Supplementary material to "Asymptotically efficient estimators for stochastic blockmodels: the naive MLE, the rank-constrained MLE, and the spectral estimator"

MINH TANG[1], JOSHUA CAPE[2] and CAREY E. PRIEBE[3]

[1]*Department of Statistics, North Carolina State University, E-mail:* mtang8@ncsu.edu

[2]*Department of Statistics, University of Pittsburgh, E-mail:* joshua.cape@pitt.edu

[3]*Applied Mathematics and Statistics, Johns Hopkins University, E-mail:* cep@jhu.edu

This document contains the technical proofs for the results stated in "Asymptotically efficient estimators for stochastic blockmodels: the naive MLE, the rank-constrained MLE, and the spectral estimator". Section A provides a proof of Theorem 2. Section B provides the proofs for Theorem 3, Theorem 4, and Lemma 1. Derivations of the covariance terms in Theorem 3 and Theorem 4 are given in Section C.

A. Proof of Theorem 2

Recall our assumption that $\mathrm{rk}(\mathbf{B}) = d$ and that, without loss of generality, the top left $d \times d$ block of $\mathbf{B}$ is invertible. Write $\mathbf{B}$ in blocks form as

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

where the dimensions of $\mathbf{B}_{11}, \mathbf{B}_{12} = \mathbf{B}_{21}^{\top}$ and $\mathbf{B}_{22}$ are $d \times d$, $d \times (K-d)$ and $(K-d) \times (K-d)$, respectively. Since $\mathbf{B}_{11}$ is invertible and $\mathrm{rk}(\mathbf{B}) = d$, we have $\mathbf{B}_{22} = \mathbf{B}_{21}\mathbf{B}_{11}^{-1}\mathbf{B}_{12}$. Let $\nu = \mathrm{vech}(\mathbf{B}_{11}) \in \mathbb{R}^{d(d+1)}$, $\eta = \mathrm{vec}(\mathbf{B}_{11}^{-1}\mathbf{B}_{12}) \in \mathbb{R}^{d(K-d)}$. Define $\theta \in \mathbb{R}^{K(K+1)/2}$ as

$$\begin{aligned} \theta &= (\mathrm{vech}(\mathbf{B}_{11}), \mathrm{vec}(\mathbf{B}_{12}), \mathrm{vech}(\mathbf{B}_{22})) \\ &= (\nu, \mathrm{vec}(\mathbf{B}_{11}\mathbf{B}_{11}^{-1}\mathbf{B}_{12}), \mathrm{vech}(\mathbf{B}_{12}^{\top}\mathbf{B}_{11}^{-1}\mathbf{B}_{11}\mathbf{B}_{11}^{-1}\mathbf{B}_{12})) \\ &= (\nu, \mathrm{vec}(\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta)), \mathrm{vech}(\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta))) \end{aligned}$$

Here $\mathrm{vech}^{-1}(\nu)$ denote the unique symmetric matrix $\mathbf{M_1}$ with $\mathrm{vech}(\mathbf{M}_1) = \nu$ and $\mathrm{vec}^{-1}(\eta)$ denote the unique $d \times (K-d)$ matrix $\mathbf{M}_2$ with $\mathrm{vec}(\mathbf{M}_2) = \eta$. The elements of θ are functions of $\nu = \mathrm{vech}(\mathbf{B}_{11})$ and η, i.e., $\theta = g(\nu, \eta)$ where g is a function mapping from $\mathbb{R}^{d(d+1)/2 + d(K-d)}$ to $\mathbb{R}^{K(K+1)/2}$. By the invariance property of maximum likelihood estimation, $\hat{\theta} = g(\hat{\nu}, \hat{\eta})$ is the maximum likelihood estimator for θ; here $(\hat{\nu}, \hat{\eta})$ are the maximum likelihood estimates of (ν, η). Suppose now that $(\hat{\nu}, \hat{\eta})$ converges to multivariate normal, i.e.,

$$n\sqrt{\rho_n}((\hat{\nu}, \hat{\eta}) - (\nu, \eta)) \longrightarrow \mathcal{N}(0, \mathbf{V}^{-1})$$

for some covariance matrix $\mathbf{V}^{-1}$. Then by the delta method

$$n\sqrt{\rho_n}(\hat{\theta} - \theta) \longrightarrow \mathcal{N}(0, \mathcal{J}\mathbf{V}^{-1}\mathcal{J}^{\top})$$

where $\mathcal{J}$ is the Jacobian matrix of partial derivatives for g.

We first evaluate $\mathcal{J}$. Writing $\mathcal{J}$ in blocks form, we have

$$\mathcal{J} = \left[\frac{d\theta}{d\nu}, \frac{d\theta}{d\eta}\right] = \begin{bmatrix} \frac{d\nu}{d\nu}, & \frac{d\nu}{d\eta} \\ \frac{d\mathrm{vec}(\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta))}{d\nu}, & \frac{d\mathrm{vec}(\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta))}{d\eta} \\ \frac{d\mathrm{vech}((\mathrm{vec}^{-1}(\eta))^{\top}\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta))}{d\nu}, & \frac{d\mathrm{vech}((\mathrm{vec}^{-1}(\eta))^{\top}\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta))}{d\eta} \end{bmatrix}$$

We have

$$\frac{d\nu}{d\nu} = \mathbf{I}_{d(d+1)/2}, \quad \frac{d\nu}{d\eta} = \mathbf{0}_{d(d+1)/2 \times d(K-d)}.$$

We now recall a relationship between Kronecker product and vectorizing a matrix,

$$\mathrm{vec}(\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3) = (\mathbf{M}_3^{\top} \otimes \mathbf{M}_1)\mathrm{vec}(\mathbf{M}_2).$$

We therefore have

$$\frac{d}{d\eta}\mathrm{vec}(\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta)) = \frac{d}{d\eta}(\mathbf{I}_{K-d} \otimes \mathrm{vech}^{-1}(\nu))\eta = (\mathbf{I}_{K-d} \otimes \mathrm{vech}^{-1}(\nu)).$$

Furthermore, using the definition of duplication matrices,

$$\begin{aligned} \frac{d}{d\nu}\mathrm{vec}(\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta)) &= \frac{d}{d\nu}(\mathrm{vec}^{-1}(\eta)^{\top} \otimes \mathbf{I}_d)\mathrm{vec}(\mathrm{vech}^{-1}(\nu)) \\ &= \frac{d}{d\nu}(\mathrm{vec}^{-1}(\eta)^{\top} \otimes \mathbf{I}_d)\mathcal{D}_d\nu = (\mathrm{vec}^{-1}(\eta)^{\top} \otimes \mathbf{I}_d)\mathcal{D}_d. \end{aligned}$$

Analogously, using the definition of elimination matrices,

$$\begin{aligned} \frac{d}{d\nu}\mathrm{vech}(\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta)) &= \frac{d}{d\nu}\mathcal{L}_{K-d}(\mathrm{vec}^{-1}(\eta)^{\top} \otimes \mathrm{vec}^{-1}(\eta)^{\top})\mathrm{vec}(\mathrm{vech}^{-1}(\nu)) \\ &= \mathcal{L}_{K-d}(\mathrm{vec}^{-1}(\eta)^{\top} \otimes \mathrm{vec}^{-1}(\eta)^{\top})\mathcal{D}_d. \end{aligned}$$

Now recall the definition of the commutation matrix $\mathcal{T}_{nm}$ $n, m \geq 1$, i.e., $\mathcal{T}_{mn}$ is the unique $mn \times mn$ matrix such that, for any $n \times m$ matrix $\mathbf{M}$,

$$\mathrm{vec}(\mathbf{A}^{\top}) = \mathcal{T}_{mn}\mathrm{vec}(\mathbf{A}). \tag{A.1}$$

Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be matrices of dimensions $p \times q$ and $m \times n$. Then

$$\mathcal{T}_{pm}(\mathbf{M}_1 \otimes \mathbf{M}_2) = (\mathbf{M}_2 \otimes \mathbf{M}_1)\mathcal{T}_{qn}. \tag{A.2}$$

Eq. (A.1), Eq. (A.2), and the chain rule for matrix differential [8, Theorem 9] together imply

$$\frac{d}{d\eta}\mathrm{vech}(\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta)) =$$

$$\mathcal{L}_{k-d}\frac{d}{d\eta}\mathrm{vec}(\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu)\mathrm{vec}^{-1}(\eta)) =$$

$$\mathcal{L}_{K-d}\Big(\big(\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu)\otimes\mathbf{I}_{K-d}\big)\frac{d}{d\eta}\mathrm{vec}(\mathrm{vec}^{-1}(\eta)^{\top}) + (\mathbf{I}_{K-d}\otimes\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu))\frac{d\eta}{d\eta}\Big) =$$

$$\mathcal{L}_{K-d}\Big(\big(\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu)\otimes\mathbf{I}_{K-d}\big)\frac{d}{d\eta}\mathcal{T}_{d(K-d)}\eta + (\mathbf{I}_{K-d}\otimes\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu))\Big) =$$

$$\mathcal{L}_{K-d}\Big(\big(\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu)\otimes\mathbf{I}_{K-d}\big)\mathcal{T}_{d(K-d)} + (\mathbf{I}_{K-d}\otimes\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu))\Big) =$$

$$\mathcal{L}_{K-d}\Big(\mathbf{I}_{(K-d)^2}+\mathcal{T}_{(K-d)^2}\Big)\Big(\mathbf{I}_{K-d}\otimes\mathrm{vec}^{-1}(\eta)^{\top}\mathrm{vech}^{-1}(\nu)\Big).$$

Collecting the above terms yields the Jacobian in Eq. (1.13) of the manuscript.

We next argue that $(\hat{\nu},\hat{\eta})-(\nu,\eta)$ converges to multivariate normal. Let $\mathbf{A}$ be a stochastic blockmodel graphs with block probabilities matrix $\mathbf{B}$. Assume that the the true vertex assignment τ is known; if τ is unknown then it can be perfectly recovered asymptotically almost surely as $n\to\infty$, provided that $n\rho_n=\omega(\log n)$. The log likelihood for (ν,η) given $\mathbf{A}$ is then

$$\ell((\nu,\eta)\mid\mathbf{A})=\sum_{r=1}^{K}\sum_{s=1}^{K}n_{rs}\log b_{rs}(\nu,\eta)+(m_{rs}-n_{rs})\log(1-b_{rs}(\nu,\eta))$$

where n_{rs} and m_{rs} are the number of observed edges and the maximum number of possible edges, respectively, between vertices in block rs and vertices in block s. By the classical theory of maximum likelihood estimation, e.g., [5, Theorem 6.5.1], $((\hat{\nu},\hat{\eta})-(\nu,\eta))$ converges to multivariate normal with covariance matrix $\mathbf{V}^{-1}$ where, recall that $\theta=\mathrm{vech}(\mathbf{B})$,

$$\mathbf{V}=\mathbb{E}\Big[\Big(\tfrac{\partial\ell}{\partial(\nu,\eta)}\Big)\Big(\tfrac{\partial\ell}{\partial(\nu,\eta)}\Big)^{\top}\Big]=\mathbb{E}\Big[\Big(\Big(\tfrac{\partial\theta}{\partial(\nu,\eta)}\Big)^{\top}\tfrac{\partial\ell}{\partial\theta}\Big)\Big(\Big(\tfrac{\partial\theta}{\partial(\nu,\eta)}\Big)^{\top}\tfrac{\partial\ell}{\partial\theta}\Big)^{\top}\Big]=\mathcal{J}^{\top}\mathbb{E}\Big[\Big(\tfrac{\partial\ell}{\partial\theta}\Big)\Big(\tfrac{\partial\ell}{\partial\theta}\Big)^{\top}\Big]\mathcal{J}$$

is the Fisher information matrix and $\mathcal{J}$ is the Jacobian matrix derived earlier. The expression for $\mathbf{D}=\mathbb{E}[(\frac{\partial\ell}{\partial\theta})(\frac{\partial\ell}{\partial\theta})^{\top}]$ follows from direct calculations.

B. Proof of Theorem 3 and Theorem 4

We first provide an outline of the main steps in the proof of Theorem 3 and Theorem 4. We derive Eq. (2.6) (and analogously Eq. (2.10)) by considering the following decomposition of $(\hat{\mathbf{B}}^{(S)}_{k\ell} - \mathbf{B}_{k\ell})$

$$
\begin{aligned}
n\rho_n^{1/2}(\hat{\mathbf{B}}^{(S)}_{k\ell} - \mathbf{B}_{k\ell}) &= \tfrac{n\rho_n^{1/2}}{\hat{n}_k\hat{n}_\ell\rho_n}\hat{\boldsymbol{s}}_k^\top \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top \hat{\boldsymbol{s}}_\ell - \tfrac{n\rho_n^{1/2}}{n_k n_\ell \rho_n}\boldsymbol{s}_k^\top \mathbb{E}[\mathbf{A}]\boldsymbol{s}_\ell \\
&= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top(\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top - \mathbf{U}\mathbf{U}^\top\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top\mathbf{U}\mathbf{U}^\top)\boldsymbol{s}_\ell \\
&+ \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \mathbf{U}(\mathbf{U}^\top\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda}\mathbf{U}^\top\hat{\mathbf{U}})\hat{\mathbf{U}}^\top\mathbf{U}\mathbf{U}^\top\boldsymbol{s}_\ell \\
&+ \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \mathbf{U}\boldsymbol{\Lambda}(\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{U}}^\top\mathbf{U} - \mathbf{I})\mathbf{U}^\top\boldsymbol{s}_\ell + o_{\mathbb{P}}(1)
\end{aligned} \tag{B.1}
$$

Our proof proceeds by writing each term on the right hand side of Eq. (B.1) as, when conditioned on $\mathbf{P}$, linear combinations of the independent random variables $\{\mathbf{A}_{ij} - \mathbf{P}_{ij}\}_{i \leq j}$ and residual terms of smaller order. More specifically, letting $\mathbf{E} = \mathbf{A} - \mathbf{P}$, $\boldsymbol{\Pi}_{\mathbf{U}} = \mathbf{U}\mathbf{U}^\top$, $\boldsymbol{\Pi}_{\mathbf{U}}^\perp = \mathbf{I} - \boldsymbol{\Pi}_{\mathbf{U}}$ and $\mathbf{P}^\dagger = \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top$ the Moore-Penrose pseudoinverse of $\mathbf{P}$, we show that

$$
\begin{aligned}
\xi^{(1)}_{k\ell} &:= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top(\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top - \mathbf{U}\mathbf{U}^\top\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top\mathbf{U}\mathbf{U}^\top)\boldsymbol{s}_\ell \\
&= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp\big(\mathbf{E}\boldsymbol{\Pi}_{\mathbf{U}} + \mathbf{E}^2\mathbf{P}^\dagger\big)\boldsymbol{s}_\ell \\
&+ \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp\big(\mathbf{E}\boldsymbol{\Pi}_{\mathbf{U}} + \mathbf{E}^2\mathbf{P}^\dagger\big)\boldsymbol{s}_k + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}),
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
\xi^{(3)}_{k\ell} &:= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \mathbf{U}\boldsymbol{\Lambda}(\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{U}}^\top\mathbf{U} - \mathbf{I})\mathbf{U}^\top\boldsymbol{s}_\ell. \\
&= -\tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}\mathbf{E}^2\mathbf{P}^\dagger\boldsymbol{s}_\ell + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}),
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
\xi^{(2)}_{k\ell} &:= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \mathbf{U}(\mathbf{U}^\top\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda}\mathbf{U}^\top\hat{\mathbf{U}})\hat{\mathbf{U}}^\top\mathbf{U}\mathbf{U}^\top\boldsymbol{s}_\ell \\
&= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}\mathbf{E}\boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_\ell - \xi^{(3)}_{k\ell} + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}).
\end{aligned} \tag{B.4}
$$

The above expressions for $\xi^{(1)}_{k\ell}, \xi^{(2)}_{k\ell}$ and $\xi^{(3)}_{k\ell}$ implies

$$
\begin{aligned}
n\rho_n^{1/2}(\hat{\mathbf{B}}^{(S)}_{k\ell} - \mathbf{B}_{k\ell}) &= \frac{n\rho_n^{-1/2}}{n_k n_\ell}\big(\boldsymbol{s}_k^\top \mathbf{E}\boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_\ell + \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \mathbf{E}\boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_k\big) \\
&+ \frac{n\rho_n^{-1/2}}{n_k n_\ell}\big(\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp\mathbf{E}^2\mathbf{P}^\dagger\boldsymbol{s}_\ell + \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp\mathbf{E}^2\mathbf{P}^\dagger\boldsymbol{s}_k\big) \\
&+ O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}).
\end{aligned} \tag{B.5}
$$

We complete the proof of by showing that

$$\begin{aligned} Z_{k\ell} &:= \frac{n\rho_n^{-1/2}}{n_k n_\ell}\Big(\boldsymbol{s}_k^\top \mathbf{E}\boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_\ell + \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \mathbf{E}\boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_k\Big) \\ &= \frac{n\rho_n^{-1/2}}{n_k n_\ell}\operatorname{tr}\,\mathbf{E}(\boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_\ell\boldsymbol{s}_k^\top + \boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_k\boldsymbol{s}_\ell^\top\boldsymbol{\Pi}_{\mathbf{U}}^\perp) \\ &= \frac{n\rho_n^{-1/2}}{n_k n_\ell}\operatorname{tr}\,\mathbf{E}(\boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_\ell\boldsymbol{s}_k^\top + \boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_k\boldsymbol{s}_\ell^\top - \boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_k\boldsymbol{s}_\ell^\top\boldsymbol{\Pi}_{\mathbf{U}}) \end{aligned} \tag{B.6}$$

converges to a normally distributed random variable, and that

$$\frac{n}{n_k n_\ell}\big(\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \mathbf{E}^2\mathbf{P}^\dagger + \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \mathbf{E}^2\mathbf{P}^\dagger \boldsymbol{s}_k\big) \xrightarrow{\text{a.s.}} \begin{cases} \theta_{k\ell} & \text{if } \rho_n \equiv 1 \\ \widetilde{\theta}_{k\ell} & \text{if } \rho_n \to 0 \end{cases} \tag{B.7}$$

as $n \to \infty$. Note the difference in scaling for the convergence of $Z_{k\ell}$ (scaling by $\frac{n\rho_n^{-1/2}}{n_k n_\ell}$) and the scaling in Eq. (B.7) (scaling by $\frac{n}{n_k n_\ell}$).

We now provide the necessary details for the proof sketch outlined above. We shall repeatedly make use of the following concentration bounds for $\|\mathbf{A} - \mathbf{P}\|$ and related quantities. We consolidated these bounds in the following lemma.

Lemma B.1. *Let $\mathbf{A} \sim \mathrm{GRDPG}_{p,q}(F)$ be a generalized random dot product graph on n vertices with sparsity factor ρ_n. Suppose $n\rho_n = \omega(\log^4(n))$. Then*

$$\|\mathbf{A} - \mathbf{P}\| = O_{\mathbb{P}}((n\rho_n)^{1/2}) \tag{B.8}$$

$$\|\mathbf{U}\mathbf{U}^\top - \hat{\mathbf{U}}\hat{\mathbf{U}}^\top\| = O_{\mathbb{P}}((n\rho_n)^{-1/2}) \tag{B.9}$$

$$\|(\mathbf{I} - \mathbf{U}\mathbf{U}^\top)\hat{\mathbf{U}}\| = O_{\mathbb{P}}((n\rho_n)^{-1/2}) \tag{B.10}$$

$$\|(\mathbf{I} - \hat{\mathbf{U}}\hat{\mathbf{U}}^\top)\mathbf{U}\| = O_{\mathbb{P}}((n\rho_n)^{-1/2}). \tag{B.11}$$

In addition, there exists an orthogonal matrix $\mathbf{W}$ such that

$$\|\mathbf{U}^\top\hat{\mathbf{U}} - \mathbf{W}\| = O_{\mathbb{P}}((n\rho_n)^{-1}) \tag{B.12}$$

The bound for $\|\mathbf{A} - \mathbf{P}\|$ in Eq. (B.8) is due to [7]. For ease of exposition, we have stated Eq. (B.8) in the context of our paper and hence the upper bound is given in terms of the factor $n\rho_n$; the original bound holds for the more general inhomogeneous random graphs model where the upper bound is now given in terms of $\sqrt{\delta}$ where $\delta = \max_i \sum_j \mathbf{P}_{ij}$ is the maximum expected degree. Similar upper bounds can be found in [6, 9, 11] with slightly different assumptions on $\mathbf{P}$. Eq. (B.9) through Eq. (B.11) are variants of the same bound for the sin-Θ distance between the subspaces spanned by $\mathbf{U}$ and $\hat{\mathbf{U}}$ and followed from Eq. (B.8) and the Davis-Kahan theorem [3, 12]. Eq. (B.12) follows from Eq. (B.9) via the following argument. Let $\sigma_1, \sigma_2, \dots, \sigma_d$ denote the singular values of $\mathbf{U}^\top\hat{\mathbf{U}}$. Then $\sigma_i = \cos(\theta_i)$ where the θ_i are the principal angles between the subspaces spanned by $\mathbf{U}^\top\hat{\mathbf{U}}$. Eq. (B.9) implies

$$\|\mathbf{U}\mathbf{U}^\top - \hat{\mathbf{U}}\hat{\mathbf{U}}^\top\| = \max_i |\sin(\theta_i)| = O_{\mathbb{P}}((n\rho_n)^{-1/2}).$$

Let $\mathbf{W}_1 \boldsymbol{\Sigma} \mathbf{W}_2^\top$ be the singular value decomposition of $\mathbf{U}^\top \hat{\mathbf{U}}$ and let $\mathbf{W} = \mathbf{W}_1 \mathbf{W}_2^\top$. We then have

$$\|\mathbf{U}^\top \hat{\mathbf{U}} - \mathbf{W}\|_F = \|\boldsymbol{\Sigma} - \mathbf{I}\|_F = \Bigl(\sum_{i=1}^{d} (1 - \sigma_i)^2\Bigr)^{1/2} \leq \sum_{i=1}^{d} (1 - \sigma_i^2) = \sum_{i=1}^{d} \sin^2(\theta_i).$$

Hence $\|\mathbf{U}^\top \hat{\mathbf{U}} - \mathbf{W}\|_F = O_{\mathbb{P}}((n\rho_n)^{-1})$ as desired.

***Remark* B.1.** *We take a brief detour to provide a direct proof of Corollary 2 for when $\mathbf{B}$ is full-rank. Comparing this proof with the subsequent proofs of Theorem 3 and Theorem 4, we see that the case when $\mathbf{B}$ is singular is much more involved.*

Recall the definition of $\hat{\mathbf{B}}^{(S)}_{k\ell}$ and let $\boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} = (\mathbf{I} - \hat{\mathbf{U}}\hat{\mathbf{U}}^\top)$. We then have

$$\begin{aligned} n\rho_n^{1/2}(\hat{\mathbf{B}}^{(S)}_{k\ell} - \mathbf{B}_{k\ell}) &= \frac{n\rho_n^{-1/2}}{n_k n_\ell}\Bigl(\boldsymbol{s}_k^\top \hat{\mathbf{U}} \hat{\boldsymbol{\Lambda}} \hat{\mathbf{U}}^\top \boldsymbol{s}_\ell - \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top \mathbf{P} \boldsymbol{s}_\ell\Bigr) \\ &= \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top (\mathbf{A} - \mathbf{P}) \boldsymbol{s}_\ell - \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \mathbf{A} \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \boldsymbol{s}_\ell. \end{aligned}$$

We now have the important fact that $\boldsymbol{s}_k = \mathbf{U}\mathbf{U}^\top \boldsymbol{s}_k$ for all $k = 1, \dots, K$. Indeed, let $\mathbf{Z}$ be the $n \times K$ matrix whose columns are the $\boldsymbol{s}_k$. Then each row of $\mathbf{Z}$ contains a single "1" and $K-1$ "0" and $z_{ij} = 1$ if and only if the ith vertex is assigned to the jth block. We then have $\mathbf{P} = \mathbf{Z}\mathbf{B}\mathbf{Z}^\top$ and thus $\mathbf{U}\mathbf{U}^\top$ is the projection onto the column space of $\mathbf{P}$ which coincides with the projection onto the column space of $\mathbf{Z}$ when $\mathbf{B}$ is full-rank. We therefore have

$$\begin{aligned} |\boldsymbol{s}_k^\top \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \mathbf{A} \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \boldsymbol{s}_\ell| &= |\boldsymbol{s}_k \mathbf{U}\mathbf{U}^\top \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \mathbf{A} \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \mathbf{U}\mathbf{U}^\top \boldsymbol{s}_\ell| \\ &\leq \|\boldsymbol{s}_k\| \times \|\mathbf{U}^\top \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}}\|^2 \times \|\boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \mathbf{A} \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}}\| \times \|\boldsymbol{s}_\ell\|. \end{aligned}$$

Eq. (B.11) implies $\|\mathbf{U}^\top \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}}\|^2 = O((n\rho_n)^{-1})$. Let $|\lambda_{d+1}(\mathbf{A})| = \|\boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \mathbf{A} \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}}\|$ be the modulus of the $d+1$ largest eigenvalue of $\mathbf{A}$ in modulus. Since $\mathbf{P}$ is of rank d, Weyl's inequality and Eq. (B.8) imply

$$\|\boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \mathbf{A} \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}}\| \leq \|\mathbf{A} - \mathbf{P}\| = O((n\rho_n)^{1/2}).$$

We therefore have

$$\frac{n\rho_n^{-1/2}}{n_k n_\ell} |\boldsymbol{s}_k^\top \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \mathbf{A} \boldsymbol{\Pi}^\perp_{\hat{\mathbf{U}}} \boldsymbol{s}_\ell| \leq \frac{n\rho_n^{-1/2}}{n_k n_\ell} \times O(n) \times O((n\rho_n)^{-1/2}) = O(n^{-1/2}\rho_n^{-1})$$

which converges to 0 for $n\rho_n = \omega(\sqrt{n})$. Thus, for $n\rho_n = \omega(\sqrt{n})$,

$$\begin{aligned} n\rho_n^{/2}(\hat{\mathbf{B}}^{(S)}_{k\ell} - \mathbf{B}) &= \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top (\mathbf{A} - \mathbf{P}) \boldsymbol{s}_\ell + o_{\mathbb{P}}(1) \\ &= \frac{n\rho_n^{-1/2}}{n_k n_\ell} \sum_{i \colon \tau_i = k} \sum_{j \colon \tau_j = \ell} (a_{ij} - p_{ij}) + o_{\mathbb{P}}(1) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \sigma^2_{k\ell}) \end{aligned}$$

where the $\sigma^2_{k\ell}$ are given in Eq. (2.4) and Eq. (2.5) for the case when $\rho_n \equiv 1$ and are given in Eq. (2.8) and Eq. (2.9) when $\rho_n \to 0$.

***Remark* B.2.** *We take another detour and prove Corollary 3 which provides a more succinct expression for the covariance matrix of $\hat{\mathbf{B}}^{(S)}$. Rewrite Eq. (B.5) as*

$$\begin{aligned} n\rho_n^{1/2}(\hat{\mathbf{B}}^{(S)}_{k\ell} - \mathbf{B}_{k\ell}) &= \frac{n\rho_n^{-1/2}}{n_k n_\ell}\big(\boldsymbol{s}_k^\top(\mathbf{A}-\mathbf{P})\boldsymbol{s}_\ell - \boldsymbol{s}_k^\top(\mathbf{I}-\mathbf{U}\mathbf{U}^\top)(\mathbf{A}-\mathbf{P})(\mathbf{I}-\mathbf{U}\mathbf{U}^\top)\boldsymbol{s}_\ell\big) \\ &+ \frac{n\rho_n^{-1/2}}{n_k n_\ell}\big(\boldsymbol{s}_k^\top(\mathbf{I}-\mathbf{U}\mathbf{U}^\top)(\mathbf{A}-\mathbf{P})^2\mathbf{P}^\dagger\boldsymbol{s}_\ell + \boldsymbol{s}_k^\top\mathbf{P}^\dagger(\mathbf{A}-\mathbf{P})^2(\mathbf{I}-\mathbf{U}\mathbf{U}^\top)\boldsymbol{s}_\ell\big) \\ &+ O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}). \end{aligned}$$

Eq. (B.7), which we will derived subsequently, implies

$$\frac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top\Big((\mathbf{I}-\mathbf{U}\mathbf{U}^\top)(\mathbf{A}-\mathbf{P})^2\mathbf{P}^\dagger\boldsymbol{s}_\ell + \mathbf{P}^\dagger(\mathbf{A}-\mathbf{P})^2(\mathbf{I}-\mathbf{U}\mathbf{U}^\top)\Big)\boldsymbol{s}_\ell \to \begin{cases} \theta_{k\ell} & \text{if } \rho_n \equiv 1 \\ \widetilde{\theta}_{k\ell} & \text{if } \rho_n \to 0 \end{cases}.$$

Now let $\mathbf{Z}$ be the $n \times K$ matrix whose columns are the $\boldsymbol{s}_k$. Rewriting the above displayed equation, for $1 \leq k \leq K$ and $1 \leq \ell \leq K$ in matrix form, we have

$$\begin{aligned} n\rho_n^{1/2}(\hat{\mathbf{B}}^{(S)} - \mathbf{B} - \frac{\boldsymbol{\Theta}}{n\rho_n}) &= n\rho_n^{-1/2}\mathbf{Z}^\dagger(\mathbf{A}-\mathbf{P})(\mathbf{Z}^\dagger)^\top \\ &- n\rho_n^{-1/2}\mathbf{Z}^\dagger(\mathbf{I}-\mathbf{U}\mathbf{U})^\top(\mathbf{A}-\mathbf{P})(\mathbf{I}-\mathbf{U}\mathbf{U}^\top)(\mathbf{Z}^\dagger)^\top + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}). \end{aligned}$$

Here $\boldsymbol{\Theta}$ denote the matrix whose elements are the $\theta_{k\ell}$ or the $\widetilde{\theta}_{k\ell}$ depending on whether $\rho_n \equiv 1$ or $\rho_n \to 0$, and $\mathbf{Z}^\dagger = (\mathbf{Z}^\top\mathbf{Z})^{-1}\mathbf{Z}^\top$ is the Moore-Penrose pseudoinverse for $\mathbf{Z}$; note that the columns of $\mathbf{Z}^\dagger$ are given by $n_k^{-1}\boldsymbol{s}_k$. Now $\mathbf{P} = \mathbf{Z}\mathbf{B}\mathbf{Z}^\top$ and hence, letting $\mathbf{V}$ denote the matrix whose columns are the eigenvectors corresponding to the non-zero eigenvalues of $\mathbf{B}$, we have $\mathbf{U}\mathbf{U}^\top = \mathbf{Z}\mathbf{V}(\mathbf{V}^\top\mathbf{Z}^\top\mathbf{Z}\mathbf{V})^{-1}\mathbf{V}^\top\mathbf{Z}^\top$ and hence

$$\mathbf{Z}^\dagger(\mathbf{I}-\mathbf{U}\mathbf{U})^\top = (\mathbf{I} - \mathbf{V}(\mathbf{V}^\top\mathbf{Z}^\top\mathbf{Z}\mathbf{V})^{-1}\mathbf{V}^\top\mathbf{Z}^\top\mathbf{Z})\mathbf{Z}^\dagger.$$

Let $\breve{\boldsymbol{\Pi}}^\perp_{\mathbf{V}} = (\mathbf{I} - \mathbf{V}(\mathbf{V}^\top\mathbf{Z}^\top\mathbf{Z}\mathbf{V})^{-1}\mathbf{V}^\top\mathbf{Z}^\top\mathbf{Z})$. Note that $\breve{\boldsymbol{\Pi}}^\perp_{\mathbf{V}}$ is an idempotent *matrix but $\breve{\boldsymbol{\Pi}}^\perp_{\mathbf{V}}$ is not necessarily symmetric , i.e., $\breve{\boldsymbol{\Pi}}^\perp_{\mathbf{V}}$ in general defines an* oblique *projection. We therefore have*

$$\begin{aligned} n\rho_n^{1/2}(\hat{\mathbf{B}}^{(S)} - \mathbf{B} - \frac{\boldsymbol{\Theta}}{n\rho_n}) &= n\rho_n^{-1/2}\mathbf{Z}^\dagger(\mathbf{A}-\mathbf{P})(\mathbf{Z}^\dagger)^\top \\ &- n\rho_n^{-1/2}\breve{\boldsymbol{\Pi}}^\perp_{\mathbf{V}}\mathbf{Z}^\dagger(\mathbf{A}-\mathbf{P})(\mathbf{Z}^\dagger)^\top(\breve{\boldsymbol{\Pi}}^\perp_{\mathbf{V}})^\top + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}). \end{aligned}$$

Recalling that $\hat{\mathbf{B}}^{(N)} = \rho_n^{-1}\mathbf{Z}^\dagger\mathbf{A}(\mathbf{Z}^\dagger)^\top$, we obtain

$$n\rho_n^{1/2}(\hat{\mathbf{B}}^{(S)} - \mathbf{B} - \frac{\boldsymbol{\Theta}}{n\rho_n}) = n\rho_n^{1/2}(\hat{\mathbf{B}}^{(N)} - \mathbf{B}) - n\rho_n^{1/2}\breve{\boldsymbol{\Pi}}^\perp_{\mathbf{V}}(\hat{\mathbf{B}}^{(N)} - \mathbf{B})(\breve{\boldsymbol{\Pi}}^\perp_{\mathbf{V}})^\top + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}).$$

Rewriting the above expression in terms of the half-vectorization of $\hat{\mathbf{B}}^{(S)}$ yields

$$n\rho_n^{1/2}\mathrm{vech}\Big(\hat{\mathbf{B}}^{(S)} - \mathbf{B} - \frac{\boldsymbol{\Theta}}{n\rho_n}\Big) = n\rho_n^{1/2}\mathcal{L}_K(\mathbf{I} - \breve{\boldsymbol{\Pi}}^\perp_{\mathbf{V}} \otimes \breve{\boldsymbol{\Pi}}^\perp_{\mathbf{V}})\mathcal{D}_K\mathrm{vech}(\hat{\mathbf{B}}^{(N)} - \mathbf{B}) + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}).$$

Now $n^{-1}\mathbf{Z}^\top\mathbf{Z} \to \mathrm{diag}(\boldsymbol{\pi})$ *almost surely and hence*

$$\breve{\mathbf{\Pi}}_{\mathbf{V}}^{\perp} = \mathbf{I} - \mathbf{V}(\mathbf{V}^\top\mathbf{Z}^\top\mathbf{Z}\mathbf{V})^{-1}\mathbf{V}^\top\mathbf{Z}^\top\mathbf{Z} \xrightarrow{\text{a.s}} \mathbf{I} - \mathbf{V}(\mathbf{V}^\top\mathrm{diag}(\boldsymbol{\pi})\mathbf{V})^{-1}\mathbf{V}^\top\mathrm{diag}(\boldsymbol{\pi})$$

Defining $\widetilde{\mathbf{\Pi}}_{\mathbf{V}} = \mathbf{I} - \mathbf{V}(\mathbf{V}^\top\mathrm{diag}(\boldsymbol{\pi})\mathbf{V})^{-1}\mathbf{V}^\top\mathrm{diag}(\boldsymbol{\pi})$*, we have, by Slutsky's theorem, that*

$$n\rho_n^{1/2}\mathrm{vech}\Big(\hat{\mathbf{B}}^{(S)} - \mathbf{B} - \frac{\mathbf{\Theta}}{n\rho_n}\Big) \xrightarrow{\text{d}} \mathcal{N}\Big(\mathbf{0}, \mathcal{L}_K(\mathbf{I} - \widetilde{\mathbf{\Pi}}_{\mathbf{V}}^{\perp} \otimes \widetilde{\mathbf{\Pi}}_{\mathbf{V}}^{\perp})\mathcal{D}_K\mathbf{D}^{-1}\mathcal{D}_K^\top(\mathbf{I} - \widetilde{\mathbf{\Pi}}_{\mathbf{V}}^{\perp} \otimes \widetilde{\mathbf{\Pi}}_{\mathbf{V}}^{\perp})^\top\mathcal{L}_K^\top\Big).$$

where $\mathbf{D}$ *is the* $\binom{K+1}{2} \times \binom{K+1}{2}$ *diagonal matrix defined in Theorem 2, i.e., the diagonal entries of* $\mathbf{D}^{-1}$ *are the variances for* $\hat{\mathbf{B}}^{(N)}$.

We now return to the proof of Theorem 3 and Theorem 4. We shall repeatedly make use of a von-Neumann expansion for $\hat{\mathbf{U}}$. More specifically, from $\mathbf{A}\hat{\mathbf{U}} = \hat{\mathbf{U}}\mathbf{\Lambda}$, we have

$$\hat{\mathbf{U}}\hat{\mathbf{\Lambda}} - (\mathbf{A} - \mathbf{P})\hat{\mathbf{U}} = \mathbf{P}\hat{\mathbf{U}}$$

which is a matrix Sylvester equation. The spectrum of $\hat{\mathbf{\Lambda}}$ and the spectrum of $\mathbf{A} - \mathbf{P}$ are disjoint with high probability and hence Theorem VII.2.1 and Theorem VII.2.2 in [1] implies

$$\hat{\mathbf{U}} = \sum_{k=0}^{\infty}(\mathbf{A} - \mathbf{P})^k\mathbf{P}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-(k+1)} = \sum_{k=0}^{\infty}(\mathbf{A} - \mathbf{P})^k\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-(k+1)}. \tag{B.13}$$

with high probability. Eq. (B.13) also implies

$$\mathbf{\Pi}_{\mathbf{U}}^{\perp}\hat{\mathbf{U}} = \mathbf{\Pi}_{\mathbf{U}}^{\perp}\sum_{k=1}^{\infty}(\mathbf{A} - \mathbf{P})^k\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-(k+1)}. \tag{B.14}$$

Several key steps in our proof of Theorem 3 and Theorem 4 proceed by using Lemma B.1 to truncate the series expansions in Eq. (B.13) and Eq. (B.14). More specifically, we have the following result.

Lemma B.2. *Let* $\mathbf{A} \sim \mathrm{GRDPG}_{p,q}(F)$ *be a generalized random dot product graph on* n *vertices with sparsity factor* ρ_n*. Then with* $\mathbf{E} = \mathbf{A} - \mathbf{P}$*, we have*

$$\begin{aligned}\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{\Lambda}} - \mathbf{\Lambda}\mathbf{U}^\top\hat{\mathbf{U}} = \mathbf{U}^\top\mathbf{A}\hat{\mathbf{U}} - \mathbf{U}^\top\mathbf{P}\hat{\mathbf{U}} &= \mathbf{U}^\top\Big(\sum_{k=1}^{\infty}\mathbf{E}^k\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-k}\Big) \\ &= \mathbf{U}^\top\mathbf{E}\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1} + \mathbf{U}^\top\mathbf{E}^2\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-2} + O_{\mathbb{P}}((n\rho_n)^{-1/2}) \qquad \text{(B.15)} \\ &= \mathbf{U}^\top\mathbf{E}\mathbf{U}\mathbf{U}^\top\hat{\mathbf{U}} + \mathbf{U}^\top\mathbf{E}^2\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^\top\hat{\mathbf{U}} + O_{\mathbb{P}}((n\rho_n)^{-1/2}) \\ &= O_{\mathbb{P}}(1).\end{aligned}$$

$$\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1} - \mathbf{\Lambda}^{-1}\mathbf{U}^\top\hat{\mathbf{U}} = O_{\mathbb{P}}((n\rho_n)^{-2}) \tag{B.16}$$

$$\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-2} - \mathbf{\Lambda}^{-2}\mathbf{U}^\top\hat{\mathbf{U}} = O_{\mathbb{P}}((n\rho_n)^{-3}). \tag{B.17}$$

In addition, we also have

$$\begin{aligned}\mathbf{\Pi}_{\mathbf{U}}^{\perp}\hat{\mathbf{U}} &= \mathbf{\Pi}_{\mathbf{U}}^{\perp}\mathbf{E}\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\top}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-2} + O_{\mathbb{P}}((n\rho_n)^{-1}) \\ &= \mathbf{\Pi}_{\mathbf{U}}^{\perp}\mathbf{E}\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^{\top}\hat{\mathbf{U}} + O_{\mathbb{P}}((n\rho_n)^{-1}) \\ &= \mathbf{E}\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^{\top}\hat{\mathbf{U}} + O_{\mathbb{P}}((n\rho_n)^{-1}),\end{aligned} \tag{B.18}$$

$$\begin{aligned}\mathbf{\Pi}_{\mathbf{U}}^{\perp}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}} &= \mathbf{\Pi}_{\mathbf{U}}^{\perp}\mathbf{E}\big(\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\top}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1} + \mathbf{E}\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\top}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-2}\big) + O_{\mathbb{P}}((n\rho_n)^{-1/2}) \\ &= \mathbf{\Pi}_{\mathbf{U}}^{\perp}\mathbf{E}\mathbf{\Pi}_{\mathbf{U}}\hat{\mathbf{U}} + \mathbf{\Pi}_{\mathbf{U}}^{\perp}\mathbf{E}^2\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^{\top}\hat{\mathbf{U}} + O_{\mathbb{P}}((n\rho_n)^{-1/2}).\end{aligned} \tag{B.19}$$

Proof. We first derive parts of Eq. (B.15). From Lemma B.1, we obtain

$$\begin{aligned}\Big\|\sum_{k=3}^{\infty}\mathbf{E}^k\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\top}\hat{\mathbf{\Lambda}}^{-k}\Big\| &\leq \sum_{k=3}^{\infty}\|\mathbf{E}^k\| \times \|\mathbf{\Lambda}\| \times \|\hat{\mathbf{\Lambda}}^{-k}\| \\ &\leq \sum_{k=3}^{\infty} O_{\mathbb{P}}(n\rho_n)^{-(k-1)/2}) = O_{\mathbb{P}}((n\rho_n)^{-1/2}).\end{aligned}$$

and hence

$$\begin{aligned}\mathbf{U}^{\top}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}} - \mathbf{\Lambda}\mathbf{U}^{\top}\hat{\mathbf{U}} = \mathbf{U}^{\top}\mathbf{E}\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\top}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1} + \mathbf{U}^{\top}\mathbf{E}^2\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\top}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-2} \\ + O_{\mathbb{P}}((n\rho_n)^{-1/2}).\end{aligned} \tag{B.20}$$

Let $\boldsymbol{u}_i$ denote the i-th column of $\mathbf{U}$. We note that $\mathbf{U}^{\top}\mathbf{E}\mathbf{U}$ is a $d \times d$ matrix whose ij-th entry can be written as $\boldsymbol{u}_i^{\top}\mathbf{E}\boldsymbol{u}_j$. Now, conditioned on $\mathbf{P}$, $\boldsymbol{u}_i^{\top}\mathbf{E}\boldsymbol{u}_j$ is a sum of independent mean 0 random variables, and hence, by Hoeffding's inequality, $\boldsymbol{u}_i^{\top}\mathbf{E}\boldsymbol{u}_j = O_{\mathbb{P}}(1)$. A union bound over the $d(d+1)/2$ upper triangular entries of $\mathbf{U}^{\top}\mathbf{E}\mathbf{U}$ then yield $\|\mathbf{U}^{\top}\mathbf{E}\mathbf{U}\| = O_{\mathbb{P}}(1)$. We therefore have

$$\begin{aligned}\|\mathbf{U}^{\top}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}} - \mathbf{\Lambda}\mathbf{U}^{\top}\hat{\mathbf{U}}\| \leq \|\mathbf{U}^{\top}\mathbf{E}\mathbf{U}\| \times \|\mathbf{\Lambda}\| \times \|\hat{\mathbf{\Lambda}}\|^{-1} + \|\mathbf{E}^2\| \times \|\mathbf{\Lambda}\| \times \|\hat{\mathbf{\Lambda}}\|^{-2} \\ + O_{\mathbb{P}}((n\rho_n)^{-1/2}) = O_{\mathbb{P}}(1).\end{aligned}$$

We next show Eq. (B.16). Let ω_{ij} denote the ij-th entry of $\mathbf{U}^{\top}\hat{\mathbf{U}}$ and let $\hat{\lambda}_i$ and λ_i denote the i-th diagonal element of $\hat{\mathbf{\Lambda}}$ and $\mathbf{\Lambda}$ respectively, i.e., $\hat{\lambda}_i$ and λ_i are the i-th largest eigenvalue, in modulus, of $\mathbf{A}$ and $\mathbf{P}$. Then the ij- th entry of $\mathbf{U}^{\top}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1} - \mathbf{\Lambda}^{-1}\mathbf{U}^{\top}\hat{\mathbf{U}}$ can be written as

$$\omega_{ij}\big(\hat{\lambda}_j^{-1} - \lambda_i^{-1}\big) = \omega_{ij}\frac{\lambda_i - \hat{\lambda}_j}{\lambda_i\hat{\lambda}_j}.$$

Therefore, letting $\mathbf{H}$ denote the $d \times d$ matrix whose ij-th entry is $\lambda_i^{-1}\hat{\lambda}_j^{-1}$, we have (with $\circ$ denoting the Hadamard product between matrices)

$$\mathbf{U}^{\top}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1} - \mathbf{\Lambda}^{-1}\mathbf{U}^{\top}\hat{\mathbf{U}} = (\mathbf{\Lambda}\mathbf{U}^{\top}\hat{\mathbf{U}} - \mathbf{U}^{\top}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}) \circ \mathbf{H} = O_{\mathbb{P}}((n\rho_n)^{-2}).$$

Eq. (B.17) is derived in an analogous manner. More specifically, let $\widetilde{\mathbf{H}}$ denote the $d \times d$ matrix whose ij-th entry is $\hat{\lambda}_j^{-2}\lambda_i^{-2}(\hat{\lambda}_j + \lambda_i)$, we have

$$\mathbf{U}^\top \hat{\mathbf{U}} \hat{\mathbf{\Lambda}}^{-2} - \mathbf{\Lambda}^{-2} \mathbf{U}^\top \hat{\mathbf{U}} = (\mathbf{\Lambda} \mathbf{U}^\top \hat{\mathbf{U}} - \mathbf{U}^\top \hat{\mathbf{U}} \hat{\mathbf{\Lambda}}) \circ \widetilde{\mathbf{H}} = O_{\mathbb{P}}((n\rho_n)^{-3}).$$

We then apply Eq. (B.16) and Eq. (B.17) to Eq. (B.20) and obtain another representation for $\mathbf{U}^\top \hat{\mathbf{U}} \hat{\mathbf{\Lambda}} - \mathbf{\Lambda} \mathbf{U}^\top \hat{\mathbf{U}}$, namely

$$\mathbf{U}^\top \hat{\mathbf{U}} \hat{\mathbf{\Lambda}} - \mathbf{\Lambda} \mathbf{U}^\top \hat{\mathbf{U}} = \mathbf{U}^\top \mathbf{E} \mathbf{U} \mathbf{U}^\top \hat{\mathbf{U}} + \mathbf{U}^\top \mathbf{E}^2 \mathbf{U} \mathbf{\Lambda}^{-1} \mathbf{U}^\top \hat{\mathbf{U}} + O_{\mathbb{P}}((n\rho_n)^{-1/2}).$$

Eq. (B.15) is thereby established. Eq. (B.18) and Eq. (B.19) is derived in a similar manner to that of Eq. (B.15). □

Deriving Eq. (B.3) and Eq. (B.4)

We start with the observation

$$\begin{aligned} \mathbf{U}^\top \hat{\mathbf{U}} \hat{\mathbf{U}}^\top \mathbf{U} - \mathbf{I} &= \mathbf{U}^\top \hat{\mathbf{U}} \mathbf{W}^\top \mathbf{W} \hat{\mathbf{U}}^\top \mathbf{U} - \mathbf{I} \\ &= -(\mathbf{U} - \hat{\mathbf{U}} \mathbf{W}^\top)^\top (\mathbf{U} - \hat{\mathbf{U}} \mathbf{W}) + \mathbf{U}^\top (\mathbf{U} - \hat{\mathbf{U}} \mathbf{W})(\mathbf{U} - \hat{\mathbf{U}} \mathbf{W})^\top \mathbf{U}. \end{aligned}$$

Now $\|\mathbf{U}^\top(\mathbf{U} - \hat{\mathbf{U}}\mathbf{W})\| = \|\mathbf{I} - \mathbf{\Sigma}\|$ where $\mathbf{\Sigma}$ is the diagonal matrix whose diagonal entries are the singular values of $\mathbf{U}^\top \hat{\mathbf{U}}$. Lemma B.1 then implies $\|\mathbf{U}^\top(\mathbf{U} - \hat{\mathbf{U}}\mathbf{W})\| = \mathcal{O}_{\mathbb{P}}((n\rho_n)^{-1})$ and hence

$$\mathbf{U}^\top \hat{\mathbf{U}} \hat{\mathbf{U}}^\top \mathbf{U} - \mathbf{I} = -(\mathbf{U} - \hat{\mathbf{U}} \mathbf{W}^\top)^\top (\mathbf{U} - \hat{\mathbf{U}} \mathbf{W}) + O_{\mathbb{P}}((n\rho_n)^{-2}). \tag{B.21}$$

We recall the following bounds

$$\|s_k\| = \sqrt{n_k} = \Theta(\sqrt{n}); \quad \|s_\ell\| = \sqrt{n_\ell} = \Theta(\sqrt{n}) \tag{B.22}$$

$$n\rho_n = \omega(\sqrt{n}); \quad \|\mathbf{\Lambda}\| = O_{\mathbb{P}}(n\rho_n). \tag{B.23}$$

Eq. (B.21) and Lemma B.2 then imply

$$\begin{aligned} \xi_{k\ell}^{(3)} &= -\frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top \mathbf{U} \mathbf{\Lambda} (\mathbf{U} - \hat{\mathbf{U}} \mathbf{W}^\top)^\top (\mathbf{U} - \hat{\mathbf{U}} \mathbf{W}) \mathbf{U}^\top \boldsymbol{s}_\ell + O_{\mathbb{P}}(n^{-1}\rho_n^{-3/2}) \\ &= -\frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top \mathbf{U} \mathbf{U}^\top (\mathbf{A} - \mathbf{P})(\mathbf{U} - \hat{\mathbf{U}} \mathbf{W}) \mathbf{U}^\top \boldsymbol{s}_\ell + O_{\mathbb{P}}(n^{-1/2}\rho_n) + O_{\mathbb{P}}(n^{-1}\rho_n^{-3/2}) \\ &= -\frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top \mathbf{U} \mathbf{U}^\top (\mathbf{A} - \mathbf{P})^2 \mathbf{U} \mathbf{\Lambda}^{-1} \mathbf{U}^\top \boldsymbol{s}_\ell + O_{\mathbb{P}}(n^{-1/2}\rho_n) \end{aligned}$$

thereby establishing Eq. (B.3).

We next derive Eq. (B.4). We recall Eq. (B.15) in Lemma B.2, namely that

$$\mathbf{U}^\top \hat{\mathbf{U}} \hat{\mathbf{\Lambda}} - \mathbf{\Lambda} \mathbf{U}^\top \hat{\mathbf{U}} = \mathbf{U}^\top \mathbf{E} \mathbf{U} \mathbf{U}^\top \hat{\mathbf{U}} + \mathbf{U}^\top \mathbf{E}^2 \mathbf{U} \mathbf{\Lambda}^{-1} \mathbf{U}^\top \hat{\mathbf{U}} + O_{\mathbb{P}}((n\rho_n)^{-1/2}).$$

We therefore have, in conjunction with Eq. (B.21), that

$$(\mathbf{U}^\top \hat{\mathbf{U}} \hat{\mathbf{\Lambda}} - \mathbf{\Lambda} \mathbf{U}^\top \hat{\mathbf{U}}) \hat{\mathbf{U}}^\top \mathbf{U} = \mathbf{U}^\top \mathbf{E} \mathbf{U} + \mathbf{U}^\top \mathbf{E}^2 \mathbf{U} \mathbf{\Lambda}^{-1} + O_{\mathbb{P}}((n\rho_n)^{-1/2}).$$

and hence

$$
\begin{aligned}
\xi_{k\ell}^{(2)} &= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}(\boldsymbol{s}_k^\top \mathbf{U}(\mathbf{U}^\top \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda}\mathbf{U}^\top \hat{\mathbf{U}})\hat{\mathbf{U}}^\top \mathbf{U}\mathbf{U}^\top \boldsymbol{s}_\ell) \\
&= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \mathbf{U}\Big(\mathbf{U}^\top \mathbf{E}\mathbf{U} + \mathbf{U}^\top \mathbf{E}^2 \mathbf{U}\boldsymbol{\Lambda}^{-1} + O_{\mathbb{P}}((n\rho_n)^{-1/2})\Big)\mathbf{U}^\top \boldsymbol{s}_\ell \\
&= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \big(\boldsymbol{\Pi}_{\mathbf{U}}\mathbf{E}\boldsymbol{\Pi}_{\mathbf{U}} + \boldsymbol{\Pi}_{\mathbf{U}}\mathbf{E}^2\mathbf{P}^\dagger\big)\boldsymbol{s}_\ell + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}) \\
&= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}\mathbf{E}\boldsymbol{\Pi}_{\mathbf{U}}\boldsymbol{s}_\ell - \xi_{k\ell}^{(3)} + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}).
\end{aligned}
$$

as desired.

Deriving Eq. (B.2)

We start with the decomposition

$$
\begin{aligned}
\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top - \mathbf{U}\mathbf{U}^\top \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top \mathbf{U}\mathbf{U}^\top &= \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top - \boldsymbol{\Pi}_{\mathbf{U}}\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}} \\
&= \boldsymbol{\Pi}_{\mathbf{U}}^\perp \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp + \boldsymbol{\Pi}_{\mathbf{U}}^\perp \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}} + \boldsymbol{\Pi}_{\mathbf{U}}\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp .
\end{aligned}
$$

Now let $\omega_{k\ell}^{(1)} = \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}^\perp \boldsymbol{s}_\ell$. By Eq. (B.19) in Lemma B.2, we have

$$
\begin{aligned}
\omega_{k\ell}^{(1)} &= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \boldsymbol{s}_\ell \\
&= \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \big(\boldsymbol{\Pi}_{\mathbf{U}}^\perp \mathbf{E}\boldsymbol{\Pi}_{\mathbf{U}}\hat{\mathbf{U}} + \boldsymbol{\Pi}_{\mathbf{U}}^\perp \mathbf{E}^2\mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top \hat{\mathbf{U}} + O_{\mathbb{P}}((n\rho_n)^{-1/2})\big)\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \boldsymbol{s}_\ell
\end{aligned}
$$

From Lemma B.1, we have $\|\boldsymbol{\Pi}_{\mathbf{U}}^\perp \hat{\mathbf{U}}\| = O_{\mathbb{P}}((n\rho_n)^{-1/2})$, and hence

$$
\omega_{k\ell}^{(1)} = \tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \mathbf{E}\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}^{-1}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \boldsymbol{s}_\ell + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1})
$$

Now, conditional on $\mathbf{P}$, $\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp (\mathbf{A} - \mathbf{P})\mathbf{U}$ is vector in $\mathbb{R}^d$ whose elements are sum of independent mean 0 random variables. Therefore, by Hoeffding's inequality and the fact that $\|\boldsymbol{s}_k\| = \Theta(\sqrt{n})$ and $\|\mathbf{U}\|_F = \sqrt{d}$, we have

$$
\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp (\mathbf{A} - \mathbf{P})\mathbf{U} = O_{\mathbb{P}}(\sqrt{n})
$$

and thus

$$
\begin{aligned}
|\omega_{k\ell}^{(1)}| &= |\tfrac{n\rho_n^{-1/2}}{n_k n_\ell}\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp \boldsymbol{s}_\ell| \\
&\le \rho_n^{-1/2} \times \|\boldsymbol{\Lambda}\| \times \|\hat{\boldsymbol{\Lambda}}^{-1}\| \times \|\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}}^\perp\| + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}) \qquad \text{(B.24)} \\
&= O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1}).
\end{aligned}
$$

Next let $\omega_{k\ell}^{(2)} := \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \hat{\mathbf{U}} \hat{\boldsymbol{\Lambda}} \hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_\ell$. Once again Eq. (B.19) implies

$$\begin{aligned}\omega_{k\ell}^{(2)} &= \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \big(\mathbf{E}\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top \hat{\boldsymbol{U}} \hat{\boldsymbol{\Lambda}}^{-1} + \mathbf{E}^2 \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top \hat{\boldsymbol{U}} \hat{\boldsymbol{\Lambda}}^{-2} + O_{\mathbb{P}}((n\rho_n)^{-1/2})\big) \hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_\ell \\ &= \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \big(\mathbf{E}\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top \hat{\boldsymbol{U}} \hat{\boldsymbol{\Lambda}}^{-1} + \mathbf{E}^2 \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top \hat{\boldsymbol{U}} \hat{\boldsymbol{\Lambda}}^{-2}\big) \hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_\ell + O_{\mathbb{P}}((n\rho_n)^{-1/2})\end{aligned}$$

Applying Eq. (B.16) and Eq. (B.17) to the above yield

$$\omega_{k\ell}^{(2)} = \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top \big(\boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \mathbf{E}\mathbf{U}\mathbf{U}^\top \hat{\mathbf{U}}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}} + \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \mathbf{E}^2 \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top \hat{\mathbf{U}}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}}\big) \boldsymbol{s}_\ell + O_{\mathbb{P}}((n\rho_n)^{-1/2}).$$

Using Eq. (B.21), we replace $\mathbf{U}\mathbf{U}^\top \hat{\mathbf{U}}\hat{\mathbf{U}}^\top$ by $\mathbf{I}$ and replace $\mathbf{U}^\top \hat{\mathbf{U}}\hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}} = \mathbf{U}^\top \hat{\mathbf{U}}\hat{\mathbf{U}}^\top \mathbf{U}\mathbf{U}^\top$ by $\mathbf{U}^\top$ in the above display, thereby obtaining

$$\omega_{k\ell}^{(2)} = \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_k^\top \big(\boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \mathbf{E} \boldsymbol{\Pi}_{\boldsymbol{U}} + \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \mathbf{E}^2 \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top\big) \boldsymbol{s}_\ell + O_{\mathbb{P}}((n\rho_n)^{-1/2}).$$

By exchanging k and ℓ, we also have

$$\begin{aligned}\omega_{k\ell}^{(3)} &:= \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \hat{\mathbf{U}} \hat{\boldsymbol{\Lambda}} \hat{\mathbf{U}}^\top \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_k \\ &= \frac{n\rho_n^{-1/2}}{n_k n_\ell} \boldsymbol{s}_\ell^\top \big(\boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \mathbf{E} \boldsymbol{\Pi}_{\mathbf{U}} + \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \mathbf{E}^2 \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top\big) \boldsymbol{s}_k + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1})\end{aligned}$$

Combining the above expressions for $\omega_{k\ell}^{(1)}, \omega_{k\ell}^{(2)}$ and $\omega_{k\ell}^{(3)}$, we obtain

$$\begin{aligned}\xi_{k\ell}^{(3)} &= \omega_{k\ell}^{(1)} + \omega_{k\ell}^{(2)} + \omega_{k\ell}^{(3)} \\ &= \frac{n\rho_n^{-1/2}}{n_k n_\ell} \big(\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \mathbf{E} \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_\ell + \boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \mathbf{E}^2 \mathbf{P}^\dagger \boldsymbol{s}_\ell + \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \mathbf{E} \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_k + \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp} \mathbf{E}^2 \mathbf{P}^\dagger \boldsymbol{s}_k\big) \\ &\quad + O_{\mathbb{P}}(n^{-1/2}\rho_n^{-1})\end{aligned}$$

as desired.

Deriving Eq. (2.4), Eq. (2.5), Eq. (2.8) and Eq. (2.9)

We first recall Eq. (B.6),

$$Z_{k\ell} = \frac{n\rho_n^{-1/2}}{n_k n_\ell} \operatorname{tr}(\mathbf{A} - \mathbf{P})(\boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_\ell \boldsymbol{s}_k^\top + \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_k \boldsymbol{s}_\ell^\top - \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_k \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}).$$

With $\mathbf{M} = \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_\ell \boldsymbol{s}_k^\top + \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_k \boldsymbol{s}_\ell^\top - \boldsymbol{\Pi}_{\mathbf{U}} \boldsymbol{s}_k \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}$, we have

$$\begin{aligned}Z_{k\ell} &= \frac{n\rho_n^{-1/2}}{n_k n_\ell} \sum_i \sum_j (\mathbf{A}_{ij} - \mathbf{P}_{ij}) \mathbf{M}_{ij} \\ &= \frac{n\rho_n^{-1/2}}{n_k n_\ell} \Big(\sum_{i<j} (\mathbf{A}_{ij} - \mathbf{P}_{ij})(\mathbf{M}_{ij} + \mathbf{M}_{ji}) + \sum_i (\mathbf{A}_{ii} - \mathbf{P}_{ii}) \mathbf{M}_{ii} \Big)\end{aligned} \tag{B.25}$$

which is a sum of $n(n+1)/2$ independent mean 0 random variables. By the Lindeberg-Feller central limit theorem, $Z_{k\ell} \xrightarrow{\mathrm{d}} N(0, \mathrm{Var}[Z_{k\ell}])$. All that remains is to evaluate $\mathrm{Var}[Z_{k\ell}]$.

Let $\mathbf{X}$ be the $n \times d$ matrix such that X_i, the i-th row of $\mathbf{X}$, is ν_k if $\tau_i = k$, i.e., $\mathbf{X}\mathbf{I}_{p,q}\mathbf{X}^\top = \mathbb{E}[\mathbf{A}]$. We observe that $\mathbf{\Pi_U} = \mathbf{U}\mathbf{U}^\top = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top$ as $\mathbf{\Pi_U}$ is the orthogonal projection onto the column space of $\mathbf{X}\mathbf{I}_{p,q}\mathbf{X}^\top$ which coincides with that of $\mathbf{X}$. Let $\tau = (\tau_1, \dots, \tau_n)$ be the vertices to block assignments of $\mathbf{A}$. Then the ij-th entries of $\mathbf{\Pi_U}\boldsymbol{s}_\ell\boldsymbol{s}_k^\top$, $\mathbf{\Pi_U}\boldsymbol{s}_\ell\boldsymbol{s}_k^\top$, and $\mathbf{\Pi_U}\boldsymbol{s}_k\boldsymbol{s}_\ell^\top\mathbf{\Pi_U}$ are

$$(\mathbf{\Pi_U}\boldsymbol{s}_\ell\boldsymbol{s}_k^\top)_{ij} = n_\ell X_i^\top(\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell * \mathbb{1}\{\tau_j = k\}, \tag{B.26}$$

$$(\mathbf{\Pi_U}\boldsymbol{s}_k\boldsymbol{s}_\ell^\top)_{ij} = n_k X_i^\top(\mathbf{X}^\top\mathbf{X})^{-1}\nu_k * \mathbb{1}\{\tau_j = \ell\}, \tag{B.27}$$

$$(\mathbf{\Pi_U}\boldsymbol{s}_k\boldsymbol{s}_\ell^\top\mathbf{\Pi_U})_{ij} = n_k n_\ell X_i^\top(\mathbf{X}^\top\mathbf{X})^{-1}\nu_k\nu_l^\top(\mathbf{X}^\top\mathbf{X})^{-1}X_j, \tag{B.28}$$

and hence

$$\begin{aligned}\mathbf{M}_{ij} + \mathbf{M}_{ji} &= n_k\nu_\ell^\top(\mathbf{X}^\top\mathbf{X})^{-1}\big(X_i\mathbb{1}\{\tau_j = k\} + X_j\mathbb{1}\{\tau_i = k\}\big) \\ &\quad + n_\ell\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\big(X_i\mathbb{1}\{\tau_j = \ell\} + X_j\mathbb{1}\{\tau_i = \ell\}\big) \\ &\quad - n_k n_\ell X_i^\top(\mathbf{X}^\top\mathbf{X})^{-1}\big(\nu_\ell\nu_k^\top + \nu_k\nu_\ell^\top\big)(\mathbf{X}^\top\mathbf{X})^{-1}X_j.\end{aligned} \tag{B.29}$$

Next, we note that

$$\begin{aligned}\mathrm{Var}[Z_{k\ell}] &= \tfrac{n^2\rho_n^{-1}}{n_k^2 n_\ell^2}\sum_{i<j}\mathbf{P}_{ij}(1-\mathbf{P}_{ij})(\mathbf{M}_{ij}+\mathbf{M}_{ji})^2 + \tfrac{n^2\rho_n^{-1}}{n_k^2 n_\ell^2}\sum_i \mathbf{P}_{ii}(1-\mathbf{P}_{ii})\mathbf{M}_{ii}^2 \\ &= \tfrac{n^2\rho_n^{-1}}{2n_k^2 n_\ell^2}\sum_i\sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij})(\mathbf{M}_{ij}+\mathbf{M}_{ji})^2 + o_{\mathbb{P}}(1) \\ &= S_{kk} + 2S_{k\ell} + S_{\ell\ell} + 2S_{ko} + 2S_{\ell o} + S_{oo} + o_{\mathbb{P}}(1)\end{aligned} \tag{B.30}$$

where each S_{**} correspond to summing $\eta_{ij} := \mathbf{P}_{ij}(1-\mathbf{P}_{ij})(\mathbf{M}_{ij}+\mathbf{M}_{ji})^2$ over some subset of the indices (i, j), namely

$$S_{kk} = \tfrac{n^2\rho_n^{-1}}{2n_k^2 n_\ell^2}\sum_{\tau_i=k}\sum_{\tau_j=k}\eta_{ij},\ S_{\ell\ell} = \tfrac{n^2\rho_n^{-1}}{2n_k^2 n_\ell^2}\sum_{\tau_i=\ell}\sum_{\tau_j=\ell}\eta_{ij},$$

$$S_{k\ell} = \tfrac{n^2\rho_n^{-1}}{2n_k^2 n_\ell^2}\sum_{\tau_i=k}\sum_{\tau_j=\ell}\eta_{ij},\ S_{ko} = \tfrac{n^2\rho_n^{-1}}{2n_k^2 n_\ell^2}\sum_{\tau_i=k}\sum_{\tau_j\notin\{k,\ell\}}\eta_{ij},$$

$$S_{\ell o} = \tfrac{n^2\rho_n^{-1}}{2n_k^2 n_\ell^2}\sum_{\tau_i=\ell}\sum_{\tau_j\notin\{k,\ell\}}\eta_{ij},\ S_{oo} = \tfrac{n^2\rho_n^{-1}}{2n_k^2 n_\ell^2}\sum_{\tau_i\notin\{k,\ell\}}\sum_{\tau_j\notin\{k,\ell\}}\eta_{ij}.$$

If $k \neq \ell$, then for (i, j) such that $\tau_i = k$ and $\tau_j = k$, Eq. (B.29) yield

$$\mathbf{M}_{ij} + \mathbf{M}_{ji} = 2n_\ell\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell - 2n_k n_\ell\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\boldsymbol{\nu}_\ell\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\nu_k.$$

and hence, since $\mathbf{P}_{ij} = \rho_n \mathbf{B}_{\tau_i,\tau_j}$,

$$\begin{aligned} S_{kk} &= \frac{2n^2\rho_n^{-1}}{n_k^2 n_\ell^2} n_k^2 \rho_n \mathbf{B}_{kk}(1-\rho_n\mathbf{B}_{kk})(n_\ell \nu_k^\top (\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell - n_k n_\ell \nu_k^\top (\mathbf{X}^\top\mathbf{X})^{-1}\boldsymbol{\nu}_\ell \nu_k^\top (\mathbf{X}^\top\mathbf{X})^{-1}\nu_k)^2 \\ &= 2\mathbf{B}_{kk}(1-\rho_n\mathbf{B}_{kk})(\boldsymbol{\nu}_k^\top (\tfrac{\mathbf{X}^\top\mathbf{X}}{n})^{-1}\boldsymbol{\nu}_\ell - \tfrac{n_k}{n}\nu_k^\top (\tfrac{\mathbf{X}^\top\mathbf{X}}{n})^{-1}\nu_\ell \boldsymbol{\nu}_k^\top (\tfrac{\mathbf{X}^\top\mathbf{X}}{n})^{-1}\boldsymbol{\nu}_k)^2 \end{aligned}$$

We therefore have

$$S_{kk} \xrightarrow{\text{a.s.}} \begin{cases} 2\mathbf{B}_{kk}(1-\mathbf{B}_{kk})\zeta_{k\ell}^2(1-\pi_k\zeta_{kk})^2 & \text{if } \rho_n \equiv 1 \text{ for all } n \\ 2\mathbf{B}_{kk}\zeta_{k\ell}^2(1-\pi_k\zeta_{kk})^2 & \text{if } \rho_n \to 0 \end{cases}$$

as $n \to \infty$. Similarly, we have

$$S_{\ell\ell} \xrightarrow{\text{a.s.}} \begin{cases} 2\mathbf{B}_{\ell\ell}(1-\mathbf{B}_{\ell\ell})\zeta_{k\ell}^2(1-\pi_\ell\zeta_{\ell\ell})^2 & \text{if } \rho_n \equiv 1 \text{ for all } n \\ 2\mathbf{B}_{\ell\ell}\zeta_{k\ell}^2(1-\pi_\ell\zeta_{\ell\ell})^2 & \text{if } \rho_n \to 0 \end{cases}$$

as $n \to \infty$. If $k \neq \ell$, then for (i,j) with $\tau_i = k$ and $\tau_j = \ell$, Eq. (B.29) yield

$$\begin{aligned} \mathbf{M}_{ij} + \mathbf{M}_{ji} &= n_\ell \nu_\ell^\top (\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell + n_k \nu_k^\top (\mathbf{X}^\top\mathbf{X})^{-1}\nu_k \\ &\quad - n_k n_\ell \nu_k^\top (\mathbf{X}^\top\mathbf{X})^{-1}\nu_k \nu_\ell^\top (\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell - n_k n_\ell (\nu_k^\top (\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell)^2 \end{aligned}$$

and hence

$$S_{k\ell} \xrightarrow{\text{a.s.}} \begin{cases} \frac{1}{2}\pi_k\pi_\ell\mathbf{B}_{k\ell}(1-\mathbf{B}_{k\ell})\big(\frac{1}{\pi_k}\zeta_{\ell\ell} + \frac{1}{\pi_\ell}\zeta_{kk} - \zeta_{kk}\zeta_{\ell\ell} - \zeta_{k\ell}^2\big)^2 & \text{if } \rho_n \equiv 1 \\ \frac{1}{2}\pi_k\pi_\ell\mathbf{B}_{k\ell}\big(\frac{1}{\pi_k}\zeta_{\ell\ell} + \frac{1}{\pi_\ell}\zeta_{kk} - \zeta_{kk}\zeta_{\ell\ell} - \zeta_{k\ell}^2\big)^2 & \text{if } \rho_n \to 0 \end{cases}$$

as $n \to \infty$. If $k \neq \ell$ then for (i,j) with $\tau_i = k, \tau_j \notin \{k,\ell\}$, Eq. (B.29) yield

$$\begin{aligned} \mathbf{M}_{ij} + \mathbf{M}_{ji} &= n_\ell X_j^\top (\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell - n_k n_\ell \nu_k^\top (\mathbf{X}^\top\mathbf{X})^{-1}\nu_k \nu_\ell^\top (\mathbf{X}^\top\mathbf{X})^{-1} X_j \\ &\quad - n_k n_\ell \nu_k^\top (\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell \nu_k^\top (\mathbf{X}^\top\mathbf{X})^{-1} X_j \end{aligned}$$

and hence

$$S_{ko} \xrightarrow{\text{a.s.}} \begin{cases} \frac{1}{2}\sum_{r\notin\{k,\ell\}} \pi_k\pi_r\mathbf{B}_{kr}(1-\mathbf{B}_{kr})(\frac{1}{\pi_k}\zeta_{\ell r} - \zeta_{kk}\zeta_{\ell r} - \zeta_{k\ell}\zeta_{kr})^2 & \text{if } \rho_n \equiv 1 \\ \frac{1}{2}\sum_{r\notin\{k,\ell\}} \pi_k\pi_r\mathbf{B}_{kr}(\frac{1}{\pi_k}\zeta_{\ell r} - \zeta_{kk}\zeta_{\ell r} - \zeta_{k\ell}\zeta_{kr})^2 & \text{if } \rho_n \to 0 \end{cases}$$

as $n \to \infty$. By symmetry, we also have

$$S_{ko} \xrightarrow{\text{a.s.}} \begin{cases} \frac{1}{2}\sum_{r\notin\{k,\ell\}} \pi_\ell\pi_r\mathbf{B}_{\ell r}(1-\mathbf{B}_{\ell r})(\frac{1}{\pi_\ell}\zeta_{kr} - \zeta_{\ell\ell}\zeta_{kr} - \zeta_{k\ell}\zeta_{\ell r})^2 & \text{if } \rho_n \equiv 1 \\ \frac{1}{2}\sum_{r\notin\{k,\ell\}} \pi_\ell\pi_r\mathbf{B}_{\ell r}(\frac{1}{\pi_\ell}\zeta_{kr} - \zeta_{\ell\ell}\zeta_{kr} - \zeta_{k\ell}\zeta_{\ell r})^2 & \text{if } \rho_n \to 0 \end{cases}$$

Finally, when (i,j) is such that $\tau_i \notin \{k,\ell\}$ and $\tau_j \notin \{k,\ell\}$, Eq. (B.29) yield

$$\mathbf{M}_{ij} + \mathbf{M}_{ji} = -n_k n_\ell X_i^\top (\mathbf{X}^\top\mathbf{X})^{-1}(\nu_k\nu_\ell^\top + \nu_\ell\nu_k^\top)(\mathbf{X}^\top\mathbf{X})^{-1} X_j$$

and thus

$$S_{oo} \xrightarrow{\text{a.s.}} \begin{cases} \frac{1}{2}\sum_{r\notin\{k,\ell\}}\sum_{s\notin\{k,\ell\}} \pi_r\pi_s\mathbf{B}_{rs}(1-\mathbf{B}_{rs})(\zeta_{kr}\zeta_{\ell s} + \zeta_{\ell r}\zeta_{ks})^2 & \text{if } \rho_n \equiv 1 \\ \frac{1}{2}\sum_{r\notin\{k,\ell\}}\sum_{s\notin\{k,\ell\}} \pi_r\pi_s\mathbf{B}_{rs}(\zeta_{kr}\zeta_{\ell s} + \zeta_{\ell r}\zeta_{ks})^2 & \text{if } \rho_n \to 0 \end{cases}$$

as $n \to \infty$. Combining the above expressions for $S_{kk}, S_{k\ell}, S_{\ell\ell}, S_{\ell o}, S_{ko}$ and S_{oo} yield the expressions for $\sigma^2_{k\ell}$ and $\widetilde{\sigma}^2_{k\ell}$ in Eq.(2.5) and Eq.(2.9). For example, with $\rho_n \equiv 1$,

$$\begin{aligned}
\sigma^2_{k\ell} &= 2\mathbf{B}_{kk}(1-\mathbf{B}_{kk})\zeta^2_{k\ell}(1-\pi_k\zeta_{kk})^2 + 2\mathbf{B}_{\ell}(1-\mathbf{B}_{\ell\ell})\zeta^2_{k\ell}(1-\pi_\ell\zeta_{\ell\ell})^2 \\
&\quad + \pi_k\pi_\ell\mathbf{B}_{k\ell}(1-\mathbf{B}_{k\ell})\big(\tfrac{1}{\pi_k}\zeta_{\ell\ell} + \tfrac{1}{\pi_\ell}\zeta_{kk} - \zeta_{kk}\zeta_{\ell\ell} - \zeta^2_{k\ell}\big)^2 \\
&\quad + \sum_{r\notin\{k,\ell\}} \pi_k\pi_r\mathbf{B}_{kr}(1-\mathbf{B}_{kr})(\tfrac{1}{\pi_k}\zeta_{\ell r} - \zeta_{kk}\zeta_{\ell r} - \zeta_{k\ell}\zeta_{kr})^2 \\
&\quad + \sum_{r\notin\{k,\ell\}} \pi_k\pi_r\mathbf{B}_{kr}(1-\mathbf{B}_{kr})(\tfrac{1}{\pi_k}\zeta_{\ell r} - \zeta_{kk}\zeta_{\ell r} - \zeta_{k\ell}\zeta_{kr})^2 \\
&\quad + \tfrac{1}{2}\sum_{r\notin\{k,\ell\}}\sum_{s\notin\{k,\ell\}} \pi_r\pi_s\mathbf{B}_{rs}(1-\mathbf{B}_{rs})(\zeta_{kr}\zeta_{\ell s} + \zeta_{\ell r}\zeta_{ks})^2
\end{aligned}$$

for when $k \neq \ell$. Straightforward manipulations then yield the form given in Eq. (2.5).

When $k = \ell$, the term $\mathrm{Var}[Z_{kk}]$ is decomposed as $\mathrm{Var}[Z_{kk}] = S_{kk} + 2S_{ko} + S_{oo}$ where we now have

$$S_{kk} = \frac{n^2\rho_n^{-1}}{2n_k^4}\sum_{\tau_i=k}\sum_{\tau_j=k}\eta_{ij},\; S_{ko} = \frac{n^2\rho_n^{-1}}{2n_k^4}\sum_{\tau_i=k}\sum_{\tau_j\neq k}\eta_{ij},\; S_{oo} = \frac{n^2\rho_n^{-1}}{2n_k^4}\sum_{\tau_i\neq k}\sum_{\tau_j\neq k}\eta_{ij}.$$

If $k = \ell$, then for (i,j) such that $\tau_i = k = \ell$ and $\tau_j = k = \ell$, Eq. (B.29) yield

$$\mathbf{M}_{ij} + \mathbf{M}_{ji} = 4n_k\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\nu_k - 2n_k^2(\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\nu_k)^2$$

from which we obtain

$$\mathbf{S}_{kk} \xrightarrow{\text{a.s.}} \begin{cases} 2\mathbf{B}_{kk}(1-\mathbf{B}_{kk})\zeta^2_{kk}(2-\pi_k\zeta_{kk})^2 & \text{if } \rho_n \equiv 1 \text{ for all } n \\ 2\mathbf{B}_{kk}\zeta^2_{kk}(2-\pi_k\zeta_{kk})^2 & \text{if } \rho_n \to 0. \end{cases}$$

If $k = \ell$, then for (i,j) such that $\tau_i = k$ and $\tau_j \neq k$, Eq. (B.29) yield

$$\mathbf{M}_{ij} + \mathbf{M}_{ji} = 2n_k\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}X_j - 2n_k^2\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\nu_k\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}X_j$$

and thus

$$\mathbf{S}_{ko} \xrightarrow{\text{a.s.}} \begin{cases} 2\sum_{r\neq k}\pi_k\pi_r\mathbf{B}_{kr}(1-\mathbf{B}_{kr})\zeta^2_{kr}(\frac{1}{\pi_k} - \zeta_{kk})^2 & \text{if } \rho_n \equiv 1 \text{ for all } n \\ 2\sum_{r\neq k}\pi_k\pi_r\mathbf{B}_{kr}\zeta^2_{kr}(\frac{1}{\pi_k} - \zeta_{kk})^2 & \text{if } \rho_n \to 0. \end{cases}$$

Finally, for $k = \ell$ and $\tau_i \neq k$, $\tau_j \neq k$, we have

$$\mathbf{M}_{ij} + \mathbf{M}_{ji} = -2n_k^2X_i(\mathbf{X}^\top\mathbf{X})^{-1}\nu_k\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}X_j$$

$$S_{oo} \xrightarrow{\text{a.s.}} \begin{cases} 2\sum_{r\neq k}\sum_{s\neq k}\pi_r\pi_s\mathbf{B}_{rs}(1-\mathbf{B}_{rs})\zeta^2_{kr}\zeta^2_{ks} & \text{if } \rho_n \equiv 1 \\ 2\sum_{r\neq k}\sum_{s\neq k}\pi_r\pi_s\mathbf{B}_{rs}\zeta^2_{kr}\zeta^2_{ks} & \text{if } \rho_n \to 0. \end{cases}$$

Combining the above expressions for S_{kk}, S_{ko} and S_{oo} and some straightforward manipulations yield us Eq. (2.4) and Eq. (2.8).

Deriving Eq. (2.3) and Eq. (2.7)

Our argument is similar to that of [10] and is based on a log-Sobolev concentration inequality of [2] which yield

$$\begin{aligned}
\tfrac{n}{n_k n_\ell} \boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp}(\mathbf{A}-\mathbf{P})^2 \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top \boldsymbol{s}_\ell &= \mathbb{E}[\tfrac{n}{n_k n_\ell} \boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp}(\mathbf{A}-\mathbf{P})^2 \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top \boldsymbol{s}_\ell] + O_{\mathbb{P}}(n^{-1/2}) \\
\tfrac{n}{n_k n_\ell} \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp}(\mathbf{A}-\mathbf{P})^2 \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top \boldsymbol{s}_k &= \mathbb{E}[\tfrac{n}{n_k n_\ell} \boldsymbol{s}_\ell^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp}(\mathbf{A}-\mathbf{P})^2 \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top \boldsymbol{s}_k] + O_{\mathbb{P}}(n^{-1/2})
\end{aligned}$$

where the expectations are taken with respect to $\mathbf{A}$, conditional on $\mathbf{P}$. We now evaluate $\theta_{k\ell}^{(1)} := \tfrac{n}{n_k n_\ell}\mathbb{E}[\boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp}(\mathbf{A}-\mathbf{P})^2 \mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top \boldsymbol{s}_\ell]$. Let $\mathbf{D} = \mathbb{E}[(\mathbf{A}-\mathbf{P})]^2$ be the diagonal matrix whose diagonal entries are

$$\mathbf{D}_{ii} = \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij}) = \rho_n \sum_{r=1}^{K} n_r X_i^\top \mathbf{I}_{p,q}\nu_r(1-\rho_n X_i^\top \mathbf{I}_{p,q}\nu_r).$$

Next, we note $\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top = \mathbf{P} = \rho_n \mathbf{X}\mathbf{I}_{p,q}\mathbf{X}^\top$ and $\mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top$ is the Moore-Penrose pseudo-inverse $\mathbf{P}^\dagger$ of $\mathbf{P}$. Since the Moore-Penrose pseudoinverse of $\mathbf{P}$ is unique, we therefore have

$$\mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top = \mathbf{P}^\dagger = (\rho_n \mathbf{X}\mathbf{I}_{p,q}\mathbf{X}^\top)^\dagger = \rho_n^{-1}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top.$$

We then have

$$\begin{aligned}
\theta_{k\ell}^{(1)} &= \tfrac{n}{n_k n_\ell} \boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp}\mathbb{E}[(\mathbf{A}-\mathbf{P})^2]\mathbf{U}\boldsymbol{\Lambda}^{-1}\mathbf{U}^\top \boldsymbol{s}_\ell \\
&= \tfrac{n}{\rho_n n_k n_\ell} \boldsymbol{s}_k^\top \boldsymbol{\Pi}_{\mathbf{U}}^{\perp}\mathbf{D}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top \boldsymbol{s}_\ell \\
&= \tfrac{n}{\rho_n n_k n_\ell} \boldsymbol{s}_k^\top (\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top)\mathbf{D}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top \boldsymbol{s}_\ell \\
&= \tfrac{n}{\rho_n n_k}\Big(\boldsymbol{s}_k^\top \mathbf{D}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}(\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell - n_k\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{D}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}(\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell\Big)
\end{aligned}$$

Letting $\zeta_{k\ell}^{(1)} = \tfrac{n}{\rho_n n_k}\boldsymbol{s}_k^\top \mathbf{D}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}(\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell$, some straightforward simplifications yield

$$\begin{aligned}
\zeta_{k\ell}^{(1)} &= \frac{n}{\rho_n n_k}\boldsymbol{s}_k^\top \mathbf{D}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}(\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell \\
&= n\sum_{r=1}^{K} n_r \nu_k^\top \mathbf{I}_{p,q}\nu_r(1-\rho_n\nu_k^\top\mathbf{I}_{p,q}\nu_r)\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}(\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell \\
&\xrightarrow{\text{a.s.}} \begin{cases} \sum_{r=1}^{K}\pi_r\mathbf{B}_{kr}(1-\mathbf{B}_{kr})\nu_k^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}\nu_\ell & \text{if } \rho_n \equiv 1 \\ \sum_{r=1}^{K}\pi_r\mathbf{B}_{kr}\nu_k^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}\nu_\ell & \text{if } \rho_n \to 0 \end{cases}
\end{aligned}$$

as $n \to \infty$. Similarly, letting $\zeta_{k\ell}^{(2)} = n\rho_n^{-1}\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{D}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}(\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell$,

$$
\begin{aligned}
\zeta_{k\ell}^{(2)} &= n\rho_n^{-1}\sum_i \nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}X_i\mathbf{D}_{ii}X_i^\top(\mathbf{X}^\top\mathbf{X})^{-2}\nu_\ell \\
&= n\sum_{s=1}^K n_s\nu_k^\top(\mathbf{X}^\top\mathbf{X})^{-1}\nu_s\sum_{r=1}^K n_r\nu_s^\top\mathbf{I}_{p,q}\nu_r(1-\rho_n\nu_s^\top\mathbf{I}_{p,q}\nu_r)\nu_s^\top(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{I}_{p,q}(\mathbf{X}^\top\mathbf{X})^{-1}\nu_\ell \\
&\xrightarrow{\text{a.s.}} \begin{cases} \sum_{s=1}^K\sum_{r=1}^K \pi_r\pi_s\nu_k^\top\Delta^{-1}\nu_s\mathbf{B}_{sr}(1-\mathbf{B}_{sr})\nu_s^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}\nu_\ell & \text{if } \rho_n \equiv 1 \\ \sum_{s=1}^K\sum_{r=1}^K \pi_r\pi_s\nu_k^\top\Delta^{-1}\nu_s\mathbf{B}_{sr}\nu_s^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}\nu_\ell & \text{if } \rho_n \to 0 \end{cases}
\end{aligned}
$$

as $n \to \infty$. Now $\theta_{k\ell}^{(1)} = \zeta_{k\ell}^{(1)} + \zeta_{k\ell}^{(2)}$. We therefore have, for $\rho_n \equiv 1$, that

$$
\begin{aligned}
\theta_{k\ell}^{(1)} &= \sum_{r=1}^K \pi_r\mathbf{B}_{kr}(1-\mathbf{B}_{kr})\nu_k^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}\nu_\ell \\
&\quad - \sum_{r=1}^K\sum_{s=1}^K \pi_r\pi_s\mathbf{B}_{sr}(1-\mathbf{B}_{sr})\nu_s^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}\nu_\ell\nu_k^\top\Delta^{-1}\nu_s.
\end{aligned}
$$

In contrast, if $\rho_n \to 0$, then

$$
\theta_{k\ell}^{(1)} = \sum_{r=1}^K \pi_r\mathbf{B}_{kr}\nu_k^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}\nu_\ell - \sum_{r=1}^K\sum_{s=1}^K \pi_r\pi_s\mathbf{B}_{sr}\nu_s^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}\nu_\ell\nu_k^\top\Delta^{-1}\nu_s.
$$

Swapping ℓ with k in the above expression yield a similar expression for $\theta_{k\ell}^{(2)} = \mathbb{E}[n\boldsymbol{s}_\ell^\top\mathbf{\Pi}_{\mathbf{U}}^\perp(\mathbf{A}-\mathbf{P})^2\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^\top\boldsymbol{s}_k]$. Since $\theta_{k\ell}^{(1)} + \theta_{k\ell}^{(2)} = \theta_{k\ell}$ for $\rho_n \equiv 1$, we conclude

$$
\begin{aligned}
\theta_{k\ell} &= \sum_{r=1}^K \pi_r\big(\mathbf{B}_{kr}(1-\mathbf{B}_{kr}) + \mathbf{B}_{\ell r}(1-\mathbf{B}_{\ell r})\big)\nu_k^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}\nu_\ell \\
&\quad - \sum_{r=1}^K\sum_{s=1}^K \pi_r\pi_s\mathbf{B}_{sr}(1-\mathbf{B}_{sr})\nu_s^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}(\nu_\ell\nu_k^\top + \nu_k\nu_\ell^\top)\Delta^{-1}\nu_s.
\end{aligned}
$$

Similarly, $\theta_{k\ell}^{(1)} + \theta_{k\ell}^{(2)} = \widetilde{\theta}_{k\ell}$ when $\rho_n \to 0$, and hence

$$
\begin{aligned}
\widetilde{\theta}_{k\ell} &= \sum_{r=1}^K \pi_r\big(\mathbf{B}_{kr} + \mathbf{B}_{\ell r}\big)\nu_k^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}\nu_\ell \\
&\quad - \sum_{r=1}^K\sum_{s=1}^K \pi_r\pi_s\mathbf{B}_{sr}\nu_s^\top\Delta^{-1}\mathbf{I}_{p,q}\Delta^{-1}(\nu_\ell\nu_k^\top + \nu_k\nu_\ell^\top)\Delta^{-1}\nu_s
\end{aligned}
$$

as desired.

Proof of Lemma 1

We recall the notion of the $2 \to \infty$ norm for matrices, namely, for a $n \times m$ matrix $\mathbf{A}$ (with $\mathbf{A}_i$ denoting the i-th row of $\mathbf{A}$)

$$\|\mathbf{A}\|_{2\to\infty} = \max_{\|\boldsymbol{x}\|_2=1} \|\mathbf{A}\boldsymbol{x}\|_\infty = \max_{i\in[n]} \|\mathbf{A}_i\|_2.$$

Eq. (3.1) in Lemma 1 can thus be rewritten as

$$\|\hat{\mathbf{U}}_n - \mathbf{U}_n\mathbf{W}\|_{2\to\infty} = O_{\mathbb{P}}\Big(\frac{\log^c n}{n\sqrt{\rho_n}}\Big) \tag{B.31}$$

for some orthogonal $\mathbf{W}$. We now derive Eq. (B.31). For ease of exposition, we shall drop the index n from our matrices $\mathbf{X}_n$, $\mathbf{A}_n$, $\hat{\mathbf{U}}_n$ and $\mathbf{U}_n$. We first note that for any matrices $\mathbf{A}$ and $\mathbf{B}$ whose product $\mathbf{AB}$ is well-defined, $\|\mathbf{AB}\|_{2\to\infty} \leq \|\mathbf{A}\|_{2\to\infty} \times \|\mathbf{B}\|$. Next, we note that $\|\mathbf{U}\|_{2\to\infty} = O_{\mathbb{P}}(n^{-1/2})$ as the rows of $\mathbf{X}$ are sampled i.i.d. from F. Recalling Lemma B.1, we then have

$$\begin{aligned}
\|\hat{\mathbf{U}} - \mathbf{U}\mathbf{W}\|_{2\to\infty} &\leq \|\hat{\mathbf{U}} - \mathbf{U}\mathbf{U}^\top\hat{\mathbf{U}}\|_{2\to\infty} + \|\mathbf{U}\mathbf{U}^\top\hat{\mathbf{U}} - \mathbf{U}\mathbf{W}\|_{2\to\infty} \\
&\leq \|\hat{\mathbf{U}} - \mathbf{U}\mathbf{U}^\top\hat{\mathbf{U}}\|_{2\to\infty} + \|\mathbf{U}\|_{2\to\infty}\|\mathbf{U}^\top\hat{\mathbf{U}} - \mathbf{W}\| \\
&\leq \|\hat{\mathbf{U}} - \mathbf{U}\mathbf{U}^\top\hat{\mathbf{U}}\|_{2\to\infty} + O_{\mathbb{P}}\Big(\tfrac{1}{n^{3/2}\rho_n}\Big).
\end{aligned}$$

Eq. (B.14) now implies (recall that $\mathbf{\Pi}_{\mathbf{U}}^{\perp} = \mathbf{I} - \mathbf{U}\mathbf{U}^\top$)

$$\begin{aligned}
\|\hat{\mathbf{U}} - \mathbf{U}\mathbf{U}^\top\hat{\mathbf{U}}\|_{2\to\infty} &\leq \sum_{k=1}^{\infty} \|\mathbf{\Pi}_{\mathbf{U}}^{\perp}(\mathbf{A}-\mathbf{P})^k\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-(k+1)}\|_{2\to\infty} \\
&\leq \sum_{k=1}^{\infty} \|(\mathbf{A}-\mathbf{P})^k\mathbf{U}\|_{2\to\infty} \times \|\mathbf{\Lambda}\| \times \|\hat{\mathbf{\Lambda}}^{-1}\|^{(k+1)} \\
&+ \sum_{k=1}^{\infty} \|\mathbf{U}\|_{2\to\infty} \times \|\mathbf{U}^\top(\mathbf{A}-\mathbf{P})^k\mathbf{U}\| \times \|\mathbf{\Lambda}\| \times \|\hat{\mathbf{\Lambda}}^{-1}\|^{(k+1)}.
\end{aligned} \tag{B.32}$$

Once again, by Lemma B.1, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \|\mathbf{U}\|_{2\to\infty} \times \|\mathbf{U}^\top(\mathbf{A}-\mathbf{P})^k\mathbf{U}\| \times \|\mathbf{\Lambda}\| \times \|\hat{\mathbf{\Lambda}}^{-1}\|^{(k+1)} &\leq \sum_{k=1}^{\infty} \mathcal{O}_{\mathbb{P}}\Big(\frac{1}{\sqrt{n}(n\rho_n)^{k/2}}\Big) \\
&= \mathcal{O}_{\mathbb{P}}\Big(\frac{1}{n\sqrt{\rho_n}}\Big).
\end{aligned} \tag{B.33}$$

We now bound $\sum_{k=1}^{\infty} \|(\mathbf{A}-\mathbf{P})^k\mathbf{U}\|_{2\to\infty} \times \|\mathbf{\Lambda}\| \times \|\hat{\mathbf{\Lambda}}^{-1}\|^{(k+1)}$. We need the following slight restatement of Lemma 7.10 from [4].

Lemma B.3. *Assume the setting and notations in Lemma 1. Let $\boldsymbol{u}_j$ be the j-th column of $\mathbf{U}$ for $j = 1, 2, \dots, d$. Then there exists constants $c > 0$ such that for all $k \leq \log n$*

$$\|(\mathbf{A}-\mathbf{P})^k\mathbf{U}\|_{2\to\infty} \leq \sqrt{d}\max_{j\in[d]} \|(\mathbf{A}-\mathbf{P})^k\boldsymbol{u}_j\|_\infty = O_{\mathbb{P}}\Big(\frac{\sqrt{d}(n\rho_n)^{k/2}\log^{kc}(n)}{\sqrt{n}}\Big).$$

We note that Lemma 7.10 from [4] was originally stated for the case when $\boldsymbol{u}_j = n^{-1/2}\mathbf{1}$ [1], but the argument used in the proof of Lemma 7.10 can be easily extended to the setting where the entries of $\boldsymbol{u}_j$ are "delocalized", i.e., $\|\boldsymbol{u}_j\|_\infty = O_{\mathbb{P}}(n^{-1/2})$. Using Lemma B.3 and Lemma 1, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} \|(\mathbf{A}-\mathbf{P})^k\mathbf{U}\|_{2\to\infty}\|\mathbf{\Lambda}\|\|\hat{\mathbf{\Lambda}}^{-1}\|^{(k+1)} &\leq \sum_{k=1}^{\log n} O_{\mathbb{P}}\Big(\frac{\sqrt{d}\log^{kc}(n)}{\sqrt{n}(n\rho_n)^{k/2}}\Big) + \sum_{k>\log n} O_{\mathbb{P}}((n\rho_n)^{-k/2}) \\ &\leq O_{\mathbb{P}}\Big(\frac{\sqrt{d}\log^{c}(n)}{n\sqrt{\rho_n}}\Big) + O_{\mathbb{P}}((n\rho_n)^{-(\frac{1}{2}\log n)}).\end{aligned}$$

If we now assume $n\rho_n = \omega(\log^2(n))$, then

$$(n\rho_n)^{-(\frac{1}{2}\log n)}) = o_{\mathbb{P}}\Big(\frac{\sqrt{d}\log n)}{n\sqrt{\rho_n}}\Big)$$

and hence

$$\sum_{k=1}^{\infty} \|(\mathbf{A}-\mathbf{P})^k\mathbf{U}\|_{2\to\infty} \times \|\mathbf{\Lambda}\| \times \|\hat{\mathbf{\Lambda}}^{-1}\|^{(k+1)} \leq O_{\mathbb{P}}\Big(\frac{\sqrt{d}\log^{c}(n)}{n\sqrt{\rho_n}}\Big). \tag{B.34}$$

Substituting Eq. (B.33) and Eq. (B.34) into Eq. (B.32) yield Eq. (B.31), as desired.

[1]There is a small typo in [4] in that for Lemma 7.10, $\boldsymbol{e}$ is defined as $\boldsymbol{e} = \mathbf{1}$, while $\boldsymbol{e} = n^{-1/2}\mathbf{1}$ is used everywhere else in the paper.

C. Covariance terms

We now derive the covariances between the entries of $\hat{\mathbf{B}}^{(S)}$. For ease of exposition, we only consider the case when $\rho_n \equiv 1$. The case where $\rho_n \to 0$ is almost identical. Let (k, ℓ) and (k', ℓ') be two given pairs of indices with $1 \leq k \leq \ell \leq K$ and $1 \leq k' \leq \ell' \leq K$. Let $\sigma_{k\ell,k'\ell'}$ denote the covariance between $\hat{\mathbf{B}}^{(S)}_{k\ell}$ and $\hat{\mathbf{B}}^{(S)}_{k'\ell'}$. We then have

$$\begin{aligned}
\sigma_{k\ell,k'\ell'} &= \Big(\mathbf{B}_{kk'}(1-\mathbf{B}_{kk'}) + \mathbf{B}_{\ell\ell'}(1-\mathbf{B}_{\ell\ell'})\Big)\zeta_{k\ell'}\zeta_{k'\ell} \\
&+ \Big(\mathbf{B}_{k\ell'}(1-\mathbf{B}_{k\ell'}) + \mathbf{B}_{k'\ell}(1-\mathbf{B}_{k'\ell})\Big)\zeta_{kk'}\zeta_{\ell\ell'} \\
&+ \frac{1}{\pi_k}\sum_r \pi_r \mathbf{B}_{kr}(1-\mathbf{B}_{kr})\Big(\zeta_{\ell r}\zeta_{\ell' r}\mathbb{1}\{k'=k\} + \zeta_{k'r}\zeta_{\ell r}\mathbb{1}\{\ell'=k\}\Big) \\
&+ \frac{1}{\pi_\ell}\sum_r \pi_r \mathbf{B}_{\ell r}(1-\mathbf{B}_{\ell r})\Big(\zeta_{kr}\zeta_{\ell' r}\mathbb{1}\{k'=\ell\} + \zeta_{kr}\zeta_{k'r}\mathbb{1}\{\ell'=\ell\}\Big) \\
&- \sum_r \pi_r\Big(\mathbf{B}_{kr}(1-\mathbf{B}_{kr}) + \mathbf{B}_{\ell' r}(1-\mathbf{B}_{\ell' r})\Big)\zeta_{k\ell'}\zeta_{k'r}\zeta_{\ell r} \\
&- \sum_r \pi_r\Big(\mathbf{B}_{kr}(1-\mathbf{B}_{kr}) + \mathbf{B}_{k' r}(1-\mathbf{B}_{k' r})\Big)\zeta_{kk'}\zeta_{\ell r}\zeta_{\ell' r} \\
&- \sum_r \pi_r\Big(\mathbf{B}_{k'r}(1-\mathbf{B}_{k'r}) + \mathbf{B}_{\ell r}(1-\mathbf{B}_{\ell r})\Big)\zeta_{k'\ell}\zeta_{kr}\zeta_{\ell' r} \\
&- \sum_r \pi_r\Big(\mathbf{B}_{\ell r}(1-\mathbf{B}_{\ell r}) + \mathbf{B}_{\ell' r}(1-\mathbf{B}_{\ell' r})\Big)\zeta_{\ell\ell'}\zeta_{kr}\zeta_{k' r} \\
&+ \sum_r\sum_s \pi_r\pi_r \mathbf{B}_{rs}(1-\mathbf{B}_{rs})\Big(\zeta_{kr}\zeta_{k'r}\zeta_{\ell s}\zeta_{\ell' s} + \zeta_{kr}\zeta_{\ell' r}\zeta_{k's}\zeta_{\ell s}\Big).
\end{aligned} \tag{C.1}$$

Eq. (C.1) follows from the same ideas as that used to derive the variances $\sigma^2_{k\ell}$ (see Eq. (B.30)) but with much more involved bookkeeping. Fix the pairs (k, ℓ) and (k', ℓ') and let $\mathbf{M}$ and $\mathbf{N}$ be the matrices

$$\begin{aligned}
\mathbf{M} &= \mathbf{\Pi}_{\mathbf{U}} \boldsymbol{s}_\ell \boldsymbol{s}_k^\top + \mathbf{\Pi}_{\mathbf{U}} \boldsymbol{s}_k \boldsymbol{s}_\ell^\top - \mathbf{\Pi}_{\mathbf{U}} \boldsymbol{s}_k \boldsymbol{s}_\ell^\top \mathbf{\Pi}_{\mathbf{U}} \\
\mathbf{N} &= \mathbf{\Pi}_{\mathbf{U}} \boldsymbol{s}_{\ell'} \boldsymbol{s}_{k'}^\top + \mathbf{\Pi}_{\mathbf{U}} \boldsymbol{s}_{k'} \boldsymbol{s}_{\ell'}^\top - \mathbf{\Pi}_{\mathbf{U}} \boldsymbol{s}_{k'} \boldsymbol{s}_{\ell'}^\top \mathbf{\Pi}_{\mathbf{U}}
\end{aligned}$$

Recall from Eq. (B.26) through Eq. (B.28) that the entries of $\mathbf{M}_{ij}$ are of the form $\mathbf{M}_{ij} = m^{(1)}_{ij} + m^{(2)}_{ij} - m^{(3)}_{ij}$ where

$$\begin{aligned}
m^{(1)}_{ij} &= n_\ell X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_\ell \mathbb{1}\{\tau_j = k\} \\
m^{(2)}_{ij} &= n_k X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_k \mathbb{1}\{\tau_j = \ell\}, \\
m^{(3)}_{ij} &= n_k n_\ell X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_k \nu_l^\top (\mathbf{X}^\top \mathbf{X})^{-1} X_j,
\end{aligned}$$

The terms for $\mathbf{N}_{ij} = n_{ij}^{(1)} + n_{ij}^{(2)} - n_{ij}^{(3)}$ are analogous. Next recall the expression for $Z_{k\ell}$ in Eq. (B.25) and note that $\sigma_{k\ell,k'\ell'} = \mathrm{Cov}(Z_{k\ell}, Z_{k'\ell'})$. We then have

$$\begin{aligned}
\sigma_{k\ell,k'\ell'} &= \tfrac{n^2}{n_k n_\ell n_{k'} n_{\ell'}} \sum_{i<j} \mathbf{P}_{ij}(1-\mathbf{P}_{ij})(\mathbf{M}_{ij}+\mathbf{M}_{ji})(\mathbf{N}_{ij}+\mathbf{N}_{ji}) + \tfrac{4n^2}{n_k n_\ell n_{k'} n_{\ell'}} \sum_i \mathbf{P}_{ii}(1-\mathbf{P}_{ii})\mathbf{M}_{ii}\mathbf{N}_{ii} \\
&= \tfrac{n^2}{2n_k n_\ell n_{k'} n_{\ell'}} \sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij})(\mathbf{M}_{ij}+\mathbf{M}_{ji})(\mathbf{N}_{ij}+\mathbf{N}_{ji}) + o_{\mathbb{P}}(1) \\
&= \tfrac{n^2}{2n_k n_\ell n_{k'} n_{\ell'}} \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \sum_i \sum_j c_{\alpha,\beta} \mathbf{P}_{ij}(1-\mathbf{P}_{ij})(m_{ij}^{(\alpha)} + m_{ji}^{(\alpha)})(n_{ij}^{(\beta)} + n_{ji}^{(\beta)}) + o_{\mathbb{P}}(1)
\end{aligned}$$

where $c_{\alpha,\beta} = -1$ if $(\alpha,\beta) \in \{(1,3),(2,3),(3,1),(3,2)\}$ and $c_{\alpha,\beta} = 1$ otherwise. All that remains is to bound the inner sums for each combination of α and β. The calculations are straightforward but tedious. We illustrate some of these calculations below.

We first consider the sums involving $m_{ij}^{(1)} n_{ij}^{(1)}$ and $m_{ji}^{(1)} n_{ji}^{(1)}$. We have

$$\begin{aligned}
m_{ij}^{(1)} n_{ij}^{(1)} &= n_\ell X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_\ell \mathbb{1}\{\tau_j = k\} n_{\ell'} X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{\ell'} \mathbb{1}\{\tau_j = k'\}, \\
m_{ji}^{(1)} n_{ji}^{(1)} &= n_\ell X_j^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_\ell \mathbb{1}\{\tau_i = k\} n_{\ell'} X_j^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{\ell'} \mathbb{1}\{\tau_i = k'\},
\end{aligned}$$

and hence, by swapping the roles of the indices i and j,

$$\begin{aligned}
\sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij}) m_{ij}^{(1)} n_{ij}^{(1)} &= \sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij}) m_{ji}^{(1)} n_{ji}^{(1)} \\
&= \sum_r n_r \mathbf{B}_{kr}(1-\mathbf{B}_{kr}) n_k n_\ell n_{\ell'} \nu_r^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_\ell \nu_{\ell'}^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_r \mathbb{1}\{k = k'\}.
\end{aligned}$$

Now recall the definition of $\zeta_{rs} = \nu_r^\top \Delta^{-1} \nu_s$. Hence, by the law of large numbers,

$$\begin{aligned}
&\tfrac{n^2}{2n_k n_{k'} n_\ell n_{\ell'}} \sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij})(m_{ij}^{(1)} n_{ij}^{(1)} + m_{ji}^{(1)} n_{ji}^{(1)}) = \\
&\tfrac{n}{n_{k'}} \sum_r \tfrac{n_r}{n} \mathbf{B}_{kr}(1-\mathbf{B}_{kr}) \nu_r^\top \Big(\tfrac{\mathbf{X}^\top \mathbf{X}}{n}\Big)^{-1} \nu_\ell \nu_{\ell'} \Big(\tfrac{\mathbf{X}^\top \mathbf{X}}{n}\Big)^{-1} \nu_r \mathbb{1}\{k = k'\} \\
&\longrightarrow \frac{1}{\pi_k} \sum_r \pi_r \mathbf{B}_{kr}(1-\mathbf{B}_{kr}) \zeta_{\ell r} \zeta_{\ell' r} \mathbb{1}\{k = k'\}.
\end{aligned}$$

We next consider the terms $m_{ij}^{(1)} n_{ji}^{(1)}$ and $m_{ji}^{(1)} n_{ij}^{(1)}$, i.e.,

$$\begin{aligned}
m_{ij}^{(1)} n_{ji}^{(1)} &= n_\ell X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_\ell \mathbb{1}\{\tau_j = k\} n_{\ell'} X_j^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{\ell'} \mathbb{1}\{\tau_i = k'\} \\
m_{ji}^{(1)} n_{ij}^{(1)} &= n_\ell X_j^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_\ell \mathbb{1}\{\tau_i = k\} n_{\ell'} X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{\ell'} \mathbb{1}\{\tau_j = k'\} = m_{ij}^{(1)} n_{ji}^{(1)}
\end{aligned}$$

We therefore have

$$\tfrac{n^2}{2n_k n_\ell n_{k'} n_{\ell'}} \sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij})(m_{ij}^{(1)} n_{ji}^{(1)} + m_{ji}^{(1)} n_{ij}^{(1)}) = n^2 \mathbf{B}_{kk'}(1-\mathbf{B}_{kk'}) \nu_k^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{\ell'} \nu_{k'}^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_\ell$$

which converges to $\mathbf{B}_{kk'}(1-\mathbf{B}_{kk'})\zeta_{k\ell'}\zeta_{k'\ell}$ as $n \to \infty$.

We now consider the term $m_{ij}^{(1)} n_{ij}^{(3)}$ and $m_{ji}^{(1)} n_{ji}^{(3)}$, i.e.,

$$\begin{aligned} m_{ij}^{(1)} n_{ij}^{(3)} &= n_\ell X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_\ell \mathbb{1}\{\tau_j = k\} n_{k'} n_{\ell'} X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{k'} \nu_{\ell'}^\top (\mathbf{X}^\top \mathbf{X})^{-1} X_j \\ m_{ji}^{(1)} n_{ji}^{(3)} &= n_\ell X_j^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_\ell \mathbb{1}\{\tau_i = k\} n_{k'} n_{\ell'} X_j^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{k'} \nu_{\ell'}^\top (\mathbf{X}^\top \mathbf{X})^{-1} X_i, \end{aligned}$$

and hence, by swapping the roles of the indices i and j,

$$\begin{aligned} \sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij}) m_{ij}^{(1)} n_{ij}^{(3)} &= \sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij}) m_{ji}^{(1)} n_{ji}^{(3)} \\ &= n_k n_{k'} n_\ell n_{\ell'} \sum_r n_r \mathbf{B}_{kr}(1-\mathbf{B}_{kr}) \nu_r^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_\ell \nu_r^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{k'} \nu_k^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{\ell'}. \end{aligned}$$

Therefore, by the law of large numbers,

$$\tfrac{n^2}{2 n_k n_\ell n_{k'} n_{\ell'}} \sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij}) m_{ij}^{(1)} n_{ij}^{(3)} + m_{ji}^{(1)} n_{ji}^{(3)} \longrightarrow \sum_r \pi_r \mathbf{B}_{kr}(1-\mathbf{B}_{kr}) \zeta_{k\ell'} \zeta_{k'r} \zeta_{\ell r}$$

We finally consider the terms $m_{ij}^{(3)} n_{ij}^{(3)}$ and $m_{ji}^{(3)} n_{ji}^{(3)}$, i.e.,

$$\begin{aligned} m_{ij}^{(3)} n_{ij}^{(3)} &= n_k n_\ell X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_k \nu_\ell^\top (\mathbf{X}^\top \mathbf{X})^{-1} X_j n_{k'} n_{\ell'} X_i^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{k'} \nu_{\ell'}^\top (\mathbf{X}^\top \mathbf{X})^{-1} X_j, \\ m_{ji}^{(3)} n_{ji}^{(3)} &= n_k n_\ell X_j^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_k \nu_\ell^\top (\mathbf{X}^\top \mathbf{X})^{-1} X_i n_{k'} n_{\ell'} X_j^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{k'} \nu_{\ell'}^\top (\mathbf{X}^\top \mathbf{X})^{-1} X_i. \end{aligned}$$

Once again, by swapping the roles of the indices i and j, we have

$$\sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij}) m_{ij}^{(3)} n_{ij}^{(3)} = \sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij}) m_{ji}^{(3)} n_{ji}^{(3)} =$$
$$n_k n_{k'} n_\ell n_{\ell'} \sum_r \sum_s n_r n_s \mathbf{B}_{rs}(1-\mathbf{B}_{rs}) \nu_r^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_k \nu_\ell^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_s \nu_r^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_{k'} \nu_{\ell'}^\top (\mathbf{X}^\top \mathbf{X})^{-1} \nu_s$$

and hence

$$\tfrac{n^2}{2 n_k n_{k'} n_\ell n_{\ell'}} \sum_i \sum_j \mathbf{P}_{ij}(1-\mathbf{P}_{ij})(m_{ij}^{(3)} n_{ij}^{(3)} + m_{ji}^{(3)} n_{ji}^{(3)}) \longrightarrow \sum_r \sum_s \pi_r \pi_s \mathbf{B}_{rs}(1-\mathbf{B}_{rs}) \zeta_{kr} \zeta_{\ell s} \zeta_{k'r} \zeta_{\ell' s}.$$

The four sums that we derived above are the representative sums i.e., the remaining sums follow the same approach by permuting the roles of the indices k, k', ℓ and ℓ'. We omit the details.

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