

- 17.1 Let  $x$  be a feasible solution of the primal solution - then it must satisfy  $Ax = b, x > 0$ . Similarly if  $\lambda$  is a feasible solution of the dual, it must satisfy  $\lambda^T A \leq c^T$ .

Then since  $x > 0$  and  $\lambda^T A \leq c^T \implies \lambda^T b = \lambda^T Ax \leq c^T x$ , which is the weak duality lemma.

- 17.3 (a) The tableau for this problem is given by

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ -2 & -3 & 0 & 0 & 0 \end{bmatrix}$$

Then we pivot about (1, 2) to get

$$\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 2 \\ \frac{3}{2} & 0 & -\frac{1}{2} & 1 & 3 \\ -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 6 \end{bmatrix}$$

Then we pivot about (2, 1) to get

$$\begin{bmatrix} 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 \\ 1 & 0 & -\frac{1}{3} & \frac{2}{3} & 2 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & 7 \end{bmatrix}$$

Here the reduced cost coefficients are all positive, so this is an optimal solution  $x_* = [2, 1]^T$

- (b) The dual of this problem is maximize  $f(\lambda) = 4\lambda_1 + 5\lambda_2$  subject to

$$\lambda_1 + 2\lambda_2 \leq -2$$

$$\lambda_2 + 2\lambda_1 \leq -3$$

We know that  $\lambda_* = c_i - r_i = [-\frac{4}{3}, -\frac{1}{3}]$

- 17.4 The dual of this problem is maximize  $f(\lambda) = 11\lambda_1 + 8\lambda_2 + 3\lambda_3$  subject to

$$5\lambda_1 + 2\lambda_2 + 1\lambda_3 \leq 4$$

$$\lambda_1 + \lambda_2 + 2\lambda_3 \leq 3$$

$$\lambda_1, \lambda_2, \lambda_3 > 0$$

- 17.5 (a) The dual of this problem is maximize  $f(\lambda) = 2\lambda_1 + 7\lambda_2 + 3\lambda_3$  subject to

$$-2\lambda_1 - 1\lambda_2 + \lambda_3 \leq -1$$

$$\lambda_1 + 2\lambda_2 \leq -2$$

$$\lambda_1, \lambda_2, \lambda_3 > 0$$

- (b) We can solve the dual by using  $\lambda_*^T = c_b^T B^{-1}$

$$\text{Here } c_b^T = [-1, -2, 0] \text{ and } B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \text{ So } \lambda_* = [0, -1, -2]^T$$

- 17.6 (a) The dual of this problem is to minimize  $\lambda^T b$  subject to  $-\lambda^T A \leq c^T$

- (b) Yes it does. Note that any feasible point in the dual is optimal if  $b = 0$ . Then  $y$  must be optimal. But then by the duality theorem, the primal also has an optimal solution with the same cost, 0. Note that  $x = 0$  is feasible and satisfies the equal cost, so it is the optimal solution to the primal.

- 17.8 (a) The dual of this problem is to max  $f(\lambda) = \lambda$  such that  $\lambda^T [a_1, \dots, a_n] \leq 1, \lambda > 0$

- (b) The duality theorem states that if the primal has an optimal solution so must the dual, and the cost of each solution should be the same. Note that the optimal solution of the dual is clearly  $\lambda = \frac{1}{a_n}$ , since  $a_n$  is the greatest, so the primal must have the same cost.

The only feasible point that has the same cost is  $[0, \dots, \frac{1}{a_n}]$  so then this must be the optimal solution of the primal.

- (c) Let us start at some nonfeasible basic solution  $[0 \dots a_i \dots 0]$   
 Then the reduced cost coefficients for column  $j$  would be  $1 - \frac{a_j}{a_i}$   
 Note that the most negative cost coefficient would be when  $j = n$ , in which case the algorithm terminates in one step.

- 17.9 (a) The dual of this problem is to minimize  $f(\lambda) = \lambda$  subject to  $\forall i : \lambda \geq c_i$ .  
 (b) This means that the optimal solution  $\lambda_* = c_4$ .  
 (c) By the duality theorem, we know there must be an optimal solution to the primal that shares the same cost  $c_4$ .  
 The only feasible solution that has this property is  $[0, 0, 0, 1, 0, \dots, 0]$ , which must be the optimal solution.

- 17.13 Note that  $\mu > 0$  and  $A^T \lambda + \mu = c \implies A^T \lambda < c$  so we know this is a feasible solution to the dual.  
 We can rewrite this as  $\lambda^T A + \mu^T = c^T$  but then multiplying by  $x$  on both sides yields

$$\lambda^T A x + \mu^T x = c^T x$$

But then since  $Ax = b$  and  $\mu^T x = c^T x$  we get  $\lambda^T b = c^T x$ , but by the duality theorem this means that  $x, \lambda$  are the optimal solutions to the primal/dual.

- 17.15 The dual of this problem is  $\max f(\lambda) = 3 * \lambda_1 + 3 * \lambda_2$  subject to

$$\lambda_1 + 2\lambda_2 \leq 1$$

$$2\lambda_1 + \lambda_2 \leq 1$$

$$\lambda_1, \lambda_2 \geq 0.$$

We know that  $\lambda^T = c_b^T B^{-1} = [1, 1] \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = [\frac{1}{3}, \frac{1}{3}]$

We can see that the duality theorem, holds as  $c^T x = 2 = \lambda^T b$ .

- 17.16 The dual to this problem is to maximize  $\lambda^T 0$  subject to  $\lambda^T 0 \leq c^T$ .

This dual only has a feasible solution if  $c^T \geq 0$ . But note that any feasible solution to the dual is optimal, and by the duality theorem we know that the  $\exists \lambda^* \implies \exists x^*$ .

So there is an optimal solution iff  $c \geq 0$ . But in this case we can see that the vector 0 satisfies  $c^T x = \lambda^T b = 0$  so it is an optimal solution.

- 17.17 Consider the primal linear programming problem  $\max f(x) = 0^T x$  s.t.  $Ax \geq b, x \geq 0$  and it's respective dual  $\min b^T \lambda$  s.t.  $\lambda^T A \leq 0, \lambda \geq 0$

( $\implies$ ) Let  $\lambda = 0$  then  $\lambda$  is feasible since  $0^T A = 0$  so the feasible set of the dual is nonempty. Since for any feasible solution  $\lambda^T A \leq 0 \implies b^T \lambda \leq 0$  any feasible solution is less than 0, so  $\lambda_* = 0$ .

By the duality theorem, the existences of an optimal solution of the dual means that there is also a optimal solution for the primal so  $\exists x \text{ s.t. } Ax = b, x \geq 0$ .

( $\Leftarrow$ ) If  $x$  satisfies these conditions, then it is a feasible solution of the primal, with cost 0. But then if  $A^T \lambda \leq 0$  then it is a feasible solution of the dual and then by weak duality theorem  $b^T \lambda < c^T x = 0$ .

- 17.18 (a) The dual is  $\max f(\lambda) = b^T \lambda$  such that  $\lambda^T A \leq 0$

(b) Let  $\lambda = 0$  then  $\lambda$  is feasible since  $0^T A = 0$  so the feasible set of the dual is nonempty

(c) Since for any feasible solution  $\lambda^T A \leq 0 \implies b^T \lambda \leq 0$  any feasible solution is less than the solution in b), which makes that solution the optimal one.

(d) By the duality theorem, the existences of an feasible solution of the dual means that there is also a feasible solution for the primal so  $\exists x \text{ s.t. } Ax = b, x \geq 0$ .

(e) If  $x$  satisfies these conditions, then it is a feasible solution of the primal, with cost 0. But then if  $A^T \lambda \leq 0$  then it is a feasible solution of the dual and then by weak duality  $b^T \lambda < c^T x = 0$ .

- 17.19 Consider the primal linear programming problem  $\max f(x) = 0^T x$  s.t.  $(-A)x \leq (-b), x \geq 0$  and it's respective dual  $\min b^T \lambda$  s.t.  $\lambda^T (-A) \geq 0, \lambda \geq 0$

( $\implies$ ) If  $\exists x : Ax \geq b$  then it is a feasible solution to the primal, and as such by the Weak Duality Theorem we know that  $c^T x \geq \lambda^T b$  so  $0 \geq -b^T \lambda \implies b^T \lambda \geq 0$  for all feasible solutions of the the dual ( $A^T \lambda = 0, \lambda \geq 0$ ).

( $\Leftarrow$ ) Note 0 is a feasible solution of the dual, and it is optimal, since from assumption we know  $b^T \lambda \geq 0$ . But again by duality theorem  $\Rightarrow \exists x : -Ax \geq -b \Rightarrow \exists x : Ax \leq b$ .

17.20 ( $\Rightarrow$ ) We can use 17.19 here, with  $A = A, b = [-1 \dots -1]$ . Then we know from above that  $\exists Ax \leq [-1 \dots -1] \iff \forall y : A^T y = 0, y \geq 0 \Rightarrow [-1 \dots -1]^T y \leq 0$ .

But since  $y \geq 0$ ,  $\Rightarrow y = 0$ , as otherwise it'd be negative because of  $b$ . So there only  $\exists x : Ax \leq [-1 \dots -1] < 0$  if  $y = 0$ .

( $\Leftarrow$ ) Again by 17.19 we can see that if  $\forall T y = 0, y \geq 0 : y = 0$  then  $\exists Ax < [-1 \dots -1] < 0$ .

17.24 (a) If  $x_0, y_0$  are feasible points of primal/dual then  $f_1(x_0) \geq f_2(y_0)$ .

Note that  $y \geq 0, Ax - b \leq 0 \Rightarrow y_0^T (Ax_0 - b) \leq 0$ , since both points are feasible.

$$f_1(x_0) \geq f_1(x_0) + y_0^T (Ax_0 - b) = \frac{1}{2} x_0^T x_0 + y_0^T Ax_0 - y^T b$$

$$f_2(y_0) = -\frac{1}{2} y_0^T (AA^T) y_0 - b^T y_0$$

$$f_1(x_0) - f_2(y_0) \geq \frac{1}{2} x_0^T x_0 + y_0^T Ax_0 - y^T b + \frac{1}{2} y_0^T (AA^T) y_0 + b^T y_0$$

$$f_1(x_0) - f_2(y_0) \geq \frac{1}{2} (x_0^T x_0 + 2y_0^T Ax_0 y_0^T (AA^T) y_0) = \frac{1}{2} \|y_0^T A + x_0\|^2 > 0$$

so  $f_1(x_0) - f_2(y_0) \geq 0$

(b) Let  $x_0, y_0$  be feasible points in the primal/dual respectively such that  $f_1(x_0) = f_2(y_0)$ . WLOG for any feasible point  $s$  in the primal, by a) we know  $f_1(s) \geq f_2(y_0) = f_1(x_0)$ . so  $x_0$  is the optimal solution.

20.2 (a) We can express this as the following matrix

$$\begin{bmatrix} 2 & 2 & 0 & 1 & 4 \\ 2 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 2 & 0 & 0 & 0 \\ 4 & 0 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ -6 \\ 3 \\ 6 \end{bmatrix}$$

Using the formulas given in Section 20.6 we get

$$[x_* \mid \lambda_*] = [\frac{16}{5}, -\frac{1}{10}, -\frac{34}{23}, -\frac{27}{5}, -\frac{6}{5}],$$

Note  $L = F = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  and the  $T(x_*) = \text{span}([-10, 5, 8])$

Let  $y \in T(x_*)$  then:

$$y^T L y = \alpha^2 [-10, 5, 6] \begin{bmatrix} 2 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -10 \\ 5 \\ 6 \end{bmatrix} = 1200\alpha^2 \geq 0$$

and the SOSC is satisfied so  $x_*$  is a min.

(b)  $\nabla f(x) = [4, 2x_2]$  and  $\nabla h(x) = [2x_1, 2x_2]$  so by  $0 = \nabla f(x) - \lambda \nabla h(x)$  we get

$$4 + 2\lambda x_1 = 0, 2x_2 + \lambda x_2 = 0, x_1^2 + x_2^2 = 9$$

We have four points that satisfy the condition,  $\lambda = \pm \frac{2}{3}, x = [\mp 3, 0]$  and  $\lambda = -1, x = [2, \mp \sqrt{5}]$

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + \lambda 2I$$

The first two are local minimizers while the last two are local maximizers.

(c)  $\nabla f(x) = [x_2, x_1]$  and  $\nabla h(x) = [2x_1, 8x_2]$  so by  $0 = \nabla f(x) - \lambda \nabla h(x)$  we get

$$x_2 + 2\lambda x_1 = 0, x_1 + 8\lambda x_2 = 0, x_1^2 + 4x_2^2 - 1 = 0$$

We have four points that satisfy the condition,  $\lambda = \frac{1}{4}, x = [\pm 1, \mp 1]/\sqrt{2}$  and  $\lambda = -\frac{1}{4}, x = \pm[1, 1]/\sqrt{2}$

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$$

The first two are local maximizers while the last two are local minimizers.

20.5 (a)  $2(x - [1, \sqrt{3}]^T) + 2\lambda_* x = 0, \|x\| = 9$ . so  $\lambda_* = \pm \frac{2}{3}$ . So either  $x_* = \pm[1, \sqrt{3}]^T$ .

(b) We get  $L = (1 + \lambda_*)I$ . For  $1 + \lambda_2 = -\frac{2}{3}$ , which means that this point is not a local minimizer, as the SONC is violated. However, the first point satisfies the SOSC, as  $1 + \lambda_1 = \frac{2}{3} > 0$ .

20.6 (a) The problem is to min  $f(x) = 2ab + 2bc + 2ca$  such that  $abc = V$ . So then we get  $\nabla f(x) = 2[b + c, a + c, a + b]$  and  $\nabla h(x) = [bc, ac, ba]$ .

From  $\nabla f - \lambda \nabla h = 0$  we get

$$2(b + c) - \lambda bc = 0$$

$$2(a + c) - \lambda ac = 0$$

$$2(b + a) - \lambda ab = 0$$

$$abc = V$$

(b) Note that  $\nabla h = 0$  means one of  $a, b, c$  are 0, but this means the volume is 0 and it is no longer a feasible point. Hence all points are regular ( $\nabla h(x) \neq 0$ )

(c) Solving the system of equations we get  $a = b = c = \sqrt[3]{V}$  with  $\lambda = -4\sqrt[3]{V}$ .

(d)

$$L = -2 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

so  $\forall y : y^T L y = -2(y_1(y_2 + y_3) + y_2(y_1 + y_3) + y_3(y_2 + y_1)) = 2(y_1^2 + y_2^2 + y_3^2) > 0$  so the SOSC is satisfied.

20.8 (a) We get  $\nabla f(x) = [2, 3], \nabla h(x) = [x_2, x_1]$ . Solving the Lagrange condition we get  $[2, 3] - \lambda_*[x_2, x_1] = [0, 0]$ . So  $x_2 = \frac{2}{\lambda}, x_1 = \frac{3}{\lambda} \implies \frac{6}{\lambda^2} = 6 \implies \lambda = \pm 1$ . There are then two candidate points  $[3, 2], [-3, -2]$ .

(b) Note that  $L = 0 + \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

Let  $v \in T(x_1) \implies v = \alpha[-3, 2]$ . Then  $v^T L v = \alpha^2 \begin{bmatrix} -3 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = 12\alpha^2 > 0$ . So this satisfies the SOSC and is therefore a minimum.

But for  $\lambda_* = 1, v^T L v = -12\alpha^2 < 0$  so this is also a classifier.

(c)  $f(x_1) = 8, f(x_2) = -16$  so neither are global min/max.

20.12 Consider the minimization problem which has the same solution as the original problem min  $-\|Ax\|^2$  subject to  $\|x\|^2 = 1$ .

The the Langrange condition is  $-2A^T A x + \lambda(2x) = 0 \implies A^T A x = \lambda x$ . But then  $\lambda$  is an eigenvalue of  $A^T A$  and  $\lambda_* = x^T A^T A x = \|Ax\|^2 = -f(x)$ .

So the max eigenvalue,  $\lambda_{\max} = \|A\|_2^2$ .

20.17 (a)  $\nabla f = (Ax - b)^T A, \nabla h(x) = C$

$$(Ax - b)^T A + \lambda^T C = 0$$

Solving for  $x = (A^T A)^{-1}(A^T b - C^T \lambda)$  we can then get.

$$Cx = d = C(A^T A)^{-1}(A^T b - C^T \lambda)$$

We can then solve for  $\lambda = (C(A^T A)^{-1}C^T)^{-1}(C(A^T A)^{-1}A^T b - d)$

Then taking  $Q = A^T A^{-1}$  we can get

$$x = QA^T b - QC^T(CQC^T)^{-1}(CQA^T b - d)$$

- (b) If we take the objective function as  $f(x) = \frac{1}{2}x^T A^T A x - b^T A x + C$  we can see it is equivalent to  $\min \frac{1}{2}y^T Q y, Q = A^T A$  subject to  $Cy = d - CQ^{-1}A^T b$   
The solution to this from Section 20.6 is

$$y = Q^{-1}C^T(CQ^{-1}C^T)^{-1}A^T b$$

which is the same thing we derived in a.

20.18 Let  $y = x - Q^{-1}c$  Then we can use our usual solution to get

$$y = x - Q^{-1}c = Q^{-1}C^T(CQ^{-1}C^T)^{-1}A^T b$$

With this, we can solve for x to get

$$x = Q^{-1}(c + A^T(AQ^{-1}A^T)^{-1}(b - AQ^{-1}c))$$

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