MATH 131 Homework 2
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1. **Prove that addition is associative.**

   We need to show $\forall a, b, c \in \mathbb{N} : (a + b) + c = a + (b + c)$.

   We will use the definition of addition: $\forall n, m \in \mathbb{N} : n + S(m) = S(n + m)$, as well as P5 of the Peanno axioms. Let a, b be fixed and $P(n) = (a + b) + n = a + (b + n)$.

   Base case: $P(0)$ is TRUE by inspection.

   $$(a + b) + 0 = a + b = a + (b) = a + (b + 0)$$

   Inductive step: Assume $P(n)$ is true, then consider $P(S(n))$

   $$(a + b) + S(n) = S((a + b) + n) =_{by P(n)} S(a + (b + n)) = a + S(b + c) = a + (b + S(n))$$

   so $P(n) \implies P(S(n))$ and by P5 $\forall n \in \mathbb{N}\, P(n)$ holds.

2. **Prove $\forall m, n, r \in \mathbb{N} : m \leq n \implies r \cdot m \leq r \cdot n$.**

   Let n, m be fixed and $P(r) = m \leq n \implies r \cdot m \leq r \cdot$ Base case (r = 0): If $n > m$ the implication holds be default otherwise $0 \cdot m = 0 = 0 \cdot n$ so this is TRUE.

   Inductive step: Assume $P(r)$ is true, then consider $P(S(r))$

   $$m \leq n \implies S(r) \cdot m \leq S(r) \cdot n$$

   $$m \leq n \implies r \cdot m + m \leq r \cdot n + n$$

   Either $m \leq n$ in which case $r \cdot m \leq r \cdot n \implies r \cdot m + m \leq r \cdot n + n$ or $m > n$, in which case the $P(S(r))$ is true by default.

   so $P(r) \implies P(S(r))$ and by P5 $\forall r \in \mathbb{N}\, P(r)$ holds.

3. **Show $\sqrt[3]{5 - \sqrt{3}}$ is not rational.**

   Assume that $\sqrt[3]{5 - \sqrt{3}} \in \mathbb{Q}$. Then, $\left(\sqrt[3]{5 - \sqrt{3}}\right)^3 \in \mathbb{Q}$, since the rationals are closed under multiplication. This means that $5 - \sqrt{3} \in \mathbb{Q}$. But since $\mathbb{Q}$ is closed under subtraction $5 - (5 - \sqrt{3}) \in \mathbb{Q} \implies \sqrt{3} \in \mathbb{Q}$ which is a contradiction of the RRT. Therefore $\sqrt[3]{5 - \sqrt{3}} \notin \mathbb{Q}$.

4. **Prove (v) $0 < 1$ and (vii) $\forall a, b, c \in \mathbb{R}\; 0 < a < b \implies 0 < b^{-1} < a^{-1}$**

   To prove $0 < 1$ we will use (iv). Let $a = 1$, then by (iv) $0 < 1^2 \implies 0 < 1$

   To prove (vii), notice by (vi) we get $0 < b^{-1}$ and $0 < a^{-1}$, so we just need to show that $b^{-1} < a^{-1}$.

   By (iii) we get $0 < a^{-1}b^{-1}$ and we can use (i) to get $-a^{-1}b^{-1} < -0$ We can apply (ii) with $c = -a^{-1}b^{-1}$ to get $bc \leq ac$ so $-a^{-1} \leq -b{-1}$. But then we can apply (i) again to get $b^{-1} \leq a^{-1}$

5. **Prove $\forall a, b, c \in \mathbb{R} |a + b + c| \leq |a| + |b| + |c|$**

   Because addition is commutative $a + b + c = (a + b) + c \implies |a + b + c| = |(a + b) + c|$ By triangle inequality we get $|(a + b) + c| \leq |a + b| + |c|$ so

   $$|a + b + c| \leq |a + b| + |c|$$

   Again by triangle inequality, $|a + b| \leq |a| + |b|$ so

   $$|a + b + c| \leq |a + b| + |c| \leq |a| + |b| + |c|$$

   Prove $|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$

   Let $P(n) = |a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$

   Base case: n=0 is trivial $|emtpy|$ = nothign

   Inductive step: Assume $P(n)$ is TRUE, then consider $P(n + 1)$

   $$|a_1 + a_2 + \ldots + a_n + a_{n+1}| = |(a_1 + a_2 + \ldots + a_n) + a_{n+1}| \leq |a_1 + a_2 + \ldots + a_n| + |a_{n+1}| = |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|$$

   So $P(n) \implies P(n + 1)$ and $\forall n \in \mathbb{N}\, P(n)$ holds.

6. **Prove** $\inf S \leq \sup S$

   By definition $\forall x \in S. \sup S > x > \inf S$ so $\inf S \leq \sup S$.

   What can you say about S if $\inf S = \sup S$.

   S is just one element.

7. **Give an example of S and T where $S \cap T \neq \emptyset$.**

   Let $S = \{1\}$ and $T = \{1, 2\}$ then $S \cap T = \{1\} \neq \emptyset$

   Give an example of S and T where $S \cap T = \emptyset$ and $\sup S = \inf T$.

   Let $S = \{r \in \mathbb{Q} : r^2 < 7\}$ and $T = \{r \in \mathbb{Q} : r^2 > 7\}$ then $S \cap T = \emptyset$ and $\sup S = \inf T = \sqrt{7}$.

8. **Prove** $a < b \implies \exists x \in I : a < x < b$

   First we show $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subset I$. By RRT we know that $\sqrt{2} \notin \mathbb{Q}$.

   Let $x \in \{r + \sqrt{2} : r \in \mathbb{Q}\}$. Suppose that $x = (r + i) \in \mathbb{Q}$, then $(r + i) - r = i \in \mathbb{Q}$, since $\mathbb{Q}$ is closed under subtraction, but this is a contradiction of RRT, so therefore $x \notin \mathbb{Q} \implies x \in I$.

   Now for any $a < b$ we can find a $r \in \mathbb{Q} : a - \sqrt{2} < r < b - \sqrt{2}$, by the denseness of $\mathbb{Q}$ (Theorem 4.7). But then if we take $r + \sqrt{2}$ we get $a < r + \sqrt{2} < b$ and we showed $r + \sqrt{2} \in I$, so there will always exists an irrational number $a < i < b$.

9. **Prove** $\sup(A + B) = \sup A + \sup B$**.**

   To show $\sup(A + B) = \sup A + \sup B$, we will first show $\sup A + \sup B \geq \sup(A + B)$.

   Let $x \in A + B : x = a + b \leq \sup A + \sup B$ since $\forall a \in A.a \leq \sup A$ and likewise $\forall b \in B.b \leq \sup B$. So $\sup(A + B) \leq \sup A + \sup B$.

   Consider the set $A + \sup B = \{\forall a \in A : a + \sup B\}$. Clearly $A + \sup B \subset A + B$ so taking the suprema yields $\sup(A + \sup B) \leq \sup(A + B)$.

   But $\sup(A + \sup B) = \sup A + \sup B \leq \sup(A + B)$. So since

   $$\sup A + \sup B \leq \sup(A + B) \wedge \sup A + \sup B \geq \sup(A + B) \implies \sup A + \sup B = \sup(A + B)$$

   Prove $\inf(A + B) = \inf A + \inf B$**.**

   To show $\inf(A + B) = \inf A + \inf B$, we will first show $\inf A + \inf B \leq \inf(A + B)$.

   Let $x \in A + B : x = a + b \geq \inf A + \inf B$ since $\forall a \in A.a \geq \inf A$ and likewise $\forall b \in B.b \geq \inf B$. So $\inf(A + B) \geq \inf A + \inf B$.

   Consider the set $A + \inf B = \{\forall a \in A : a + \inf B\}$. Clearly $A + \inf B \subset A + B$ so taking the suprema yields $\inf(A + \inf B) \geq \inf(A + B)$.

   But $\inf(A + \inf B) = \inf A + \inf B \geq \inf(A + B)$. So since

   $$\inf A + \inf B \leq \inf(A + B) \wedge \inf A + \inf B \geq \inf(A + B) \implies \inf A + \inf B = \inf(A + B)$$