MATH 131 Homework 5
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1. **Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{(n+1)^2}](t_n)$**

   (a) t_n is bounded from below and decreasing so the limit must exist
   Proof via induction: Let $P(n) = (t_n \leq 1) \wedge (t_{n+1} < t_n)$
   Base Case: Consider $P(1) = (t_1 < 1) \wedge (t_2 < t_1)$. Clearly $1 \leq 1$, and $t_2 = \ \leq 1 = t_n$ So $P(1)$ is TRUE.
   Inductive Step: Assume $P(n)$, so $(t_n \leq 1) \wedge (t_{n+1} < t_n)$
   Since $n \in \mathbb{N} \implies n \geq 0 \implies (n+1)^2 > 1 \implies \frac{1}{(n+1)^2} < 1 \implies 1 - \frac{1}{(n+1)^2} < 1$
   Then since $[1 - \frac{1}{(n+1)^2}] < 1$ and $t_n \leq 1 \implies t_{n+1} \leq 1$.
   But also $[1 - \frac{1}{(n+1)^2}] < 1 \implies t_n[1 - \frac{1}{(n+1)^2}] = t_{n+1} < 1(t_n) = t_n$ so $P(n) \implies P(n+1)$.

   (b) I think $\lim t_n$ is $\frac{1}{2}$

   (c) Let $P(n) = t_n = \frac{n+1}{2n}$
   Base Case: Consider $P(1)$, then $t_1 = 1 = \frac{1+1}{2(1)}$ is TRUE by inspection.
   Inductive Step: Assume $P(n)$, so $t_n = \frac{n+1}{2n}$. Multiplying by $[1 - \frac{1}{(n+1)^2}]$ yields

   $$\left[1 - \frac{1}{(n+1)^2}\right] t_n = t_{n+1} = \frac{n+1}{2n}\left[1 - \frac{1}{(n+1)^2}\right] = \frac{(n+1)^3}{2n(n+1)^2} - \frac{n+1}{2n(n+1)^2} = \frac{(n+1)(n)(n+2)}{2n(n+1)^2} = \frac{n+2}{2(n+1)}$$

   so $P(n) \implies P(n+1)$.

   (d) After proving this, I think $\lim t_n$ is $\frac{1}{2}$

2. **Prove that $\lim \frac{a_{n+1}}{a_n}$ exists and identify it's value**

   $$a_0 = 1 \wedge a_1 = a \wedge (\forall n \in \mathbb{N} : a_{n+2} = 2a_n + a_{n+1})$$

   Let $x = \frac{a_{n+1}}{a_n}$ then $\forall n \in \mathbb{N} : \frac{a_{n+2}}{a_n+1} = \frac{2a_n}{a_{n+1}} + 1$
   Taking the limit of both sides yields

   $$\lim \frac{a_{n+2}}{a_{n+1}} = \lim \left(2\frac{a_n}{a_{n+1}} + 1\right) = 2 \lim \frac{a_n}{a_{n+1}} + 1$$

   So $L = \frac{2}{L} + 1$ and solving for L we get $L = 2, -1$, but L cannot be negative because this sequence is strictly increasing and starts greater than 0, so $L = \lim \frac{a_{n+1}}{a_n} = 2$.

3. **Show every subsequence of a subsequence is a subsequence of the original sequence.**

   Let s_n be a sequence and t_k be a subsequence, and u_j be a subsequence of t_k. Then by definition of subsequence, $t = s(\sigma(k))$ and $u = t(\gamma(j))$. We can then define a new function, $\sigma(\gamma(j))$, which is a mapping from $\mathbb{N}$ to s. But by definition, this means that u_j is a subsequence of s_n, as $u = s(\sigma(\gamma(j)))$.

   So a subsequence of a subsequence is a subsequence of the original sequence.

4. **Find the set S of subsequential limits.**

   Note that each column is a monotone sequence, which is just $\frac{1}{i}$ repeated infinitely, so for each column, we can define a subsequence whose limit is $\frac{1}{n}$.

   But also along each row, we can define a subsequence $(s_n) = \frac{1}{n}$ which converges to 0.

   So S contains $\{\forall n \in \mathbb{N} : k\frac{1}{n}\} \cup \{0\}$

   Determine $\limsup s_n$ and $\liminf s_n$.

   We know from Theorem 8 that $\sup S = \limsup s_n$ and $\inf S = \liminf s_n$. So $\limsup s_n = 1$ and $\liminf s_n = 0$

5. **Let r_n be an enumeration of $\mathbb{Q}$. Show there is a subsequence whose limit is ∞.**

   Take r_n as a sequence of $\mathbb{Q}$. Then $\forall a \in \mathbb{R}\, \exists (r_{n_k})$ that converges to a based on the density of $\mathbb{Q}$ on $\mathbb{R}$.

   Then the subsequential limits of $(r_n) = \mathbb{R} \cup \{-\infty, \infty\} \implies \limsup r_n = \infty$. Then by Theorem 11.7 there exists a monotonic subsequence whose limit is $\limsup r_n = \infty$.

6. **Show** $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$ **for bounded sequences** (s_n) **and** (t_n)

   Let $\sup(s_n + t_n), \sup(s_n), \sup(t_n)$ be z, x, y respectively. Since all the sequences are bounded, $x, y, z \in \mathbb{R}$ and by definition of sup $\forall n \in \mathbb{N}\, s_n < x, t_n < y$.

   But this implies that $\forall n \in \mathbb{N} : s_n + t_n \leq x + y$ so $x + y$ is an upper bound, but this means that $z \leq x + y$, since it is the least upper bound.

   So $\sup(s_n + t_n) \leq \sup s_n + \sup t_n$ and taking the limit on both sides yields

$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$$

7. **Show** $\limsup(s_n + t_n) \leq (\limsup s_n)(\limsup t_n)$ **for bounded sequences of nonegative numbers** (s_n) **and** (t_n)

   Let $\sup(s_n t_n), \sup(s_n), \sup(t_n)$ be z, x, y respectively. Since all the sequences are bounded, $x, y, z \in \mathbb{R}$ and by definition of sup $\forall n \in \mathbb{N}\, s_n < x, t_n < y$.

   But also since $(s_n), (t_n)$ are nonnegative $\implies \sup(s_n) > 0, \sup(t_n) > 0$.

   But this means that that $\forall n \in \mathbb{N} : s_n t_n \leq (x)(y)$ so $(x)(y)$ is an upper bound, but this means that $z \leq x + y$, since it is the least upper bound.

   So $\sup(s_n t_n) \leq (\sup s_n)(\sup t_n)$ and taking the limit on both sides yields

$$\limsup(s_n t_n) \leq (\limsup s_n)(\limsup t_n)$$

8. **Prove** (s_n) **is bounded if and only if** $\limsup|s_n| < +\infty$

   Proof by contradiction, suppose $\limsup|s_n| = +\infty$ and (s_n) is bounded. Then we know that $\limsup|s_n| = +\infty = \lim|s_n|$

   A sequence (s_n) is said to be bounded if $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N} : |s_n| \leq M$. so $(|s_n|)$ is bounded $\implies \exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N} : ||s_n|| \leq M$ but $||s_n|| = |s_n|$. But if $|s_n|$ is bounded by M then so must $\lim|s_n|$, but this is a contradiction, because by definition $\forall x \in \mathbb{R} : +\infty > x$. So therefore (s_n) is bounded if $\limsup|s_n| < +\infty$.

9. **Calculate** $\lim(n!)^{\frac{1}{n}}$

   Let $(s_n) = n!$, then

$$\lim\left|\frac{s_{n+1}}{s_n}\right| = \lim \frac{(n+1)!}{n!} = \lim n + 1 = \infty$$

   .

   The by Corollary 12.3 if $\lim\left|\frac{s_{n+1}}{s_n}\right| = L \implies \lim|s_n|^{\frac{1}{n}} = L$

   So $\lim n!^{\frac{1}{n}} = \infty$.