MATH 131 Homework 7
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1. **Find a series which diverges by the Root Test but for which the Ratio Test gives no information.**

   Consider the series $\sum a_n = \sum (2+(-1)^n)^n$. Then the ratio test $|\frac{a_{n+1}}{a_n}| = |\frac{(2+(-1)^{n+1})^{n+1}}{(2+(-1)^n)^n}|$

   If $n = 2k$ is even then $|\frac{a_{n+1}}{a_n}| = |\frac{(2+-1)^{2k+1}}{(3)^{2k}}| = |\frac{1}{3^{2k}}|$ which converges to 0.

   If $n = 2k+1$ is odd then $|\frac{a_{n+1}}{a_n}| = |\frac{(2+1)^{2k+2}}{(2-1)^{2k+1}}| = |3^{2k+2}|$ which diverges to ∞.

   So the set of subsequential limits is $\{0, \infty\}$ and so $\liminf |\frac{a_{n+1}}{a_n}| = 0 < 1 < \limsup |\frac{a_{n+1}}{a_n}| = \infty$ so the Ratio test gives no information.

   The Root test $|a_n|^{\frac{1}{n}} = 2 + (-1)^n$. So if n is even this is 3 and if n is odd this is 1. Thus $\limsup |a_n|^{\frac{1}{n}} = 3 > 1$ so this series diverges.

2. **Prove that $\sum_1^\infty \frac{1}{n^2}$ converges by comparing it to $\sum_1^\infty \frac{1}{n(n+1)}$**

   Note that $a_n < b_n$ and $b_n = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ but then $\lim b_n = \lim(\frac{1}{n} - \frac{1}{n+1}) = \lim \frac{1}{n} - \lim \frac{1}{n+1} = 0$ so $\{b_n\}$ is Cauchy and thus $\sum b_n$ converges.

   But then by the Comparison Test $n(n+1) = n^2 + n \geq n \implies |a_n| < b_n \implies \sum a_n \leq \sum b_n$ so $\sum a_n$ converges as well.

3. **Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.**

   Take the series $a_n = \frac{1}{n}$. This is the first harmonic, which diverges, but $\sum a_n^2 = \sum \frac{1}{n^2}$ is the second harmonic which converges.

   Observe that if $\sum a_n$ is a convergent series of nonnegative terms then $\sum a_n^2$ converges.

   We know that for $\sum a_n$ to be convergent, $\lim a_n = 0$ so for $\epsilon = 1 \exists N \forall n > N : |a_n| < 1 \implies \forall n > N : |a_n^2| < a_n$. But then via the comparison test $\sum a_n^2$ converges as well.

   Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

   Consider the series $a_n = \frac{(-1)^n}{\sqrt{n}}$. This series converges due to the Alternating Series Theorem, as $\lim \frac{1}{\sqrt{n}} = 0$. But $\sum a_n^2 = \sum \frac{1}{n}$ which is the first harmonic series and diverges.

4. **Prove if (a_n) is a decreasing sequence of real numbers and $\sum a_n$ converges then $\lim na_n = 0$.**

   Note that $a_n \geq 0$. Assume $\exists i : a_i < 0 \implies \forall x > i : a_x < 0 \implies \lim a_n < 0$ which is a contradiction as $\sum a_n$ converges.

   Since $\sum a_n$ converges it must be Cauchy so $\forall \epsilon > 0 \exists m \in \mathbb{N} : \forall n > m : |\sum_{k=m}^n a_k| < \epsilon$.

   Fix $\epsilon > 0$ then we know $\exists x \in \mathbb{N} \forall n > x : |\sum_{k=x}^n a_k| = a_x + \ldots + a_n < \frac{\epsilon}{4}$ since $\sum a_n$ converges. Similarily we know $\exists y \in \mathbb{N} \forall n > y : |\sum_{k=x}^n a_k| = a_x + \ldots + a_n < \frac{\epsilon}{4x}$ since $\sum a_n$ converges.

   Then take $N = \max(x, y)$. Then $na_n \leq ma_m + \sum_{k=m}^n a_k \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon$.

   Use a to prove $\sum \frac{1}{n}$ diverges.

   Assume $\sum \frac{1}{n}$ converges. Then a) implies that $\lim n\frac{1}{n} = 0$, but this is false. Therefore $\sum \frac{1}{n}$ must diverge.

5. **Write the following repeating decimals as rationals.**

   (a) $.2 = \frac{1}{5}$
   (b) $.0\bar{2} = \frac{.2}{9} = \frac{1}{45}$
   (c) $.\bar{02} = \frac{2}{99}$
   (d) $3.\bar{14} = 3 + \frac{14}{99} = \frac{311}{99}$
   (e) $.\bar{10} = \frac{10}{99} = \frac{10}{90}$
   (f) $.14\bar{92} = 1 + \frac{142}{999} = \frac{1141}{999}$

6. **Prove $f(x) = \sqrt{x}$ is continuous inside it's domain $Dom(f) = \{x \in \mathbb{R} : x \geq 0\}$**

   To prove continuity we will show $\forall x_0 \forall \epsilon > 0 \exists \delta > 0 : |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.
   Fix $x_0 \in Dom(f)$ and $\epsilon > 0$. Take $\delta = \epsilon\sqrt{x_0}$.
   Then $|x - x_0| = |(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})| = |\sqrt{x} - \sqrt{x_0}||\sqrt{x} + \sqrt{x_0}| < \epsilon\sqrt{x}_0 = \delta$.
   But $|\sqrt{x} - \sqrt{x}_0||\sqrt{x}_0| < |\sqrt{x} - \sqrt{x}_0||\sqrt{x}_0| < \epsilon\sqrt{x}_0 \implies |\sqrt{x} - \sqrt{x}_0| < \epsilon$.

7. **Prove that f is an injection with $Ran(f)$ being an interval in $\mathbb{R}$, and so f^{-1} is defined.**

   Supose f was not an injection, so $\exists x, y \in Dom(f) : f(x) = f(y) \wedge x \neq y$. Then either $x < y$ or $y < x$.

   WLOG consider $x < y$. Then since f is continuous and strictly increasing $\implies f(x) < f(y)$ which is a contradiction.

   Prove f^{-1} is continuous.

   To prove continuity we will show $\forall x_0 \in Ran(f) \forall \epsilon > 0 \exists \delta > 0 : |x - x_0| < \delta \implies |f^{-1}(x) - f^{-1}(x_0)| < \epsilon$.

   Fix $x_0 \in Ran(f)$ and $\epsilon > 0$. Take $\delta = \epsilon$.

8. **Prove $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0, x_0 = 0$ is discontinuous.**

   Assume $f(x)$ was continuous at $x_0 = 0 : \lim x_n = x_0 \implies \lim f(x_n) = f(x_0)$ But then take $x_n = \frac{1}{n}$. $\lim x_n = 0$, but $\lim f(x) = 1 \neq f(0) = 0$.

   Prove $g(x) = sin(\frac{1}{x})$ for $x \neq 0$ and $g(x) = 0$ for $x = 0, x_0 = 0$ is discontinuous.

   Assume $g(x)$ was continuous at $x_0 = 0 : \lim x_n = x_0 \implies \lim g(x_n) = g(x_0)$

   But then take $x_n = \frac{1}{2\pi n + \frac{\pi}{2}}$. $\lim x_n = \lim \frac{1}{2\pi n + \frac{\pi}{2}} = 0$, but $\lim g(x) = \lim sin(2\pi n + \frac{\pi}{2}) = sin(\frac{\pi}{2}) = 1 \neq g(0) = 0$.

   Prove $sgn(x) = -1$ for $x < 0$, $sgn(x) = 0$ for $x = 0$, $sgn(x) = 1$ for $x > 0, x + 0 = 0$ is discontinuous.

   Assume $sgn(x)$ was continuous at $x_0 = 0 : \lim x_n = x_0 \implies \lim sgn(x_n) = sgn(x_0)$ But then take $x_n = \frac{1}{n}$. $\lim x_n = 0$, but $\lim sgn(x) = 1 \neq sgn(0) = 0$.

9. **Prove if $\forall r \in \mathbb{R} : f(r) = 0 \implies \forall x \in (a, b) : f(r) = 0$.**

   Take $x \in (a, b)$. If $x \ in \mathbb{R}$, then $f(r) =$ and we are done. If $x \in \mathbb{I}$ then by denseness we know that exists a sequence of rational numbers that converges to x. But f is continuous so $lim r_n = x_0 \implies \lim f(r_n) = \lim f(x_0) = 0$.

   Let f, g be continuous real valued functions on (a, b) such that $f(r) = g(r) \forall r \in \mathbb{R}$, prove $f(x) = g(x) \forall x$.

   Let $f - g = f(x) - g(x)$. Then since $\forall r \in \mathbb{R} \cap (a, b) : (f - g)(x) = f(r) - g(r) = 0$ by a) $\forall x \in (a, b) : (f - g)(x) = 0 \implies f(x) - g(x) = 0 \implies f(x) = g(x)$.

10. **Prove $\lim(a_n + b_n)$ exists and equals $A + B$**

    Fix ϵ. Then for $\epsilon_1 = \frac{\epsilon}{3} \exists N \forall n > N : |a_n - A| < \frac{epsilon}{3}$. And also for $\epsilon_2 = \frac{\epsilon}{3} \exists N \forall n > N : |b_n - L| < \frac{\epsilon}{3}$.

    But then by triangle inequality $|a_n + b_n - A - B| \leq |a_n - A| + |b_n - B| \leq \frac{2\epsilon}{3}$. So then $\lim(a_n + b_n) = A + B$.

    Prove $\lim c(a_n)$ exists and equals cA

    Note that $c(a_n) = \sum_{i=1}^{c} a_n$. But then by the sum rule above $\lim c(a_n) = \sum_{i=1}^{c} \lim a_n = cA$.

    Prove $\lim(a_n b_n)$ exists and equals AB

    Note $(a_n - A)(b_n - B) = a_n b_n - A(b_n) - B(a_n) + AB$. Solving for $a_n b_n = (a_n - A)(b_n - B) + A(b_n) + B(a_n) - AB$.

    Taking the sum rule yields $\lim a_n(b_n) = \lim(a_n - A)(b_n - B) + \lim A(b_n) + \lim B(a_n) + \lim -AB$

    But then we can apply the a to get that $\lim(a_n - A) = \lim a_n - \lim A = A - A = 0 \wedge \lim(b_n - B) = \lim b_n - \lim B = B - B = 0$.

    so $\lim a_n(b_n) = 0 + AB + BA - AB = AB$.

    Prove $\lim \frac{a_n}{b_n} = \frac{A}{B}$, assuming $b_n, B \neq 0$.

    First we'll prove $\lim \frac{1}{b_n} = \frac{1}{B}$ and then use the product rule. Fix $\epsilon > 0$. Take $\epsilon_1 = \frac{|B|}{2}$ then $\exists N_1 \forall n > N_1 : |b_n - B| < \frac{|B|}{2}$.

    But then $|B| = |B - b_n + b_n| \leq |B - b_n| + |b_n| \leq \frac{|B|}{2} + |b_n| \implies \frac{1}{b_n} < \frac{2}{|B|}$.

    Then take $\epsilon_2 = \frac{|B|^2}{2}\epsilon$. $\exists N_2 \forall n > N_2 : |b_n - B| < \frac{|B|^2}{2}\epsilon$.

    Then take $N = \min(N_1, N_2), \forall n > N : |\frac{1}{b_n} - \frac{1}{B}| = |\frac{B - b_n}{B(b_n)}| = \frac{1}{|B|}\frac{1}{|b_n|}|B - b_n| < \frac{1}{|B|}\frac{2}{|B|}\frac{|B|^2}{2}\epsilon < \epsilon$.