MATH 131 Homework 6
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1. **Prove** $\forall x \in \mathbb{R} : x \in A = \left(\forall k \in \mathbb{N} : a_k := \{n \in \mathbb{N} : |a_n - x| < \frac{1}{k+1}\}\right)$
2. **Prove** $b, c \in A$.

   We will prove this via contradiction. We know from Theorem 11.8 in the book that $\sup A = \limsup a_n = b$ and likewise $\inf A = \liminf a_n = c$
3. **Prove** (s_n) **is Cauchy and hence convergent.**

   (s_n) is Cauchy if $\forall \epsilon > 0, \exists N, \forall m, n > N : |s_m - s_n| < \epsilon$

   WLOG assume $m > n$.

   Then $|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \cdots + s_{n+1} - s_n|$. So by triangle inequality $|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n|$

   But we know $\forall n \in \mathbb{N} : |s_{n+1} - s_n| < 2^{-n}$ so

   $|s_m - s_n| \leq 2^{-m+1} + 2^{-m+2} + \cdots + 2^{-n} = 2^{-m+!}$

   So for any ϵ choose $N =$ so (s_n) is Cauchy and hence convergent.

   Is this true if $|s_{n+1} - s_n| < \frac{1}{n}$

   No, take $(s_n) = \sum_{i=1} n \frac{1}{i}$. Then $|s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n}$ but (s_n) is not Cauchy.

   Fix $\epsilon > 0$. For any N we pick, $\exists m, n > N$, such that /abs
4. **Consider the following sequences**

$$a_n = (-1)^n, b_n = \frac{1}{n}, c_n = n^2, d_n = \frac{6n+4}{7n-3}$$

   Monotone subsequences: $(a_{2n}), (b_n), (c_n), (d_n)$.

   Set of subsequential limits: $\{-1, 1\}, \{0\}, \{\infty\}, \{\frac{6}{7}\}$

   lim sup and lim inf: $1, -1; 0, 0; \infty, \infty; \frac{6}{7}, \frac{6}{7}$

   a_n does not converge, while c_n diverges to ∞. b_n, d_n both converge.

   a_n, b_n, d_n are all bounded, while c_n is unbounded.
5. **Consider the following sequences**

$$w_n = (-2)^n, x_n = 5^{(-1)^n}, y_n = 1 + (-1)^n, z_n = n \cos\left(\frac{n\pi}{4}\right)$$

   Monotone subsequences: $(w_{2n}), (x_{2n}), (y_{2n}), (z_{4n-2})$.

   Set of subsequential limits: $\{-\infty, \infty\}, \{\frac{1}{5}, 5\}, \{0, 2\}, \{-\infty, 0, \infty\}$

   lim sup and lim inf: $-\infty, \infty; \frac{1}{5}, 5; 0, 2; -\infty, \infty$

   None of these 4 sequences converge

   w_n, z_n are bounded, while x_n, y_n are unbounded.
6. **Prove** $\liminf s_n = -\limsup(-s_n)$

   We know from Theorem 10.6 $\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$

   But from Exercise 5.6 we know that $\inf\{s_n : n > N\} = -\sup\{-s_n : n > N\}$ so long as S is a nonempty subseto of $\mathbb{R}$, but this is exactly what $\{s_n : n > N\}$ is.

   So $\liminf s_n = \lim_{N\to\infty} - \sup\{-s_n : n > N\} = -\limsup(-s_n)$.
7. **Determine which of the following series converge.**

   $\sum \frac{n-1}{n^2}$ Note that $n > 10 \implies frac{n-1}n^2 > \frac{1}{2n}$, a divergent harmonic series, so this sum diverges by the comparison test.

   $\sum(-1)^n$ Note that $a_n = (-1)^n$ so $\lim a_n \neq 0 \implies$ divergence.

$\sum \frac{3n}{n^3} = 3\sum \frac{1}{n^2}$ but this is a convergent harmonic series, so this sequence also converges.

$\sum \frac{n^3}{3^n}$. By the Ratio test, $\lim|\frac{(n+1)^3}{3n^3}| = \frac{1}{3} < 1$. so this series converges.

$\sum \frac{n^2}{n!}$. By the Ratio test, $\lim|\frac{1}{n} + \frac{1}{n^2}| = 0 < 1$ so this series converges.

$\sum \frac{1}{n^n}$. By the Root test, $\lim|a_n|^{\frac{1}{n}} = \lim|\frac{1}{n^n}|^{\frac{1}{n}} = \lim \frac{1}{n} = 0 < 1$ so this series converges.

$\sum \frac{n}{2^n}$. By the Ratio test, $\lim|\frac{n+1}{2n}| = \frac{1}{2} < 1$ so this series converges.

8. **Prove that if $\sum|a_n|$ converges and (b_n) is bounded then $\sum a_n b_n$ converges.**

   We just need to show that $\sum a_n b_n$ is Cauchy to show that it converges.

9. **Predicate Calculus**

   Let $m|n = \exists k \in \mathbb{N} : (k)(n) = m$

   (a) $\forall n \in \mathbb{N} : (n|3) \wedge (n|2) \implies (n|7)$
   (b) $\forall n \in \mathbb{N} : (n|3) \wedge (n|2) \implies (n|7)$
   (c) $\forall n \in \mathbb{N} : (n|6) \wedge (n|5) \implies (n|20)$
   (d) $\forall n \in \mathbb{N} : (n|3) \wedge (n|2) \implies (n|7)$
   (e) $\forall n \in \mathbb{N} : (n|3) \wedge (n|2) \implies (n|7)$