recover Z perfectly if matrix X is available. Scenario 3 is remarkably different from 1 and 2 since vsp does not recover Z and Y exactly from matrix X, and there is no 'yardstick' algorithm for comparison. Hence, for Scenario 3, we study performance of vsp only, for both X and A, which corresponds to $\sigma = 0$ and $\sigma > 0$.

Results of simulations are presented in Tables 1–4. The errors are measured as Frobenius norms $\Delta_Z = \|\hat{Z} - Z\|_F/\sqrt{n\,k}$ and $\Delta_Y = \|\hat{Y} - Y\|_F/\sqrt{n\,d}$, averaged over 1,000 runs. The standard deviations of the means are reported in parentheses.

Tables 1 and 2 confirm that the algorithms designed specifically for SBM and DCBM have better precision that vsp since they 'know' that matrix Z has only one nonzero element per row. However, adjusted vsp, which makes use of this information, performs very similarly to algorithms specifically designed for SBM and DCBM, with clustering algorithm of Gao et al. (2018) being slightly more precise in the case of DCBM. Hence, adjusted vsp can be used for clustering in the SBM (with average miss-classification proportion Δ_Z^2). The errors grow as a decreases and w increases due, respectively to sparsity increase and decline of the signal-to-noise ratio.

Table 3 shows that, as ν grows and kurtosis $\kappa = 3 + 6/(\nu - 4)$ decreases, precision of the vsp declines, even when exact matrix X is available. Therefore, for $\sigma > 0$, we carry out simulations only with $\nu = 5$ ($\kappa = 9$). We set $d = 2n$ for various choices of n. Tables 3 and 4 demonstrate that small kurtosis can be as much of a problem for recovering Z and X as noise. Indeed, errors for small κ do not decline as n and d grow as they do for larger κ and σ.

Conflict of interest: None declared.

References

Gao C., Ma Z., Zhang A. Y., & Zhou H. H. (2018). Community detection in degree-corrected block models. *Annals of Statistics*, *46*(5), 2153–2185. https://www.jstor.org/stable/26542860

Lei J., & Rinaldo A. (2015). Consistency of spectral clustering in stochastic block models. *Annals of Statistics*, *43*(1), 215–237. https://doi.org/10.1214/14-AOS1274

The vote of thanks was passed by acclamation.

https://doi.org/10.1093/jrsssb/qkad031
Advance access publication 6 April 2023

Joshua Cape's contribution to the Discussion of 'Vintage Factor Analysis with Varimax Performs Statistical Inference' by Rohe & Zeng

Joshua Cape

Department of Statistics, University of Wisconsin–Madison

Address for correspondence: Joshua Cape, 1250A Medical Sciences Center, 1300 University Avenue, Madison, WI 53706, USA. Email: jrcape@wisc.edu

I congratulate Professor Rohe and Dr Zeng on their illuminating paper. Their broad contributions will no doubt redouble contemporary research activity in multivariate analysis for years to come. Even the paper's appendices are full of valuable gems, not to be overlooked by readers.

Rohe and Zeng contextualize their findings vis-à-vis classical developments in psychometrics and statistics. This choice is apt, for the authors champion a newfound understanding of varimax factor rotations as inferential, not merely exploratory. In a secondary capacity, the paper further complements a burgeoning body of contemporary work dedicated to the entrywise study of eigenvector estimation, perturbations, and asymptotics, in network analysis (e.g., Abbe et al., 2020; Tang & Priebe, 2018) and in high-dimensional statistics (e.g., Cape et al., 2019b; Fan et al., 2018). Under certain data-generating conditions, these works characterize the behaviour of individual vectors and their coordinates in low-dimensional point cloud representations of data, in a much more precise manner than was considered in past decades, let alone during the time of Thurstone or Kaiser.

Rohe and Zeng emphasize two forms of sparsity in their treatment of stochastic blockmodel random graphs (SBMs) in the appendix: network sparsity, and latent variable sparsity. The contemporary literature has heretofore largely overlooked the latter and its implications, notably in papers dedicated to adjacency spectral embedding in the time since (Sussman et al., 2012).

For SBMs, Rohe and Zeng establish a high-probability uniform error bound which I summarize as $\max_{1\leq i\leq n}\|\widehat{Z}-ZP_n\|_{i,\ell_2}=\mathrm{o}_{\mathbb{P}}(1)$ in the moderately sparse regime $n\rho_n \gtrsim (\log n)^c$. Here $\|\cdot\|_{i,\ell_2}$ denotes the ℓ_2 vector norm of the i-th row of a given matrix, while $n\rho_n$ reflects nodal expected degree. Consequently, perfect clustering (discrimination) is achieved using $\widehat{Z}$; more interestingly, the sparse latent variable matrix Z can be element-wise directly estimated uniformly well.

More can be said. The eigenvector analysis in Cape et al. (2019a) and Rubin-Delanchy et al. (2022) hints that asymptotic normality might hold for the varimax-based estimator at the scaling $\sqrt{n\rho_n}$. Indeed, it has recently been established in Cape (2022) that for certain SBM graphs, loosely speaking, the i-th row vector of $\sqrt{n\rho_n}(\widehat{Z}-ZP_n)$ is conditionally asymptotically multivariate Gaussian, with block-specific asymptotic covariance matrix.

Unanswered questions abound. Among them, it would be interesting to quantify the estimation performance of varimax rotations in dimension $\widehat{k}\neq k_{\text{true}}$ and when the coefficient matrix B is rank degenerate.

Conflict of interest: None declared.

References

Abbe E., Fan J., Wang K., & Zhong Y. (2020). Entrywise eigenvector analysis of random matrices with low expected rank. *Annals of Statistics*, *48*(3), 1452–1474. https://doi.org/10.1214/19-AOS1854

Cape J. (2022). On varimax asymptotics in network models and spectral methods for dimensionality reduction. Under review.

Cape J., Tang M., & Priebe C. E. (2019a). Signal-plus-noise matrix models: Eigenvector deviations and fluctuations. *Biometrika*, *106*(1), 243–250. https://doi.org/10.1093/biomet/asy070

Cape J., Tang M., & Priebe C. E. (2019b). The two-to-infinity norm and singular subspace geometry with applications to high-dimensional statistics. *Annals of Statistics*, *47*(5), 2405–2439. https://doi.org/10.1214/18-AOS1752

Fan J., Wang W., & Zhong Y. (2018). An ℓ_∞ eigenvector perturbation bound and its application to robust covariance estimation. *Journal of Machine Learning Research*, *18*(207), 1–42.

Rubin-Delanchy P., Cape J., Tang M., & Priebe C. E. (2022). A statistical interpretation of spectral embedding: The generalised random dot product graph. *Journal of the Royal Statistical Society, Series B*, *84*(4), 1446–1473. https://doi.org/10.1111/rssb.12509

Sussman D. L., Tang M., Fishkind D. E., & Priebe C. E. (2012). A consistent adjacency spectral embedding for stochastic blockmodel graphs. *Journal of the American Statistical Association*, *107*(499), 1119–1128. https://doi.org/10.1080/01621459.2012.699795

Tang M., & Priebe C. E. (2018). Limit theorems for eigenvectors of the normalized Laplacian for random graphs. *Annals of Statistics*, *46*(5), 2360–2415. https://doi.org/10.1214/17-AOS1623

https://doi.org/10.1093/jrsssb/qkad032
Advance access publication 5 April 2023