MATH 131 Homework 10
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1. **Calculate the upper and lower Darboux integrals for f on $[0, b]$**

   Let a partition π be defined so $t_k = \frac{kb}{n}$. Then $\sup\{f(x) : x \in [t_{k-1}, t_k]\} = t_k$ and $\inf\{f(x) : x \in [t_{k-1}, t_k]\} = 0$ due to the denseness of $\mathbb{R}$ on $\mathbb{Q}$.

$$U(f,\pi) = \sum_{k=1}^{n} t_k(t_k - t_{k-1}) = \frac{kb}{n}\frac{b}{n} = \left(\frac{b}{n}\right)^2 \sum_{k=1}^{n} k = \left(\frac{b}{n}\right)^2 \frac{n(n+1)}{2}$$

$$U(f) = \inf\{U(f,\pi) : \pi \text{ partition of } [0,b]\} = \lim_{n\to\infty} \left(\frac{b}{n}\right)^2 \frac{n(n+1)}{2} = \frac{b^2}{2}$$

   .

$$L(f,\pi) = \sum_{k=1}^{n} \inf\{f(x) : x \in [t_{k-1}, t_k]\} = \sum_{k=1}^{n} 0(t_k - t_{k-1}) = 0$$

$$L(f) = \sup\{L(f,\pi) : \pi \text{ partition of } [0,b]\} = 0$$

   Is f integrable on $[0, b]$?

   $U(f) \neq L(f)$ so f is not integrable over $[0, b]$, except when $b = 0$.

2. **Show that the associated Reimann sums converge to the upper integral**

   Plugging, we get that the upper Darboux integral is $\frac{1}{2}$.

   The Reimann sum is given by

$$\sum_{k=1}^{n} f(\frac{k-1}{n})(t_k - t_{k-1}) = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)\left(\frac{1}{n}\right) = \left(\frac{1}{n^2}\right)\sum_{k=1}^{n}(k-1) = \left(\frac{1}{n^2}\right)\sum_{i=0}^{n-1} i^2 = \left(\frac{1}{n^2}\right)\left(\frac{(n-1)(n)}{2}\right) = \frac{n^2-n}{2n^2}$$

   . but then $\lim_{n\to\infty} \frac{n^2-n}{2n^2} = \frac{1}{2}$. So the Reinman sum converges to the uppder Darboux integral.

3. **Prove that if f and g are integrable then so is $f + g$.**

   Take π to be some partitoin of $[a, b]$.

   Note that $(f + g)(x) = f(x) + g(x)$ and that $\sup(A + B) \leq \sup(A) + \sup(B)$ and $\inf(A + B) \geq \inf(A) + \inf(B)$. Then

$$U(f+g,\pi) = \sum_{k=1}^{n} \sup\{f(x) + g(x) : x \in [t_{k-1}, t_k]\}(t_k - t_{k-1})$$

$$\leq \sum_{k=1}^{n} \sup\{f(x) : x \in [t_{k-1}, t_k]\}(t_k - t_{k-1}) + \sum_{k=1}^{n} \sup\{g(x) : x \in [t_{k-1}, t_k]\}(t_k - t_{k-1})$$

$$\leq U(f,\pi) + U(g,\pi)$$

   WLOG this also holds for inf and thus the lower Darboux sum.

   Since $U(f + g, \pi) \leq U(f, \pi) + U(g, \pi)$ and $L(f + g, \pi) \geq L(f, \pi) + L(g, \pi)$

   $U(f + g) = U(f) + U(g) = L(f) + L(g) = L(f + g) \implies f + g$ is integrable and is equal to the sum of the twon integrals respectively.

4. **Show** $U(f^2, \pi) - L(f^2, \pi) \leq 2B[U(f, \pi) - L(f, \pi)]$

   Note that $\forall x, y \in [a, b] : f(x)^2 - f(y)^2 = (f(x) - f(y))(f(x) + f(y)) \leq (f(x) - f(y))(|f(x)| + |f(y)|) \leq 2B(f(x) - f(y)$.

   Fix some partition π of $[a, b]$. Note that if

   Show that f^2 is integrable if f is.

5.