MATH 131 Homework 3
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1. **Let $a, b \in R$. Show if $a \le b_1$ for every $b_1 > b$, then $a \le b$.**

   Suppose $a > b$. Then by the denseness of $\mathbb{Q}$ (Thm 4.7) $\exists b_1 : a > b_1 > b$ but this is a contradiction, as we said $a \le b_1$ for every $b_1 > b$. Therefore $a \le b$ if $a \le b_1$ for every $b_1 > b$.

2. **Prove that for any $A, B \subset \mathbb{R} : \sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.**

   WLOG Suppose $\sup(A) > \sup(B)$ then $\forall b \in B : \sup(B) > b \implies \forall b \in B : \sup(A) > \sup(B) > b$.

   So $\forall x \in A \cup B : \sup(A) > x \implies \sup(A)$ is an upper bound on $A \cup B$. Now we will prove that $\sup(A)$ is the least upper bound.

   Suppose $\exists x : x$ is a upper bound $\wedge\, x < \sup A$. But then $x < \sup(A) \wedge \forall a \in A : x > a$. But this is a contradiction, as by definition $\sup(A)$ is the least upper bound. So therefore $\sup(A) = \sup(A \cup B)$.

   Note when $\sup(A) = \sup(B)$ either choice satisfies max.

3. **Prove every nonempty set $F \subset P(E)$ admits $\sup F$ and $\inf F$ and that $\sup(F) = \cap F$**

   Let $F \subset P(E)$. Then take $\forall f \in F : f \subset \cup F$ so $\cup F$ is an uppper bound. Suppose $\exists x \in P(E) : (\forall f \in F : f \subset x) \wedge x \subset \cup F$. But then this suggests $\exists a \in x$ s.t. $a \notin \cup F$ but this is a contradiction, as all elements in x are in all sets of $f \implies \in \cup F$. So $\cup F$ is the least upper bound and thus the suprema.

   Likewise $\forall f \in F : \cap F \subset f$ so $\cap F$ is a lower bound. Suppose $\exists x \in P(E) : (\forall f \in F : x \subset f) \wedge \cap F \subset x$. But then this suggests $\exists a \in xa \notin \cup F$ but this is a contradiction, as all elements in x are in all sets of $f \implies \in \cap F$. So $\cap F$ is the least upper bound and thus the suprema.

   Prove that $\lim_{n\to\infty} \inf A_n \subset \lim_{n\to\infty} \sup A_n$

   $$\lim_{n\to\infty} \inf A_n = \bigcup_{i=0}^{\infty} \bigcap_{m=n}^{\infty} A_n$$

   $$\lim_{n\to\infty} \sup A_n = \bigcap_{i=0}^{\infty} \bigcup_{m=n}^{\infty} A_n$$

   Let $a \in \liminf A_n \implies \exists n$ s.t. $\forall m > n : a \in A_m$ but then $a \in \bigcup_{m=n}^{\infty} A_m$. So we just need to show that this holds $\forall n$. Suppose there was a n where this did not hold, so $a \notin \bigcup_{m=n}^{\infty} A_m$. But if it is not in the union it cannot be in the intersection, so we have a contradiction and thus $a \in \limsup A_n$.

4. **Determine $\lim s_n$ where $s_n = \sqrt{n^2+1} - n$**

   Claim: $\lim s_n = 0$

   Proof: Fix $k \in \mathbb{N}$. Then take $n_0 = k + 1$.

   If $n \ge n_0$ then $\left|\sqrt{n^2+1} - n\right| = \left|\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+n)}\right| = \left|\frac{1}{\sqrt{n^2+1}+n}\right|$

   But note that $\sqrt{n^2+1} \ge n$ so $\left|\frac{1}{\sqrt{n^2+1}+n}\right| = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{2n} < \frac{1}{n} < \frac{1}{n_0} = \frac{1}{k+1}$

5. **Find** $\lim \frac{4n+3}{7n-5}$

   Claim: $\lim \frac{4n+3}{7n-5} = \frac{4}{7}$

   Proof: Fix $k \in \mathbb{N}$. Then take $n_0 = \max(5, 1+k)$

   If $n \ge n_0$ then $|\frac{4n+3}{7n-5} - \frac{4}{7}| = \frac{7(4n+3)-4(7n-5)}{7(7n-5)} = \frac{41}{7(7n-5)}$ but since $n \ge 5 \implies \frac{41}{7(7n-5)} < \frac{1}{n} < \frac{1}{1+k}$. So this is indeed a limit.

6. **Find** $\lim \frac{1}{n}\sin(x)$

   Claim: This limit is 0

   Proof: Fix k. Then take $n_0 = 1 + k$.

   If $n \ge n_0$ then $|\frac{1}{n}\sin(x)| = |\frac{1}{n}||\sin(x)| < |\frac{1}{n}| = \frac{1}{n} < \frac{1}{n_0} = \frac{1}{k+1}$

7. **Prove** $\lim s_n = 0$ **iff** $\lim|s_n| = 0$**.**

   By definition we say that $\lim s_n = 0$ if

$$\forall k \in \mathbb{N}\, \exists n_0 \forall n \in \mathbb{N} : n \geq n_0 \implies |s_n - 0| < \frac{1}{k}$$

   But note that $||s_n|| = |s_n|$ so via subsitution

$$\left[\forall k \in \mathbb{N}\, \exists n_0 \forall n \in \mathbb{N} : n \geq n_0 \implies |s_n - 0| < \frac{1}{k}\right] \implies \text{ the limit for } ||s_n|| = 0.$$

   and likewise

$$\left[\forall k \in \mathbb{N}\, \exists n_0 \forall n \in \mathbb{N} : n \geq n_0 \implies ||s_n - 0|| < \frac{1}{k}\right] \implies \text{ the limit for } |s_n| = 0.$$

   Observer for $s_n = (-1)^n$ $\lim|s_n| = 0$ **but** $\lim s_n = 0$ **does not exist.**

   $\lim|s_n| = 1$, as $|(-1)^n|$ is always equal to 1. However, s_n will oscillate between -1 and 1 indefinitely, and as we showed in class, a sequence can only have one value as a limit, so therefore the limit does not exist.

8. **Find** $\lim \sqrt[3]{n^3 + n^2 + 1} - \sqrt[3]{n^3 + 1}$

   Claim: This limit is 0

   Proof: Let $a = \sqrt[3]{n^3 + n^2 + 1}$ and $b = \sqrt[3]{n^3 + 1}$. Fix k. Then take $n_0 = k + 1$.

   If $n \geq n_0$ then $|a - b| = |\frac{(a-b)(a^2+ab+b^2)}{a^2+ab+b^2}| = \frac{n^2}{a^2+ab+b^2}$

   But since $a, b > n \implies ab > n^2, aa > n^2, bb > n^2$:

$$\frac{n^2}{a^2 + ab + b^2} < \frac{n^2}{3n^2} < \frac{n^2}{n^3} = \frac{1}{n} < \frac{1}{n_0} = \frac{1}{k+1}$$

9. **Determine if** $\{a_n\}_{n \in \mathbb{N}}$ **exists.**

$$a_0 := 1 \wedge \left(\forall n \in \mathbb{N} : a_{n+1} = \sqrt{2 + a_n}\right)$$

   Claim: The limit is 2

   Proof: First by induction, we will show that $\{a_n\}$ is increasing. So let $P(n) = a_{n+1} \geq a_n$

   Base case: We can see that $a_1 = \sqrt{3} \geq a_0 = 1$ so this is TRUE by inspection.

   Inductive Step: Assume $P(n)$ and we will consider $P(n + 1)$

   If $a_{n+1} \geq a_n$ then $\sqrt{2 + a_{n+1}} \geq \sqrt{2 + a_n} \implies a_{n+2} > a_{n+1} \implies P(n + 1)$

   Next we will show that $\{a_n\}$ is bounded, again via induction. Let $P(n) = a_n < 3$

   Base Case: $a_0 = 1 < 3$ so $P(0)$ is TRUE. Inductive Step: Assume $P(n)$ the consider $P(n + 1)$ $a_n < 3 \implies \sqrt{2 + a_n} < 3$ since at most on the LHS we can get is $\sqrt{3} \implies a_{n+1} < 3$

   But since $f(x) = \sqrt{2 + x}$ is continuous:

$$L = \lim_{n \to \inf} a_{n+1} = \lim_{n \to \inf} \sqrt{2 + a_n} = \sqrt{2 + \lim_{n \to\ inf} a_n} = \sqrt{2 + L}$$

   Solving this equation we get $L^2 = 2 + L$ so $L = 2$.