MATH 131 Homework 9
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1. **Let $f(x) = x^{\frac{1}{3}}$ show $f'(x) = \frac{1}{3}x^{\frac{-2}{3}}$**

   For $a \neq 0$:

$$f'(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^{\frac{1}{3}}-a^{\frac{1}{3}}}{x-a} = \lim_{x\to a}\frac{x^{\frac{1}{3}}-a^{\frac{1}{3}}}{(x^{\frac{1}{3}}-a^{\frac{1}{3}})(x^{\frac{2}{3}}+x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}})} = \lim_{x\to a}\frac{1}{x^{\frac{2}{3}}+x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}} = \frac{1}{3}a^{-\frac{2}{3}}$$

2. **Let $f(x) = x^2$ rational and $f(x) = 0$ irrational.**

   **Prove $f$ is continuous at $x = 0$**

   $f$ is continuous at 0 if $\forall\delta 0 \forall x \in \mathbb{R}\, \exists\epsilon > 0 : |x-0| < \delta \implies |f(x)-f(0)| < \epsilon$

   When $x \in \mathbb{Q} : |f(x)-f(0)| = |x^2-0| < |x^2|$, and $x \notin \mathbb{Q} : |f(x)-f(0)| = |0-0| = 0$ so $|f(x)-f(0)| < |x|^2$.

   Fix $\delta$. Take $\epsilon = \sqrt{\delta}$.

   Then $|x-0| < \delta \implies |x| < \delta \implies |x|^2 < \delta^2$.

   But then $|f(x)-f(0)| < |x|^2 = \delta^2 = \epsilon$, so $f$ is continuous at 0.

   **Prove $f$ is not continuous $\forall x \neq 0$**

   Pick $a \neq 0$.

   WLOG If $a$ is irrational then there exists a sequence of rational numbers that converges to $a$. but then $f(a_n) = a^2 \implies \lim f(a_n) = a^2 \neq 0$. But $f(x) = 0$ by definition so $f$ is discontinuous at $a$.

   **Prove $f$ is differentiable at $x = 0$.**

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)-0}{x-0} = \lim_{x\to 0}\frac{f(x)}{x}$$

   When $x \in \mathbb{Q} : \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0} x = 0$.

   Simlarly when $x \notin \mathbb{Q} : \lim_{x\to 0}\frac{0}{x} = 0$.

   So $\lim_{x\to 0}\frac{f(x)}{x} = 0 = f'(0)$.

3. **Prove if $f$ and $g$ are differentiable and $f(0) = g(0)$ and $\forall x : f'(x) \leq g'(x)$ then $\forall x \geq 0 : f(x) < g(x)$.**

   Consider $h(x) = f(x) - g(x)$. Then since $f'(x) \leq g'(x) \implies h'(x) = f'(x) - g'(x) \leq 0$, and $h(x)$ is differentiable on $\mathbb{R}$.

   Fix $x$. By Mean Value Theorem $\exists y \in (0, x) : h'(x) = \frac{h(x)-h(0)}{x-0}$.

   But then $h(x) - h(0) < h'(x) < 0 \implies h(x) < h(0)$.

   But then $f(x) - g(x) < f(0) - g(0) \implies f(x) \leq g(x)$.

4. **Show $\forall x \in (0, \frac{\pi}{2})x < \tan x$** Let $f(x) = x - \tan x$. Then $f'(x) = 1 - \sec^2 x = 1 - (1 + \tan^2 x) = \tan^2 x$, which is $> 0$ for all $x \in (0, \frac{\pi}{2})$. So $f$ is strictly increasing and note $f(0) = 0 - \tan 0 = 0$ so $\forall x \in (0, \frac{\pi}{2} : f(x) > 0 \implies x < \tan x$

   **Show that $\frac{x}{\sin x}$ is strictly increasing.**

   Let $f(x) = \frac{x}{\sin x}$, then $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$.

   But from above we know $x < \tan x \implies \sin x > x\cos x$. So $\forall x \in (0, \frac{\pi}{2} : f'(x) > 0$, which means $f(x)$ is strictly increasing.

   **Show that $\forall x \in [0, \frac{\pi}{2}] : x \leq \frac{\pi}{2}\sin x$**

   Note that at the endpoints $x = \frac{\pi}{2}\sin x$ and $\frac{x}{\sin x}$ is stricty increasing as shown above.,

$$\forall x, y : 0 < x < y < \frac{\pi}{2} \implies \frac{x}{\sin x} < \frac{y}{\sin y} < \lim_{y\to\frac{\pi}{2}}\frac{y}{\sin y} = \frac{\pi}{2}$$

5. **Let $f$ be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$.Show $(s_n)$ converges**

   Note $f$ is continuous and differentiable on $\mathbb{R}$, so we can use the MVT on the interval $(s_{n-1}, s_n)$ to know that $\exists y \in (s_{n-1}, s_n) : f'(y) = \frac{f(s_n)-f(s_{n-1})}{s_n - s_{n-1}}$.

   But then using the fact that $a < 1$ we get

$$|\frac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}}| \leq \implies |f(s_n) - f(s_{n-1})| \leq a|s_n - s_{n-1}| \implies |s_{n+1} - s_n| \leq a|s_n - s_{n-1}|$$

   But we can apply this recursive defintion to get $|s_{n+1} - s_n| \leq a^n|s(1) - s(0)|$.

   $(s_n)$ converges if $\forall\epsilon > 0 \exists N \forall m > n > N : |s_m - s_n| < \epsilon$

   Fix $s(0) \in \mathbb{R}, \epsilon > 0$. Then we pick. **Prove $f(x) = x$ has a fixed point.**

   From above, we know $(s_n)$ converges, so let $s = \lim s_n$. Since $(s_n)$ converges, so does $(f(s_n))$, also to $s$, as it is the same sequences, just delayed.

   Then because $f$ is differentiable and continuous on $\mathbb{R}$ $\lim s_n = s \implies \lim f(s_n) = f(s) \implies f(s) = s$ and $f$ has a fixed point.

6. **Find $\lim_{x\to 0} \frac{x^3}{\sin x - x}$**

   Since $\lim x^3 = \lim g(x) = 0$ we can apply LHopital's rule.

   $\lim_{x\to 0} \frac{x^3}{\sin x - x} = \lim_{x\to 0} \frac{3x^2}{\cos x - 1} = \frac{0}{0}$

   Note again $\lim x^2 = \lim \cos x - 1 = 0$ so we can apply LHoptial's rule twice again.

$$\lim_{x\to 0} \frac{x^3}{\sin x - x} = \lim_{x\to 0} \frac{3x^2}{\cos x - 1} = \lim_{x\to 0} \frac{6x}{-\sin x} = \lim_{x\to 0} \frac{6}{-\cos x} = -6$$

   **Find $\lim_{x\to 0} \frac{\tan x - x}{x^3}$**

   Again we recursively apply LHopitals rule

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 -1}{3x^2} = \lim_{x\to 0} \frac{2\tan x \sec^2 x}{6x} = \lim_{x\to 0} \frac{2\sec^2 x + 6\tan^2 x \sec^2 x}{6} = \frac{1}{3}$$

   **Find $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x}$**

   We can rewrite this as $\lim_{x\to 0} \frac{x - \sin x}{x \sin x}$, then apply LHopital's rule.

$$\lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = 0$$

7. **Find $\lim_{x\to 0}(1 + 2x)^{\frac{1}{x}}$.**

   We can rewrite $(1 + 2x)^{\frac{1}{x}} = e^{\frac{\log 1+2x}{x}}$ so $\lim_{x\to 0}(1 + 2x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{\log 1+2x}{x}} = e^{\lim_{x\to 0} \frac{\log 1+2x}{x}}$.

   Then by LHopital's rule:

$$\lim_{x\to 0} \frac{\log 1 + 2x}{x} = \lim_{x\to 0} \frac{2}{1 + 2x} = 2 \implies \lim_{x\to 0}(1 + 2x)^{\frac{1}{x}} = e^2$$

   **Find $\lim_{y\to\infty}(1 + 2/y)^y$**

   We can rewrite $(1 + 2/y)^y$ as $e^{\frac{\log(1+2/y)}{y^{(-1)}}}$

   Then by LHopital's rule: $\lim_{y\to\infty} \frac{\log 1+\frac{2}{y}}{\frac{1}{y}} = \frac{2}{1+\frac{2}{y}} = 2$

8. **Let $f$ be differentiable on some interval $(c, \infty)$ and suppose $\lim_{x\to\infty}[f(x) + f'(x)]$. Prove $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$.**

   Note that $f(x) = \frac{f(x)e^x}{e^x}$. Then by LHopital's rule

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{f(x)e^x}{e^x} = \lim_{x\to\infty} \frac{f(x)e^x + f'(x)e^x}{e^x} = \lim_{x\to\infty} \frac{[f(x) + f'(x)]e^x}{e^x} = L$$

   But then if $\lim_{x\to\infty} f(x) = L$ by the sum rule of limits $\lim_{x\to\infty} f(x) + f'(x) = \lim f(x) + \lim f'(x)$ so $\lim f'(x) = 0$.