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MATH 164 Homework 3
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15.1 Minimize $f(x_1, x_2) = -2x_1 - x_2$ such that

$x_1 - s_1 = 2$

$x_1 + x_2 + s_2 = 3$

$x_1 + 2x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

15.3 We can rewrite the original problem as follows

Minimize $\sum_{i=1}^{n} c_i(x_i^+ - x_i^-)$ such that

$$A(x^+ - x^-) = b, x^+, x^- \geq 0$$

However for it to be true standard form we'll need to rewrite A and c as $A' = [A, -A], c' = [c, c]$.

15.5 Let number of units shipped be x_1, x_2, x_3, x_4 respectively for AC, AD. BC, BD.

Minimize $f(x) = x_1 + 2x_2 + 3x_3 + 4x_4$ s.t.

$x_1 + x_2 + s_1 = 70$ $x_3 + x_4 + s_2 = 80$ $x_1 + x_3 = 50$ $x_2 + x_4 = 60$

15.7 Let x_i be the weight of item i used.

Then our goal is to minimize the total cost $f(x) = 2x_1 + 4x_2 + x_3 + 2x_4$ such that.

$x_1 + x_2 + x_3 + x_4 = 1000$

$3x_1 + 8x_2 + 16x_3 + 4x_4 = 10000$

$6x_1 + 46x_2 + 9x_3 + 9x_4 = 2000$

$20x_1 + 5x_2 + 4x_3 = 5000$

However the only solution is $\begin{bmatrix} 179 \\ -175 \\ 573 \\ 422 \end{bmatrix}$ which is infeasible, so there is no solution.

15.9 The matrix has full rank 3, so there are $\binom{5}{3} = 10$ basic solutions.

They are as follows:

$x^*_{1,2,3} = \frac{1}{17}\begin{bmatrix} -4 & -80 & 83 & 0 & 0 \end{bmatrix}$

$x^*_{1,2,4} = \begin{bmatrix} -10 & 49 & 0 & -83 & 0 \end{bmatrix}$

$x^*_{1,2,5} = \frac{1}{31}\begin{bmatrix} 105 & 25 & 0 & 0 & 83 \end{bmatrix}$

$x^*_{1,3,4} = \frac{1}{11}\begin{bmatrix} -12 & 0 & 49 & -80 & 0 \end{bmatrix}$

$x^*_{1,3,5} = \frac{1}{35}\begin{bmatrix} 100 & 0 & 25 & 0 & 80 \end{bmatrix}$

$x^*_{1,4,5} = \frac{1}{18}\begin{bmatrix} 65 & 0 & 0 & 25 & 49 \end{bmatrix}$

$x^*_{2,3,4} = \begin{bmatrix} 0 & -6 & 5 & 2 & 0 \end{bmatrix}$

$x^*_{2,3,5} = \frac{1}{23}\begin{bmatrix} 0 & -100 & 105 & 0 & 4 \end{bmatrix}$

$x^*_{2,4,5} = \begin{bmatrix} 0 & 13 & 0 & -21 & 2 \end{bmatrix}$

$x^*_{3,4,5} = \frac{1}{19}\begin{bmatrix} 0 & 0 & 65 & -100 & 12 \end{bmatrix}$

15.10

16.2 (a)

$$A = \begin{bmatrix} 3 & 1 & 0 & 1 \\ 6 & 2 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, c = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

(b) The tableau is given by

$$\begin{bmatrix} 3 & 1 & 0 & 1 & 4 \\ 6 & 2 & 1 & 1 & 5 \\ 2 & -1 & -1 & 0 & 0 \end{bmatrix}$$

we need to pivot about $(2,3)$ and $(1,4)$ and to get

$$\begin{bmatrix} 3 & 1 & 0 & 1 & 4 \\ 3 & 1 & 1 & 0 & 1 \\ 5 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) The basic feasible solution corresponding to this is $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}$ with cost -1.

(d) The coefficients are $\begin{bmatrix} 5 & 0 & 0 & 0 \end{bmatrix}$.

(e) Yes it is optimal, as all the reduced row coefficients are positive.

(f) Yes, there is a basic feasible solution, since the phase I of the two-phase method returned a valid basic solution.

(g)

$$\begin{bmatrix} 0 & 0 & -1 & 1 & 2 & -1 & 3 \\ 1 & \frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 2 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

16.3 The tableau for this problem is given by

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ -1 & -1 & -3 & 0 \end{bmatrix}$$

By $R_3 + R_2 + R_1 \to R_3$ we can get the cannonical form.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

And then we pivot about -1 by $R_1 + R_3 \to R_3; R_2 - R_1 \to R_1$.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 5 \end{bmatrix}$$

Then the solution given is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ for a total cost of 5.

16.4 We know there will be 3 slack variables because we have three inequalities. The tableau for this problem is given by

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 & 7 \\ 1 & 1 & 0 & 0 & 1 & 9 \\ -2 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There's a negative reduced cost coefficient, so we pivot along $(1,1)$ via $R_3 - R_1 \to R_3$ and $R_4 + 2R_1 \to R_4$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 & 7 \\ 0 & 1 & -1 & 0 & 1 & 4 \\ 0 & -1 & 2 & 0 & 0 & 10 \end{bmatrix}$$

Again we pivot along $(3, 2)$ via $R_2 + R_4 \to R_2$ and $R_4 + R_3 \to R_4$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 2 & 1 & 0 & 17 \\ 0 & 1 & -1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 1 & 14 \end{bmatrix}$$

Now our reduced row coefficients are all positive so our solution is $x = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ for a total cost of 14.

16.5 (a) We can get the basis from the cannonical tableau

$$\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 3 & 4 \end{bmatrix}$$

Then we can try to get the identity matrix on the right via $R_2 - 2R_1 \to R_2$ and $R_1 - R_2 \to R_2$ and $R_2/2 \to R_2$

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

so $B = \frac{1}{2} \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$

(b) We know that $c_D^T = r_D^T + c_B^T B^{-1} D$

$$= [-1, 1] + [8, 7]^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = [30, 47]$$

so the missing part of $c = [30, 47]^T$.

(c) The basic feasible solution is given by $B^{-1}b = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 38 \end{bmatrix}$.

(d) The first two elements are given above and the last element is given by $-c_B^T B^{-1} b = -\begin{bmatrix} 7 & 8 \end{bmatrix} \begin{bmatrix} 16 \\ 38 \end{bmatrix} =$ -426 The missing values are $[16, 38, 416]^T$.

16.6 The columns in the constraint matrix A corresponding to x^+, x^- are linearly dependent. Hence they cannot both enter a basis at the same time. This means that only one variable, can assume a nonnegative value; the nonbasic variable is necessarily zero.

16.8 (a) $\begin{bmatrix} 6 \\ 0 \\ 7 \\ 5 \\ 0 \end{bmatrix}$ is the basic feasible solution for this tableau, with $f(x) = 8$

(b) $\begin{bmatrix} 0 & 4 & 0 & 0 & -4 \end{bmatrix}$

(c) Yes, as we can get any negative value since the last column is all negative.

(d) We would neeed to pivot about $(3, 2)$ to get

$$\begin{bmatrix} 0 & 0 & -\frac{1}{3} & 1 & 0 & \frac{8}{3} \\ 1 & 0 & -\frac{2}{3} & 0 & 0 & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} & 0 & -1 & \frac{7}{3} \\ 0 & 0 & -\frac{4}{3} & 0 & 0 & -\frac{4}{3} \end{bmatrix}$$

(e) $x = [52, 0, 76, 28, 23]^T$

(f) Note from the tableau $a_2 = a_4 + 2a_1 + 3a_3$ and $a_5 = -a_4 - 2a_1 - 3a_3$ so therefore $\begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ -1 \\ -1 \end{bmatrix} \in$ $kernel(A)$. By RN theorem we know that $nullity(A) = 2$ Since these two vectors are linearly independent, they form a basis for the kernel of A.

16.9 (a) We can rewrite this as minimize $f(x) = x_1 + 2x_2$ s.t. $x_2 - s_1 = 1$

(b) First we create our tableau

$$\begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

Then for phase 1 we try to minimize $f(x) = y_1$

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We zero out the bottom row via $R_2 - R_1 \to R_2$

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -1 \end{bmatrix}$$

Then we pivot about $(1, 2)$ via $R_2 + R_1 \to R_2$ and get back our original matrix

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The we start phase 2 by pivoting about $(1, 2)$ to get

$$\begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 2 & -2 \end{bmatrix}$$

so our solution is $x = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

16.10 (a) A basic solution is found when $x_2 = 0$ which forces $x_1 = 1$ for $[1, 0]^T$.

(b) $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$

(c) The algorithm terminates since the problem is unbounded, as the element at $(1, 2) < 0$

(d) Note that any $x_1 \in \mathbb{R}$ the vector of the form $[x_1, x_1 - 1]$ is feasible, so the objective function can take any arbitrary value in $\mathbb{R}$.

16.11 First we construct the cannonical tableau

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 2 & 1 & 0 & -1 & 3 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

With x_1, x_2 as basic variables we get $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \implies B^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$

so then $\lambda_d = \frac{1}{3} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$ and $r_d = c_d^T - \lambda^T D = 0 - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

Since these are all non-negative, this is an optimal solution $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ with cost 2.

16.12 (a) The cannonical tableau has the form

$$\begin{bmatrix} 5 & 1 & -1 & 0 & 0 & 11 \\ -2 & -1 & 0 & 1 & 0 & -8 \\ 1 & 2 & 0 & 0 & -1 & 7 \\ 4 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There is no basic feasible solution, so we can add in y_1, y_2, y_3 to get an updated tableau

$$\begin{bmatrix} 5 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 11 \\ 2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Our $B^{-1} = I_3, y_0 = \begin{bmatrix} 11 \\ 8 \\ 7 \end{bmatrix}$

with y_1, y_2, y_3 as our basic variables we get $\lambda = [1, 1, 1]^T$ and $r_D = [-8, -4, 1, 1, 1]^T$. So we pivot about $(1, 1)$ to bring x_1 into the basis.

After this our $B^{-1} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{2}{5} & 1 & 0 \\ -\frac{1}{5} & 0 & 1 \end{bmatrix}, y_0 = [\frac{11}{5}, \frac{18}{5}, \frac{24}{5}]$

with x_1, y_2, y_3 as our basis we get $\lambda = [-\frac{3}{5}, 1, 1]^T$ and $r_D = [-\frac{12}{5}, -\frac{3}{5}, 1, 1, \frac{8}{5}]^T$.
So we pivot about $(2, 3)$ to bring x_2 into the basis.

After this our $B^{-1} = \begin{bmatrix} \frac{2}{9} & 0 & -\frac{1}{9} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{9} & 0 & \frac{5}{9} \end{bmatrix}, y_0 = [\frac{5}{3}, 2, \frac{8}{3}]$

with x_1, x_2, y_3 as our basis we get $\lambda = [-\frac{1}{3}, 1, -\frac{1}{3}]^T$ and $r_D = [-\frac{1}{3}, 1, -\frac{1}{3}, \frac{4}{3}, \frac{4}{3}]^T$.
So we pivot about $(3, 2)$ to bring x_3 into the basis.

After this our $B^{-1} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ -1 & 3 & -1 \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}, y_0 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$

with x_1, x_2, x_3 as our basis we get $\lambda = 0$ and $r_D = [0, 0, 1, 1, 1]^T$.
So our initial solution $= [3, 2, 6, 0, 0]^T$
Going back to our original tableau with x_1, x_2, x_3 as our basis we get $\lambda = [0, \frac{5}{3}, \frac{2}{3}]$ which are the same ad $r_d > 0$
so the optimal solution is $[3, 2]^T$.

(b) The cannonical tableau for this problem is

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 0 & 0 & 20 \\ 6 & 5 & 3 & 2 & 0 & 1 & 0 & 100 \\ 3 & 4 & 9 & 12 & 0 & 0 & 1 & 75 \\ -6 & -4 & -7 & -5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Our $B^{-1} = I_3, y_0 = \begin{bmatrix} 20 \\ 100 \\ 75 \end{bmatrix}$

with y_1, y_2, y_3 as our basic variables we get $\lambda = 0$ and $r_D = [-6, -4, -7, -5]^T$. So we pivot about $(3, 2)$ to bring x_2 into the basis.

After this our $B^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{9} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & \frac{1}{9} \end{bmatrix}, y_0 = [\frac{35}{3}, 75, \frac{25}{3}]$

with x_1, x_2, y_3 as our basis we get $\lambda = [0, 0, -\frac{7}{9}]^T$ and $r_D = [-\frac{11}{3}, -\frac{8}{9}, \frac{13}{3}, \frac{7}{9}]^T$.
So we pivot about $(2, 1)$ to bring x_1 into the basis.

After this our $B^{-1} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ -1 & 3 & -1 \\ 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}, y_0 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$

with x_1, x_2, x_3 as our basis we get $r_D = [\frac{27}{15}, \frac{43}{15}, \frac{11}{15}, \frac{8}{15}]^T$.
so the optimal solution is $[15, 0, \frac{10}{3}, 0]^T$.

16.14 (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(b) any vector in the $span(\begin{bmatrix} 2 \\ 1 \end{bmatrix})$ will be a valid c_1, c_2.

(c) These are all 0 since the basic feasible solutions are optimal.