MATH 131 Homework 9
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1. **Let** $f(x) = x^{\frac{1}{3}}$ **show** $f'(x) = \frac{1}{3}x^{\frac{-2}{3}}$

   For $a \neq 0$:

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^{\frac{1}{3}} - a^{\frac{1}{3}}}{x - a} = \lim_{x \to a} \frac{x^{\frac{1}{3}} - a^{\frac{1}{3}}}{(x^{\frac{1}{3}} - a^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}})} = \lim_{x \to a} \frac{1}{x^{\frac{2}{3}} + x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}} = \frac{1}{3}a^{-\frac{2}{3}}$$

2. **Let** $f(x) = x^2$ **rational and** $f(x) = 0$ **irrational.**

   Prove f **is continuous at** $x = 0$

   f is continuous at 0 if $\forall \delta 0 \forall x \in \mathbb{R}\, \exists \epsilon > 0 : |x - 0| < \delta \implies |f(x) - f(0)| < \epsilon$

   When $x \in \mathbb{Q} : |f(x) - f(0)| = |x^2 - 0| < |x^2|$, and $x \notin \mathbb{Q} : |f(x) - f(0)| = |0 - 0| = 0$ so $|f(x) - f(0)| < |x|^2$.

   Fix δ. Take $\epsilon = \sqrt{\delta}$.

   Then $|x - 0| < \delta \implies |x| < \delta \implies |x|^2 < \delta^2$.

   But then $|f(x) - f(0)| < |x|^2 = \delta^2 = \epsilon$, so f is continuous at 0.

   Prove f **is not continuous** $\forall x \neq 0$

   Pick $a = \sqrt{2}0$.

   Prove f **is differentiable at** $x = 0$**.**

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - 0}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

   When $x \in \mathbb{Q} : \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} x = 0$.

   Simlarly when $x \notin \mathbb{Q} : \lim_{x \to 0} \frac{0}{x} = 0$.

   So $\lim_{x \to 0} \frac{f(x)}{x} = 0 = f'(0)$.

3. **Prove if** f **and** g **are differentiable and** $f(0) = g(0)$ **and** $\forall x : f'(x) \leq g'(x)$ **then** $\forall x \geq 0 : f(x) < g(x)$**.**

   Consider $h(x) = f(x) - g(x)$. Then since $f'(x) \leq g'(x) \implies h'(x) = f'(x) - g'(x) \leq 0$, and $h(x)$ is differentiable on $\mathbb{R}$.

   Fix x. By Mean Value Theorem $\exists y \in (0, x) : h'(x) = \frac{h(x) - h(0)}{x - 0}$.

   But then $h(x) - h(0) < h'(x) < 0 \implies h(x) < h(0)$.

   But then $f(x) - g(x) < f(0) - g(0) \implies f(x) \leq g(x)$.

4. **Show** $\forall x \in (0, \frac{\pi}{2}) x < \tan x$ Let $f(x) = x - \tan x$. Then $f'(x) = 1 - \sec^2 x = 1 - (1 + \tan^2 x) = \tan^2 x$, which is > 0 for all $x \in (0, \frac{\pi}{2})$.

   Then by MVT $x \exists y \in (0, x) : f'(x) = \frac{f(x) - f(0)}{x - 0}$

5. **Placeholder**

6. **Find** $\lim_{x \to \infty} \frac{x - \sin x}{x}$

$$\lim_{x \to \infty} \frac{x - \sin x}{x} = \lim_{x \to \infty} \frac{x}{x} - \frac{\sin x}{x} = \lim_{x \to \infty} 1 - \lim_{x \to \infty} \frac{\sin x}{x} = 1 - 0 = 1$$

7. **Placeholder**

8. **Placeholder**