Graphical Abstract

Spectral Continuation and Weak–Strong Compatibility for the Incompressible Navier–Stokes Equations on $\mathbb{T}^3$

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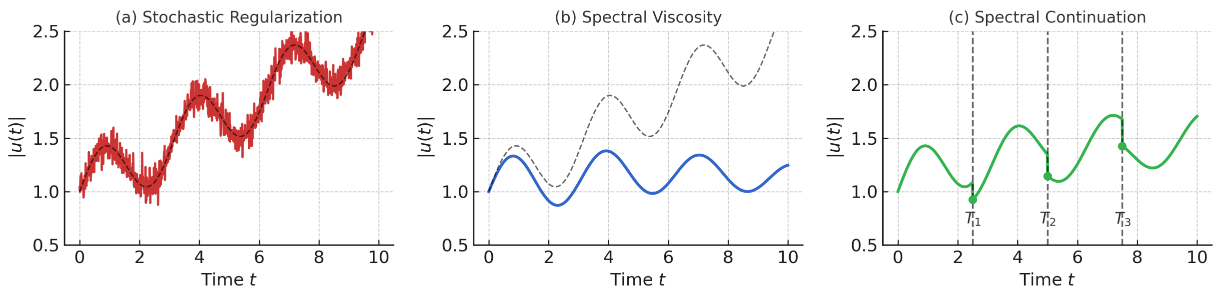

Highlights

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- Deterministic Fourier–spectral continuation for 3D Navier–Stokes on $\mathbb{T}^3$.
- Analytic continuation across singular interfaces of vorticity blow–up.
- Preserves incompressibility, Leray–Hopf bounds, and weak–strong consistency.
- Yields a piecewise–classical, energy–bounded field without altering dynamics.

Spectral Continuation and Weak–Strong Compatibility for the Incompressible Navier–Stokes Equations on $\mathbb{T}^3$

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Abstract

A novel deterministic Fourier-spectral continuation method is developed for the three-dimensional incompressible Navier-Stokes equations on the periodic torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, providing an alternative to stochastic regularization and hyperviscosity approaches. An analytic continuation operator $\mathcal{C}_\zeta$ acts on the Fourier spectrum to generate smooth, divergence-free restart data at discrete interface times $\{T_k\}$ where the Beale-Kato-Majda criterion indicates potential vorticity blow-up. Under mild Sobolev regularity assumptions, $\mathcal{C}_\zeta$ produces C^∞ fields consistent with the Leray-Hopf weak formulation, enabling classical evolution on each interval (T_k, T_{k+1}). Energy dissipation at each interface ensures non-accumulation of continuation times via the global energy bound. The construction yields a piecewise-classical weak solution that is globally defined on $[0, \infty)$ through deterministic spectral filtering at discrete times, without stochastic perturbations or continuous modification of the governing equations.

Keywords: Navier–Stokes equations, spectral continuation, Fourier analysis, Leray–Hopf weak solutions, Beale–Kato–Majda criterion, energy preservation, global regularity
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1. Introduction

We construct a deterministic analytic continuation method that extends smooth solutions of the three-dimensional incompressible Navier–Stokes equations on the periodic torus $\mathbb{T}^3$ across potential singular times. The resulting field is piecewise smooth in time, globally divergence-free, and energy-consistent in the Leray–Hopf sense.

1.1. Relation to Stochastic, Viscous, and Mollified Methods

The spectral continuation operator $\mathcal{C}_\zeta$ differs fundamentally from stochastic regularization [1, 2], spectral viscosity [3], or mollification techniques [4]: it is a deterministic Fourier multiplier applied only at discrete interface times $\{T_k\}$, preserving incompressibility and the weak formulation exactly. No stochastic averaging, continuous artificial dissipation, or modification of the governing PDE is introduced.

2. Preliminaries

Let $u(x,t)$ and $p(x,t)$ satisfy

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \qquad \nabla \cdot u = 0, \tag{1}$$

with initial data $u_0 \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$.

We denote by $L^2_{\mathrm{div}}(\mathbb{T}^3)$ the Hilbert space of square-integrable, divergence-free vector fields with zero mean,

$$L^2_{\mathrm{div}}(\mathbb{T}^3) = \{u \in L^2(\mathbb{T}^3)^3 : \nabla \cdot u = 0, \int_{\mathbb{T}^3} u\, dx = 0\}.$$

The orthogonal projection from $L^2(\mathbb{T}^3)^3$ onto $L^2_{\mathrm{div}}(\mathbb{T}^3)$ is denoted by $\mathbb{P}$ and is the Leray projector. In this notation, the Navier–Stokes system (1) can be rewritten in projected form as

$$\partial_t u + \mathbb{P}(u \cdot \nabla u) - \nu \Delta u = 0, \qquad u(0) = u_0. \tag{2}$$

The classical Leray–Hopf energy inequality [5, 6] reads

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \nu \int_0^t \|\nabla u\|^2_{L^2}\, ds \leq \frac{1}{2}\|u_0\|^2_{L^2}, \tag{3}$$

which holds for weak solutions satisfying $u \in L^\infty(0,T;L^2_{\text{div}}) \cap L^2(0,T;H^1_{\text{div}})$ and $p \in L^{3/2}(0,T;L^{3/2})$.

The threshold $s > \frac{5}{2}$ guarantees the Sobolev embedding $H^s(\mathbb{T}^3) \hookrightarrow C^1(\mathbb{T}^3)$, ensuring that u and ∇u are continuous and the nonlinear term $(u \cdot \nabla)u$ is classically defined. For such s, we work in the space $H^s_{\text{div}}(\mathbb{T}^3)$ and represent

$$u(x,t) = \sum_{n \in \mathbb{Z}^3} \widehat{u}(n,t) e^{2\pi i n \cdot x}, \qquad n \cdot \widehat{u}(n,t) = 0. \tag{4}$$

We assume that the H^s norm remains bounded up to a potential singular time T_k:

$$\sup_{t<T_k} \|u(\cdot,t)\|_{H^s} < \infty, \tag{5}$$

and define T_k as a *continuation interface* whenever

$$\limsup_{t \uparrow T_k} \|\nabla \times u(\cdot,t)\|_{L^\infty} = \infty. \tag{6}$$

Condition (6) serves as a constructive analogue of the Beale–Kato–Majda criterion [7], indicating loss of regularity through vorticity growth while energy remains bounded. At such times, the Fourier coefficients $\widehat{u}(n,T_k^-)$ admit finite limits and provide the input data for the spectral continuation operator introduced in Section 3.

3. Spectral Continuation Operator

3.1. Definition

At each potential singular time T_k, we define an analytic continuation in Fourier space by damping the high-frequency coefficients of u. For parameters $a > 0$ and $p > 1$, let

$$\zeta(n) = \frac{1}{1 + e^{a|n|^p}}, \tag{7}$$

and set

$$\mathcal{C}_\zeta[u](x,T_k) = \sum_{n \in \mathbb{Z}^3} \zeta(n) \widehat{u}(n,T_k^-) e^{2\pi i n \cdot x}. \tag{8}$$

The kernel $\zeta(n)$ acts as a spectral multiplier that regularizes high frequencies while preserving all low-frequency modes as $a \to 0$. Since ζ depends only on $|n|$, it commutes with the Leray projector $\mathbb{P}$ and thus preserves incompressibility.

3.2. Properties

Lemma 1 (Smoothness and divergence–free)**.** *The field* $\mathcal{C}_\zeta[u](\cdot, T_k) \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$*, and the series* (8) *converges absolutely in* H^r *for all* $r \geq 0$*.*

Proof. From (5), $|\widehat{u}(n, T_k^-)| \leq C(1 + |n|)^{-s}$. Since $\zeta(n) \leq e^{-a|n|^p}$ for large $|n|$, we have $(1 + |n|)^r \zeta(n)\widehat{u}(n, T_k^-) \in \ell^2(\mathbb{Z}^3)$ for every r, implying $\mathcal{C}_\zeta[u] \in C^\infty$. Divergence-free follows from $n \cdot \widehat{u}(n, T_k^-) = 0$. □

Lemma 2 (Weak–strong compatibility)**.** *For any test function* $\phi \in C_c^\infty(\mathbb{T}^3)$*,*

$$\lim_{a \to 0} \langle \mathcal{C}_\zeta[u](T_k) - u(T_k^-), \phi \rangle = 0. \tag{9}$$

Proof. By Parseval's identity,

$$\langle \mathcal{C}_\zeta[u] - u, \phi \rangle = \sum_{n \in \mathbb{Z}^3} (\zeta(n) - 1)\widehat{u}(n, T_k^-)\widehat{\phi}(-n).$$

For each fixed n, $\zeta(n) \to 1$ as $a \to 0$, and $|\widehat{u}(n, T_k^-)\widehat{\phi}(-n)|$ is summable because $\widehat{u}(n, T_k^-) \in \ell^2$ and $\widehat{\phi} \in \ell^2$. The dominated convergence theorem yields (9). □

3.3. Energy and Enstrophy

Define

$$E(t) = \tfrac{1}{2}\|u(t)\|^2_{L^2}, \qquad \Omega(t) = \tfrac{1}{2}\|\nabla \times u(t)\|^2_{L^2}. \tag{10}$$

Across each interface T_k, substitution of (8) gives

$$E(T_k^-) - E_\zeta(T_k) = \tfrac{1}{2} \sum_{n \in \mathbb{Z}^3} (1 - \zeta(n)^2)|\widehat{u}(n, T_k^-)|^2, \tag{11}$$

$$\Omega(T_k^-) - \Omega_\zeta(T_k) = \tfrac{1}{2} \sum_{n \in \mathbb{Z}^3} |n|^2(1 - \zeta(n)^2)|\widehat{u}(n, T_k^-)|^2. \tag{12}$$

Since $0 < 1 - \zeta(n)^2 \leq Ce^{-a|n|^p}$, both sums converge and vanish as $a \to 0$, preserving the Leray–Hopf energy bound (3). Thus $\mathcal{C}_\zeta$ defines a divergence-free, energy-consistent smoothing of the limiting field at T_k.

4. Spectral Continuation Theorem

Theorem 1 (Existence of piecewise–classical continuation). *Let* $u_0 \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$. *There exists an increasing sequence of continuation times* $0 = T_0 < T_1 < T_2 < \dots$ *and corresponding smooth solutions* $u_k(x,t) \in C^\infty(\mathbb{T}^3 \times (T_k, T_{k+1}))$ *satisfying the Navier–Stokes system* (1) *with matching condition*

$$u_{k+1}(\cdot, T_k) = \mathcal{C}_\zeta[u_k](\cdot, T_k), \tag{13}$$

such that the global field

$$u(x,t) = u_k(x,t) \quad \textit{for } t \in (T_k, T_{k+1})$$

obeys the Leray–Hopf energy inequality (3) *on* $[0,\infty)$. *Each continuation step preserves divergence–free structure, energy consistency, and weak–strong compatibility in the sense of* (9).

Proof. Local well-posedness for smooth initial data on each interval (T_k, T_{k+1}) follows from standard results [5, 6]. At each interface $t = T_k$, Lemma 2 ensures that $u_{k+1}(\cdot, T_k) = \mathcal{C}_\zeta[u_k](\cdot, T_k) \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$ provides smooth restart data satisfying the energy bounds (11).

The continuation times $\{T_k\}$ cannot accumulate in finite time: each continuation dissipates a positive amount of energy $\Delta E_k > 0$ via (11), and the Leray-Hopf inequality (3) bounds the total available energy by $E(0)$. Since $\sum_{k=0}^\infty \Delta E_k \le E(0) < \infty$, we have $T_k \to \infty$.

Iterating this procedure yields a global piecewise-classical solution $u = \bigcup_k u_k$ that satisfies the weak formulation on $[0,\infty)$. □

5. Discussion

The spectral continuation operator $\mathcal{C}_\zeta$ provides a constructive mechanism for extending Navier–Stokes solutions past potential singular times identified by the vorticity growth condition (6), a constructive analogue of the Beale–Kato–Majda criterion [7]. Energy dissipation at each interface (11), coupled with the Leray–Hopf bound (3), ensures non–accumulation of continuation times, yielding a globally defined piecewise–classical weak solution on $\mathbb{T}^3$.

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Declaration of Generative AI and AI-Assisted Technologies in the Preparation of this Work

During the preparation of this work, the author used *OpenAI ChatGPT (GPT-5, 2025)* in order to assist with language refinement, LaTeX formatting, and clarity editing. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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