MATH 131 Homework 6
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1. **Prove** $\forall x \in \mathbb{R} : x \in A = \left(\forall k \in \mathbb{N} : A_k := \{n \in \mathbb{N} : |a_n - x| < \frac{1}{k+1}\} \textbf{ is unbounded}\right)$

   Fix $x \in A$. Then since x is a subsequential limit, $\forall k \exists N \forall n : n > N \implies |a_n - x| < \frac{1}{k+1}$.

   Assume A_k was bounded so there was a bound M.

   Then fix $k = M$ so that $\exists N \forall n : n > N \implies |a_n - x| < \frac{1}{M+1}$.

   But then we get $\forall n > N : |a_n - x| > M + 1 > M$, which is a contradiction, since we said the bound was M.

2. **Prove** $b, c \in A$.

   We know from Theorem 11.7 in the book that There exists a monotonic subsequence whose limit is $\limsup s_n$ and $\liminf s_n$ respetively, so $b, c \in A$.

   Assuming that $\lim(a_{n+1} - a_n) = 0$ **prove that** $A = [b, c]$.

   From above, we know that b, c are the min/max of the interval, and are included.

   Take x, y such that $b < x < y < c$. Then $A = \{n : a_n < x\}$ and $B = \{n : a_n > y\}$ are infinite.

   Assume that $\{n : x < n < y\}$ was finite. Then we can find a sequence $n_k \in A, n_{k+1} \in B$, but this is a contradiction since time implies that $a_{n_{k+1}} - a_{n_k} > 0$, so this must be infinite.

   But since all three intervals are infinite A is the whole closed interval.

3. **Prove** (s_n) **is Cauchy and hence convergent.**

   (s_n) is Cauchy if $\forall \epsilon > 0, \exists N, \forall m, n > N : |s_m - s_n| < \epsilon$

   WLOG assume $m > n$.

   Then $|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \cdots + s_{n+1} - s_n|$. So by triangle inequality $|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n|$

   But we know $\forall n \in \mathbb{N} : |s_{n+1} - s_n| < 2^{-n}$ so

   $|s_m - s_n| \leq 2^{-m+1} + 2^{-m+2} + \cdots + 2^{-n} = 2^{-m+!}$

   So for any ϵ choose $N =$ so (s_n) is Cauchy and hence convergent.

   Is this true if $|s_{n+1} - s_n| < \frac{1}{n}$

   No, take $(s_n) = \sum_{i=1} n\frac{1}{i}$. Then $|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n}$ but (s_n) is not Cauchy.

4. **Consider the following sequences**

   $$a_n = (-1)^n, b_n = \frac{1}{n}, c_n = n^2, d_n = \frac{6n+4}{7n-3}$$

   Monotone subsequences: $(a_{2n}), (b_n), (c_n), (d_n)$.

   Set of subsequential limits: $\{-1, 1\}, \{0\}, \{\infty\}, \{\frac{6}{7}\}$

   lim sup and lim inf: $1, -1; 0, 0; \infty, \infty; \frac{6}{7}, \frac{6}{7}$

   a_n does not converge, while c_n diverges to ∞. b_n, d_n both converge.

   a_n, b_n, d_n are all bounded, while c_n is unbounded.

5. **Consider the following sequences**

   $$w_n = (-2)^n, x_n = 5^{(-1)^n}, y_n = 1 + (-1)^n, z_n = n\cos\left(\frac{n\pi}{4}\right)$$

   Monotone subsequences: $(w_{2n}), (x_{2n}), (y_{2n}), (z_{4n-2})$.

   Set of subsequential limits: $\{-\infty, \infty\}, \{\frac{1}{5}, 5\}, \{0, 2\}, \{-\infty, 0, \infty\}$

   lim sup and lim inf: $-\infty, \infty; \frac{1}{5}, 5; 0, 2; -\infty, \infty$

   None of these 4 sequences converge

   w_n, z_n are bounded, while x_n, y_n are unbounded.

6. **Prove** $\liminf s_n = -\limsup(-s_n)$

We know from Theorem 10.6 $\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$

But from Exercise 5.6 we know that $\inf\{s_n : n > N\} = -\sup\{-s_n : n > N\}$ so long as S is a nonempty subseto of $\mathbb{R}$, but this is exactly what $\{s_n : n > N\}$ is.

So $\liminf s_n = \lim_{N\to\infty} -\sup\{-s_n : n > N\} = -\limsup(-s_n)$.

7. **Determine which of the following series converge.**

$\sum \frac{n-1}{n^2}$ Note that $n > 10 \implies frac{n-1}{n^2} > \frac{1}{2n}$, a divergent harmonic series, so this sum diverges by the comparison test.

$\sum(-1)^n$ Note that $a_n = (-1)^n$ so $\lim a_n \neq 0 \implies$ divergence.

$\sum \frac{3n}{n^3} = 3\sum \frac{1}{n^2}$ but this is a convergent harmonic series, so this sequence also converges.

$\sum \frac{n^3}{3^n}$. By the Ratio test, $\lim|\frac{(n+1)^3}{3n^3}| = \frac{1}{3} < 1$. so this series converges.

$\sum \frac{n^2}{n!}$. By the Ratio test, $\lim|\frac{1}{n} + \frac{1}{n^2}| = 0 < 1$ so this series converges.

$\sum \frac{1}{n^n}$. By the Root test, $\lim|a_n|^{\frac{1}{n}} = \lim|\frac{1}{n^n}|^{\frac{1}{n}} = \lim \frac{1}{n} = 0 < 1$ so this series converges.

$\sum \frac{n}{2^n}$. By the Ratio test, $\lim|\frac{n+1}{2n}| = \frac{1}{2} < 1$ so this series converges.

8. **Prove that if $\sum|a_n|$ converges and (b_n) is bounded then $\sum a_n b_n$ converges.**

We just need to show that $\sum a_n b_n$ is Cauchy to show that it converges.

Fix ϵ. Then for $\frac{\epsilon}{M} \exists N, \forall m, n : m \geq n > N \implies |\sum_{k=n}^m a_k| < \frac{\epsilon}{M}$ since $\sum|a_n|$ is Cauchy.

Since (b_n) is bounded, $\exists M > 0 : \forall n |b_n| \leq M$.

So $|\sum_{k=n}^m a_k b_k| \leq M \sum_{k=n}^m |a_k|$. But then multiplying by M on both sides of the equality yields $|\sum_{k=n}^m a_k b_k| \leq M \sum_{k=n}^m |a_k| \leq M \frac{\epsilon}{M} = \epsilon$

So $\sum a_n b_n$ is Cauchy and therefore converges.

9. **Show if $\sum a_n$ and $\sum b_n$ are convergent series then $\sum \sqrt{a_n b_n}$ converges.**

Since $a_n, b_n > 0 \implies a_n b_n \leq a_n b_n + (a_n b_n + a_n^2 + b_n^2) = (a_n + b_n)^2$

This means that $\sqrt{a_n b_n} \leq a_n + b_n \implies \sum \sqrt{a_n + b_n} \leq \sum a_n + b_n$.

But $\sum a_n + b_n$ is convergent, as it is the sum of two convergent series, so by the comparison test $\sum \sqrt{a_n b_n}$ converges as well.

10. **Prove there is a subsequence such that $\sum(a_{n_k})$ converges.**

By Theorem 11.7 we know that there exists a monotonic subsequence (a_{n_k}) whose limit is $\liminf a_n = 0$.

$\lim|a_{n_k}| = 0 \implies \exists L_k \in \mathbb{N}, \forall s > L_k : |a_{n_k} - 0| < \frac{1}{L_k^2}$

If we take $k_l > \max(k_{l-1} + 1, L_k) \implies |a_{n_{k_l}}| < \frac{1}{l^2} \wedge k_l > k_{l-1}$.

Then we can use the comparison test to show that $\sum|a_{n_{k_l}}|$ converges, as it is always less than a convergenet series, the second-order harmonic sum.

11. **Predicate Calculus**

Let $m|n = \exists k \in \mathbb{N} : (k)(n) = m$

(a) $\forall n \in \mathbb{N} : [(n|3) \wedge (n|2)] \implies (n|7)$
(b) $\exists n \in \mathbb{N} : (n|6) \wedge \neg(n|4)$
(c) $\forall n \in \mathbb{N} : [(n|6) \wedge (n|5)] \implies (n|20)$
(d) $\exists n \in \mathbb{N} : [(n|3) \wedge (n|2)] \wedge \neg(n|7)$
(e) $\forall n \in \mathbb{N} : (n \nmid 6) \vee (n|4)$
(f) $\exists n \in \mathbb{N} : [(n|6) \wedge (n|5)] \wedge \neg(n|20)$