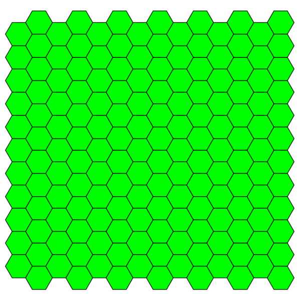
Definitions:

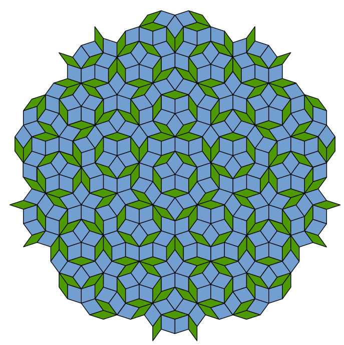
Tiling: A tiling is a collection of specific polygons that collectively "tile" the 2D plane.

A tiling T is a projection of an n-dimensional mother lattice w/ init. cond. $\vec{\gamma} \in \mathbb{R}^n$, denoted:

$T = \mathbb{D}_n(G_n^{\vec{\gamma}})$ // Where $\mathbb{D}_n$ is the projection operator acting on the mother lattice $G_n^{\vec{\gamma}}$

A tiling of this nature is a unitary quasi-crystal. $\mathbb{D}_n$ projects a series of vertices whose edges all have unit length h





These two tiles are unitary, yet the first is a regular repeating crystal, while the second is a non-repeating quasi-crystal.

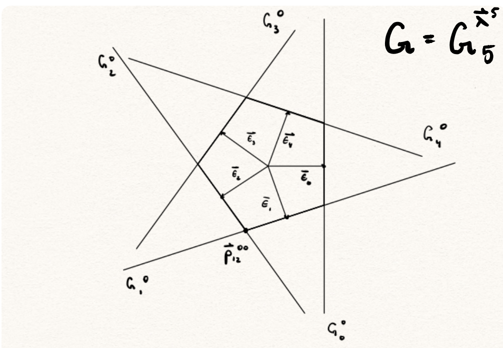
$G_n^{\vec{\gamma}}$ is pseudo n-fold symmetric to the extent that $\vec{\gamma} = \vec{0}^n$, T has $\begin{cases} \frac{n-1}{2} & \text{if odd} \\ \frac{n}{2}-1 & \text{if even} \end{cases}$ unique tile types. Furthermore, any subset of T: $T_s \subset T$ exists an infinite number of times but not periodically.

We wish to derive T:

$\lim\limits_{k\to\infty} \overline{T}^{\circ} = \lim\limits_{k\to\infty} \overline{\mathbb{D}_n(G_n^{\vec{\gamma}}(k))}^{\circ} \cong \mathbb{R}^2$ where k is the range of $G_n^{\vec{\gamma}}$

About $G_n^{\vec{\gamma}}$:

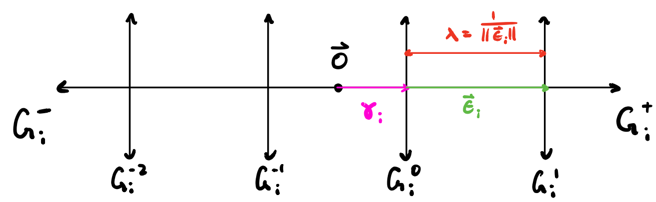
$G_n^{\vec{\gamma}}$ comprises n orthogonal hyper-planes $G_i \subset G_n^{\vec{\gamma}}$, where each G_i themselves comprise $2k+1$ lines $G_i^t \subset G_i$. The 0^{th} index of each hyper-plane G_i^0 is offset from the origin some amount $\gamma_i \in \vec{\gamma}$ along the normal direction of G_i, $\vec{e}_i = e^{\sqrt{-1}\,\zeta_i}$ for $\zeta_i = \frac{2}{n}\pi i$ $\forall i \in \mathbb{N} : i < n$



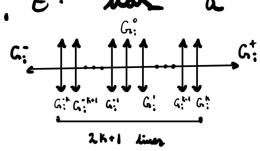
$G = G_5^{\vec{\lambda}^5}$ $(k=0)$

$$G_i := \bigcup_{h \in \omega} G_i^h \quad \forall\, G_i^h \{\vec{x} \in \mathbb{R}^2 : \vec{x}\cdot\vec{e}_i + \gamma_i = h + \tfrac{1}{2}\}$$

$$\omega = \{x \in \mathbb{Z} : -k \le x \le k\}$$



Each cardinal direction, $\vec{e}_i$ has a hyperplane G_i, where G_i is a set of $2\cdot k+1$ lines



If $G_n^{\vec{\gamma}}$ is non-singular, then no more than two lines intersect at a point.

Thus for any point $p_{rs}^{\alpha\beta} \in G_n^{\vec{\gamma}}$

$$p_{rs}^{\alpha\beta} = G_r^{\alpha} \cap G_s^{\beta} \quad \text{for} \quad \underset{0 \le r < s \le n}{G_r^{\alpha} \in G_r,\, G_s^{\beta} \in G_s} \;;\; \underset{0 \le \alpha, \beta \le k}{G_r, G_s \in G_n^{\vec{\gamma}}}$$

Let's note that the tiling T can be constructed by taking the union of the projection of all points $p_{rs}^{\alpha\beta} \in G_n^{\vec{\gamma}}(k)$

$$T = \mathbb{D}_n(G_n^{\vec{\gamma}}(k)) = \bigcup_{0 \le r \le n}\left[\bigcup_{r+1 \le s \le n}\left[\bigcup_{-k \le a \le k}\left[\bigcup_{-k \le b \le k} \mathbb{D}'(p_{rs}^{\alpha\beta})\right]\right]\right]$$

Let's examine the projective operator $\mathbb{D}'(\rho)$.

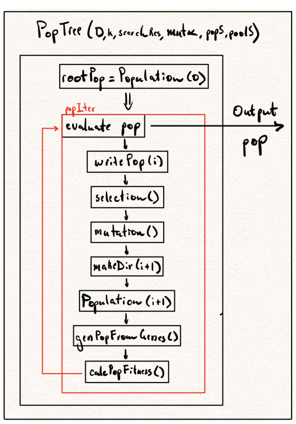
Every $\vec{\rho}_{rs}^{\,\alpha\beta}$ has some $\begin{bmatrix} r \\ s \\ \alpha \\ \beta \end{bmatrix}$ coordinate in $G_n^{\vec{\gamma}}$. We take $\begin{bmatrix} r \\ s \\ \alpha \\ \beta \end{bmatrix} \xrightarrow{\vec{P}} \vec{\rho} = \begin{bmatrix} \rho_x \\ \rho_y \end{bmatrix} \in \mathbb{R}^2$ where $\vec{\rho}$ is the vector location of $\rho_{rs}^{\alpha\beta}$ in $\mathbb{R}^2$. We then take $\begin{bmatrix} \rho_x \\ \rho_y \end{bmatrix} \xrightarrow{\vec{K}} \vec{K}(\vec{\rho}) = \begin{bmatrix} S_0(\vec{\rho}) \\ S_1(\vec{\rho}) \\ \vdots \\ S_{n-1}(\vec{\rho}) \\ S_n(\vec{\rho}) \end{bmatrix} \in \mathbb{R}^n$

Where $\vec{K}(\vec{\rho})$ is the superpositional coordinate of the tile $T_{rs}^{\alpha\beta} \in T$, where the superposition of $\vec{K}(\vec{\rho})$ and the normal vectors mark the coordinates

of $T_{rs}^{\alpha\beta}$: $\begin{bmatrix} S_0(\vec{\rho}) \\ S_1(\vec{\rho}) \\ \vdots \\ S_{n-1}(\vec{\rho}) \\ S_n(\vec{\rho}) \end{bmatrix} \xrightarrow{\vec{E}} \vec{E}(\vec{K}(\vec{\rho}))$, more precisely:

$$T = \bigcup_{r,s,\alpha,\beta} T_{rs}^{\alpha\beta} = \bigcup_{0 \le r \le n} \left[\bigcup_{r+1 \le s \le n} \left[\bigcup_{-K \le a \le K} \left[\bigcup_{-K \le b \le K} \vec{E}(K(\vec{\rho}_{rs}^{\,\alpha\beta})) \right] \right] \right]$$

- for $\Upsilon_i = \|\vec{\gamma}_i\|$ for $P_i^{\theta} = \theta + \tfrac{1}{2} - \Upsilon_i$ and $\zeta_i = \frac{2}{n}\pi i$
- for $\vec{P}(\rho_{rs}^{\alpha\beta}) = \left(\dfrac{P_r^{\alpha} \sin\zeta_s - P_s^{\beta} \sin\zeta_r}{\cos\zeta_r \sin\zeta_s - \cos\zeta_s \sin\zeta_r} \quad \dfrac{\frac{P_r^{\alpha}}{\cos\zeta_r} - \frac{P_s^{\beta}}{\cos\zeta_s}}{\tan\zeta_r - \tan\zeta_s} \right)^T = \begin{bmatrix} \rho_x \\ \rho_y \end{bmatrix}$
- for $K_i(\rho_{rs}^{\alpha\beta}) = \mathcal{A}(r, s, \alpha, \beta, \gamma_i)$ for $\gamma_i \in \vec{\gamma}^{\,n}$
- for $\vec{E}(K(\vec{\rho}_{rs}^{\,\alpha\beta})) = \left[\begin{array}{c|c|c|c} \sum_i (\epsilon_{i,x} \cdot K_{-\rho_i}) & \epsilon_{r,x} + \sum_i (\epsilon_{i,x} \cdot K_{-\rho_i}) & \epsilon_{s,x} + \epsilon_{r,x} + \sum_i (\epsilon_{i,x} \cdot K_{-\rho_i}) & \epsilon_{s,x} + \sum_i (\epsilon_{i,x} \cdot K_{-\rho_i}) \\ \sum_i (\epsilon_{i,y} \cdot K_{-\rho_i}) & \epsilon_{r,y} + \sum_i (\epsilon_{i,y} \cdot K_{-\rho_i}) & \epsilon_{s,y} + \epsilon_{r,y} + \sum_i (\epsilon_{i,y} \cdot K_{-\rho_i}) & \epsilon_{s,y} + \sum_i (\epsilon_{i,y} \cdot K_{-\rho_i}) \end{array} \right]$



With this genetic algorithm we generate several

$\mathbb{D}_n : G_n^{\vec{\gamma}} \to T$, each w/ params:

$$\text{gene} : [\overbrace{n,\ sC,\ K,\ \gamma_{\text{method}},\ \vec{\gamma}}^{\text{grid}},\ \overbrace{\text{tileSize},\ \text{outline}\ \alpha}^{\text{display}},\ \overbrace{\mathcal{A}}^{\substack{\text{projective}\\ \text{functions}}},$$
$$\underbrace{\text{isRgb},\ \text{numStates},\ \text{method},\ \text{Colors},\ G_0}_{\text{init tile values}},\ \underbrace{b \to KC,\ b \to PC,\ \text{maxgen}}_{\text{automata rules}}]$$

After several weeks of running we find that

$\mathcal{A} = \text{round}(\vec{P}_i \cdot \vec{E}_i + \Upsilon_i)$ for $\vec{P}_i \in \vec{P}$ & $\vec{E}_i \in \vec{E}$ satisfies $\mathbb{D}$ universally
$\mathcal{A} = \text{round}(\vec{P}_i \cdot \vec{E}_i - \Upsilon_i)$ satisfies $\mathbb{D}$ generally ($sC = \tfrac{1}{2}$)
$\mathcal{A} = \text{ceil}(\vec{P}_i \cdot \vec{E}_i - \Upsilon_i)$ satisfies $\mathbb{D}$ partially ($sC = 0$)

https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons#/media/File:1-uniform_n1.svg
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