Gor'kov's functions and superconductivity

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Introduction to Quantum Field Theory

Spring 2021

BCS Hamiltonian

Gorkov's work starts with the BCS Hamiltonian, which depends on field operators for the electrons (Dirac spinors)

$$\hat{H}_{BCS} = \int d^3x \left[-\hat{\psi}^\dagger_\alpha \left(\frac{\nabla^2}{2m} \right) \hat{\psi}_\alpha + \frac{g}{2} \left(\hat{\psi}^\dagger_\alpha \hat{\psi}^\dagger_\beta \hat{\psi}_\beta \hat{\psi}_\alpha \right) \right]$$

1. The first term is the non-relativistic energy for fermions.
2. The second term, was used with the insight of the BCS theory published several months prior to Gorkov's calculation.
3. g is the coupling associated with the four electron interaction e.g. the creation and destruction of Cooper pairs.

field operators and canonical quantization

Gorkov's work uses the Dirac representation in the limiting case that the momentum is small relative to the speed of light. Consequently, the dependence on $\gamma^{1,2,3}$ in the spinor portion of the solution vanishes. In the Dirac representation, γ^0 is just the identity, so we drop all γ-matrices from the calculations.

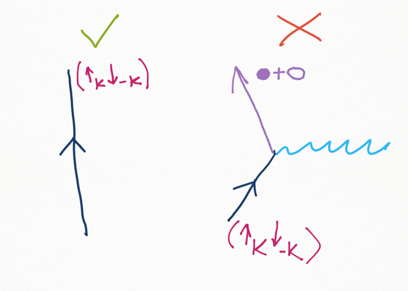
$$\psi_\alpha(x) = V^{-1/2} \sum_{k,s} a(k,s) u_\alpha(s) e^{ikx}$$

$$\psi_\beta^\dagger(x') = V^{-1/2} \sum_{k,s} a^\dagger(k,s) u_\beta^\dagger(s) e^{-ikx}$$

$$\{\psi_\alpha(x), \psi_\beta^\dagger(x')\} = \delta_{\alpha\beta}\delta(x - x')$$

why does the quartic term suffice?

1. no scattering of Cooper pairs: there is an energy gap, Δ, between the BCS ground state (BEC-like state) and the excited state.
2. This energy gap was determined to be on the order of 100mK
3. at cryogenic temperatures we don't allow for thermally excited cooper pairs, therefore no scattering (**this is why conventional superconductors only work at low T**)

¿¿ 

four-point function

We need to simplify the four-point function to calculate the 2-point green's function, as it appears in the equations of motion for ψ and $\psi^\dagger$.

$$\langle T\psi_\alpha(x_1)\psi_\beta(x_2)\psi^\dagger_\gamma(x_3)\psi^\dagger_\delta(x_4)\rangle|_{BCS}$$

we just add... *ZERO*?

$$\begin{aligned} &= \langle T\psi_\alpha(x_1)\psi^\dagger_\delta(x_4)\rangle \times \langle T\psi_\beta(x_2)\psi^\dagger_\gamma(x_3)\rangle \propto N \\ &- \langle T\psi_\alpha(x_1)\psi^\dagger_\gamma(x_3)\rangle \times \langle T\psi_\beta(x_2)\psi^\dagger_\delta(x_4)\rangle \propto N \\ &+ \big[\langle N|T\psi_\alpha(x_1)\psi_\beta(x_2)|N+2\rangle \\ &\times\langle N+2|T\psi^\dagger_\gamma(x_3)\psi^\dagger_\delta(x_2)|N+2\rangle\big] \end{aligned}$$

grand canonical ensemble $\Phi(T, V, \mu)$

Implicitly, we will write our equations in a form that is compatible with our system being in contact with a thermal bath and a **particle sink/source**.

1. statistical ensembles are formed by maximizing entropy, with a constraint
2. constraints are very conveniently implemented through lagrange multipliers in the maximization equation of entropy.
3. it then follows the average energy of the system is offset by the chemical potential μ, which takes the role of the lagrange multiplier in the original constraint.

$$\frac{\partial}{\partial p_i}\left[S - \lambda_N \sum_i p_i * N - \lambda_E \sum_i p_i * E - \sum_i p_i\right]$$

$$d\langle E\rangle = TdS + \mu d\langle N\rangle$$

absorbing particle number

$$\overline{E}'/V = \langle \psi_\alpha(x_1)\psi_\beta(x_2)\psi^\dagger_\gamma(x_3)\psi^\dagger_\delta(x_4)\rangle$$

$$\langle T\psi_\alpha(x_1)\psi_\beta(x_2)\psi^\dagger_\gamma(x_3)\psi^\dagger_\delta(x_4)\rangle =$$
$$\left[\langle N|T\psi_\alpha(x_1)\psi_\beta(x_2)|N+2\rangle \times \langle N+2|T\psi^\dagger_\gamma(x_3)\psi^\dagger_\delta(x_2)|N+2\rangle\right]$$

Gor'kov's Functions $F(x-x')$ and $G(x-x')$

$$
\begin{aligned}
iG_{\alpha\beta}(x-x') &= \langle T\psi_\alpha(x)\psi^\dagger_\beta(x')\rangle \\
F_{\alpha\beta}(x-x') &= \langle T\psi_\alpha(x)\psi_\beta(x')\rangle \\
F^\dagger_{\alpha\beta}(x-x') &= \langle T\psi^\dagger_\alpha(x)\psi^\dagger_\beta(x')\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\langle T\psi_\alpha(x_1)\psi_\beta(x_2)\psi^\dagger_\gamma(x_3)\psi^\dagger_\delta(x_4)\rangle \\
&= \left[\langle N|T\psi_\alpha(x_1)\psi_\beta(x_2)|N+2\rangle \times \langle N+2|T\psi^\dagger_\gamma(x_3)\psi^\dagger_\delta(x_2)|N+2\rangle\right] \\
&= e^{-i2\mu t}F(x-x') \times e^{i2\mu t}F^\dagger(x-x')
\end{aligned}
$$

field operator EOM

using the anti-commutator expression

$$[A, BC] = \{A, B\}C - B\{A, C\}$$

we apply the Heisenberg equations of motion:

$$i\frac{\partial}{\partial t}\psi_\gamma(x') = [\psi_\gamma(x'), H(x)]$$

$$= -\int d^3x[\psi_\gamma(x'), \psi_\alpha(x)^\dagger * \nabla^2/2m\ \psi_\alpha(x)]\ (A)$$

$$+\int d^3x\ g/2[\psi_\gamma(x'), \psi_\alpha^\dagger(x)\psi_\beta^\dagger(x)\psi_\beta(x)\psi_\alpha(x)]\ (B)$$

$$\begin{aligned}(A) \quad &= -\int d^3x\delta(x - x')\delta_{\alpha\gamma}(\nabla^2/2m)\psi_\alpha(x) \\ &= -(\nabla^2/2m)\psi_\gamma(x')\end{aligned}$$

field operator EOM

$$[A, BC] = \{A, B\}C - B\{A, C\}$$

$$(B) \quad = g/2 \int d^3x \{\psi_\gamma(x'), \psi^\dagger_\alpha(x)\}\psi^\dagger_\beta(x)\psi_\beta(x)\psi_\alpha(x)$$

$$-(g/2) \int d^3x \psi^\dagger_\alpha(x)\{\psi_\gamma(x'), \psi^\dagger_\beta(x)\}\psi_\beta(x)\psi_\alpha(x)$$

$$= (g/2) \int d^3x \delta(x - x')\delta_{\gamma\alpha}\psi^\dagger_\beta(x)\psi_\beta(x)\psi_\alpha(x)$$

$$-(g/2) \int d^3x \psi^\dagger_\alpha(x)\delta(x - x')\delta_{\gamma\beta}\psi_\beta(x)\psi_\alpha(x)$$

field operator EOM

applying $\psi^\dagger_\beta \psi_\alpha = -\psi_\alpha \psi^\dagger_\beta$ and substituting $x \leftarrow x'$, $\alpha \leftarrow \gamma$ we find:

$$\implies \left[i\frac{\partial}{\partial t} + \nabla^2/2m \right] \psi_\alpha(x) - g\psi^\dagger_\beta(x)\psi_\beta(x)\psi_\alpha(x) = 0$$

an analogous calcuation can be done for the conjugate field $\psi^\dagger$

$$\left[i\frac{\partial}{\partial t} - \nabla^2/2m \right] \psi^\dagger_\alpha(x) + g\psi^\dagger_\alpha(x)\psi^\dagger_\beta(x)\psi_\beta(x) = 0$$

Expression for the Green's function

We consider the time derivative of the Green's function and use both the equations of motion and the approximated four-point function to write down an analytical expression.

$$i\frac{\partial}{\partial t}G(x-x') = \frac{\partial}{\partial t}\langle T\psi_\alpha(x)\psi_\beta^\dagger(x')\rangle$$

$$= \frac{\partial}{\partial t}\left[\Theta(t-t')\langle\psi_\alpha(x)\psi_\beta^\dagger(x')\rangle - \Theta(t'-t)\langle\psi_\beta^\dagger(x')\psi_\alpha(x)\rangle\right]$$

$$= \delta(t-t')\langle\{\psi_\alpha(x),\psi_\beta^\dagger(x')\}\rangle$$

$$+\Theta(t-t')\langle\frac{\partial}{\partial t}\psi_\alpha(x)\psi_\beta^\dagger(x')\rangle$$

$$-\Theta(t'-t)\langle\psi_\beta^\dagger(x')\frac{\partial}{\partial t}\psi_\alpha(x)\rangle$$

Expression for the Green's function

Now we can insert the equation of motion for $\psi_\alpha(x)$

$$\frac{\partial}{\partial t}\psi_\alpha(x) = -ig\psi^\dagger_\gamma(x)\psi_\gamma(x)\psi_\alpha(x) + i(\nabla^2/2m)\psi_\alpha(x)$$

$$\begin{gathered}
i\frac{\partial}{\partial t}G(x-x') = \delta(x-x') \\
+\Theta(t-t')\big[(-ig)\langle\psi^\dagger_\gamma(x)\psi_\gamma(x)\psi_\alpha(x)\psi^\dagger_\beta(x')\rangle \\
+i(\nabla^2/2m)\langle\psi_\alpha(x)\psi^\dagger_\beta(x')\rangle)\big] \\
-\Theta(t'-t)\big[(-ig)\langle\psi^\dagger_\beta(x')\psi^\dagger_\gamma(x)\psi_\gamma(x)\psi_\alpha(x)\rangle \\
+i(\nabla^2/2m)\langle\psi^\dagger_\beta(x')\psi_\alpha(x)\rangle\big]
\end{gathered}$$

We can simplify the term for $t > t'$ by using the anti-commutator relations for the spinor fields

$$
\begin{aligned}
&\langle \psi^\dagger_\gamma(x)\psi_\gamma(x)\psi_\alpha(x)\psi^\dagger_\beta(x')\rangle \\
&= \langle \left[\delta(0) - \psi_\gamma(x)\psi^\dagger_\gamma(x)\right]\psi_\alpha(x)\psi^\dagger_\beta(x')\rangle \\
&= \langle \delta(0)\psi_\alpha(x)\psi^\dagger_\beta(x')\rangle \\
&-\langle \psi_\gamma(x)\left[\delta_{\gamma\alpha}\delta(0) - \psi_\alpha(x)\psi^\dagger_\gamma(x)\right]\psi^\dagger_\beta(x')\rangle
\end{aligned}
$$

The delta function terms cancel, and what remains is

$$
\langle \psi_\gamma(x)\psi_\alpha(x)\psi^\dagger_\gamma(x)\psi^\dagger_\beta(x')\rangle
$$

Relations for G(x-x') and F(x-x')

It now follows that our differential equation has terms that have the proper form of a time-ordered product: therefore we can identify G and F in our equations. (the differential equation for F follows a similar derivation shown for G)

G and F coupled equations

$$[i\tfrac{\partial}{\partial t} + \nabla^2/2m]G(x-x') - igF(0)F^\dagger(x-x') = \delta(x-x')$$

$$[i\tfrac{\partial}{\partial t} - \nabla^2/2m - 2\mu]F^\dagger(x-x') - igF^\dagger(0)G(x-x') = 0$$

definitions in terms of ψ

$$iG_{\alpha\beta}(x-x') = \langle T\psi_\alpha(x)\psi_\beta^\dagger(x')\rangle$$

$$F_{\alpha\beta}(x-x') = \langle T\psi_\alpha(x)\psi_\beta(x')\rangle$$

$$F_{\alpha\beta}^\dagger(x-x') = \langle T\psi_\alpha^\dagger(x)\psi_\beta^\dagger(x')\rangle$$

Now that we have differential equations defining F and G we can proceed to solve for them; taking the fourier transform of the equations simplifies this analysis.
We start by changing variables $\xi_k = k^2/2m - \mu$ and $\omega' = \omega - \mu$

$$(\omega - \xi_k)G(\omega, k) - igF(0)F^\dagger(\omega, k) = 1$$
$$(\omega + \xi_k)F^\dagger(\omega, k) + igF(0)G(\omega, k) = 0$$

We also take note of the general matrix form for F. Since $F_{\alpha\beta}(0) = e^{i2\mu t}\langle\psi_\alpha(x)\psi_\beta(x)\rangle$, it also follows that:

$$(F^\dagger)^* = -F \implies F^\dagger(0) = J\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then our differential equation in terms of $\Delta_{gap} = gJ$ is:

$$(\omega^2 - \xi_k^2 - \Delta^2)F^\dagger = -i\Delta$$
$$(\omega - \xi_k)G = 1 + i\Delta F^\dagger$$

The propagators for electrons in a superconductor

$$iG_{\alpha\beta}(x - x') = \langle T\psi_\alpha(x)\psi^\dagger_\beta(x')\rangle$$
$$F_{\alpha\beta}(x - x') = \langle T\psi_\alpha(x)\psi_\beta(x')\rangle$$

$$F^\dagger(\omega, k) = -i\frac{\Delta}{\omega^2 - \xi_k^2 - \Delta^2}$$
$$G(\omega, k) = \frac{\omega + \xi_k}{\omega^2 - \xi_k^2 - \Delta^2}$$

Calculation of the gap energy

We'll start by assuming the energy of the electrons to be close to the Fermi surface $\xi_k \approx v_f(k - k_f)$. We'll also rewrite the function F in a form that avoids the poles.

$$F^\dagger = -i \frac{\Delta}{(\omega - \sqrt{\xi_k^2 + \Delta^2} + i\delta)(\omega + \sqrt{\xi_k^2 + \Delta^2} - i\delta)}$$

We consider the equation relating F and it's own Fourier transform

$$F(x - x') = (2\pi)^{-4} \int d\omega d^3 k F(\omega, k) e^{ik(x-x'} e^{iw(t-t')}$$

and $F(0) \propto J$, So therefore:

$$J = (2\pi)^{-4} \int d\omega d^3 k F^\dagger(\omega, k)$$

Calculation of the gap energy

$$1 = -\frac{g}{2(2\pi)^3} \int d^3k \frac{1}{\sqrt{\xi_k^2 + \Delta^2}}$$

$$1 = -\frac{g * 4\pi}{2(2\pi)^3} \int_{-k_0}^{+k_0} dk \frac{1}{\sqrt{\xi_k^2 + \Delta^2}}$$

$$\Delta(T \to 0) = 2k_0 e^{-\pi^2/k_f g m}$$

what about finite temperatures?

We framed the problem to be invariant to particle exchange for two fold reasons: 1) quantum field theory does not care about particle number conservation. 2) the grand canonical ensemble also allows for a fluctuating total particle number.

$$iG_{\alpha\beta}(x - x', T) = \sum_n e^{(\Phi(T,V,\mu)+\mu N - E_n)/T} (G_{\alpha\beta})_n$$

$$1 = \frac{|g|}{2(2\pi)^3} \int d^3k \frac{1 - 2f(\epsilon_k)}{\sqrt{\xi_k^2 + \Delta^2(T)}}$$

how can this theory be improved?

Since 1950, physicists knew the attractive potential responsible for Cooper pairing was due to lattice vibrations. Shortly following this paper by Gor'Kov, another Russian Physicist, Eliashberg, would do a calculation of the superconducting gap energy using the following interaction Hamiltonian:

$$H_I = \int d^3x \psi_\alpha^\dagger \psi_\beta \varphi$$