}\ \beta_\Leftrightarrow$$

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$$

$$
\cfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{\cfrac{P(a) \lor Q(a), \forall y.\ S(a)}{\cfrac{P(a), \forall y.\ S(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma \qquad \cfrac{Q(a), \forall y.\ S(a)}{\cfrac{S(a)}{\cfrac{P(X_2) \lor Q(X_2), \forall y.\ S(X_2)}{\cfrac{P(X_2), \forall y.\ S(X_2)}{\cdots} \qquad \cfrac{Q(X_2), \forall y.\ S(X_2)}{\cdots}}\ \beta_\lor}\ \gamma_\forall}\ \gamma_\forall}\ \beta_\Leftrightarrow}\ \gamma_\forall
$$

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\end{array}
$$

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Exploring Branches in Parallel?

Approach

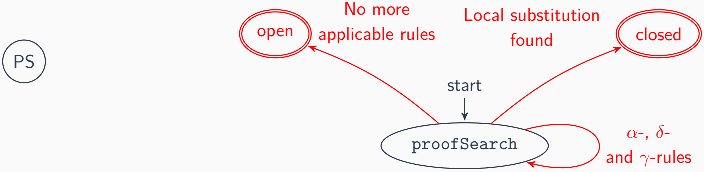
- Each branch searches for a local solution
- Management of multiple solutions with successive attempts and backtracking
- Forbid previously tried solutions
- Iterative deepening, limit of γ-rule and rules ordering

New Challenges

- Free variable dependency
- Communication between branches

Procedures Interactions

Procedures Interactions



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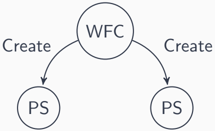

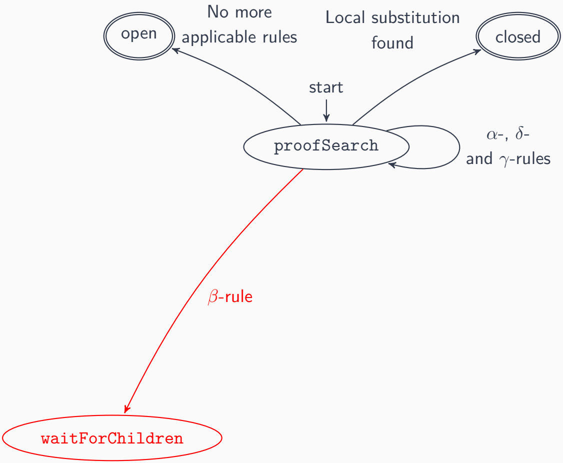

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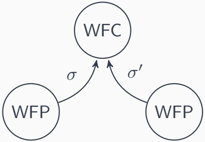

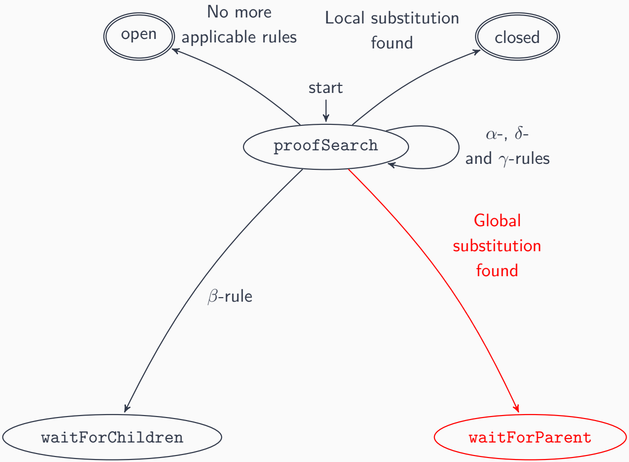

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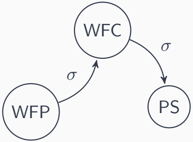

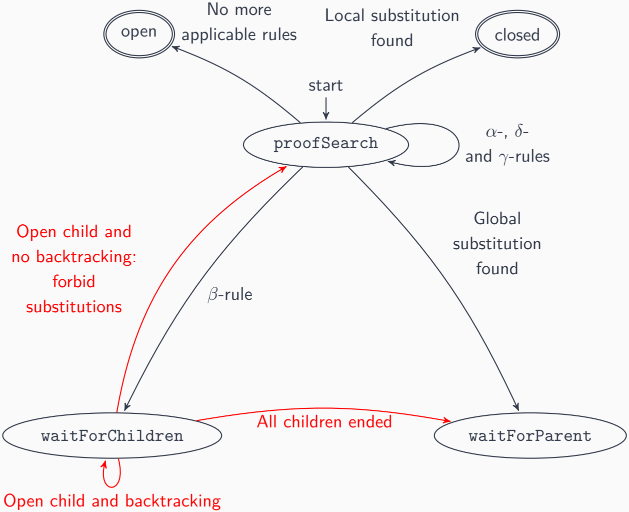

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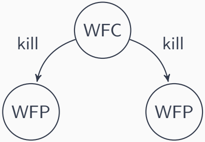

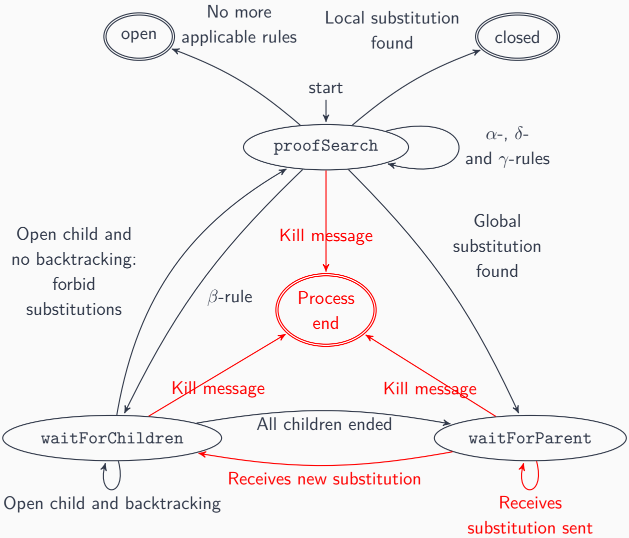

Solving Fairness Issues with Concurrency

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$$\neg Q(b)$$
$$\neg S(c)$$
$$\dfrac{\dfrac{\forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)}{P(X) \vee Q(X), \forall y.\ S(X)}\ \gamma_\forall}{\dfrac{P(X), \forall y.\ S(X)}{\odot}\ \odot_\sigma \qquad \dfrac{Q(X), \forall y.\ S(X)}{\odot}\ \odot_\sigma}\ \beta_\Leftrightarrow$$

$$\sigma = \{X \mapsto a\} \qquad \sigma = \{X \mapsto b\}$$

Solving Fairness Issues with Concurrency

$$
\begin{array}{c}
\neg P(a)\\
\neg Q(b)\\
\neg S(c)\\
\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{\dfrac{P(X) \lor Q(X), \forall y.\ S(X)}{\dfrac{P(X), \forall y.\ S(X)}{\odot}\ \odot_\sigma \qquad \dfrac{Q(X), \forall y.\ S(X)}{\odot}\ \odot_\sigma}\ \beta_\Leftrightarrow}\ \gamma_\forall
\end{array}
$$

$\sigma = \{X \mapsto a\}$ $\sigma = \{X \mapsto b\}$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(X) \lor Q(X), \forall y.\ S(X)}\ \gamma_\forall}{\dfrac{P(a), \forall y.\ S(a)}{} \qquad \dfrac{Q(a), \forall y.\ S(a)}{}}\ \beta_\Leftrightarrow$$

$$\sigma = \{X \mapsto a\} \qquad \sigma = \{X \mapsto a\}$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\cfrac{\cfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(X) \lor Q(X), \forall y.\ S(X)}\ \gamma_{\forall}}{\cfrac{P(a), \forall y.\ S(a)}{\odot}\ \odot_{\sigma} \qquad \cfrac{Q(a), \forall y.\ S(a)}{}}\ \beta_{\Leftrightarrow}$$

Closed

Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\dfrac{\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(X) \lor Q(X), \forall y.\ S(X)}\ \gamma_{\forall}}{\dfrac{P(a), \forall y.\ S(a)}{\odot}\ \odot_{\sigma} \qquad \dfrac{Q(a), \boxed{\forall y.\ S(a)}}{S(a)}\ \gamma_{\forall}}\ \beta_{\Leftrightarrow}$$

Solving Fairness Issues with Concurrency

$$\begin{array}{c}
\neg P(a) \\
\neg Q(b) \\
\neg S(c) \\
\dfrac{\forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)}{\dfrac{P(X) \vee Q(X), \forall y.\ S(X)}{\dfrac{P(a), \forall y.\ S(a)}{\odot}\ \odot_\sigma \qquad \dfrac{Q(a), \forall y.\ S(a)}{\dfrac{S(a)}{\ldots}}\ \gamma\forall}\ \beta_\Leftrightarrow}\ \gamma\forall
\end{array}$$

Open

Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\cfrac{\cfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(X) \lor Q(X), \forall y.\ S(X)}\ \gamma_\forall}{\cfrac{P({\color{red}b}), \forall y.\ S({\color{red}b})}{} \qquad \cfrac{Q({\color{red}b}), \forall y.\ S({\color{red}b})}{}}\ \beta_\Leftrightarrow$$

$${\color{blue}\sigma = \{X \mapsto b\}} \qquad\qquad {\color{blue}\sigma = \{X \mapsto b\}}$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\dfrac{\dfrac{\forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)}{P(X) \vee Q(X), \forall y.\ S(X)}\ \gamma_{\forall}}{\dfrac{P(b), \forall y.\ S(b)}{} \qquad \dfrac{Q(b), \forall y.\ S(b)}{\odot}\ \odot_{\sigma}}\ \beta_{\Leftrightarrow}$$

Closed

Solving Fairness Issues with Concurrency

$$\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \dfrac{\dfrac{\forall x.\ (P(x)\lor Q(x))\land \forall y.\ S(x)}{P(X)\lor Q(X), \forall y.\ S(X)}\,\gamma_\forall}{\dfrac{P(b), \forall y.\ S(b)}{S(b)}\,\gamma_\forall \qquad \dfrac{Q(b),\forall y.\ S(b)}{\odot}\,\odot_\sigma}\,\beta_\Leftrightarrow\end{array}$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$

$$\neg Q(a)$$

$$\neg S(c)$$

$$\cfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{\cfrac{P(X) \lor Q(X), \forall y.\ S(X)}{\cfrac{P(b), \forall y.\ S(b)}{\cfrac{S(b)}{\ldots}}\ \gamma_\forall \qquad \cfrac{Q(b), \forall y.\ S(b)}{\odot}\ \odot_\sigma}\ \beta_\Leftrightarrow}\ \gamma_\forall$$

Open

Solving Fairness Issues with Concurrency

$$\begin{array}{c} \neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \dfrac{\dfrac{\forall x.\ (P(x)\lor Q(x))\land \forall y.\ S(x)}{P(X)\lor Q(X), \forall y.\ S(X)}\,\gamma_\forall}{\dfrac{P(X), \forall y.\ S(X)}{} \qquad \dfrac{Q(X), \forall y.\ S(X)}{}}\,\beta_\Leftrightarrow \\ X \notin \{a,b\} \qquad\qquad X \notin \{a,b\} \end{array}$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\dfrac{\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(X) \lor Q(X), \forall y.\ S(X)}\ \gamma_\forall}{\dfrac{P(X), \forall y.\ S(X)}{S(X)}\ \gamma_\forall \qquad \dfrac{Q(X), \forall y.\ S(X)}{S(X)}\ \gamma_\forall}\ \beta_\Leftrightarrow$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\cfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{\cfrac{P(X) \lor Q(X), \forall y.\ S(X)}{\cfrac{P(X), \forall y.\ S(X)}{\cfrac{S(X)}{\odot}\ \odot_\sigma}\ \gamma_\forall \qquad \cfrac{Q(X), \forall y.\ S(X)}{\cfrac{S(X)}{\odot}\ \odot_\sigma}\ \gamma_\forall}\ \beta_\Leftrightarrow}\ \gamma_\forall$$

$$\sigma = \{X \mapsto c\} \qquad\qquad \sigma = \{X \mapsto c\}$$

Solving Fairness Issues with Concurrency

$$
\begin{array}{c}
\neg P(a)\\
\neg Q(b)\\
\neg S(c)\\
\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{\dfrac{P(c) \lor Q(c), \forall y.\ S(c)}{\begin{array}{c}\dfrac{P(c), \forall y.\ S(c)}{\dfrac{S(c)}{\odot}\ \odot_\sigma}\ \gamma_\forall \\ \sigma = \{X \mapsto c\}\end{array} \qquad \begin{array}{c}\dfrac{Q(c), \forall y.\ S(c)}{\dfrac{S(c)}{\odot}\ \odot_\sigma}\ \gamma_\forall \\ \sigma = \{X \mapsto c\}\end{array}}\ \beta_\Leftrightarrow}\ \gamma_\forall
\end{array}
$$

Contributions

- A Concurrent tableau-based proof-search procedure
- Concurrent exploration of branches
- Eager closure
- Backtrack and forbidden substitutions
- Tackle fairness challenges
- Completeness proof of the procedure
- Implemented into a tool: Goéland

3. Goéland: Implementation, Experiments, and Analysis

Goéland Tool

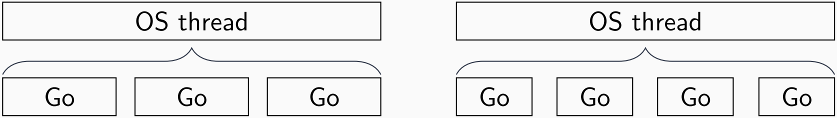
Functionnalities

- Concurrent proof-search procedure
- Equality reasoning
- Deduction modulo theory (+ polarized)
- Polymorphic types
- Alternative modes: incomplete, interactive,...

Goéland Tool

Implementation

- 40 000 lines of code
- Go programming language
- Designed for concurrency
- Goroutines: N:M lightweight threads



Theory Reasoning

Motivation and Challenges

- Reason within specific contexts (arithmetic, industrial problems, ...)
- Deal with a large number of axioms
- Handle multiple theories

Background Reasoners

- Equality
- Deduction modulo theory (DMT)

Deduction Modulo Theory

Principle

- Turns axioms into rewrite rules
- Triggers only relevant axioms
- Produces shorter proof
- Not limited to one theory

Main Heuristic

$(\forall \vec{x}.)\ A \Leftrightarrow F$ where:

- A is an atomic formula
- F is a non-atomic formula

Polarized DMT

$(\forall \vec{x}.)\ A \Rightarrow F$ where:

- A is an atomic formula
- F is a non-atomic formula

Axiom: $\forall x.\ P(x) \Leftrightarrow \forall y.\ Q(x, y) \wedge S(x, y)$

Rule: $P(X) \rightarrow \forall y.\ Q(X, y) \wedge S(X, y)$

Protocol of the Experiments

- Thousand of Problems for Theorem Provers (TPTP) library (v8.1.2)
- Syntactic (SYN) and set theory (SET) categories
- First-order logic (FOL)
- Goéland and its variants, Zenon (+ modulo), Princess, Vampire and E
- 300 seconds of timeout
- Intel Xeon E5-2680 v4 2.4GHz 2$\times$14-core processor with 128GB

Goéland Variants over SYN and SET

	SYN (288 problems)		SET (464 problems)	
Goéland	**209**	(1.2 s)	**124**	(18.6 s)
Goéland+EQ	**213**	(0.3 s)	**101**	(15.6 s)
Goéland+DMT	**209**	(1.3 s)	**217**	(5.9 s)
Goéland+DMT+EQ	**213**	(0.5 s)	**192**	(10.2 s)
Goéland+DMT +Polarized	**202**	(0.3 s)	**164**	(1.5 s)

All Provers on FOF

	FOF (5396 problems)
Goéland	**613** (10 482 s — 17.1 s)
Goéland+DMT	**770** (6 935 s — 9 s)
Goéland+DMT+EQ	**801** (10 060 s — 12.5 s)
Zenon	**1 382** (9 026 s — 6.5 s)
Zenon Modulo	**1 389** (10 028 s — 7.2 s)
Princess	**1 621** (23 200 s — 14.3 s)
Vampire	**3 342** (42 873 s — 12.8 s)
E	**3 939** (39 638 s — 10.1 s)

Analysis

- Promising results
- Less problems solved than other ATP
- Scaling issue
- Memory management
- Equality reasoning performances
- Good results with DMT

4. Toward Certification: an Output for Checkable Proofs

Advanced Skolemization Strategies

Motivations

- Shorter proofs
- Faster proof search

Inner Skolemization (δ^+-rule)

- Extension of δ-rule
- Uses only the free variables of the formula

Pre-Inner Skolemization (δ^{+^+}-rule)

- Extension of δ^+-rule
- Reuses the same Skolem symbol if they come from α-equivalent formulas

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(f(X))}\ \delta_{\neg\forall}}{\neg(D(X_2) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X_2), \neg\forall y\ D(y)}\ \alpha_{\neg\Rightarrow}}{\sigma = \{X_2 \mapsto f(X)\}}\ \odot_\sigma
$$

(a) Outer Skolemization tableau.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^+_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_\sigma
$$

(b) Inner Skolemization tableau.

Translation to Machine-Checkable Proofs

Gentzen-Schütte Calculus (GS3)

- Equivalent to tableaux
- Easily translatable to proof assistants
- Only supports outer skolemization

$$
\begin{array}{cl}
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) & \\ \hline
\neg(D(X) \Rightarrow \forall y\ D(y)) & \gamma_{\neg\exists} \\ \hline
D(X), \neg(\forall y\ D(y)) & \alpha_{\neg\Rightarrow} \\ \hline
\neg D(f(X)) & \delta_{\neg\forall} \\ \hline
\neg(D(X_2) \Rightarrow \forall y\ D(y)) & \gamma_{\neg\exists} \\ \hline
D(X_2), \neg\forall y\ D(y) & \alpha_{\neg\Rightarrow} \\ \hline
\sigma = \{X_2 \mapsto f(X)\} & \odot_\sigma
\end{array}
$$

(a) Outer Skolemization tableau proof.

$$
\begin{array}{cl}
\hline
\ldots, \neg D(c'), D(c'), \neg(\forall y\ D(y)) \vdash & \text{ax} \\ \hline
\ldots, \neg(D(c') \Rightarrow \forall y\ D(y)) \vdash & \neg\Rightarrow \\ \hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \ldots, \neg D(c') \vdash & \neg\exists \\ \hline
\ldots, D(c), \neg(\forall y\ D(y)) \vdash & \neg\forall \\ \hline
\ldots, \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash & \neg\Rightarrow \\ \hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash & \neg\exists
\end{array}
$$

(b) Equivalent GS3.

Outer Skolemisation

$$
\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}
$$

(a) Inner Skolemization tableau proof.

$$
\cfrac{\cfrac{\cfrac{\cfrac{}{\ldots, D(c), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \text{ax}}{\ldots, D(c), \neg(\forall y\ D(y)) \vdash}\ \neg_{\forall}\ (\star)}{\ldots, \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\exists}
$$

(b) Incorrect equivalent GS3.

A Deskolemization Strategy

Idea

Perform all the Skolemization steps before the other rules, so the Skolem symbol is necessarily fresh.

Key Notions

- Formulas that depend on a Skolem symbol
- Formulas that descend from a Skolem symbol
- A formula F needs to be processed before another formula G iff G makes use of a Skolem symbol generated by F

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y))}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\frac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y))}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg_{\exists}$$

Example

$$
\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}
$$

$$
\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg_{\exists}
$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg_{\exists}$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg_{\exists}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathrm{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg_{\exists}$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists
$$

Example

$$
\begin{array}{cl}
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) & \\
\hline
\neg(D(X) \Rightarrow \forall y\ D(y)) & \gamma_{\neg\exists} \\
\hline
D(X), \neg(\forall y\ D(y)) & \alpha_{\neg\Rightarrow} \\
\hline
\neg D(c) & \delta^{+}_{\neg\forall} \\
\hline
\sigma = \{X \mapsto c\} & \odot_{\sigma}
\end{array}
$$

$$
\begin{array}{cl}
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash & \\
\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash & \neg\forall \\
\hline\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash & \mathsf{W} \times 2 \\
\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash & \neg\Rightarrow \\
\hline
\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) & \neg\exists
\end{array}
$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists
$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$
\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}
$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists}\ \neg\Rightarrow
$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathrm{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg_{\exists}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg_{\forall}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathrm{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg_{\exists}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg_{\exists}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg_{\forall}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathrm{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg_{\exists}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\ \gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\ \alpha_{\neg\Rightarrow}}{\neg D(c)}\ \delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\ \odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \mathsf{ax}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\ \neg_{\exists}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\ \neg_{\forall}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\ \mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\ \neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\ \neg_{\exists}$$

Implementation

Implementation

- δ, δ^+ and δ^{+^+} Skolemization strategies
- GS3 proofs
- Deskolemization algorithms
- Coq & Lambdapi outputs

Evaluation Protocol

- Same setup as previous tests
- 3 Skolemization strategies + DMT
- Number of problems solved
- Size of the proof (number of branches)

Results

	Problems Proved	Percentage Certified	Avg. Size Increase	Max. Size Increase
Goéland	261	100 %	0 %	-
Goéland+δ^+	272	100 %	8.1 %	5.3
Goéland+δ^{+^+}	274	100 %	10.6 %	10.3
Goéland+DMT	363	100 %	0 %	-
Goéland+DMT+δ^+	375	100 %	4.5 %	3.9
Goéland+DMT+δ^{+^+}	377	100 %	7.4 %	5.2

Contributions

- An optimization of the deskolemization algorithm for δ^+
- A deskolemization algorithm for δ^{+^+}
- Soundness proof for both translations
- Output of GS3 proof into Coq and Lambdapi
- Promising results
- 100% of the proofs are certified
- Far below the theoretical bound

Conclusion

Contributions

Fairness in Tableau-Based Proof Search

- Fairness between branches managed by concurrency
- Completeness of the procedure

Theory Reasoning in Tableaux

- Implementation of two background reasoners
- Study of parallelization points and interaction with the proof search

Proof Certification

- A sound generic deskolemization algorithm
- Output of the proofs into Coq & Lambdapi

Future Work

Fairness in Tableau-Based Proof Search

- Improvement of the performances of Goéland
- Heuristics in formula computation order, closure management
- Simulate "intuition" with learning methods
- Modular and generic prover

Theory Reasoning

- Improvements of DMT (term rewriting, narrowing, manually designed rewrite rules)
- More experiments on polymorphic problems
- SMT solvers: string equations, quantum circuits

Proof Certification

- Reduce the number of branches by the use of lemmas
- Framework for verification of tableau proofs

Thank you! ☺

First-Order Logic

Conventions

- Constant symbols: a, b, c
- Function symbols: f, f'
- Bound and free variables: x, y, X, X_2, Y
- Predicate symbols: P, Q, S
- Connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- Quantifiers: $\exists, \forall$

- $Socrates$
- $Human(x)$
- $Mortal(x)$

- All humans are mortals
- Socrates is a human
- Then Socrates is mortal

$$(\forall x.\ Human(x) \Rightarrow Mortal(x)) \wedge Human(Socrates) \Rightarrow Mortal(Socrates)$$

Reasoning Methods in FOL

Resolution

- Breaks the initial formula into clauses
- Derivation step and saturation
- More efficient

Tableaux

- Works with the unaltered original formula
- Reduces the goal into subgoals
- Better interoperability

CNF : $\{\neg A\}, \{A \vee A\}, \{\neg B \vee A\}$

$\{\neg A\}$

$\{A \vee A\}$

$\{\neg B \vee A\}$

$$\cfrac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{\cfrac{(A \Rightarrow B) \Rightarrow A, \neg A}{\cfrac{\neg(A \Rightarrow B)}{\cfrac{A, \neg B}{\odot}\ \odot}\ \alpha_{\neg\Rightarrow} \qquad \cfrac{A}{\odot}\ \odot}\ \beta_{\Rightarrow}}\ \alpha_{\neg\Rightarrow}$$

Reasoning Methods in FOL

Resolution

- Breaks the initial formula into clauses
- Derivation step and saturation
- More efficient

Tableaux

- Works with the unaltered original formula
- Reduces the goal into sub-goals
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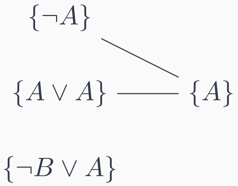
CNF : $\{\neg A\}, \{A \vee A\}, \{\neg B \vee A\}$



$$\dfrac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{\dfrac{(A \Rightarrow B) \Rightarrow A, \neg A}{\dfrac{\neg(A \Rightarrow B)}{\dfrac{A, \neg B}{\odot}\ \odot}\ \alpha_{\neg\Rightarrow} \qquad \dfrac{A}{\odot}\ \odot}\ \beta_{\Rightarrow}}\ \alpha_{\neg\Rightarrow}$$

Reasoning Methods in FOL

Resolution

- Breaks the initial formula into clauses
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Tableaux

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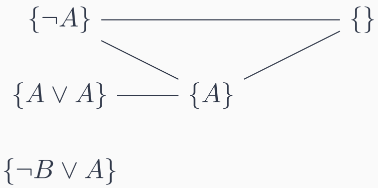
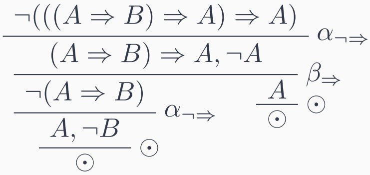
CNF : $\{\neg A\}, \{A \vee A\}, \{\neg B \vee A\}$





Select Branch

$$\dfrac{(P(a) \land \neg P(a)) \lor \bot}{\dfrac{P(a) \land P(a)}{\dfrac{P(a), \neg P(a)}{\odot}\,\odot}\,\alpha_{\land} \qquad \dfrac{\bot}{\odot}\,\odot}\,\beta_{\lor}$$

(a) Proof of $(P(a) \land \neg P(a)) \lor \bot$.

$$\dfrac{(P(a) \land \neg P(a)) \lor \bot}{\dfrac{P(a) \land P(a)}{\dfrac{P(a), \neg P(a)}{\odot}\,\odot}\,\alpha_{\land} \qquad \dfrac{\bot}{}\,\ldots}\,\beta_{\lor}$$

(b) Incompleteness caused by an unfair *select branch*.

Select Formula

$$
\begin{array}{cl}
P(a) \land \neg P(a) & \\
\forall x \; Q(X) & \\
\hline
P(a), \neg P(a) & \alpha_{\land} \\
\hline
\odot & \odot
\end{array}
$$

(a) Proof of $P(a) \land \neg P(a), \forall x \; Q(X)$.

$$
\begin{array}{cl}
P(a) \land \neg P(a) & \\
\forall x \; Q(X) & \\
\hline
Q(X) & \gamma_{\forall} \\
\hline
Q(X') & \gamma_{\forall} \\
\hline
\dots & \gamma_{\forall}
\end{array}
$$

(b) Incompleteness caused by an unfair *select formula*.

Select Pair (Right Branch First)

$$P(a)$$
$$\neg P(b)$$
$$\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))$$

Select Pair (Right Branch First)

$$\frac{\begin{array}{c} P(a) \\ \neg P(b) \\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y)) \end{array}}{P(X) \Leftrightarrow (\forall y\ P(y))}\ \gamma\forall$$

Select Pair (Right Branch First)

$$
\dfrac{\dfrac{\begin{array}{c} P(a) \\ \neg P(b) \\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y)) \end{array}}{P(X) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall}{P(X), \forall y\ P(y) \qquad \neg P(X), \neg(\forall y\ P(y))}\ \beta_\Leftrightarrow
$$

Select Pair (Right Branch First)

$$\dfrac{\dfrac{\begin{array}{c}P(a)\\ \neg P(b)\\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))\end{array}}{P(X) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall}{P(X), \forall y\ P(y) \qquad \dfrac{\neg P(X), \neg(\forall y\ P(y))}{\sigma = \{{\color{red}X} \mapsto a\}}\ \odot_\sigma}\ \beta_\Leftrightarrow$$

Select Pair (Right Branch First)

$$
\begin{array}{c}
P(a) \\
\neg P(b) \\
\dfrac{\dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(a) \Leftrightarrow (\forall y\ P(y))}\ \gamma_{\forall}}{P(a), \forall y\ P(y) \qquad \dfrac{\neg P(a), \neg(\forall y\ P(y))}{\sigma = \{X \mapsto a\}}\ \odot_{\sigma}}\ \beta_{\Leftrightarrow}
\end{array}
$$

Select Pair (Right Branch First)

$$\dfrac{\begin{array}{c}P(a)\\ \neg P(b)\\ \dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(a) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall\end{array}}{\dfrac{P(a), \forall y\ P(y)}{P(Y)}\ \gamma_\forall \qquad \dfrac{\neg P(a), \neg(\forall y\ P(y))}{\sigma = \{X \mapsto a\}}\ \odot_\sigma}\ \beta_\Leftrightarrow$$

Select Pair (Right Branch First)

$$
\begin{array}{c}
P(a) \\
\neg P(b) \\
\dfrac{\dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(a) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall}{\dfrac{\dfrac{P(a), \forall y\ P(y)}{P(Y)}\ \gamma_\forall}{\sigma = \{Y \mapsto b\}}\ \odot_\sigma \qquad \dfrac{\neg P(a), \neg(\forall y\ P(y))}{\sigma = \{X \mapsto a\}}\ \odot_\sigma}\ \beta_\Leftrightarrow
\end{array}
$$

Select Pair (Right Branch First)

$$
\begin{array}{c}
P(a) \\
\neg P(b) \\
\dfrac{\dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(a) \Leftrightarrow (\forall y\ P(y))}\ \gamma_{\forall}}{\dfrac{\dfrac{P(a), \forall y\ P(y)}{P(b)}\ \gamma_{\forall}}{\sigma = \{Y \mapsto b\}}\ \odot_{\sigma} \qquad \dfrac{\neg P(a), \neg(\forall y\ P(y))}{\sigma = \{X \mapsto a\}}\ \odot_{\sigma}}\ \beta_{\Leftrightarrow}
\end{array}
$$

Select Pair (Left Branch First)

$$P(a)$$
$$\neg P(b)$$
$$\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))$$

Select Pair (Left Branch First)

$$\frac{\begin{array}{c} P(a) \\ \neg P(b) \\ \forall x.\, P(x) \Leftrightarrow (\forall y\ P(y)) \end{array}}{P(X) \Leftrightarrow (\forall y\ P(y))}\ \gamma\forall$$

Select Pair (Left Branch First)

$$
\dfrac{\dfrac{\begin{array}{c} P(a) \\ \neg P(b) \\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y)) \end{array}}{\colorbox{pink}{$P(X) \Leftrightarrow (\forall y\ P(y))$}}\ \gamma_\forall}{P(X), \forall y\ P(y) \qquad\qquad \neg P(X), \neg(\forall y\ P(y))}\ \beta_\Leftrightarrow
$$

Select Pair (Left Branch First)

$$
\dfrac{\dfrac{P(X), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\ \odot_\sigma \qquad \neg P(X), \neg(\forall y\ P(y))}{\begin{array}{c} P(a) \\ \neg P(b) \\ \dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(X) \Leftrightarrow (\forall y\ P(y))}\ \gamma\forall \end{array}}\ \beta_{\Leftrightarrow}
$$

Select Pair (Left Branch First)

$$
\dfrac{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\ \odot_\sigma \qquad \neg P(b), \neg(\forall y\ P(y))}{\dfrac{\begin{array}{c} P(a) \\ \neg P(b) \\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y)) \end{array}}{P(b) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall}\ \beta_\Leftrightarrow
$$

Select Pair (Left Branch First)

$$\dfrac{\dfrac{\begin{array}{c}P(a)\\ \neg P(b)\\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))\end{array}}{P(b) \Leftrightarrow (\forall y\ P(y))}\,\gamma_{\forall}}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\,\odot_{\sigma} \qquad\qquad \dfrac{\neg P(b), \neg(\forall y\ P(y))}{\neg P(f(b))}\,\delta_{\neg\forall}}\,\beta_{\Leftrightarrow}$$

Select Pair (Left Branch First)

$$\dfrac{\dfrac{\begin{array}{c}P(a)\\ \neg P(b)\\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))\end{array}}{P(b) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\ \odot_\sigma \qquad\qquad \dfrac{\dfrac{\neg P(b), \neg(\forall y\ P(y))}{\neg P(f(b))}\ \delta_{\neg\forall}}{P(X_2) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall}\ \beta_\Leftrightarrow$$

Select Pair (Left Branch First)

$$
\dfrac{\dfrac{\begin{array}{c}P(a)\\ \neg P(b)\\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))\end{array}}{P(b) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\ \odot_\sigma \qquad \dfrac{\dfrac{\dfrac{\neg P(b), \neg(\forall y\ P(y))}{\neg P(f(b))}\ \delta_{\neg\forall}}{P(X_2) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall}{P(X_2), \forall y\ P(y) \qquad \neg P(X_2), \neg(\forall y\ P(y))}\ \beta_\Leftrightarrow}\ \beta_\Leftrightarrow
$$

Select Pair (Left Branch First)

$$
\begin{array}{c}
P(a)\\
\neg P(b)\\
\dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(b) \Leftrightarrow (\forall y\ P(y))}\ \gamma_{\forall}
\end{array}
$$

$$
\dfrac{P(b) \Leftrightarrow (\forall y\ P(y))}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\ \odot_{\sigma} \qquad \dfrac{\neg P(b), \neg(\forall y\ P(y))}{\dfrac{\neg P(f(b))}{\dfrac{P(X_2) \Leftrightarrow (\forall y\ P(y))}{\dfrac{P(X_2), \forall y\ P(y)}{\sigma = \{X_2 \mapsto b\}}\ \odot_{\sigma} \qquad \neg P(X_2), \neg(\forall y\ P(y))}\ \beta_{\Leftrightarrow}}\ \gamma_{\forall}}\ \delta_{\neg\forall}}\ \beta_{\Leftrightarrow}
$$

$$
\sigma' = \{X_2 \mapsto f(b)\}
$$

Select Pair (Left Branch First)

$$
\dfrac{\dfrac{\begin{array}{c} P(a) \\ \neg P(b) \\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y)) \end{array}}{P({\color{red}b}) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall}{\dfrac{P({\color{red}b}), \forall y\ P(y)}{\sigma = \{{\color{red}X} \mapsto b\}}\ \odot_\sigma \qquad \dfrac{\neg P({\color{red}b}), \neg(\forall y\ P(y))}{\dfrac{\neg P(f({\color{red}b}))}{\dfrac{P({\color{blue}b}) \Leftrightarrow (\forall y\ P(y))}{\begin{array}{c}\dfrac{P({\color{blue}b}), \forall y\ P(y)}{\sigma = \{{\color{blue}X_2} \mapsto b\}}\ \odot_\sigma \\ \sigma' = \{{\color{blue}X_2} \mapsto f(b)\}\end{array} \qquad \neg P({\color{blue}b}), \neg(\forall y\ P(y))}\ \beta_\Leftrightarrow}\ \gamma_\forall}\ \delta_{\neg\forall}}\ \beta_\Leftrightarrow
$$

Select Pair (Left Branch First)

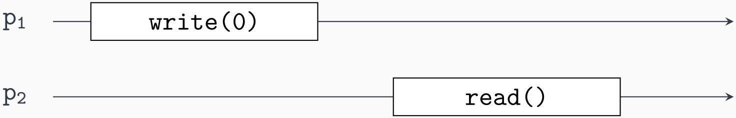
$$
\dfrac{\dfrac{\begin{array}{c}P(a)\\ \neg P(b)\\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))\end{array}}{P(b) \Leftrightarrow (\forall y\ P(y))}\ \gamma_{\forall}}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\ \odot_{\sigma} \qquad \dfrac{\dfrac{\dfrac{\neg P(b), \neg(\forall y\ P(y))}{\neg P(f(b))}\ \delta_{\neg\forall}}{P(b) \Leftrightarrow (\forall y\ P(y))}\ \gamma_{\forall}}{\begin{array}{c}\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X_2 \mapsto b\}}\ \odot_{\sigma} \\ \sigma' = \{X_2 \mapsto f(b)\}\end{array} \qquad \dfrac{\neg P(b), \neg(\forall y\ P(y))}{\neg P(f'(b))}\ \delta_{\neg\forall}}\ \beta_{\Leftrightarrow}}\ \beta_{\Leftrightarrow}
$$

Select Pair (Left Branch First)

$$
\dfrac{\dfrac{\begin{array}{c} P(a) \\ \neg P(b) \\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y)) \end{array}}{P(b) \Leftrightarrow (\forall y\ P(y))}\ \gamma_{\forall}}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\ \odot_{\sigma} \qquad \dfrac{\neg P(b), \neg(\forall y\ P(y))}{\dfrac{\neg P(f(b))}{\dfrac{P(b) \Leftrightarrow (\forall y\ P(y))}{\dfrac{P(b), \forall y\ P(y)}{\begin{array}{c} \sigma = \{X_2 \mapsto b\} \\ \sigma' = \{X_2 \mapsto f(b)\} \end{array}}\ \odot_{\sigma} \qquad \dfrac{\neg P(b), \neg(\forall y\ P(y))}{\dfrac{\neg P(f'(b))}{\cdots}\ \gamma_{\forall}}\ \delta_{\neg\forall}}\ \beta_{\Leftrightarrow}}\ \gamma_{\forall}}\ \delta_{\neg\forall}}\ \beta_{\Leftrightarrow}
$$

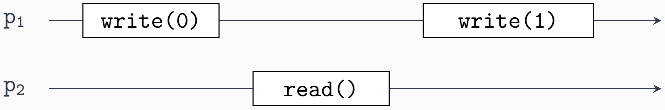
Sequential and Concurrent Executions

(a) Sequential execution of two operations on a resource `R` by the process $\mathtt{p}_1$.

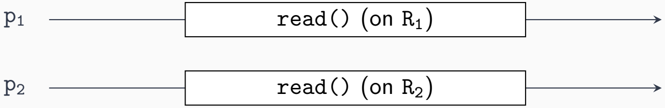


(b) Concurrent execution of two operations on a shared resource `R`: `write(0)` by $\mathtt{p}_1$ and `read()` by $\mathtt{p}_2$.

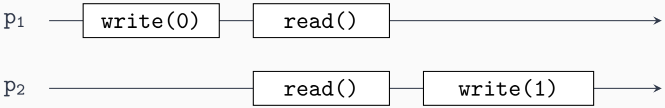
Parallelism and Concurrency



(a) *Concurrent* but not *parallel*.



(b) *Parallel* but not *concurrent*.



(c) *Concurrent* and *parallel*.

Interactions with the Proof-Search Procedure

Equality Reasoning

- Capture equality predicate $\approx$
- Extract terms that have to be equals
- $\mathcal{BSE}$ calculus
- New closure rule: $\neg(X \approx X)$

Integration

- Triggered when a predicate or an equality is generated
- Backtrack if the chosen solution does not fit with other branches
- Parallelization if multiple rules are applicable

$\mathcal{BSE}$-Calculus

Rigid E-unification Problem

A rigid E-unification problem

$$\langle E, s, t \rangle$$

consists of a finite set E of equalities of the form $(l \approx r)$ and two terms s and t such that $r, l, s, t \in \mathscr{T}$.

Constraint

An (ordering) constraint is a (finite) set of expressions of the form $s \simeq t$ or $s \succ t$ where s and t are terms. A substitution σ is a solution to a constraint $\mathcal{C}$ if and only if :

- $\sigma(s) = \sigma(t)$ for all $s \simeq t \in \mathcal{C}$, i.e., σ is an unifier for s and t.
- $\sigma(s) > \sigma(t)$ for all $s \succ t \in \mathcal{C}$, where $>$ is an arbitrary but fixed term reduction ordering.
- σ instantiates all variables occurring in $\mathcal{C}$ with ground terms.

Basic Rigid Superposition Rules

Right Basic Rigid Superposition Rule

Let $l \approx r$ or $r \approx l$ be an equality of E and l' is sub-term of s or t. Thus, the application of the right rigid basic superposition (*rbrs*) rule results in one of the following, regarding the term on which it is applied:

- s become $s_{[l' \mapsto r]}$ and the constraints $l \succ r$, $s \succ t$, $l \simeq l'$ are added to C
- t become $t_{[l' \mapsto r]}$ and the constraints $l \succ r$, $t \succ s$, $l \simeq l'$ are added to C

$$\frac{\langle E \cup \left\{ l \approx r \right\}, s, t \rangle \cdot \mathcal{C}}{\langle E \cup \{ l \approx r \}, s_{[l' \mapsto r]}, t \rangle \cdot \mathcal{C} \cup \left\{ l \succ r, s \succ t, l \simeq l' \right\}}\ rbrs$$

Basic Rigid Superposition Rules

Left Basic Rigid Superposition Rule

Let $l \approx r$ or $r \approx l$ and $u \approx v$ or $v \approx u$ be two equalities of E. Let l' be a sub-term of u. Thus, the application of the left rigid basic superposition (*lbrs*) rule results in $u \approx v$ becoming $u_{[l' \mapsto r]} \approx v$ and the constraints $l \succ r$, $u \succ v$ and $l \simeq l'$ are added to C.

$$\frac{\langle E \cup \left\{l \approx r, u \approx v\right\}, s, t\rangle \cdot \mathcal{C}}{\langle E \cup \left\{l \approx r, u_{[l' \mapsto r]} \approx v\right\}, s, t\rangle \cdot \mathcal{C} \cup \left\{l \succ r, u \succ v, l \simeq l'\right\}} \; lbrs$$

Example using Equality Reasoning

$$a \approx b$$
$$a \approx c$$
$$\neg P(c, c)$$
$$\forall x.\ P(c, c) \lor \neg(x \approx c)$$

Example using Equality Reasoning

$$\dfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \neg P(a,b) \\ \boldsymbol{\forall x.\ P(c,c) \vee \neg(x \approx c)} \end{array}}{P(c,c) \vee \neg(X \approx c)}\ \gamma\forall$$

Example using Equality Reasoning

$$
\begin{array}{c}
a \approx b \\
a \approx c \\
\neg P(c,c) \\
\dfrac{\dfrac{\forall x.\ P(c,c) \lor \neg(x \approx c)}{\boldsymbol{P(c,c) \lor \neg(X \approx c)}}\ \gamma_\forall}{P(c,c) \qquad \neg(X \approx c)}\ \beta_\lor
\end{array}
$$

Example using Equality Reasoning

$$a \approx b$$

$$a \approx c$$

$$\boldsymbol{\neg P(a, b)}$$

$$\cfrac{\cfrac{\forall x.\ P(c,c) \lor \neg(x \approx c)}{P(c,c) \lor \neg(X \approx c)}\ \gamma_\forall}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\ \beta_\lor$$

Example using Equality Reasoning

$$
\begin{array}{c}
\boxed{a \approx b} \\
\boxed{a \approx c} \\
\boldsymbol{\neg P(a, b)} \\
\dfrac{\dfrac{\forall x.\ P(c, c) \vee \neg(x \approx c)}{P(c, c) \vee \neg(X \approx c)}\ \gamma_\forall}{\boldsymbol{P(c, c)} \qquad \neg(X \approx c)}\ \beta_\vee
\end{array}
$$

$$\langle \{a \approx b, a \approx c\}, a, c \rangle \cdot \emptyset$$

Example using Equality Reasoning

$$
\begin{array}{c}
a \approx b \\
a \approx c \\
\boldsymbol{\neg P(a, b)} \\
\dfrac{\dfrac{\forall x.\ P(c, c) \lor \neg(x \approx c)}{P(c, c) \lor \neg(X \approx c)}\ \gamma_\forall}{\boldsymbol{P(c, c)} \qquad \neg(X \approx c)}\ \beta_\lor
\end{array}
$$

$$
\frac{\langle \{a \approx b, a \approx c\}, a, c\rangle \cdot \emptyset}{\langle \{a \approx b, a \approx c\}, c, c\rangle \cdot \{a \succ c, a \simeq a\}}\ rbrs
$$

Example using Equality Reasoning

$$a \approx b$$

$$a \approx c$$

$$\boldsymbol{\neg P(a, b)}$$

$$\dfrac{\dfrac{\forall x.\ P(c,c) \vee \neg(x \approx c)}{P(c,c) \vee \neg(X \approx c)}\ \gamma_\forall}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\ \beta_\vee$$

$$\dfrac{\langle \{a \approx b, a \approx c\}, a, c \rangle \cdot \emptyset}{\langle \{a \approx b, a \approx c\}, c, c \rangle \cdot \{a \succ c, a \simeq a\}}\ rbrs$$

Example using Equality Reasoning

$$
\begin{array}{c}
a \approx b \\
a \approx c \\
\boldsymbol{\neg P(c, b)} \\
\dfrac{\dfrac{\forall x.\ P(c, c) \vee \neg(x \approx c)}{P(c, c) \vee \neg(X \approx c)}\ \gamma_\forall}{\boldsymbol{P(c, c)} \qquad \neg(X \approx c)}\ \beta_\vee
\end{array}
$$

Example using Equality Reasoning

$$
\begin{array}{c}
a \approx b \\
a \approx c \\
\boldsymbol{\neg P(c, b)} \\
\dfrac{\dfrac{\forall x.\ P(c,c) \vee \neg(x \approx c)}{P(c,c) \vee \neg(X \approx c)}\ \gamma_\forall}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\ \beta_\vee
\end{array}
$$

$$\langle \{a \approx b, a \approx c\}, b, c \rangle \cdot \emptyset$$

Example using Equality Reasoning

$$a \approx b$$
$$a \approx c$$
$$\boldsymbol{\neg P(c, b)}$$
$$\dfrac{\dfrac{\forall x.\ P(c,c) \vee \neg(x \approx c)}{P(c,c) \vee \neg(X \approx c)}\ \gamma_\forall}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\ \beta_\vee$$

$$\dfrac{\langle \{a \approx b, a \approx c\}, b, c\rangle \cdot \emptyset}{\langle \{a \approx b, a \approx c\}, a, c\rangle \cdot \{b \succ a, b \simeq b\}}\ rbrs$$

Example using Equality Reasoning

$$\begin{array}{c} a \approx b \\ a \approx c \\ \boldsymbol{\neg P(c, b)} \\ \dfrac{\dfrac{\forall x.\ P(c,c) \lor \neg(x \approx c)}{P(c,c) \lor \neg(X \approx c)}\ \gamma_\forall}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\ \beta_\lor \end{array}$$

$$\frac{\langle \{a \approx b, a \approx c\}, b, c\rangle \cdot \emptyset}{\langle \left\{a \approx b, a \approx c\right\}, a, c\rangle \cdot \{b \succ a, b \simeq b\}}\ rbrs$$

Example using Equality Reasoning

$$
\begin{array}{c}
a \approx b \\
a \approx c \\
\boldsymbol{\neg P(c, b)} \\
\dfrac{\dfrac{\forall x.\ P(c,c) \vee \neg(x \approx c)}{P(c,c) \vee \neg(X \approx c)}\ \gamma_\forall}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\ \beta_\vee
\end{array}
$$

$$
\dfrac{\dfrac{\langle \{a \approx b, a \approx c\}, b, c\rangle \cdot \emptyset}{\langle \{a \approx b, a \approx c\}, a, c\rangle \cdot \{b \succ a, b \simeq b\}}\ rbrs}{\langle \{a \approx b, a \approx c\}, c, c\rangle \cdot \left\{a \succ c, a \simeq a, b \succ a, b \simeq b\right\}}\ rbrs
$$

Example using Equality Reasoning

$$a \approx b$$

$$a \approx c$$

$$\boldsymbol{\neg P(c, c)}$$

$$\cfrac{\cfrac{\forall x.\ P(c,c) \lor \neg(x \approx c)}{P(c,c) \lor \neg(X \approx c)}\ \gamma_\forall}{\cfrac{\boldsymbol{P(c,c)}}{\odot}\ \odot \qquad \neg(X \approx c)}\ \beta_\lor$$

Example using Equality Reasoning

$$
\cfrac{\cfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \neg P(c,c) \\ \forall x.\ P(c,c) \lor \neg(x \approx c) \end{array}}{P(c,c) \lor \neg(c \approx c)}\ \gamma_\forall}{\cfrac{P(c,c)}{\odot}\ \odot \qquad \boldsymbol{\neg(X \approx c)}}\ \beta_\lor
$$

$$\neg(X \approx c)$$

Example using Equality Reasoning

$$
\begin{array}{c}
a \approx b \\
a \approx c \\
\neg P(c,c) \\
\dfrac{\dfrac{\forall x.\ P(c,c) \vee \neg(x \approx c)}{P(c,c) \vee \neg(X \approx c)}\ \gamma_\forall}{\dfrac{P(c,c)}{\odot}\ \odot \qquad \dfrac{\boldsymbol{\neg(c \approx c)}}{\odot}\ \odot}\ \beta_\vee
\end{array}
$$

$$
\dfrac{\neg(X \approx c)}{\{X \mapsto c\}}\ \odot
$$

Reasoning Modulo Theory

Simple Set Theory

- A_1: $\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b$
- A_2: $\forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a$
- C: $\forall a.\ a \subseteq a$

$$A_1 \wedge A_2 \wedge \neg C$$

$$\begin{aligned}
&(\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b) \\
&\wedge (\forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a) \\
&\wedge \neg(\forall a.\ a \subseteq a)
\end{aligned}$$

Reasoning Modulo Theory

$$\dfrac{\begin{array}{c}(\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b) \wedge (\forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a) \\ \wedge \neg(\forall a.\ a \subseteq a)\end{array}}{\begin{array}{c}\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b,\ \forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a, \\ \neg(\forall a.\ a \subseteq a)\end{array}}\ \alpha_{\wedge}$$

$$\dfrac{}{\neg(a \subseteq a)}\ \delta_{\neg\forall}$$

$$\dfrac{\dfrac{}{\forall b.\ A \subseteq b \Leftrightarrow \forall x.\ x \in A \Rightarrow x \in b}\ \gamma_{\forall}}{A \subseteq B \Leftrightarrow \forall x.\ x \in A \Rightarrow x \in B}\ \gamma_{\forall}$$

$$\dfrac{}{\dfrac{A \subseteq B, x \in A \Rightarrow x \in B}{\sigma = \{A \mapsto a, B \mapsto a\}}\ \odot_{\sigma} \qquad \dfrac{\neg(A \subseteq B), \neg(\forall x.\ x \in A \Rightarrow x \in B)}{\dfrac{\neg(a \subseteq a), \neg(\forall x.\ x \in a \Rightarrow x \in a)}{\dfrac{\neg(s \in a \Rightarrow s \in a)}{\dfrac{\neg(s \in a), (s \in a)}{\odot}\ \odot}\ \alpha_{\neg\Rightarrow}}\ \delta_{\neg\forall}}\ \sigma}\ \beta_{\Leftrightarrow}$$

Reasoning Modulo Theory

$$\dfrac{(\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b) \wedge (\forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a) \wedge \neg(\forall a.\ a \subseteq a)}{\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b,\ \forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a,\ \neg(\forall a.\ a \subseteq a)}\ \alpha_{\wedge}$$

$$\dfrac{\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b,\ \forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a,\ \neg(\forall a.\ a \subseteq a)}{\neg(a \subseteq a)}\ \delta_{\neg\forall}$$

$$\dfrac{\dfrac{\neg(a \subseteq a)}{\forall b.\ A \subseteq b \Leftrightarrow \forall x.\ x \in A \Rightarrow x \in b}\ \gamma_{\forall}}{A \subseteq B \Leftrightarrow \forall x.\ x \in A \Rightarrow x \in B}\ \gamma_{\forall}$$

$$\dfrac{A \subseteq B \Leftrightarrow \forall x.\ x \in A \Rightarrow x \in B}{\dfrac{A \subseteq B, x \in A \Rightarrow x \in B}{\sigma = \{A \mapsto a, B \mapsto a\}}\ \odot_{\sigma} \qquad \dfrac{\neg(A \subseteq B), \neg(\forall x.\ x \in A \Rightarrow x \in B)}{\dfrac{\neg(a \subseteq a), \neg(\forall x.\ x \in a \Rightarrow x \in a)}{\dfrac{\neg(s \in a \Rightarrow s \in a)}{\dfrac{\neg(s \in a), (s \in a)}{\odot}\ \odot}\ \alpha_{\neg\Rightarrow}}\ \delta_{\neg\forall}}\ \sigma}\ \beta_{\Leftrightarrow}$$

Deduction Modulo Theory (DMT)

Main Heuristic

$(\forall \vec{x}.)\ A \Leftrightarrow F$ where:

- A is an atomic formula
- F is a non-atomic formula

$$\text{Axiom: } \forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b$$
$$\text{Rule: } A \subseteq B \rightarrow \forall x.\ x \in A \Rightarrow x \in B$$

$$\text{Axiom: } \forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a$$
$$\text{Rule: } A = B \rightarrow A \subseteq B \wedge B \subseteq A$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \rightarrow \forall x.\ x \in A \Rightarrow x \in B$$

$$A = B \rightarrow A \subseteq B \land B \subseteq A$$

$$\neg(\forall a.\ a \subseteq a)$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \rightarrow \forall x.\ x \in A \Rightarrow x \in B$$

$$A = B \rightarrow A \subseteq B \land B \subseteq A$$

$$\frac{\neg(\forall a.\ a \subseteq a)}{\neg(a \subseteq a)}\ \delta_{\neg\forall}$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \rightarrow \forall x.\ x \in A \Rightarrow x \in B$$

$$A = B \rightarrow A \subseteq B \land B \subseteq A$$

$$\frac{\neg(\forall a.\ a \subseteq a)}{\neg(a \subseteq a)}\ \delta_{\neg\forall}$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \to \forall x.\ x \in A \Rightarrow x \in B$$

$$A = B \to A \subseteq B \wedge B \subseteq A$$

$$\cfrac{\cfrac{\neg(\forall a.\ a \subseteq a)}{\neg(a \subseteq a)}\ \delta_{\neg\forall}}{\neg(\forall x.\ x \in a \Rightarrow x \in a)} \to (A \mapsto a, B \mapsto a)$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \rightarrow \forall x.\ x \in A \Rightarrow x \in B$$

$$A = B \rightarrow A \subseteq B \land B \subseteq A$$

$$\cfrac{\cfrac{\neg(\forall a.\ a \subseteq a)}{\neg(a \subseteq a)}\ \delta_{\neg\forall}}{\neg(\forall x.\ x \in a \Rightarrow x \in a)} \rightarrow (A \mapsto a, B \mapsto a)$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \rightarrow \forall x.\ x \in A \Rightarrow x \in B$$

$$A = B \rightarrow A \subseteq B \wedge B \subseteq A$$

$$\cfrac{\cfrac{\cfrac{\neg(\forall a.\ a \subseteq a)}{\neg(a \subseteq a)}\ \delta_{\neg\forall}}{\neg(\forall x.\ x \in a \Rightarrow x \in a)}\ \rightarrow (A \mapsto a, B \mapsto a)}{\neg(s \in a \Rightarrow s \in a)}\ \delta_{\neg\forall}$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \to \forall x.\ x \in A \Rightarrow x \in B$$

$$A = B \to A \subseteq B \wedge B \subseteq A$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\forall a.\ a \subseteq a)}{\neg(a \subseteq a)}\ \delta_{\neg\forall}}{\neg(\forall x.\ x \in a \Rightarrow x \in a)}\ \to (A \mapsto a, B \mapsto a)}{\neg(s \in a \Rightarrow s \in a)}\ \delta_{\neg\forall}}{\neg(s \in a), (s \in a)}\ \alpha_{\neg\Rightarrow}$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \rightarrow \forall x.\, x \in A \Rightarrow x \in B$$

$$A = B \rightarrow A \subseteq B \wedge B \subseteq A$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\forall a.\, a \subseteq a)}{\neg(a \subseteq a)}\,\delta_{\neg\forall}}{\neg(\forall x.\, x \in a \Rightarrow x \in a)}\rightarrow (A \mapsto a, B \mapsto a)}{\neg(s \in a \Rightarrow s \in a)}\,\delta_{\neg\forall}}{\neg(s \in a), (s \in a)}\,\alpha_{\neg\Rightarrow}}{\odot}\,\odot$$

Deduction Modulo Theory (DMT)

Benefits

- Avoid combinatorial explosion
- "Useless" axioms aren't triggered
- Shorter proof
- Not limited to one theory

Integration

- Triggered when a predicate is generated
- Backtrack if multiples rules are available

Scale-Up Experimental Results

	SYN (207 problems)	SET (113 problems)
2	**1.5 s**	**20 s** (+4)
4	**0.6 s**	**15 s** (+5)
8	**0.4 s**	**12 s** (+8)
16	**0.8 s**	**8.7 s** (+10)
28	**0.3 s** (+ 2)	**8.7 s** (+11)

Table 1: Scale-up experimental results of Goéland.

	SYN (207 problems)	SET (208 problems)
2	**1.4 s** (+ 1)	**6.1 s** (+ 5)
4	**1.3 s**	**5.3 s** (+ 8)
8	**1.1 s**	**4.7 s** (+ 7)
16	**0.6 s** (+ 1)	**4.2 s** (+ 9)
28	**0.4 s** (+ 2)	**3.1 s** (+ 9)

Table 2: Scale-up experimental results of Goéland+DMT.

Proof Tree and Segments

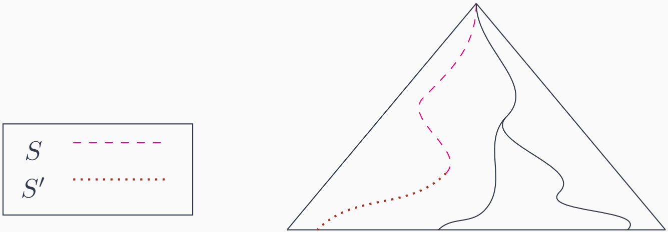


Figure 8: S is an initial segment, S' is a branch, and $S \sqsubseteq S'$.

Mapping

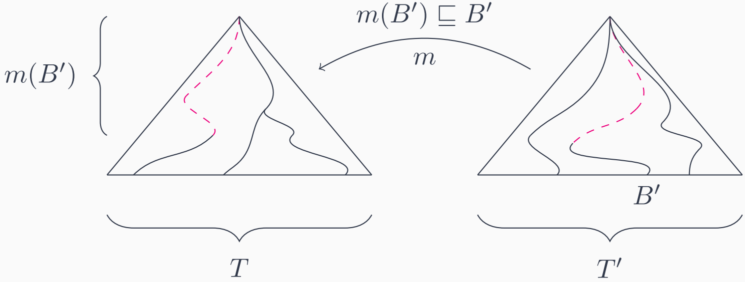


Figure 9: The branch B' is mapped to the initial segment $m(B')$, which means B' contains at least all the formulas of $m(B')$.

Proof

Key Ideas of the Proof

- We consider a proof (T, σ) for a formula F with a reintroduction limit l
- We consider the proof (T', σ') generated by Goéland with the same limit
- We build a mapping between T and T' and show that every branch is T is going to have at least all the formulas than the equivalent one in T

Critical Points

- The agreement mechanism terminates
- A "good" substitution cannot be forbidden