MATH 131 Homework 4
Jesse Cai
304634445

1. **Prove** $\exists N : n > N \implies s_n > a$

   Let $\lim s_n = L$. Fix $\epsilon = L - a$ ($L > a \implies L - a > 0$) then by limit definition $\exists N \in \mathbb{N} \forall n \in \mathbb{N} : n > N \implies |s_n - L| < \epsilon$.

   So we know that

   $$\exists N \in \mathbb{N} \forall n \in \mathbb{N} : n > N \implies |s_n - L| < (L - a) \implies -(L-a) < s_n - L < L - a \underset{\text{add } L \text{ to both sides}}{\implies} a < s_n$$

2. **Show** $\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 \textbf{ if } |a| \le 1 \\ +\infty \textbf{ if } a > 1 \\ DNE \textbf{ if } a < -1 \end{cases}$

   Consider $\lim|\frac{s_{n+1}}{s_n}|$.

   $$\frac{s_{n+1}}{s_n} = \frac{a^{n+1}}{(n+1)^p}\left(\frac{n^p}{a^n}\right) = \frac{an^p}{(n+1)^p}$$

   Then $\lim|\frac{s_{n+1}}{s_n}| = \lim|\frac{an^p}{(n+1)^p}| = |a| \lim \frac{n^p}{(n+1)^p} = |a|$

   Then if $|a| \le 1$ by 9.12a we get $\lim \frac{a^n}{n^p} = 0$ and likewise if $a > 1$ by 9.12b we get $\lim \frac{a^n}{n^p} = \infty$

   In the case of $a < -1$ we see that $\forall$ even $n \in \mathbb{N} : \lim \frac{a^n}{n^p} = \infty$ but $\forall$ odd $n \in \mathbb{N} : \lim \frac{a^n}{n^p} = -\infty$. But if a limit exists it must be unique, so therefore it does not exist.

3. **Show** $\lim \frac{a^n}{n!} = 0 \forall a \in \mathbb{R}$**.**

   Fix $a \in \mathbb{R}$. Consider $\lim|\frac{s_{n+1}}{s_n}|$.

   $$\frac{s_{n+1}}{s_n} = \frac{a^{n+1}}{(n+1)!}\left(\frac{n!}{a^n}\right) = \frac{a}{n}$$

   Then $\lim|\frac{s_{n+1}}{s_n}| = \lim|\frac{a}{n}| = |a| \lim \frac{1}{n} = |a|0 = 0 < 1$.

   Then by 9.12a we get $\lim \frac{a^n}{n!} = 0$. Since we did not specify a particular a, this holds $\forall a \in \mathbb{R}$.

4. **Prove** (σ_n) **is an increasing sequence.**

   We will prove this using induction. Let $P(n) = \sigma_{n+2} > \sigma_{n+1}$.

   Base Case: Consider $P(0)$. Note that $s_2 > s_1 \implies s_2 + s_1 > s_1 + s_1 \implies \frac{1}{2}(s_1 + s_2) > s_1$. So $P(0)$ is TRUE by inspection.

   Inductive Step: Assume $P(n)$ holds, then consider $P(n+1)$.

   $$P(n) = \frac{1}{n+2}\sum_{i=1}^{n+2} s_i > \frac{1}{n+1}\sum_{i=1}^{n+1} s_i \implies (n+1)\sum_{i=1}^{n+2} s_i > (n+2)\sum_{i=1}^{n+1} s_i \implies (n+1)s_{n+2} > \sum_{i=1}^{n+1} s_i$$

   Since s_{n+2} is positive, we can add it to both sides to get $(n+2)s_{n+2} > \sum_{i=1}^{n+2} s_i$

   But s_n increasing so $(n+2)s_{n+3} > (n+2)s_{n+2} > \sum_{i=1}^{n+2} s_i$

   Then adding $(n+2)\sum_{i=1}^{n+2} s_i$ to both sides yields

   $$(n+2)\sum_{i=1}^{n+3} s_i > (n+3)\sum_{i=1}^{n+2} s_i \implies P(n+1)$$

   So $P(n) \implies P(n+1)$, so $P(n)$ holds $\forall n \in \mathbb{N}$.

5. (a) **Find** s_2, s_3, s_4

   $s_2 = \frac{1}{3}(1+1) = \frac{2}{3}$, $s_3 = \frac{1}{3}(\frac{2}{3}+1) = \frac{5}{9}$, $s_4 = \frac{1}{3}(\frac{5}{9}+1) = \frac{14}{27}$

(b) **Show** $s_n > \frac{1}{2}$
Let $P(n) = s_{n+1} > \frac{1}{2}$.
Base Case: Consider $P(0) = s_1 > \frac{1}{2} = 1 > \frac{1}{2}$. This is TRUE by observation.
Inductive Step: Assume $P(n)$ then

$$s_{n+1} > \frac{1}{2} \implies s_{n+1} + 1 > \frac{1}{2} + 1 \implies \frac{1}{3}(s_{n+1} + 1) > \left(\frac{1}{3}\right)\frac{3}{2} = \frac{1}{2}$$

So $P(n) \implies P(n+1)$, so $P(n)$ holds $\forall n \in \mathbb{N}$.

(c) **Show** (s_n) **is a decreasing sequence.**
We will prove this by contradiction. Assume $\exists n \in \mathbb{N} : s_n \leq s_{n+1} \implies s_n \geq \frac{1}{3}(s_n + 1)$

$$s_n \leq \frac{1}{3}(s_n + 1) \implies 3s_n \geq s_n + 1 \implies s_n \leq \frac{1}{2}$$

But this is a contradiction of (b), so therefore $\forall n \in \mathbb{N} : s_n > s_{n+1} \implies (s_n)$ is decreasing.

(d) **Show** $\lim s_n$ **exists and find it.**
Note that (s_n) is bounded from below by $\frac{1}{2}$ and decreasing $\implies$ limit exists.
To find the limit, note by limit theorems in the book

$$\lim s_n = \lim s_{n+1} = \lim \frac{1}{3}(s_n + 1) = \frac{1}{3}\lim(s_n + 1) = \frac{1}{3}\lim s_n + 1$$

Let $L = \lim s_n$, solving for L we get $L = \frac{1}{3}(L+1) \implies L = \frac{1}{2}$.

6. (a) **Show** $\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$**.**
We know that $\liminf \sigma_n \leq \limsup \sigma_n$, so we just need to show that $\liminf s_n \leq \liminf \sigma_n \wedge \limsup \sigma_n \leq \limsup s_n$. Assuming $M > N$:

$$\sigma_M = \frac{1}{M}(s_1 + \ldots + s_m) = \frac{s_1 + \ldots + s_N}{M} + \frac{s_{N+1} + \ldots + s_M}{M} \leq \frac{s_1 + \ldots + s_N}{M} + \frac{M-N}{M}\sup\{s_n : n > N\}$$

But this holds $\forall M$ so this is an upper bound so since *sup* is the least upper bound:

$$\sup\{\sigma_n : n > M\} \leq \frac{s_1 + \ldots + s_N}{M} + \frac{M-N}{M}\sup\{s_n : n > N\}$$

Taking the limit wrt M on both sides we get $\limsup \sigma_n \leq \sup\{s_n : n > N\}$ and then taking the limit wrt n yields $\limsup \sigma_n \leq \limsup s_n$.
But $\limsup \sigma_n \leq \limsup s_n \implies \limsup -\sigma_n \geq \limsup -s_n \implies \liminf \sigma_n \geq \liminf s_n$. So now we have our full inequality.

(b) **Show that** $\lim s_n = \lim \sigma_n$ **if the limit exists**
If $\lim s_n$ exists then $\limsup s_n = \liminf s_n = \lim s_n$. But then we can see from

$$\lim s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \lim s_n$$

that $\liminf \sigma_n = \limsup \sigma_n = \lim s_n \implies \lim \sigma_n = \lim s_n$

(c) **Give an example where** $\lim \sigma_n$ **exists but** $\lim s_n$ **doesn't.**
Let $s_n = \{1, 2, 1, 2, 1, 2, 1, 2 \ldots\}$ s_n doesn't converge, so the limit does not exist, but $\lim \sigma_n = \frac{3}{2}$.