MATH 131 Homework 2
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1. Prove that addition is associative.

   We need to show $\forall a, b, c \in \mathbb{N} : (a + b) + c = a + (b + c)$. To do this, we fix a, b and do induction on c. when $c = 0$ we get
   $$(a + b) + 0 = a + (b + 0) = a + b = a + (b) = a$$
   Assume that it is true for 0, then
   $$(a + b) + c = a + (b + c)$$
   $$(a + b) + S(c) = S((a + b) + c) = S(a + (b + c)) = a + S(b + c) = a + (b + S(c))$$
   so it holds by induction

2. Prove that multiplication is commutative.

3. Show $\sqrt[3]{5 - \sqrt{3}}$ is not rational.

   Assume that $\sqrt[3]{5 - \sqrt{3}} \in \mathbb{Q}$. Then, $\left(\sqrt[3]{5 - \sqrt{3}}\right)^3 \in \mathbb{Q}$, since the rationals are closed under multiplication. This means that $5 - \sqrt{3} \in \mathbb{Q}$. But since $\mathbb{Q}$ is closed under addition and multiplication $-1(-5 + 5 - \sqrt{3}) \in \mathbb{Q} \implies \sqrt{3} \in \mathbb{Q}$ which is a contradiction. Therefore $\sqrt[3]{5 - \sqrt{3}} \notin \mathbb{Q}$.

4. Prove (v) $0 < 1$ and (vii) $\forall a, b, c \in \mathbb{R}\, 0 < a < b \implies 0 < b^{-1} < a^{-1}$

   To prove $0 < 1$ we will use (iv). Let $a = 1$, then by (iv) $0 < 1^2 \implies 0 < 1$

   To prove (vii), notice by (vi) we get $0 < b^{-1} \wedge 0 < a^{-1}$, so we just need to show that $b^{-1} < a^{-1}$.

   By (iii) we get $0 < a^{-1}b^{-1}$ and we can use (i) to get $-a^{-1}b^{-1} < -0$ We can apply (ii) with $c = -a^{-1}b^{-1}$ to ge $b(c) \leq ac$ so $-a^{-1} \leq -b{-}1$. But then we can apply (i) again to get $b^{-1} \leq a^{-1}$

5. Prove $|a+b+c| \leq |a|+|b|+|c| \forall a, b, c \in \mathbb{R}$ We showed that $a+b+c = (a+b)+c \implies |a+b+c| = |(a+b)+c|$ By triangle inequality we get $|(a + b) + c| \leq |a + b| + |c|$ so
   $$|a + b + c| \leq |a + b| + |c|$$
   Again by triangle inequality, $|a + b| \leq |a| + |b|$ so
   $$|a + b + c| \leq |a + b| + |c| \leq |a| + |b| + |c|$$
   TODO: part b

6. Prove $\inf S \leq \sup S$ b) S is just one element

7. Let $S = \{1\}$ and $T = \{1, 2\}$ then $S \cap T = \{1\} \neq \emptyset$

   Let $S = \{r \in \mathbb{Q} : r^2 < 7\}$ and $T = \{r \in \mathbb{Q} : r^2 > 7\}$ then $S \cap T = \emptyset$ and $\sup S = \inf T = \sqrt{7}$.

8. First we show $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subset I$

   Let $x \in \{r + \sqrt{2} : r \in \mathbb{Q}\}$ then x must be irrational, since the sum of a rational r and an irrational i is irrational $i + r$.

   Suppose that it were rational, then $-r + (r + i) = i$ is a rational, but this is a contradiction.

   now for any $a < b$ we can find a $r \in \mathbb{Q} : a - \sqrt{2} < r < b - \sqrt{2}$, so if we take $r + \sqrt{2}$ we get $a < r + \sqrt{2} < b$ and $r + \sqrt{2} \in I$.

9. To show $\sup(A + B) = \sup A + \sup B$, we will first show