MATH 131 Homework 10
Jesse Cai
304634445

1. **Calculate the upper and lower Darboux integrals for $f$ on $[0, b]$**

   Let a partition $\pi$ be defined so $t_k = \frac{kb}{n}$. Then $\sup\{f(x) : x \in [t_{k-1}, t_k]\} = t_k$ and $\inf\{f(x) : x \in [t_{k-1}, t_k]\} = 0$ due to the denseness of $\mathbb{R}$ on $\mathbb{Q}$.

   $$U(f,\pi) = \sum_{k=1}^{n} t_k(t_k - t_{k-1}) = \frac{kb}{n}\frac{b}{n} = \left(\frac{b}{n}\right)^2 \sum_{k=1}^{n} k = \left(\frac{b}{n}\right)^2 \frac{n(n+1)}{2}$$

   $$U(f) = \inf\{U(f,\pi) : \pi \text{ partition of } [0,b]\} = \lim_{n\to\infty} \left(\frac{b}{n}\right)^2 \frac{n(n+1)}{2} = \frac{b^2}{2}$$

   .

   $$L(f,\pi) = \sum_{k=1}^{n} \inf\{f(x) : x \in [t_{k-1}, t_k]\} = \sum_{k=1}^{n} 0(t_k - t_{k-1}) = 0$$

   $$L(f) = \sup\{L(f,\pi) : \pi \text{ partition of } [0,b]\} = 0$$

   **Is $f$ integrable on $[0, b]$?**

   $U(f) \neq L(f)$ so $f$ is not integrable over $[0,b]$, except when $b = 0$.

2. **Show that the associated Reimann sums converge to the upper integral**

   Plugging, we get that the upper Darboux integral is $\frac{1}{2}$.

   The Reimann sum is given by

   $$\sum_{k=1}^{n} f\left(\frac{k-1}{n}\right)(t_k - t_{k-1}) = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)\left(\frac{1}{n}\right) = \left(\frac{1}{n^2}\right)\sum_{k=1}^{n}(k-1) = \left(\frac{1}{n^2}\right)\sum_{i=0}^{n-1} i^2 = \left(\frac{1}{n^2}\right)\left(\frac{(n-1)(n)}{2}\right) = \frac{n^2-n}{2n^2}$$

   . but then $\lim_{n\to\infty} \frac{n^2-n}{2n^2} = \frac{1}{2}$. So the Reinman sum converges to the uppder Darboux integral.

3. **Prove that if $f$ and $g$ are integrable then so is $f + g$.**

   Take $\pi$ to be some partitoin of $[a, b]$.

   Note that $(f + g)(x) = f(x) + g(x)$ and that $\sup(A + B) \leq \sup(A) + \sup(B)$ and $\inf(A + B) \geq \inf(A) + \inf(B)$. Then

   $$U(f+g,\pi) = \sum_{k=1}^{n} \sup\{f(x) + g(x) : x \in [t_{k-1}, t_k]\}(t_k - t_{k-1})$$

   $$\leq \sum_{k=1}^{n} \sup\{f(x) : x \in [t_{k-1}, t_k]\}(t_k - t_{k-1}) + \sum_{k=1}^{n} \sup\{g(x) : x \in [t_{k-1}, t_k]\}(t_k - t_{k-1})$$

   $$\leq U(f,\pi) + U(g,\pi)$$

   WLOG this also holds for inf and thus the lower Darboux sum.

   Since $U(f + g, \pi) \leq U(f, \pi) + U(g, \pi)$ and $L(f + g, \pi) \geq L(f, \pi) + L(g, \pi)$

   $U(f + g) = U(f) + U(g) = L(f) + L(g) = L(f + g) \implies f + g$ is integrable and is equal to the sum of the twon integrals respectively.

4. **Show** $U(f^2, \pi) - L(f^2, \pi) \leq 2B[U(f, \pi) - L(f, \pi)]$

   Note that $\forall x, y \in [a, b] : f(x)^2 - f(y)^2 = (f(x) - f(y))(f(x) + f(y)) \leq (f(x) - f(y))(|f(x)| + |f(y)|) \leq 2B(f(x) - f(y)$.

   Fix some partition $\pi$ of $[a, b]$.

   Then for some segment $[t_{k-1}, t_k]$

   $$\sup_{x,y\in S} [f(x)^2 - f(y)^2] \leq 2B \sup_{x,y\in S} [f(x) - f(y)]$$

$$\sup_{x \in S} f(x)^2 - \inf_{y \in S} f(y)^2 \leq 2B \sup_{x \in S} f(x) - \inf_{y \in S} f(y)$$

So for any segment $M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \leq 2B[M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])]$

$U(f^2, \pi) - L(f^2, \pi) \leq 2B[U(f, \pi) - L(f, \pi)]$

**Show that $f^2$ is integrable if $f$ is.**

Fix $\epsilon$. Then take $\epsilon' = \frac{\epsilon}{2B}$. Then since $f$ is integrable $\exists \delta > 0 \forall \pi$ partition of $[a, b] : mesh(\Pi) < \delta \implies U(f, \pi) - L(f, \pi) < \epsilon'$.

But then $U(f^2, \pi) - L(f^2, \pi) \leq 2B(U(f, \pi) - L(f, \pi)) \implies U(f^2, \pi) - L(f^2, \pi) \leq \epsilon$. So $f^2$ is integrable if $f$ is.