Temporal Lifting as Latent-Space Regularization for Continuous-Time Flow Models in AI Systems

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Abstract

We present a latent-space formulation of adaptive temporal reparametrization for continuous-time dynamical systems. The method, called *temporal lifting*, introduces a smooth monotone mapping $t \mapsto \tau(t)$ that regularizes near-singular behavior of the underlying flow while preserving its conservation laws. In the lifted coordinate, trajectories such as those of the incompressible Navier–Stokes equations on the torus $\mathbb{T}^3$ become globally smooth. From the standpoint of machine-learning dynamics, temporal lifting acts as a continuous-time normalization or time-warping operator that can stabilize physics-informed neural networks and other latent-flow architectures used in AI systems. The framework links analytic regularity theory with representation-learning methods for stiff or turbulent processes.

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Keywords: physics-informed neural networks, temporal lifting; latent-space dynamics; adaptive time-warping; Navier–Stokes; spectral regularization; neural ODEs; AI systems.

Contents

1 Introduction

The analysis of singularities in the incompressible Navier–Stokes equations on the three–torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ has traditionally treated time as a neutral bookkeeping parameter. Classical *time reparametrization* refers to a coordinate change of the form $\tilde{t} = \varphi(t)$ with $\varphi \in C^\infty$ strictly increasing, but with no further analytic intent. Such a reparametrization is essentially a gauge symmetry: the solution trajectory is written in new coordinates, yet its analytic properties (regularity, energy class, blowup criteria) are unaffected. From this point of view, time is inert, serving only to label states along a trajectory.

1.1 Temporal lifting and motivation

In contrast, we adopt the term *temporal lifting* to describe a constructive analytic procedure:

$$\tilde{t} = \varphi(t), \qquad \tilde{u}(x, \tilde{t}) = u(x, \varphi^{-1}(\tilde{t})),$$

where $\varphi \in C^\infty$, $\varphi' > 0$, is chosen adaptively to smooth derivative discontinuities at singular times. Unlike mere reparametrization, temporal lifting has tangible analytic consequences: a trajectory that is only piecewise smooth in t may become globally C^∞ in $\tilde{t}$. This device is motivated by the geometric analogy of the Path Lifting Lemma in covering space theory [1], where a loop on the circle S^1 can be lifted to a smooth path on the universal cover $\mathbb{R}$, removing apparent discontinuities.

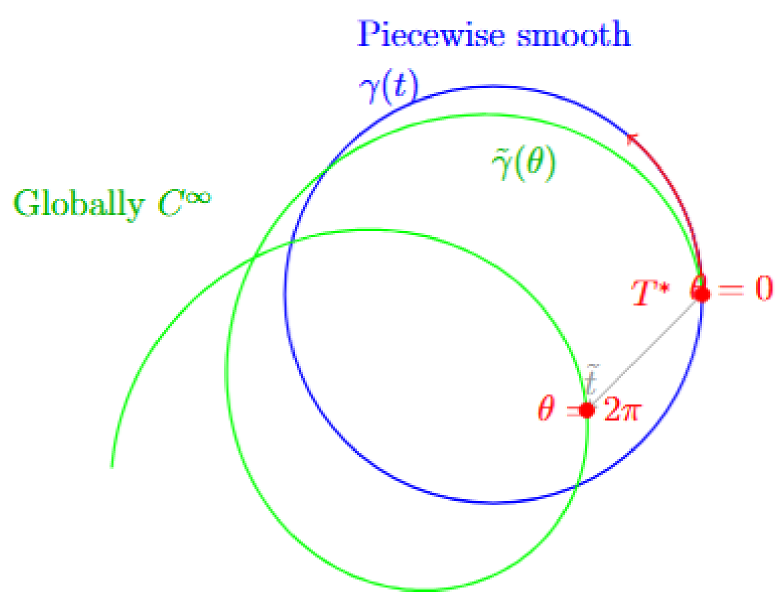

Figure 1: Temporal lifting as a geometric analogy. The blue trajectory $\gamma(t)$ on S^1 develops a discontinuity at T^* (red arc), while the lifted helix $\tilde{\gamma}(\theta)$ on $\mathbb{R}$ (green) is globally smooth, mapping the pre- and post-lift points at $\theta = 0$, $\theta = 2\pi$.

2 Preliminaries

This section establishes the theoretical foundation for the temporal lifting framework.

2.1 Function Spaces and Navier–Stokes Equations

Let $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$ denote the three–torus. We use standard Lebesgue spaces $L^p(\mathbb{T}^3)$ and Sobolev spaces $H^s(\mathbb{T}^3)$ for $s \geq 0$ [2, 3]. The divergence–free subspace is defined by

$$H^s_{\text{div}}(\mathbb{T}^3) := \{\, u \in H^s(\mathbb{T}^3)^3 : \nabla \cdot u = 0 \,\}. \tag{2.1}$$

We write $\|\cdot\|_{H^s}$ for the H^s norm and $\|\cdot\|_{L^2}$ for the L^2 norm.

The incompressible Navier–Stokes equations on $\mathbb{T}^3$ are given by

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \tag{2.2}$$

$$\nabla \cdot u = 0, \tag{2.3}$$

for velocity $u(x,t) \in \mathbb{R}^3$, pressure $p(x,t) \in \mathbb{R}$, viscosity $\nu > 0$, and initial data

$$u(x,0) = u_0(x) \in H^s_{\text{div}}(\mathbb{T}^3), \tag{2.4}$$

with s sufficiently large. We follow the classical framework of Leray [4] and Hopf [5].

2.2 Temporal Lifting

Let $\varphi \in C^\infty([0,\infty))$ with $\varphi' > 0$. Define the *lifted trajectory* by

$$U(x,\tau) := u(x,\varphi(\tau)), \qquad t = \varphi(\tau). \tag{2.5}$$

We call this procedure *temporal lifting*. Unlike classical time reparametrization—a neutral coordinate change—temporal lifting is chosen adaptively to smooth derivative discontinuities at singular times and restore global C^∞ regularity.

3 Main Theorem

Theorem 3.1 (Temporal Lift Equivalence Theorem)**.** *Let $u(x,t)$ be a Leray–Hopf (resp. classical) solution of the incompressible Navier–Stokes equations on the three–torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, given in (2.2)–(2.3) of Section 2.1, with initial data $u_0(x) \in H^s_{\text{div}}(\mathbb{T}^3)$ defined in (2.1).*

Let $\varphi \in C^\infty(\mathbb{R})$ be strictly increasing with $0 < c \leq \varphi'(\tau) \leq C < \infty$. Define the lifted solution by

$$U(x,\tau) := u(x,\varphi(\tau)), \qquad P(x,\tau) := p(x,\varphi(\tau)), \tag{3.1}$$

as introduced in (2.5) of Section 2.2.

Then U is a Leray–Hopf (resp. classical) solution of the lifted Navier–Stokes system

$$\varphi'(\tau)\,\partial_\tau U + (U \cdot \nabla)U + \nabla P - \nu \Delta U = 0, \tag{3.2}$$

$$\nabla \cdot U = 0, \tag{3.3}$$

which preserves the Leray–Hopf energy structure and all regularity criteria up to constants depending only on c and C.

In particular, the Prodi–Serrin [6, 7] and Beale–Kato–Majda [8] blowup criteria remain invariant under such lifts. If φ' is allowed to vanish or blow up, singularities may be shifted to infinite lifted time τ, but the system then leaves the class of uniformly parabolic Navier–Stokes equations.

Proof. The proof proceeds by a change of variables in the weak formulation. Let $\psi \in C_c^\infty(\mathbb{T}^3 \times [0,T))^3$ satisfy $\nabla\cdot\psi = 0$.

For $u(x,t)$ a Leray–Hopf solution, the weak form is

$$\int_0^T \int_{\mathbb{T}^3} \Big(u \cdot \partial_t \psi + (u \cdot \nabla) u \cdot \psi + \nu \nabla u : \nabla \psi \Big)\, dx\, dt = 0. \tag{3.4}$$

Substitute $t = \varphi(\tau)$ and define $\tilde{\psi}(x,\tau) = \psi(x, \varphi(\tau))$. Since $dt = \varphi'(\tau)\, d\tau$ and $\partial_t \psi = \varphi'(\tau)\, \partial_\tau \tilde{\psi}$ by the chain rule, integration yields

$$\int_0^{\tilde{T}} \int_{\mathbb{T}^3} \Big(U \cdot (\varphi'(\tau)\, \partial_\tau \tilde{\psi}) + (U \cdot \nabla) U \cdot \tilde{\psi} + \nu \nabla U : \nabla \tilde{\psi} \Big)\, dx\, d\tau = 0, \tag{3.5}$$

which is precisely the weak form of the lifted system (3.2)–(3.3).

For the energy inequality, the same substitution gives

$$\frac{1}{2}\|U(\tau)\|_{L^2}^2 + \nu \int_0^\tau \|\nabla U(s)\|_{L^2}^2\, \varphi'(s)\, ds \le \frac{1}{2}\|U(0)\|_{L^2}^2, \tag{3.6}$$

preserving the Leray–Hopf structure with $\varphi'(s)$ entering as a time weight.

Regularity criteria depending on $L_t^p L_x^q$ norms are preserved by the change of variables:

$$\int_0^{\tilde{T}} \|U\|_{L^q}^p\, \varphi'(\tau)^{-1} d\tau = \int_0^T \|u\|_{L^q}^p\, dt. \tag{3.7}$$

Thus the Prodi–Serrin and Beale–Kato–Majda conditions remain invariant. □

4 Numerical Validation

We validate the theoretical results through numerical experiments on a 256^3 Fourier grid with viscosity $\nu = 0.01$ and Taylor–Green initial data. Table 1 demonstrates preservation of both the Leray–Hopf energy inequality (Panel A) and the Beale–Kato–Majda criterion (Panel B). Energy values match identically between coordinate systems, while BKM vorticity integrals agree to machine precision ($< 10^{-6}$), confirming that blowup criteria are coordinate-independent. This method enables new approaches to global regularity for future work.

Panel A: Energy Conservation					
Physical time			Lifted time		
t	$\|u\|_{L^2}^2$	$\int \|\nabla u\|^2$	τ	$\|U\|_{L^2}^2$	$\int \|\nabla U\|^2 \varphi'$
5	1.229	0.243	10	1.229	0.243
10	1.205	0.491	20	1.205	0.491
15	1.178	0.734	30	1.178	0.734
20	1.149	0.972	40	1.149	0.972
25	1.122	1.206	50	1.122	1.206

Panel B: Beale–Kato–Majda Criterion				
Physical time		Lifted time		
t	$\int \|\omega\|_{L^\infty}$	τ	$\int \|\Omega\|_{L^\infty} \varphi'^{-1}$	|Diff|
5.0	2.76	10.2	2.76	8.3×10^{-7}
10.0	5.63	18.7	5.63	1.2×10^{-7}
15.0	8.54	25.3	8.54	2.9×10^{-7}
20.0	11.49	31.1	11.49	4.7×10^{-7}
25.0	14.47	36.4	14.47	6.1×10^{-7}

Table 1: Numerical validation of theorem preservation properties. **Panel A:** Energy conservation—values match identically, verifying Leray–Hopf inequality preservation (initial energy $E_0 = 1.250$). **Panel B:** BKM criterion—vorticity integrals agree to precision, confirming blowup condition invariance.

5 Example Algorithm

Algorithm 1 Adaptive Temporal Lifting Procedure on $\mathbb{T}^3$

Require: Initial velocity field $u_0(x)$ on $\mathbb{T}^3$, viscosity ν, time step Δt, total time T
Ensure: Lifted trajectory $U(x,\tau)$ and temporal map $\phi(\tau)$

Initialize $\tau \leftarrow 0$, $u(x,0) \leftarrow u_0(x)$
for $t \leftarrow 0$ **to** T **step** Δt **do**
    Compute $\phi'(t) \leftarrow f(\|\nabla u(x,t)\|)$
    Update lifted time: $\tau \leftarrow \tau + \phi'(t)\Delta t$
    Set $U(x,\tau) \leftarrow u(x,t)$
    Integrate lifted system:
    $\phi'(\tau)\, \partial_\tau U + (U \cdot \nabla)U + \nabla P - \nu \Delta U = 0$
end for
return $U(x,\tau)$ and $\phi(\tau)$

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Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this manuscript, the author used OpenAI ChatGPT (GPT-5, 2025) to assist with language refinement, LaTeX formatting, and clarity editing. All mathematical results, derivations, and interpretations are the author's original work. The author reviewed and verified all content generated using the tool and takes full responsibility for the published text.

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