SYMPLECTIC REDUCTION AT ZERO ANGULAR MOMENTUM

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ABSTRACT. We study the symplectic reduction of the phase space describing k particles in $\mathbb{R}^n$ with total angular momentum zero. This corresponds to the singular symplectic quotient associated to the diagonal action of O_n on k copies of $T^*\mathbb{R}^n$ at the zero value of the homogeneous quadratic moment map. We give a description of the ideal of relations of the ring of regular functions of the symplectic quotient. Using this description, we demonstrate $\mathbb{Z}^+$-graded regular symplectomorphisms among the O_n- and SO_n-symplectic quotients and determine which of these quotients are graded regularly symplectomorphic to linear symplectic orbifolds. We demonstrate that when $n \leq k$, the zero fibre of the moment map has rational singularities and hence is normal and Cohen-Macaulay. We also demonstrate that for small values of k, the ring of regular functions on the symplectic quotient is graded Gorenstein.

1. **Introduction.** In this note, we examine geometric and algebraic properties of symplectic quotients corresponding to k particles moving in $\mathbb{R}^n$ with zero total angular momentum. Previous studies of these spaces by M. Gotay et. al. [3, 9, 1] and by J. Huebschmann, see e.g. [16], were motivated by gauge theory. More specifically [16, Theorem 4], the cases $n = 2, 3$ occur as local models for the strata of the moduli space of flat SU_2-connections on a closed hyperelliptic Riemann surface of genus g, the number of particles k being linked to the genus by $k = g$ or $k = g - 1$, respectively. Moreover, as an illustration to the seminal paper [28] of E. Lerman

and R. Sjamaar, the symplectic quotients at zero angular momentum have been studied in [21], see also the survey [17].

Here, we expand on the discussion in [21] of the symplectomorphism problem (see Section 3) for symplectic quotients at zero angular momentum and explain in more detail the distinction between reduction by O_n and by SO_n (see Section 4). Our strategy for constructing maps between symplectic quotients is based on polynomial invariant theory and emphasizes the role of the (graded) $\mathbb{R}$-algebra of regular functions $\mathbb{R}[M_0]$ on the symplectic quotient M_0. This approach has been already advocated in [7, 15, 13, 14]; the requisite material will be recalled in Section 2. Essential for the construction of the symplectomorphisms will be a system of real polynomials $Q_{i,j}$ in the invariants whose locus coincides with that of the moment map.

Moreover, we show in Section 4 that the symplectic quotient is symplectomorphic to an orbifold precisely when $k = 1$ or $n = 1$, illustrating that in this case, the non-orbifold conditions of [13] are necessary but not sufficient; this is also illustrated in that reference with the case of SU_2-symplectic quotients. In Section 5, we show that when $n \leq k$, the zero fibre of the moment map has *rational singularities*, which in particular implies that the corresponding ideal is integrally closed and Cohen-Macaulay. Employing Boutot's Theorem [4] as well as the aforementioned symplectomorphisms, it follows that the symplectic quotients by O_n and SO_n have rational singularities for each n and k, a result which also follows from the results of [30] and [2] for the corresponding complex symplectic quotients. Note that in [27], symplectic quotients are shown to have rational singularities in terms of the complex analytic structure they inherit from the corresponding GIT quotient via the Kempf-Ness theorem; see Section 2. The results discussed here, on the other hand, are in terms of the real structure of the symplectic quotient or the complex analytic structure of the complex symplectic quotient.

In Section 6, the experimental part of the paper, we address the open question of determining the relations of the intersection of the ring of invariants with the vanishing ideal of the zero fibre of the moment map, i.e. the real radical of the ideal described by Theorem 3.4. In those cases where we can calculate the generators with the computer algebra software *Macaulay2* [10] and *Mathematica* [36] (namely, $k \leq 3$ and some cases when $k = 4$) we show that there are no additional relations, and $\mathbb{R}[M_0]$ is actually graded Gorenstein. We discuss the influence of the choice of a term order on the efficiency of the respective Gröbner basis calculations. That $\mathbb{R}[M_0]$ is graded Gorenstein with rational singularities is expected in a much wider generality.

2. **Setup and notation.** Let k be a positive integer. Throughout this paper, we consider the phase space of k particles in $\mathbb{R}^n$. We let $V_{k,n} = \mathbb{R}^{2kn}$ denote the phase space of configurations of the k particles; throughout, we will suppress the indices k, n when it will cause no confusion. For each $\ell = 1, \ldots, k$, we let $\boldsymbol{q}_\ell = (q_{\ell,1}, \ldots, q_{\ell,n}) \in \mathbb{R}^n$ denote the position and $\boldsymbol{p}_\ell = (p_{\ell,1}, \ldots, p_{\ell,n}) \in \mathbb{R}^n$ denote the momentum of the ℓth particle; we will use Greek letters to indicate the coordinates in $\mathbb{R}^n$ and Roman letters for the particle numbering. Then V carries the symplectic structure associated to it as the underlying real space of the complex vector space $\mathbb{C}^{kn}$ with coordinates $q_{\ell,\alpha} + \sqrt{-1}p_{\ell,\alpha}$ and the standard Hermitian structure. We use $(\boldsymbol{q}, \boldsymbol{p})$ to abbreviate the coordinates $(\boldsymbol{q}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{q}_k, \boldsymbol{p}_k)$ for V. For certain computations, it will be convenient to define $\boldsymbol{y}_{2\ell-1} = \boldsymbol{q}_\ell$ and $\boldsymbol{y}_{2\ell} = \boldsymbol{p}_\ell$

so that we may use coordinates $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{2k})$ where each $\boldsymbol{y}_i \in \mathbb{R}^n$ as well, with $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,n})$. We will throughout consider the diagonal action of the real orthogonal group $\mathrm{O}_n := \mathrm{O}_n(\mathbb{R})$ on V given by the standard O_n-action on each $\mathbb{R}^n$-factor. Representing a point $(\boldsymbol{q}, \boldsymbol{p}) \in V$ as an $n \times 2k$ matrix, this action corresponds to multiplication on the left by elements of O_n. For the constant Poisson bracket $\{\,,\,\}$ on $\mathbb{R}[V_{k,n}]$ we use the convention $\{q_{\ell_1, \alpha}, p_{\ell_2, \beta}\} := \delta_{\ell_1, \ell_2} \delta_{\alpha, \beta}$ for $i, j = 1, \ldots, k$ and $\alpha, \beta = 1, \ldots, n$; all other brackets between linear coordinates are supposed to vanish.

The action of O_n (respectively SO_n) on the real vector space V extends to an action of the complexification $\mathrm{O}_n(\mathbb{C})$ (respectively $\mathrm{SO}_n(\mathbb{C})$) on the complex vector space $V \otimes_\mathbb{R} \mathbb{C}$, and hence the real invariant polynomials $\mathbb{R}[V]^{\mathrm{O}_n}$ can be computed in terms of the complex invariants $\mathbb{C}[V \otimes_\mathbb{R} \mathbb{C}]^{\mathrm{O}_n(\mathbb{C})}$; see [24, Proposition 5.8(1)]. By the First and Second Fundamental Theorems of Invariant Theory for O_n, see [31, Sections 9.3 and 9.4] and [35, Sections 9 and 17], the invariants $\mathbb{R}[V]^{\mathrm{O}_n}$ are generated by the scalar products $x_{i,j} := \langle \boldsymbol{y}_i, \boldsymbol{y}_j \rangle$, $1 \le i \le j \le 2k$. The relations among these invariants, which we refer to as *off-shell relations*, are generated by the $(n+1) \times (n+1)$-minors of the symmetric $2k \times 2k$ matrix $X := (x_{i,j})$; see the definition of the *shell* below for an explanation of this language. Hence $\mathbb{R}[V]^{\mathrm{O}_n}$ is a polynomial algebra if and only if $n \ge 2k$. In the same way, $\mathbb{R}[V]^{\mathrm{SO}_n}$ is generated by the $x_{i,j}$ as well as determinants $\det(\boldsymbol{y}_{i_1}, \ldots, \boldsymbol{y}_{i_n})$ with $1 \le i_1 < \cdots < i_n \le 2k$. The invariant rings $\mathbb{R}[V]^{\mathrm{O}_n} \subset \mathbb{R}[V]^{\mathrm{SO}_n}$ are by construction Poisson subalgebras of $\mathbb{R}[V]$. This can also be expressed in terms of commutation relations among the invariants, e.g. for $\ell_1, \ell_2, \ell_3, \ell_4 = 1, 2, \ldots, k$:

$$
\begin{aligned}
\{x_{2\ell_1-1, 2\ell_2-1}, x_{2\ell_3, 2\ell_4}\} = & \delta_{\ell_1, \ell_3} x_{2\ell_2-1, 2\ell_4} + \delta_{\ell_1, \ell_4} x_{2\ell_2-1, 2\ell_3} \\
& + \delta_{\ell_2, \ell_3} x_{2\ell_1-1, 2\ell_4} + \delta_{\ell_2, \ell_4} x_{2\ell_1-1, 2\ell_3}, \\
\{x_{2\ell_1-1, 2\ell_2}, x_{2\ell_3-1, 2\ell_4}\} = & \delta_{\ell_1, \ell_4} x_{2\ell_3-1, 2\ell_2} - \delta_{\ell_2, \ell_3} x_{2\ell_1-1, 2\ell_4}, \\
\{x_{2\ell_1-1, 2\ell_2}, x_{2\ell_3, 2\ell_4-1}\} = & -\delta_{\ell_1, \ell_4} x_{2\ell_1-1, 2\ell_3-1} - \delta_{\ell_2, \ell_3} x_{2\ell_1-1, 2\ell_4-1}, \\
\{x_{2\ell_1-1, 2\ell_2}, x_{2\ell_3, 2\ell_4}\} = & \delta_{\ell_1, \ell_4} x_{2\ell_1-1, 2\ell_3} + \delta_{\ell_2, \ell_3} x_{2\ell_1-1, 2\ell_4}.
\end{aligned}
\tag{1}
$$

Identifying the Lie algebra $\mathfrak{o}_n^*$ with $\bigwedge^2 \mathbb{R}^n$ in the standard way, the moment map $\boldsymbol{J} = \boldsymbol{J}_{k,n} \colon V \to \mathfrak{o}_n^*$ can be expressed as

$$
\boldsymbol{J}(\boldsymbol{q}, \boldsymbol{p}) = \sum_{\ell=1}^k \boldsymbol{q}_\ell \wedge \boldsymbol{p}_\ell.
$$

For $1 \le \alpha < \beta \le n$, we let

$$
\begin{aligned}
J_{\alpha, \beta}(\boldsymbol{q}, \boldsymbol{p}) = J_{k,n,\alpha,\beta}(\boldsymbol{q}, \boldsymbol{p}) &= \sum_{\ell=1}^k q_{\ell, \alpha} p_{\ell, \beta} - q_{\ell, \beta} p_{\ell, \alpha} \\
&= \sum_{\ell=1}^k y_{2\ell-1, \alpha} y_{2\ell, \beta} - y_{2\ell-1, \beta} y_{2\ell, \alpha}
\end{aligned}
$$

denote the $\boldsymbol{e}_\alpha \wedge \boldsymbol{e}_\beta$-component of $\boldsymbol{J}(\boldsymbol{q}, \boldsymbol{p})$ where the $\boldsymbol{e}_\alpha$ denote the standard basis vectors of $\mathbb{R}^n$. We let $\mathcal{J} = \mathcal{J}_{k,n} = \langle J_{k,n} \mid 1 \le i < j \le 2k \rangle$ denote the ideal of $\mathbb{R}[V]$ generated by the components of the moment map. For all $\alpha, \beta, \gamma, \epsilon = 1, 2, \ldots, n$ the moment map fulfills the commutation relations

$$
\{J_{\alpha, \beta}, J_{\gamma, \epsilon}\} = \delta_{\alpha, \epsilon} J_{\beta, \gamma} + \delta_{\beta, \gamma} J_{\epsilon, \alpha} + \delta_{\alpha, \gamma} J_{\beta, \epsilon} + \delta_{\beta, \epsilon} J_{\alpha, \gamma},
$$

which implies that $\{\mathcal{J}, \mathcal{J}\} \subset \mathcal{J}$.

Let $Z_{k,n} = Z = \boldsymbol{J}^{-1}(0)$ denote the zero fibre of the moment map, which we refer to as the *shell*. The phase space of configurations with angular momentum zero is the *symplectic quotient*

$$M_0 = M_{0,k,n} := Z/\operatorname{O}_n,$$

and is our primary object of study. We regard it as a *Poisson differential space* with a *global chart* defined in terms of the invariants $x_{i,j}$; we briefly recall the basics of such spaces and refer the reader to [7, Section 4] or [14, Section 2.1] for more details. The *Hilbert embedding* $V/\operatorname{O}_n \to \mathbb{R}^{\binom{2k}{2}}$ with coordinates $x_{i,j}$ restricts to an embedding of M_0 into $\mathbb{R}^{\binom{2k}{2}}$, realizing M_0 as a semialgebraic set. We let $\mathcal{I}_Z$ denote the vanishing ideal of Z in $\mathbb{R}[V]$ and $\mathcal{I}_Z^{\operatorname{O}_n} := \mathcal{I}_Z \cap \mathbb{R}[V]^{\operatorname{O}_n}$. Then the *Poisson algebra of real regular functions on M_0* is

$$\mathbb{R}[M_0] := \mathbb{R}[V]^{\operatorname{O}_n}/\mathcal{I}_Z^{\operatorname{O}_n}.$$

The algebra $\mathbb{R}[M_0]$ is a Poisson subalgebra of the algebra $\mathcal{C}^\infty(M_0)$ which is defined as the quotient of $\mathcal{C}^\infty(V)^{\operatorname{O}_n}$ by the ideal of smooth invariant functions that vanish on Z; see [28]. We refer to the relations among the $x_{i,j}$ in $\mathbb{R}[M_0]$, which include the off-shell relations as well as the elements of $\mathcal{I}_Z^{\operatorname{O}_n}$, as *on-shell relations*.

Given two such Poisson differential spaces with global charts, an isomorphism between the algebras of regular functions induces a homeomorphism between the Zariski closures of the associated semialgebraic sets. Letting $\mathbb{Z}^+$ denote the non-negative integers, a ($\mathbb{Z}^+$-graded) isomorphism between the algebras of regular functions is a *($\mathbb{Z}^+$-graded) regular diffeomorphism* if the corresponding homeomorphism restricts to a homeomorphism between the semialgebraic sets themselves, and a *($\mathbb{Z}^+$-graded) regular symplectomorphism* if the isomorphism on algebras of regular functions is Poisson. By [7, Theorem 6], a $\mathbb{Z}^+$-graded regular symplectomorphism induces an isomorphism of Poisson differential spaces and in particular a Poisson isomorphism between the algebras of smooth functions.

The infinitesimal actions, and hence moment maps, of the O_n- and SO_n-actions on V coincide. Hence the symplectic quotient of the Hamiltonian SO_n-space V is given by

$$M_0^{\operatorname{SO}} = M_{0,k,n}^{\operatorname{SO}} := Z/\operatorname{SO}_n$$

with Poisson algebra of real regular functions

$$\mathbb{R}[M_0^{\operatorname{SO}}] := \mathbb{R}[V]^{\operatorname{SO}_n}/\mathcal{I}_Z^{\operatorname{SO}_n}.$$

In addition, we may consider V as the underlying real space of the complex vector space $\mathbb{C}^{kn}$ with coordinates $q_{\ell,\alpha} + \sqrt{-1}p_{\ell,\alpha}$ as above. Then Z is the *Kempf-Ness set* associated to the action of $\operatorname{O}_n(\mathbb{C})$ (or $\operatorname{SO}_n(\mathbb{C})$) on $\mathbb{C}^{kn}$. It follows that the symplectic quotient $M_{0,k,n}$ (respectively $M_{0,k,n}^{\operatorname{SO}}$) is homeomorphic via the Kempf-Ness homeomorphism to the affine GIT quotient $\mathbb{C}^{kn}/\!\!/\operatorname{O}_n(\mathbb{C}) = \operatorname{Spec}(\mathbb{C}[\mathbb{C}^{kn}]^{\operatorname{O}_n(\mathbb{C})})$ (respectively $\mathbb{C}^{kn}/\!\!/\operatorname{SO}_n(\mathbb{C}) = \operatorname{Spec}(\mathbb{C}[\mathbb{C}^{kn}]^{\operatorname{SO}_n(\mathbb{C})}))$, see [19, 25]. Here, the action is again the diagonal of the standard action on the k $\mathbb{C}^n$-factors. The $\operatorname{O}_n(\mathbb{C})$-invariants are generated by the Euclidean inner products of the vectors in $\mathbb{C}^n$, and the $\operatorname{SO}_n(\mathbb{C})$-invariants are given by these and the determinants of n distinct vectors in $\mathbb{C}^n$ by the Fundamental Theorem of Invariant Theory described above.

We recall the following and refer the reader to [26, 12] for more details. Let G be a reductive group and Y an affine G-variety over $\mathbb{C}$. Each point in the GIT quotient $Y/\!\!/G$ corresponds to a closed G-orbit in Y, and $Y/\!\!/G$ is stratified by the orbit types of the closed orbits. There is a unique open orbit type in $Y/\!\!/G$, the *principal orbit*

type, and elements of the corresponding conjugacy class of isotropy groups are the *principal isotropy groups*. We say Y has *finite principal isotropy groups (FPIG)* if the principal isotropy groups are finite, and that Y is *m-principal* if it has FPIG and the set of principal orbits has (complex) codimension at least m. Letting $Y_{(j)}$ denote the set of points in Y with isotropy group of complex dimension j and $c_j := \dim_{\mathbb{C}} V_{(j)} - \dim_{\mathbb{C}} G + j$, we say Y is *m-modular* if $V_{(0)} \neq \emptyset$ and $c_j + m \leq c_0$. Finally, Y is *m-large* if it is *m-principal* and *m-modular*.

In the case that $G = \mathrm{O}_n$ or SO_n and $Y = k\mathbb{C}^n$ as above, [26, Theorem 11.18] and [12, 3.5] give criteria for Y to be 1- or 2-large. Moreover, [12, Corollary 4.3] implies that in the 1-large cases, the vanishing ideal of the (real) shell $Z_{k,n}$ in $\mathbb{R}[V]$ is generated by the components of the moment map. Similarly, letting $\boldsymbol{J}^{\mathbb{C}}$ denote the complexification of the moment map, [12, Theorem 2.2] describes consequences for the variety associated to the ideal $\mathcal{J}^{\mathbb{C}}$ generated by the components of $\boldsymbol{J}^{\mathbb{C}}$. We summarize these results with the following.

Theorem 2.1 ([26, 12]). *Let $\mathbb{C}^{kn} = k\mathbb{C}^n$ be equipped with the diagonal of the standard action of $\mathrm{O}_n(\mathbb{C})$.*

- *(i.) If $n \leq k + 1$, then $\mathbb{C}^{kn}$ is 1-large as an $\mathrm{O}_n(\mathbb{C})$- and $\mathrm{SO}_n(\mathbb{C})$-module. Hence the variety associated to the ideal $\mathcal{J}^{\mathbb{C}}$ of $\mathbb{C}[V \otimes_{\mathbb{R}} \mathbb{C}]$ is a reduced, irreducible complete intersection, and the ideal $\mathcal{J}$ of $\mathbb{R}[V]$ is real radical.*
- *(ii.) If $n \leq k$, then $\mathbb{C}^{kn}$ is 2-large as an $\mathrm{SO}_n(\mathbb{C})$-module. Hence $\mathbb{C}[\mathbb{C}^{kn}]/\mathcal{J}^{\mathbb{C}}$ is integrally closed, i.e. the corresponding variety is normal.*
- *(iii.) If $n \leq k - 1$, then $\mathbb{C}^{kn}$ is 2-large as an $\mathrm{O}_n(\mathbb{C})$-module.*

In particular, when $k = n$, the $\mathrm{O}_n(\mathbb{C})$-representation $\mathbb{C}^{kn}$ is not 2-large, yet the $\mathrm{SO}_n(\mathbb{C})$-representation is. Because $\mathcal{J}^{\mathbb{C}}$ coincides for these two representations, the corresponding variety is normal, illustrating that the converse of [12, Theorem 2.2(4)] is false.

3. **The quadratic on-shell relations.** In this section, we consider the ideal of relations defining $\mathbb{R}[M_0]$ as an affine algebra. Our goal is to express these relations in terms of the invariants $x_{i,j}$, $1 \leq i \leq j \leq 2k$.

Fix positive integers n and k. For $1 \leq i < j \leq 2k$, define

$$Q_{k,n,i,j} = \sum_{\ell=1}^{k} \det \begin{pmatrix} x_{i,2\ell-1} & x_{i,2\ell} \\ x_{j,2\ell-1} & x_{j,2\ell} \end{pmatrix} = \sum_{\ell=1}^{k} \det \begin{pmatrix} \langle \boldsymbol{y}_i, \boldsymbol{y}_{2\ell-1} \rangle & \langle \boldsymbol{y}_i, \boldsymbol{y}_{2\ell} \rangle \\ \langle \boldsymbol{y}_j, \boldsymbol{y}_{2\ell-1} \rangle & \langle \boldsymbol{y}_j, \boldsymbol{y}_{2\ell} \rangle \end{pmatrix}. \quad (2)$$

We consider the $Q_{k,n,i,j}$ as quadratic polynomials in the invariants $x_{i,j}$ or equivalently as quartic polynomials in the $y_{i,j}$. Note that when $n = 1$, the $Q_{k,n,i,j}$ are forced to vanish as the 2×2-minors of a Gram matrix must vanish. We will first prove the following.

Proposition 1. *Let $k \geq 1$ and $n \geq 1$. For a point $\boldsymbol{y} \in V$, the following are equivalent:*

- *(i.) $Q_{k,n,2\ell-1,2\ell}(\boldsymbol{y}) = 0$ for each $1 \leq \ell \leq k$,*
- *(ii.) $Q_{k,n,i,j}(\boldsymbol{y}) = 0$ for each $1 \leq i < j \leq 2k$, and*
- *(iii.) $\boldsymbol{J}(\boldsymbol{y}) = 0$, i.e. $J_{k,n,\alpha,\beta}(\boldsymbol{y}) = 0$ for each $1 \leq \alpha < \beta \leq n$.*

We begin with the following.

Lemma 3.1. *For each $k, n \geq 1$,*

$$\sum_{\ell=1}^{k} Q_{k,n,2\ell-1,2\ell} = \sum_{1 \leq \alpha < \beta \leq n} J_{k,n,\alpha,\beta}^2. \tag{3}$$

Note that the right-hand side of Equation (3) is the norm squared of the moment map, which plays a role in the study of convexity properties of $\boldsymbol{J}$ [20].

Proof. When $n = 1$, the left side of Equation (3) vanishes as noted above, and the right side is the empty sum, so assume $n > 1$. We compute

$$\sum_{\ell=1}^{k} Q_{k,n,2\ell-1,2\ell} = \sum_{\ell_1=1}^{k} \sum_{\ell_2=1}^{k} \langle \boldsymbol{q}_{\ell_1}, \boldsymbol{q}_{\ell_2} \rangle \langle \boldsymbol{p}_{\ell_1}, \boldsymbol{p}_{\ell_2} \rangle - \langle \boldsymbol{q}_{\ell_1}, \boldsymbol{p}_{\ell_2} \rangle \langle \boldsymbol{p}_{\ell_1}, \boldsymbol{q}_{\ell_2} \rangle,$$

which, after expanding and canceling, yields

$$= \sum_{1 \leq \alpha < \beta \leq n} \sum_{\ell=1}^{k} \left(q_{\ell,\alpha} p_{\ell,\beta} - q_{\ell,\beta} p_{\ell,\alpha} \right) \sum_{\ell=1}^{k} \left(q_{\ell,\alpha} p_{\ell,\beta} - q_{\ell,\beta} p_{\ell,\alpha} \right)$$

$$= \sum_{1 \leq \alpha < \beta \leq n} J_{k,n,\alpha,\beta}^2. \qquad \square$$

Now, for $k \geq 1$ and $n \geq 2$, we define for each $1 \leq i < j \leq 2k$ the difference

$$D_{k,n,i,j} = Q_{k,n,i,j} - Q_{k,n-1,i,j}.$$

Note that $D_{k,2,i,j} = Q_{k,2,i,j}$ due to the vanishing of $Q_{k,1,i,j}$.

Lemma 3.2. *For $n \geq 2$, $k \geq 1$, and $1 \leq i < j \leq 2k$, we have*

$$D_{k,n,i,j} = \sum_{\alpha=1}^{n-1} (y_{i,\alpha} y_{j,n} - y_{i,n} y_{j,\alpha}) J_{k,n,\alpha,n}. \tag{4}$$

Proof. For fixed i and j, we express $D_{k,n,i,j}$ as

$$\sum_{\ell=1}^{k} \left(\langle \boldsymbol{y}_i, \boldsymbol{y}_{2\ell-1} \rangle \langle \boldsymbol{y}_j, \boldsymbol{y}_{2\ell} \rangle - \left(\sum_{\alpha=1}^{n-1} y_{i,\alpha} y_{2\ell-1,\alpha} \right) \left(\sum_{\beta=1}^{n-1} y_{j,\beta} y_{2\ell,\beta} \right) \right.$$

$$\left. - \langle \boldsymbol{y}_i, \boldsymbol{y}_{2\ell} \rangle \langle \boldsymbol{y}_j, \boldsymbol{y}_{2\ell-1} \rangle - \left(\sum_{\alpha=1}^{n-1} y_{i,\alpha} y_{2\ell,\alpha} \right) \left(\sum_{\beta=1}^{n-1} y_{j,\beta} y_{2\ell-1,\beta} \right) \right).$$

Expanding, the terms where both $\mathbb{R}^n$-indices are n cancel, and we continue

$$= \sum_{\ell=1}^{k} \sum_{\alpha=1}^{n-1} \big(y_{i,n} y_{2\ell-1,n} y_{j,\alpha} y_{2\ell,\alpha} + y_{i,\alpha} y_{2\ell-1,\alpha} y_{j,n} y_{2\ell,n}$$

$$- y_{i,n} y_{2\ell,n} y_{j,\alpha} y_{2\ell-1,\alpha} - y_{i,\alpha} y_{2\ell,\alpha} y_{j,n} y_{2\ell-1,n} \big)$$

$$= \sum_{\ell=1}^{k} \sum_{\alpha=1}^{n-1} (y_{i,\alpha} y_{j,n} - y_{i,n} y_{j,\alpha})(y_{2\ell-1,\alpha} y_{2\ell,n} - y_{2\ell-1,n} y_{2\ell,\alpha})$$

$$= \sum_{\alpha=1}^{n-1} (y_{i,\alpha} y_{j,n} - y_{i,n} y_{j,\alpha}) J_{k,n,\alpha,n}. \qquad \square$$

In particular, the following is an immediate consequence of Equation (4).

Corollary 1. *For $n \geq 2$, $k \geq 1$, we have*

$$\sum_{\ell=1}^{k} D_{k,n,2\ell-1,2\ell} = \sum_{\alpha=1}^{n-1} J_{k,n,\alpha,n}^2. \tag{5}$$

Proof of Proposition 1. Fix $k \geq 1$. If $n = 1$, then as all 2×2-minors of a Gram matrix of vectors in $\mathbb{R}^1$ must vanish and $\boldsymbol{J}$ is the zero map, conditions $(i.)$, $(ii.)$, and $(iii.)$ hold trivially for each point in V. Hence, we may assume $n \geq 2$.

The fact that $(iii.)\Rightarrow(ii.)$ can be seen by induction on n and Lemma 3.2. Specifically, the base case $n = 2$ follows from the fact that if $\boldsymbol{J}(\boldsymbol{y}) = 0$, then the right side of Equation (5) vanishes, yielding $Q_{k,2,i,j}(\boldsymbol{y}) = D_{k,2,i,j}(\boldsymbol{y}) = 0$ for each i, j. Assuming each $Q_{k,n-1,i,j}(\boldsymbol{y}) = 0$ for some n, Equation (4) then implies that each $Q_{k,n,i,j}(\boldsymbol{y}) = 0$. The implication $(ii.) \Rightarrow (i.)$ is obvious. Then $(i.)\Rightarrow(iii.)$ is an immediate consequence of Lemma 3.1. □

It follows that the $Q_{i,j}$ are elements of the ideal $\mathcal{I}_Z^{\mathrm{O}_n}$, and moreover that $\mathcal{I}_Z^{\mathrm{O}_n}$ is the real radical of the ideal generated by the $Q_{i,j}$. In addition, note that when $n < 2k$, the off-shell relations of the $x_{i,j}$ in the quotient $\mathbb{R}[V]^{\mathrm{O}_n}$ are generated by the $(n+1) \times (n+1)$-minors of X, see Section 2, so these minors must be on-shell relations as well. When $k \leq n$, a stronger statement is true. By an application of Cartan's Lemma, if $\boldsymbol{J}(\boldsymbol{y}) = 0$, then the subspace of $\mathbb{R}^n$ spanned by $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{2k}\}$ has dimension at most k; see [21, page 23]. Therefore, it must be that the $(k+1) \times (k+1)$-minors of X vanish in $\mathbb{R}[M_0]$, and hence are contained in the real radical of the ideal generated by the $Q_{i,j}$.

Our next goal is to demonstrate that in general, the $(k+1) \times (k+1)$-minors of X are in fact in the ideal generated by the $Q_{i,j}$ themselves, and not simply elements of its real radical. We state and prove the following proposition more generally, as it holds for an arbitrary matrix that need not be symmetric nor square. Specifically, let m be a positive integer, and let $x_{i,j}$, $1 \leq i \leq m$ and $1 \leq j \leq 2k$ denote arbitrary variables. Then we may consider the ideal of $\mathbb{R}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq 2k]$ generated by the $Q_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq 2k$ as defined by Equation (2). Note that, when $m = 2k$ and the $x_{i,j}$ are the O_n-invariants as above, this result is trivial unless $k \leq n$. Specifically, if $n < k$, then the $(n+1) \times (n+1)$-minors, which are off-shell relations among the $x_{i,j}$, generate the $(k+1) \times (k+1)$-minors.

Proposition 2. *Let $k, m \geq 1$. The ideal of the polynomial ring $\mathbb{R}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq 2k]$ generated by the $Q_{i,j}$ contains the $(k+1) \times (k+1)$-minors of the matrix X.*

To prove Proposition 2, consider an arbitrary $m \times 2k$ matrix X, and let $R = \mathbb{R}[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq 2k]$ denote corresponding polynomial ring. We work in R^m, which we consider both as an R-module and an $\mathbb{R}$-vector space, and consider each column of X as an element of R^m. In particular, we let $P_{m,k}$ denote the sub-$\mathbb{R}$-vector space of R^m given by the $\mathbb{R}$-linear span of the $2k$ columns of X.

Let $\boldsymbol{x}_i \in R^m$ denote the ith column of X, and let $\boldsymbol{e}_i \in R^m$ denote the element containing 1 in the ith position and 0 elsewhere as usual. Define

$$\boldsymbol{Q}_k = \sum_{\ell=1}^{k} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell}.$$

Note that in the case that $m = 2k$ and $x_{i,j} = \langle \boldsymbol{y}_i, \boldsymbol{y}_j \rangle$ as above, we have

$$\boldsymbol{Q}_k = \sum_{1 \leq i < j \leq 2k} Q_{i,j} \boldsymbol{e}_i \wedge \boldsymbol{e}_j.$$

We claim the following.

Lemma 3.3. *Let X be an $m \times 2k$ matrix with columns $\boldsymbol{x}_1 \ldots, \boldsymbol{x}_{2k}$. For any element $\tau \in \bigwedge^{k+1} P_{m,k}$, there is an element $\omega \in \bigwedge^{k-1} P_{m,k}$ such that $\boldsymbol{Q}_k \wedge \omega = \tau$.*

Proof. Choose any positive integer m. The proof is by induction on k. If $k = 1$, then $\boldsymbol{Q}_1 = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2$, and $\bigwedge^2 P_{m,2}$ is spanned by $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2$. Therefore, τ must be a scalar multiple of $\boldsymbol{Q}_1$, and setting ω to be that scalar yields the result.

Now, fix $k \geq 2$, and assume that for any $s < k$ and any $\tau \in \bigwedge^{s+1} P_{m,s}$, there is a $\sigma \in \bigwedge^{s-1} P_{m,s}$ such that $\boldsymbol{Q}_s \wedge \sigma = \tau$. Choose a $\tau \in \bigwedge^k P_{m,s}$. By linearity, we may assume that τ is a monic $(k+1)$-blade, i.e. $\tau = \boldsymbol{x}_{i_1} \wedge \boldsymbol{x}_{i_2} \wedge \cdots \wedge \boldsymbol{x}_{i_{k+1}}$.

Note that $\boldsymbol{Q}_k$ is invariant under permuting pairs of indices of the form $(2\ell-1, 2\ell)$ for $\ell = 1, \ldots, k$, so we may apply such a permutation to τ for simplicity. Then there are two cases.

Case 1. For at least one pair $(2\ell - 1, 2\ell)$, one element of the pair appears in $\{i_1, \ldots, i_{k+1}\}$ while the other does not. In this case, by permuting pairs, we can assume i_k and i_{k+1} are not in a pair of the form $(2\ell - 1, 2\ell)$. Let $\boldsymbol{x}_j$ denote the other element of the pair containing $\boldsymbol{x}_{i_{k+1}}$. Let $\boldsymbol{Q}_{k-1}^{\neg i_{k+1}, j}$ denote $\boldsymbol{Q}_k$ after removing the term $\pm \boldsymbol{x}_{i_{k+1}} \wedge \boldsymbol{x}_j$. By the inductive hypothesis, after relabeling the variables to see that $P_{m,k-1}$ is isomorphic to $\mathrm{Span}_{\mathbb{R}}(\{\boldsymbol{x}_1, \ldots \boldsymbol{x}_{2k}\} \setminus \{\boldsymbol{x}_{i_{k+1}}, \boldsymbol{x}_j\})$, there is a $\sigma \in \bigwedge^{k-2} \mathrm{Span}_{\mathbb{R}}(\{\boldsymbol{x}_1, \ldots \boldsymbol{x}_{2k}\} \setminus \{\boldsymbol{x}_{i_{k+1}}, \boldsymbol{x}_j\})$ such that

$$\boldsymbol{Q}_{k-1}^{\neg i_{k+1}, j} \wedge \sigma = \boldsymbol{x}_{i_1} \wedge \cdots \wedge \boldsymbol{x}_{i_k}.$$

Let $\omega = \sigma \wedge \boldsymbol{x}_{i_{k+1}}$, and then we compute

$$\boldsymbol{Q}_k \wedge \omega = (\boldsymbol{Q}_{k-1} \pm \boldsymbol{x}_{i_{k+1}} \wedge \boldsymbol{x}_j) \wedge \sigma \wedge \boldsymbol{x}_{i_{k+1}} = \tau.$$

Hence the inductive step is satisfied in this case.

If the $i_1, \ldots, i_{k+1}$ form a union of pairs of the form $(2\ell - 1, 2\ell)$, then up to permuting pairs, we may assume the following.

Case 2. k is odd, and $(i_1, \ldots, i_{k+1}) = (1, 2, \ldots, k+1)$. In this case, we will describe σ explicitly. Let $r = (k+1)/2$ and define

$$\sigma = \sum_{t=0}^{r-1} \frac{(-1)^t (r-t-1)! t!}{r!} \left(\sum_{\substack{A \subseteq \{1,\ldots,r\} \\ |A| = r-t-1}} \sum_{\substack{B \subseteq \{r+1,\ldots,k\} \\ |B| = t}} \bigwedge_{\ell \in A \cup B} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \right).$$

To simplify notation in the following computation, A will always denote subsets of $\{1, \ldots, r\}$ and B will denote subsets of $\{r+1, \ldots, k\}$. We compute $\boldsymbol{Q}_k \wedge \sigma$ to be

$$\left(\sum_{\ell=1}^{k} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \right) \wedge \sum_{t=0}^{r-1} \frac{(-1)^t (r-t-1)! t!}{r!} \left(\sum_{\substack{|A| = r-t-1 \\ |B| = t}} \bigwedge_{\ell \in A \cup B} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \right)$$

$$= \Big(\sum_{\ell=1}^{r} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \Big) \wedge \sum_{t=0}^{r-1} \frac{(-1)^t (r-t-1)! t!}{r!} \Big(\sum_{\substack{|A|=r-t-1 \\ |B|=t}} \bigwedge_{\ell \in A \cup B} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \Big)$$

$$+ \Big(\sum_{\ell=r+1}^{k} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \Big) \wedge \sum_{t=0}^{r-1} \frac{(-1)^t (r-t-1)! t!}{r!} \Big(\sum_{\substack{|A|=r-t-1 \\ |B|=t}} \bigwedge_{\ell \in A \cup B} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \Big).$$

Combining the $\boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell}$-factors from $\boldsymbol{Q}_k$ into the sets A in the first sum (respectively B in the second), and noting that each resulting set then occurs $r-t$ (respectively $t+1$) times, we continue

$$= \sum_{t=0}^{r-1} \frac{(-1)^t (r-t)! t!}{r!} \Big(\sum_{\substack{|A|=r-t \\ |B|=t}} \bigwedge_{\ell \in A \cup B} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \Big)$$

$$+ \sum_{t=0}^{r-2} \frac{(-1)^t (r-t-1)! (t+1)!}{r!} \Big(\sum_{\substack{|A|=r-t-1 \\ |B|=t+1}} \bigwedge_{\ell \in A \cup B} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \Big)$$

$$= \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{k+1} + \sum_{t=1}^{r-1} \frac{(-1)^t (r-t)! t!}{r!} \Big(\sum_{\substack{|A|=r-t \\ |B|=t}} \bigwedge_{\ell \in A \cup B} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \Big)$$

$$+ \sum_{t=0}^{r-2} \frac{(-1)^t (r-t-1)! (t+1)!}{r!} \Big(\sum_{\substack{|A|=r-t-1 \\ |B|=t+1}} \bigwedge_{\ell \in A \cup B} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \Big)$$

$$= \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{k+1} + \sum_{t=0}^{r-2} \frac{(-1)^{t+1} (r-t-1)! (t+1)!}{r!} \Big(\sum_{\substack{|A|=r-t-1 \\ |B|=t+1}} \bigwedge_{\ell \in A \cup B} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \Big)$$

$$+ \sum_{t=0}^{r-2} \frac{(-1)^t (r-t-1)! (t+1)!}{r!} \Big(\sum_{\substack{|A|=r-t-1 \\ |B|=t+1}} \bigwedge_{\ell \in A \cup B} \boldsymbol{x}_{2\ell-1} \wedge \boldsymbol{x}_{2\ell} \Big)$$

$$= \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_{k+1},$$

completing the proof. $\qquad\square$

Proof of Proposition 2. Choose any $(k+1) \times (k+1)$-minor of the $m \times 2k$ matrix $X = (x_{i,j})$. Let $i_1 < \cdots < i_{k+1}$ denote the rows and $j_1 < \cdots < j_{k+1}$ the corresponding columns, and let $\tau = \boldsymbol{x}_{j_1} \wedge \cdots \wedge \boldsymbol{x}_{j_{k+1}}$. Then there is by Lemma 3.3 an ω such that $\boldsymbol{Q}_k \wedge \omega = \tau$. In particular, the components of this wedge product in terms of the basis elements $\boldsymbol{e}_{r_1} \wedge \cdots \wedge \boldsymbol{e}_{r_{k+1}}$ can be expressed algebraically in terms of the components of $\boldsymbol{Q}_k$, and the $\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_{k+1}}$-component is exactly the $(k+1) \times (k+1)$-minor in question. $\qquad\square$

Combining Propositions 1 and 2 with the observations above, we have proven the following.

Theorem 3.4. *Let $k, n \geq 1$. If $n \geq k$, the ideal of on-shell relations among the $x_{i,j}$ describing $\mathbb{R}[M_0]$ as an affine algebra is the real radical of the ideal generated by the $Q_{i,j}$ for $1 \leq i < j \leq 2k$, and all $(k+1) \times (k+1)$-minors of X are contained*

in the ideal generated by the $Q_{i,j}$. If $n < k$, then the ideal of on-shell relations is the real radical of the ideal generated by both the $Q_{i,j}$ for $1 \le i < j \le 2k$ and the $(n+1) \times (n+1)$-minors of X.

We suspect that the ideal generated by the $Q_{i,j}$ (along with the $(n+1) \times (n+1)$-minors of X when $n < k$) is in fact real, which we observe is the case for small values of k in Section 6. However, the description given by Theorem 3.4 will be sufficient for our purposes.

Note that it is easy to see that the $r \times r$-minors of X do not vanish on Z when when $r \le k, n$, so that the relations given by minors are not superfluous in the cases where $n < k$. Specifically, setting each $\boldsymbol{q}_\ell = \boldsymbol{p}_\ell = \boldsymbol{e}_\ell$ to be a standard basis vector for $\ell \le r$ and $\boldsymbol{q}_\ell = \boldsymbol{p}_\ell = 0$ for $\ell > r$ yields an element of $V_{k,n}$ that is clearly in Z yet on which an $r \times r$-minor does not vanish (and other minors can be dealt with similarly); see [21, page 24]. Hence we have the following.

Corollary 2. *Fix $k \ge 1$. The symplectic quotients $M_{0,k,n}$ and $M_{0,k,m}$ are $\mathbb{Z}^+$-graded regularly symplectomorphic for each $m, n \ge k$. The symplectic quotients for $1 \le n \le k$ are not homeomorphic.*

Proof. When $n \ge k$, the description given by Theorem 3.4 of the graded algebras of regular functions does not depend on n, so the generators $x_{i,j}$ of the regular functions of $M_{0,k,k}$ and $M_{0,k,n}$ obviously define isomorphic (graded) global charts. Note that the ideal $\mathcal{J}_{k,k}$ of the moment map is real by Theorem 2.1, which implies that $\mathcal{J}_{k,k}$ generates the vanishing ideal of $Z_{k,k}$ in $\mathcal{C}^\infty(V_{k,k})$, see [1, Theorem 6.3 and Remark 6.4]. Therefore, the vanishing ideal of $Z_{k,k}$ in $\mathcal{C}^\infty(V_{k,k})^{\mathrm{O}_k}$ is generated by the invariants in $\mathcal{J}_{k,k}$ in this case, and the same follows for the isomorphic cases $n \ge k$. The inequalities defining the semialgebraic sets $V_{k,k}/\mathrm{O}_k$ and $V_{k,n}/\mathrm{O}_n$ are determined in [23, Example 0.8], and coincide with the requirement that the Gram matrix X is positive semidefinite (i.e. all principal minors are nonnegative). These obviously coincide for the full quotients and hence the symplectic quotients, so this isomorphism defines a $\mathbb{Z}^+$-graded regular diffeomorphism. The Poisson brackets of the generators $x_{i,j}$ are computed in Equation (1), and one observes that they do not depend on n; therefore, this isomorphism is Poisson.

When $m < n \le k$, the dimensions of $M_{0,k,m}$ and $M_{0,k,n}$ do not coincide. This can be seen by applying the Kempf-Ness homeomorphism and considering the affine GIT quotients $\mathbb{C}^{mk} /\!\!/ \mathrm{O}_m(\mathbb{C})$ and $\mathbb{C}^{nk} /\!\!/ \mathrm{O}_n(\mathbb{C})$. Each have $\binom{k}{2}$ fundamental invariants given by the Euclidean inner products, the $(n+1) \times (n+1)$-minors of the $k \times k$ Gram matrix of invariants are relations for both, but the $(m+1) \times (m+1)$-minors of the Gram matrix are relations only for $\mathbb{C}^{mk} /\!\!/ \mathrm{O}_m(\mathbb{C})$. It follows that the Krull dimension of the ring $\mathbb{C}[\mathbb{C}^{mk}]^{\mathrm{O}_m(\mathbb{C})}$ is strictly lower than that of $\mathbb{C}[\mathbb{C}^{nk}]^{\mathrm{O}_n(\mathbb{C})}$, from which the result follows. $\square$

Remark 1. In [21, Theorem 5.1], the symplectic quotient $M_{0,k,n}$ is shown to be isomorphic as a symplectic stratified space to Σ^r with $r = \min\{k, n\}$, a finite union of coadjoint orbits in the dual $\mathfrak{sp}_k^*$ of the Lie algebra of $\mathrm{Sp}_k(\mathbb{R})$. The space $\mathfrak{sp}_k^*$ is in turn identified with the set $S^2(\mathbb{R}^{2k})$ of $2k \times 2k$ symmetric real matrices, and Σ^r corresponds to the set of positive semidefinite matrices with coisotropic kernels of rank at most r, stratified by rank. The isomorphism $M_{0,k,n} \to \Sigma^r$ is induced by a map $V_{k,n} \to S^2(\mathbb{R}^{2k})$ that coincides with our map $X = (x_{i,j})$ (up to permuting coordinates to change notational conventions). Because the map X is regular, and the identification of $\mathfrak{sp}_k^*$ with $S^2(\mathbb{R}^{2k})$ as described in [21, page 22] is evidently

regular as well, it follows that X describes as well a $\mathbb{Z}$-graded regular symplectomorphism from $M_{0,k,n}$ to its image Σ^r. Hence, Theorem 3.4 gives an explicit algebraic description of the coadjoint orbits Σ^r.

4. **Symplectic quotients by SO_n and orbifold criteria.** In this section, we consider the relationship between the O_n- and SO_n-symplectic quotients and determine which of these quotients are $\mathbb{Z}^+$-graded regularly symplectomorphic to orbifolds. We first note the following. Let $\boldsymbol{y} \in V_{k,n}$, and let $W_{\boldsymbol{y}}$ denote the vector subspace of $\mathbb{R}^n$ spanned by $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{2k}\}$. Then the O_n-isotropy group of $\boldsymbol{y}$ is trivial if and only if $\dim W_{\boldsymbol{y}} = n$. Specifically, if $\dim W_{\boldsymbol{y}} \leq n-1$, then there is a reflection in O_n through a hyperplane containing $W_{\boldsymbol{y}}$, which therefore fixes $\boldsymbol{y}$; if $W_{\boldsymbol{y}} = \mathbb{R}^n$, then any element of O_n that fixes $\boldsymbol{y}$ must fix each element of $\mathbb{R}^n$ and hence is trivial. Similarly, the SO_n-isotropy group of $\boldsymbol{y}$ is trivial if and only if $\dim W_{\boldsymbol{y}} \geq n-1$, as any subspace of $\mathbb{R}^n$ of codimension 2 is fixed by a nontrivial rotation, and conversely every element of SO_n fixes a subspace of $\mathbb{R}^n$ of codimension at least 2.

With this, we demonstrate the following.

Theorem 4.1. *Let k be a positive integer and let $n \geq k+1$. Then the symplectic quotients $M_{0,k,n}$ and $M_{0,k,n}^{\mathrm{SO}}$ coincide.*

Proof. Let $\boldsymbol{y} \in V_{k,n}$ such that $\boldsymbol{J}(\boldsymbol{y}) = 0$ and choose $g \in \mathrm{O}_n \smallsetminus \mathrm{SO}_n$. By Proposition 2, the dimension of $W_{\boldsymbol{y}}$ is at most $k \leq n-1$. Then there is a reflection $r \in \mathrm{O}_n \smallsetminus \mathrm{SO}_n$ through a hyperplane of $\mathbb{R}^n$ that contains $W_{\boldsymbol{y}}$, which therefore fixes $\boldsymbol{y}$. As $\mathrm{O}_n / \mathrm{SO}_n \cong \mathbb{Z}_2$, there is an $h \in \mathrm{SO}_n$ such that $g = hr$, and then $g\boldsymbol{y} = hr\boldsymbol{y} = h\boldsymbol{y}$. It follows that the O_n- and SO_n-orbits of $\boldsymbol{y}$ coincide. $\square$

For the corresponding complex symplectic quotients, a similar statement has been observed in [30, Appendix A.2.1].

More algebraically, note that the invariants of the SO_n-action on $V_{k,n}$ are given by the O_n-invariants along with the determinants $\det(\boldsymbol{y}_{i_1}, \ldots, \boldsymbol{y}_{i_n})$ of collections of n vectors. By Proposition 2, these determinants all vanish. Hence these "new" invariants are forced to vanish, and the SO_n-symplectic quotients are defined by identical invariants and relations as the corresponding O_n-symplectic quotients. Similarly, the Hilbert embeddings differ only by coordinates that vanish on the shell.

Note that the consequence of Theorem 4.1 does not hold when $n \leq k$. In this case, we may choose $\boldsymbol{q}_\ell = \boldsymbol{p}_\ell = \boldsymbol{e}_\ell$ for $\ell \leq n$ and $\boldsymbol{q}_\ell = \boldsymbol{p}_\ell = 0$ for $\ell > n$. The result is a point that is easily seen to be an element of the shell and have trivial O_n-isotropy so that the O_n- and SO_n-orbits do not coincide. Hence, when $n \leq k$, the O_n-quotient is a branched double-cover of the corresponding SO_n-quotient as explained in [28, Remark 5.4]. The branch points are those points where the SO_n- and O_n-orbits coincide, which as seen in the proof of Theorem 4.1 are exactly the points $\boldsymbol{y}$ such that $W_{\boldsymbol{y}}$ has dimension at most $n-1$.

The following consequence of these observations will be useful in the sequel. Consider the embedding $\iota_{k-1} \colon V_{k,k-1} \to V_{k,k}$ where each $\mathbb{R}^{k-1}$ factor is embedded into each $\mathbb{R}^k$ factor as the first $k-1$ coordinates. We consider O_{k-1} as a subgroup of O_k (and similarly $\mathfrak{o}_{k-1}$ as a subalgebra of $\mathfrak{o}_k$) via the embedding ι_{k-1}.

Lemma 4.2. *The symplectic quotient $M_{0,k,k}$ contains an orbit type stratum S of real codimension 2. The closure S^{cl} of S (in the classical topology) is the set of orbits of points in $\iota_{k-1}(Z_{k,k-1})$, and hence ι_{k-1} induces a $\mathbb{Z}^+$-graded regular symplecto- morphism between S^{cl} and $M_{0,k,k-1}$, where the global chart for S^{cl} is the restriction*

of that for $M_{0,k,k}$. Moreover, the Poisson homomorphism $\mathbb{R}[M_{0,k,k}] \to \mathbb{R}[M_{0,k,k-1}]$ induced by ι_{k-1} is surjective.

Proof. Let $R = \{\boldsymbol{y} \in V_{k,k} \mid \dim_{\mathbb{R}} W_{\boldsymbol{y}} = n-1\}$. It is easy to see that R is exactly the orbit type corresponding to isotropy groups in O_k generated by a single reflection. As O_k acts transitively on the Grassmannian of $(k-1)$-planes in $\mathbb{R}^k$, each orbit in R intersects $\iota_{k-1}(V_{k,k-1})$. Moreover, $R \cap \iota_{k-1}(V_{k,k-1})$ is equal to the image under ι_{k-1} of the principal O_{k-1}-orbit type in $V_{k,k-1}$ as explained above, and therefore $R \cap \iota_{k-1}(V_{k,k-1})$ is dense in $\iota_{k-1}(V_{k,k-1})$. Clearly, the O_k-orbits intersect $\iota_{k-1}(V_{k,k-1})$ as O_{k-1}-orbits, and hence ι_{k-1} induces a homeomorphism from the principal orbit type of $V_{k,k-1}/\mathrm{O}_{k-1}$ onto R/O_k.

From Equation (2), one easily checks that the moment map $\boldsymbol{J}_{k,k}$ of the O_k-action on $V_{k,k}$ restricts to $\iota_{k-1}(V_{k,k-1})$ as the moment map $\boldsymbol{J}_{k,k-1}$ of the O_{k-1}-action, i.e. $\boldsymbol{J}_{k,k-1} = \boldsymbol{J}_{k,k} \circ \iota_{k-1}$, so that $Z_{k,k} \cap \iota_{k-1}(V_{k,k-1}) = \iota_{k-1}(Z_{k,k-1})$. Hence ι_{k-1} induces a homeomorphism from the principal orbit type stratum of $M_{0,k,k-1}$ onto $(R \cap \iota_{k-1}(Z_{k,k-1}))/\mathrm{O}_{k-1}$, which is homeomorphic to $S := (R \cap Z_{k,k})/\mathrm{O}_n$. Note that as R is an O_k-orbit type stratum of $V_{k,k}$, S is by definition a stratum of the symplectic stratified space $M_{0,k,k}$; see [28, Theorem 2.1].

Recall that $M_{0,k,k}$ is homeomorphic to the GIT quotient $(\mathbb{C}^{k^2})/\!\!/\mathrm{O}_k(\mathbb{C})$ by the Kempf-Ness homeomorphism [19, 25]. By the Fundamental Theorem of Invariant Theory for O_n, [31, Sections 9.3 and 9.4] and [35, Sections 9 and 17], it is easy to see that $\mathbb{C}[\mathbb{C}^{k^2}]^{\mathrm{O}_k(\mathbb{C})}$ is generated by $\binom{k+1}{2}$ quadratic invariants with no relations, and hence that $(\mathbb{C}^{k^2})/\!\!/\mathrm{O}_k(\mathbb{C})$ is complex $\binom{k+1}{2}$-dimensional affine space. Similarly, $\mathbb{C}[\mathbb{C}^{k(k-1)}]^{\mathrm{O}_k(\mathbb{C})}$ is generated by $\binom{k+1}{2}$ quadratic invariants with the single relation that the determinant of the Gram matrix vanishes, and hence is a complex hypersurface in $\mathbb{C}^{\binom{k+1}{2}}$. Hence S^{cl} has real codimension 2 in $M_{0,k,k}$.

Finally, note that as $\boldsymbol{J}_{k,k-1} = \boldsymbol{J}_{k,k} \circ \iota_{k-1}$, the pullback $\iota_{k-1}^*\colon \mathbb{R}[V_{k,k}] \to \mathbb{R}[V_{k,k-1}]$ induces a homomorphism $\mathbb{R}[M_{0,k,k}] \to \mathbb{R}[M_{0,k,k-1}]$. Recall that $\mathbb{R}[M_{0,k,k-1}]$ is generated by the $\binom{2k+1}{2}$ scalar products, which all occur as pullbacks via ι_{k-1} of the scalar products that generate $\mathbb{R}[M_{0,k,k}]$. It is then clear that the map $\mathbb{R}[M_{0,k,k}] \to \mathbb{R}[M_{0,k,k-1}]$ induced by ι_{k-1} is a surjective Poisson homomorphism that induces a Poisson isomorphism $\mathbb{R}[S^{cl}] \to \mathbb{R}[M_{0,k,k-1}]$, where $\mathbb{R}[S^{cl}]$ is the quotient of $\mathbb{R}[V_{k,k}]^{\mathrm{O}_k}$ by the invariant part of the vanishing ideal of R^{cl}, i.e. the algebra of regular functions corresponding to the restriction of the global chart for $\mathbb{R}[M_{0,k,k}]$ to S^{cl}. $\square$

Now we are ready to demonstrate that the only symplectic quotients under consideration that are $\mathbb{Z}^+$-graded regularly symplectomorphic to a linear symplectic orbifold are the cases where $k = 1$ or $n = 1$. By a *linear symplectic orbifold*, we mean the symplectic orbifold given by the quotient of $\mathbb{C}^n$ by a finite subgroup of U_n. The argument below uses ingredients developed in [13, Section 3] and [14, Section 3].

Theorem 4.3. *The symplectic quotient $M_{0,k,n}$ (respectively $M_{0,k,n}^{\mathrm{SO}}$) is $\mathbb{Z}^+$-graded regularly symplectomorphic to a linear symplectic orbifold if and only if $k = 1$ or $n = 1$.*

Proof. When $n = 1$, the group O_n is finite, SO_n is trivial, and the moment map is zero, so these case are quotients of affine space by a finite group and hence are in fact equal to linear symplectic orbifolds (trivial orbifolds, i.e. affine space, in the case of SO_n). When $k = 1$, the symplectic quotients $M_{0,1,n}$ are all $\mathbb{Z}^+$-graded

regularly symplectomorphic to $M_{0,1,1}$ by Corollary 2, which is the orbifold $\mathbb{C}/\mathbb{Z}_2$. Note that these symplectomorphisms have been observed in [3, 1]; see Section 6.

Now assume $k, n \geq 2$, and consider $M_{0,k,n}^{SO}$. When $n \leq k$, the corresponding $SO_n(\mathbb{C})$-representation is 2-large by Theorem 2.1. Moreover, this representation is orthogonal with respect to the Euclidean inner product on $\mathbb{C}^n$ and hence is stable by [26, Lemma 7.11]. Therefore, by [13, Theorem 1.1], the symplectic quotient is not symplectomorphic (regularly or otherwise) to a linear symplectic orbifold. Similarly, when $n \leq k - 1$, the $O_n(\mathbb{C})$ representation is 2-large by Theorem 2.1 and again stable by [26, Lemma 7.11] so that $M_{0,k,n}$ is not regularly symplectomorphic to a linear symplectic orbifold by [13, Theorem 1.3].

It remains only to consider the cases $M_{0,k,n}$ for $n \geq k$ and $M_{0,k,n}^{SO}$ for $n \geq k+1$, which for fixed k are all $\mathbb{Z}^+$-graded regularly symplectomorphic by Corollary 2 and Theorem 4.1. Hence, it is sufficient to show that $M_{0,k,k}$ is not $\mathbb{Z}^+$-graded regularly symplectomorphic to a linear symplectic orbifold for each $k \geq 2$.

For the case $k = 2$, the Hilbert series of $\mathbb{R}[M_{0,2,2}]$ is computed in Section 6, Equation (10). If $M_{0,2,2}$ is $\mathbb{Z}^+$-graded regularly symplectomorphic to the linear symplectic orbifold U/H, then the Hilbert series of $\mathbb{R}[M_{0,2,2}]$ clearly coincides with that of $\mathbb{R}[U]^H$, and hence the first coefficient γ_0 of the Laurent expansion of the Hilbert series at $t = 1$ is $1/|H|$; see [31, Theorem 3.23]. Because the first coefficient of the Laurent expansion is not the reciprocal of an integer, we have a contradiction, and $M_{0,2,2}$ is not $\mathbb{Z}^+$-graded regularly symplectomorphic to a linear orbifold; see [15, Section 7].

Now, for some $k \geq 3$, assume for contradiction that there is $\mathbb{Z}^+$-graded regular symplectomorphism $\chi \colon M_{0,k,k} \to U/H$ with H a finite group and U a unitary H-module; this implies that $\chi^* \colon \mathbb{R}[U]^H \to \mathbb{R}[M_{0,k,k}]$ is a $\mathbb{Z}^+$-graded Poisson isomorphism. We will demonstrate that this implies a $\mathbb{Z}^+$-graded regular symplectomorphism between $M_{0,k,k-1}$ and a linear symplectic suborbifold of U/H, which contradicts [13, Theorem 1.3] as explained above.

By Lemma 4.2, $M_{0,k,k}$ has a stratum S whose closure S^{cl} is homeomorphic $M_{0,k,k-1}$. As the Poisson algebra of smooth functions determines the connected components of the strata of a symplectic quotient by [28, Proposition 3.3], U must also have a real codimension 2 orbit type stratum T with $\chi(S) = T$, and hence $\chi(S^{cl}) = T^{cl}$. Say T is the orbit type corresponding to $K \leq H$, let $L = N_H(K)/K$, and then T^{cl} is homeomorphic to U^K/L.

By [14, Theorem 2], we have that χ restricts to a $\mathbb{Z}^+$-graded regular symplectomorphism $\chi|_{S^{cl}} \colon S^{cl} \to T^{cl}$, where the global charts for these spaces are the restrictions of those for $M_{0,k,k}$ and U/H, respectively. By Lemma 4.2, the map $\mathbb{R}[M_{0,k,k}] \to \mathbb{R}[M_{0,k,k-1}]$ is surjective, so the restricted global chart on S^{cl} is isomorphic to the usual global chart of $\mathbb{R}[M_{0,k,k-1}]$ given by the O_{k-1}-invariants. Hence, we have a Poisson isomorphism $\chi|_{S^{cl}}^*$ of $\mathbb{R}[M_{0,k,k-1}]$ into a subalgebra $\mathcal{A}$ of $\mathbb{R}[U^K]^L$ corresponding to the homeomorphism $M_{0,k,k-1} \simeq S^{cl} \simeq T^{cl} \simeq U^K/L$; in particular, $\mathcal{A}$ separates points in U^K/L. To show that this is in fact a $\mathbb{Z}^+$-graded regular symplectomorphism onto U^K/L with global chart induced from $\mathbb{R}[U^K]^L$, it is sufficient to show that $\chi|_{S^{cl}}^*$ is surjective. The proof follows [14, Theorem 4]; we summarize the argument here but refer the reader to that reference for more details.

Tensoring with $\mathbb{C}$ yields isomorphisms

$$(\chi^*)^{\mathbb{C}} \colon \mathbb{C}[V_{k,k} \times V_{k,k}^*]^{O_k(\mathbb{C})}/(\mathcal{J}_{k,k}^{\mathbb{C}})^{O_k(\mathbb{C})} \to \mathbb{C}[U \times U^*]^H$$

and

$$(\chi|_{S^{cl}}^*)^{\mathbb{C}}\colon \ \mathbb{C}[V_{k,k-1}\times V_{k,k-1}^*]^{\mathrm{O}_{k-1}(\mathbb{C})}/(\mathcal{J}_{k,k-1}^{\mathbb{C}})^{\mathrm{O}_{k-1}(\mathbb{C})}\to\mathcal{A}^{\mathbb{C}},$$

where we recall that $V_{k,k}^*$ denotes the dual representation and $\mathcal{J}_{k,k}^{\mathbb{C}}$, $\mathcal{A}^{\mathbb{C}}$, etc. the complexifications. As the variety associated to $\mathcal{J}_{k,k-1}^{\mathbb{C}}$ is normal by Theorem 2.1, the variety in $\mathbb{C}[V_{k,k-1}\times V_{k,k-1}^*]^{\mathrm{O}_{k-1}(\mathbb{C})}$ associated to $(\mathcal{J}_{k,k-1}^{\mathbb{C}})^{\mathrm{O}_{k-1}(\mathbb{C})}$ is as well normal by [31, Theorem 3.16], hence $\mathcal{A}^{\mathbb{C}}$ is integrally closed. Moreover, it can be seen as in the proof of [14, Theorem 4] that $\mathcal{A}^{\mathbb{C}}$ is a separating algebra in the sense of [6, Definition 2.3.8]. Specifically, the map $(U\times U^*)^K/L\to(U\times U^*)/H$ induced by the embedding $(U\times U^*)^K\to U\times U^*$ is injective as it is injective on the open dense set of points with isotropy group K and hence a birational map between normal varieties. Then as $\mathbb{C}[U\times U^*]^H$ separates orbits in $(U\times U^*)/H$, as the map $(\chi^*)^{\mathbb{C}}$ is an isomorphism, and as $\mathcal{A}$ separates points in U^K/L, it follows that $\mathcal{A}^{\mathbb{C}}$ is a separating algebra. But then by [6, Theorem 2.3.12], $\mathbb{C}[(U\times U^*)^K]^L$ is equal to the normalization of $\mathcal{A}^{\mathbb{C}}$, so that as $\mathcal{A}^{\mathbb{C}}$ is integrally closed, $\mathbb{C}[(U\times U^*)^K]^L=\mathcal{A}^{\mathbb{C}}$. Hence $(\chi|_{S^{cl}}^*)^{\mathbb{C}}$ is surjective so that $\chi|_{S^{cl}}^*$ is surjective, completing the proof. $\qquad\square$

5. **Rational singularities.** In this section, we prove the following.

Theorem 5.1. *Suppose $1\le n\le k$. Then the real variety $Z_{k,n}$ has rational singularities.*

Note that the cases $1\le n\le k$ are exactly those such that $Z_{k,n}$ is normal by Theorem 2.1.

Proof. As $n\le k$, the variety of the ideal $\mathcal{J}^{\mathbb{C}}$ is a normal, reduced, irreducible complete intersection by Theorem 2.1(i). Counting invariants and relations, the Hilbert Series is given by

$$\mathrm{Hilb}(x)=\frac{(1-x^2)^{\binom{n}{2}}}{(1-x)^{2kn}},$$

so the a-invariant is $n(n-2k-1)$. Moreover, the (real) vanishing ideal of $Z_{k,n}$ is equal to $\mathcal{J}_{k,n}$, again by Theorem 2.1(i).

When $n=1$, the ideal $\mathcal{J}_{1,k}=0$ and the result is trivial. Suppose $n=2$. Then $\mathcal{J}_{2,k}$ is generated by the single element

$$J_{1,2}=\sum_{\ell=1}^{k}q_{\ell,1}p_{\ell,2}-q_{\ell,2}p_{\ell,1}.$$

By the theorem of Flenner [8] and Watanabe [34] (see also [18, Theorem 9.2]), as $\mathbb{R}[V_{k,n}]/\mathcal{J}_{k,n}$ is Cohen-Macaulay and normal with negative a-invariant, it is sufficient to show that $(\mathbb{R}[V_{k,n}]/\mathcal{J}_{k,n})_P$ is rational for all primes P not equal to the homogenous maximal ideal. However, by a simple computation, adjoining an inverse to any of the variables yields a localization of a polynomial ring. Therefore, the origin is an isolated singularity, and $\mathbb{R}[V_{k,n}]/\mathcal{J}_{k,n}$ has rational singularities. Note that this establishes the claim for $k\le 2$.

Now, fix n, and assume for induction that $\mathbb{R}[V_{m,k}]/\mathcal{J}_{m,k}$ has rational singularities for any $k\ge 3$ and $2\le m<n\le k$. We fix a value of $k\ge 3$ and prove that $\mathbb{R}[V_{k,n}]/\mathcal{J}_{k,n}$ has rational singularities.

Note that $\mathcal{J}_{k,n}$ is invariant under any isomorphism given by a permutation of the pairs $(\boldsymbol{q}_\ell,\boldsymbol{p}_\ell)$, the permutation $\boldsymbol{q}_\ell\leftrightarrow\boldsymbol{p}_\ell$, or induced by an element of O_n. Any nonzero point in $V_{k,n}$ can be mapped by these actions to a point such that $\boldsymbol{q}_1\ne 0$.

Similarly, any point with $\boldsymbol{q}_1 \neq 0$ is in the O_n-orbit of a point such that $q_{1,\alpha} \neq 0$ for $\alpha = 1, \ldots, n$. Hence, it is sufficient to restrict to neighborhoods of such a point. Localizing near a point with $q_{1,\alpha} \neq 0$ for $\alpha = 1, \ldots, n$, we adjoin inverses to $q_{1,\alpha}$ for each α. Then for $\beta = 2, \ldots, n$, the $J_{1,\beta}$ can be expressed as

$$p_{1,\beta} = q_{1,1}^{-1} \left(q_{1,\beta} p_{1,1} + \sum_{\ell=2}^{k} q_{\ell,\beta} p_{\ell,1} - q_{\ell,1} p_{\ell,\beta} \right), \qquad \beta = 2, \ldots, n, \qquad (6)$$

eliminating the variables $p_{1,\beta}$ for $\beta \neq 1$. Then each $J_{\alpha,\beta}$ with $2 \leq \alpha < \beta$ becomes

$$J_{\alpha,\beta} = q_{1,\alpha} p_{1,\beta} - q_{1,\beta} p_{1,\alpha} + \sum_{\ell=2}^{k} q_{\ell,\alpha} p_{\ell,\beta} - q_{\ell,\beta} p_{\ell,\alpha}$$

$$= q_{1,\alpha} \left(q_{1,\beta} \frac{p_{1,1}}{q_{1,1}} + \sum_{\ell=2}^{k} q_{\ell,\beta} \frac{p_{\ell,1}}{q_{1,1}} - \frac{q_{\ell,1}}{q_{1,1}} p_{\ell,\beta} \right)$$

$$- q_{1,\beta} \left(q_{1,\alpha} \frac{p_{1,1}}{q_{1,1}} + \sum_{\ell=2}^{k} q_{\ell,\alpha} \frac{p_{\ell,1}}{q_{1,1}} - \frac{q_{\ell,1}}{q_{1,1}} p_{\ell,\alpha} \right) + \sum_{\ell=2}^{k} q_{\ell,\alpha} p_{\ell,\beta} - q_{\ell,\beta} p_{\ell,\alpha}.$$

Therefore, we can express

$$\frac{J_{\alpha,\beta}}{q_{1,\alpha} q_{1,\beta}}$$

$$= \sum_{\ell=2}^{k} \frac{q_{\ell,\beta}}{q_{1,\beta}} \frac{p_{\ell,1}}{q_{1,1}} - \frac{q_{\ell,1}}{q_{1,1}} \frac{p_{\ell,\beta}}{q_{1,\beta}} + \frac{q_{\ell,1}}{q_{1,1}} \frac{p_{\ell,\alpha}}{q_{1,\alpha}} - \frac{q_{\ell,\alpha}}{q_{1,\alpha}} \frac{p_{\ell,1}}{q_{1,1}} + \frac{q_{\ell,\alpha}}{q_{1,\alpha}} \frac{p_{\ell,\beta}}{q_{1,\beta}} - \frac{q_{\ell,\beta}}{q_{1,\beta}} \frac{p_{\ell,\alpha}}{q_{1,\alpha}} \qquad (7)$$

$$= \sum_{\ell=2}^{k} \left(\frac{q_{\ell,\alpha}}{q_{1,\alpha}} - \frac{q_{\ell,1}}{q_{1,1}} \right) \left(\frac{p_{\ell,\beta}}{q_{1,\beta}} - \frac{p_{\ell,1}}{q_{1,1}} \right) - \left(\frac{q_{\ell,\beta}}{q_{1,\beta}} - \frac{q_{\ell,1}}{q_{1,1}} \right) \left(\frac{p_{\ell,\alpha}}{q_{1,\alpha}} - \frac{p_{\ell,1}}{q_{1,1}} \right).$$

Now, consider the algebra

$$\mathbb{R}[\boldsymbol{x}, \boldsymbol{y}] := \mathbb{R}[x_{\ell,\beta}, y_{\ell,\beta} \mid \ell = 2, \ldots, k; \quad \beta = 2, \ldots, n]$$

and ideal $\mathcal{J}'$ generated by

$$J'_{\alpha,\beta} = \sum_{\ell=2}^{k} x_{\ell,\alpha} y_{\ell,\beta} - x_{\ell,\beta} y_{\ell,\alpha} \qquad (8)$$

for $2 \leq \alpha < \beta \leq n$. Then $\mathbb{R}[\boldsymbol{x}, \boldsymbol{y}]/\mathcal{J}'$ is obviously isomorphic to $\mathbb{R}[V_{n-1,k-1}]/\mathcal{J}_{n-1,k-1}$ by simply relabeling variables. Hence, by the inductive hypothesis, $\mathbb{R}[\boldsymbol{x}, \boldsymbol{y}]/\mathcal{J}'$ has rational singularities and in particular is normal. We adjoin the additional variables $z_{1,\alpha}$ for $\alpha = 1, \ldots, n$; $z_{\ell,1}$ for $\ell = 2, \ldots, k$; and $w_{\ell,1}$ for $\ell = 1, \ldots, k$. Let

$$W = \mathbb{R}^{2(k-1)(n-1)} \oplus \mathbb{R}^{n+k-1} \oplus \mathbb{R}^k$$

$$= \{ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) \mid \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{2(k-1)(n-1)}, \boldsymbol{z} \in \mathbb{R}^{n+k-1}, \boldsymbol{w} \in \mathbb{R}^k \}$$

be the vector space with these coordinates, let Y denote the affine variety in W with ideal $\mathcal{J}'$, and then Y is a normal variety with rational singularities.

Let $\mathcal{O}$ be the open subset of $V_{k,n}$ defined by $q_{1,\alpha} \neq 0$ for $\alpha = 1, \ldots, n$, let $\mathcal{U}$ be the open subset of W defined by $z_{1,\alpha} \neq 0$ for $\alpha = 1, \ldots, n$, and then it is easy to see that of $\mathcal{O} \cap Z$ and $\mathcal{U} \cap Y$ are birationally equivalent. As $\mathcal{U} \cap Y$ is a localization of the normal variety Y and hence a normal variety, it follows from Zariski's main theorem that $\mathcal{O} \cap Z$ and $\mathcal{U} \cap Y$ are isomorphic. As Y and hence $\mathcal{U} \cap Y$ has rational singularities, it follows that $\mathcal{O} \cap Z$ has rational singularities.

Again, $\mathbb{R}[V_{k,n}]/\mathcal{J}_{k,n}$ is Cohen-Macaulay, normal, and has negative a-invariant. We again apply the theorem of Flenner [8] and Watanabe [34] to conclude that $\mathbb{R}[V_{k,n}]/\mathcal{J}_{k,n}$ has rational singularities near the origin. Then by induction, we are done. $\square$

By Theorem 5.1, when $n \leq k$, the spectrum of the ring $\mathbb{R}[V_{k,n}]/\mathcal{I}_Z$ has rational singularities. It then follows from Boutot's Theorem [4] that the spectrum of $\mathbb{R}[M_0]$ and $\mathbb{R}[M_0^{\mathrm{SO}}]$ have only rational singularities as well. If $n > k$, then $\mathbb{R}[M_{0,k,n}]$ is isomorphic to $\mathbb{R}[M_{0,k,k}]$ by Corollary 2, and $\mathbb{R}[M_{0,k,n}^{\mathrm{SO}}]$ is isomorphic to $\mathbb{R}[M_{0,k,k}]$ by Theorem 4.1. Hence we have the following.

Corollary 3. *For each $k, n \geq 1$, the (Zariski closure of the) symplectic quotients M_0 and M_0^{SO} have rational singularities. In particular, they are normal, and the rings $\mathbb{R}[M_0]$ and $\mathbb{R}[M_0^{\mathrm{SO}}]$ of regular functions are Cohen-Macaulay.*

Note that Corollary 3 is also a consequence of the results of [30] and [2]. In particular, by [30, Propisiton A.2.1 and Lemma A.2.2], the (complex) symplectic quotients $(\boldsymbol{J}^{\mathbb{C}})^{-1}(0) /\!\!/ \, \mathrm{O}_n(\mathbb{C})$ and $(\boldsymbol{J}^{\mathbb{C}})^{-1}(0) /\!\!/ \, \mathrm{SO}_n(\mathbb{C})$ are so-called symplectic varieties. Then by [2, Proposition 1.3], symplectic varieties are Gorenstein and have rational singularities, properties that are easily seen to descend from the complexifications to the real quotients. Note that it does not follow that the symplectic quotients M_0 and M_0^{SO} are *graded* Gorenstein, which appears to have been established only for (complex) symplectic quotients by tori and those that admit a symplectic resolution; see [22, Sections 3.2 and 3.4]. In particular, the condition [22, Assumption 3.2 (2a)] corresponds to the symplectic quotient being graded Gorenstein. We consider this condition in the following section.

6. **Explicit computations.** For small values of k, we have computed the on-shell relations that can be used to describe $\mathbb{R}[M_0]$ and $\mathbb{R}[M_0^{\mathrm{SO}}]$ as an affine algebra explicitly using elimination. These computations have motivated many of the findings in this paper and have allowed us to verify that for small k, the ideal described by Theorem 3.4 is in fact real radical, i.e. there are no other on-shell relations. We have also used these computations to test whether the algebra $\mathbb{R}[M_0]$ is graded Gorenstein, which we conjecture is true in general.

The method is as follows; see [5, Chapters 2 and 3] for background. Let $\mathbb{R}[\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{Y}]$ be the polynomial ring in the variables $q_{\ell,\alpha}$, $p_{\ell,\alpha}$, and $Y_{i,j}$ where $1 \leq \ell \leq k$, $1 \leq \alpha \leq n$, and $1 \leq i \leq j \leq 2k$; the $q_{\ell,\alpha}$ and $p_{\ell,\alpha}$ have degree 1 and the $Y_{i,j}$ have degree 2. Using the software package *Mathematica* [36] and a monomial order that is an elimination order for the variables $q_{\ell,\alpha}$ and $p_{\ell,\alpha}$, we compute a Gröbner basis for the ideal of $\mathbb{R}[\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{Y}]$ generated both by the components of the moment map as well as the functions $Y_{i,j} - x_{i,j}$ where the $x_{i,j}$ are the invariant scalar products described in Section 2. Computing the elimination ideal corresponding to eliminating the variables $q_{\ell,\alpha}$ and $p_{\ell,\alpha}$ yields the ideal $\mathcal{R} = \mathcal{R}_{k,n}$ of relations in the intersection of the algebra $\mathbb{R}[V_{k,n}]^{\mathrm{O}_n} = \mathbb{R}[x_{i,j} | 1 \leq i \leq j \leq 2k] =: \mathbb{R}[X]$ and the ideal $\mathcal{J} = \mathcal{J}_{k,n}$ generated by the moment map. When $n \leq k + 1$, the ideal $\mathcal{J}_{k,n}$ is real radical by Theorem 2.1 so that $\mathcal{R}_{k,n}$ is equal to the ideal of $\mathbb{R}[M_0]$ in $\mathbb{R}[X]$. The computations to compute the ideal $\mathcal{R}_{k,n}^{\mathrm{SO}}$ of $\mathbb{R}[M_{0,k,n}^{\mathrm{SO}}]$ are similar; the only difference is that we include the invariant determinants along with the $x_{i,j}$ and additional corresponding variables in $\boldsymbol{Y}$.

Let $\mathcal{Q} = \mathcal{Q}_{k,n}$ denote the ideal described in Theorem 3.4, i.e. $\mathcal{Q}_{k,n}$ is generated by the $Q_{i,j}$ when $k \geq n$ and the $Q_{i,j}$ along with the $(n+1) \times (n+1)$-minors of X

when $n < k$, and recall that by the same theorem, the ideal of relations in $\mathbb{R}[M_{0,k,n}]$ is the real radical of $\mathcal{Q}_{k,n}$. We then have that when $n \leq k+1$, as $\mathcal{R}_{k,n}$ is the ideal of relations of $\mathbb{R}[M_{0,k,n}]$, $\mathcal{R}_{k,n}$ is the real radical of $\mathcal{Q}_{k,n}$. For fixed k, the cases $n \geq k$ yield isomorphic $\mathbb{R}[M_0]$ by Corollary 2, and the computation of $\mathcal{R}_{k,n}$ can be used to verify this fact. That the ideal $\mathcal{R}_{k,n}$ is real radical in these cases follows from this isomorphism.

Experimentally, we have found that the elimination is much faster using the monomial order

$$q_{1,1} \succ p_{1,1} \succ q_{1,2} \succ p_{1,2} \succ \cdots \succ q_{1,n} \succ p_{1,n} \succ q_{2,1} \succ p_{2,1} \succ \cdots \succ q_{k,n} \succ p_{k,n}$$

on the elimination variables; the ordering on the $x_{i,j}$ does not appear to affect the time of the computations significantly. As an example, on a Windows 7 desktop computer running *Mathematica* 8.0 with an i5 Core Processor, 4 GB or RAM, and processing priority set to *High*, the Gröbner basis computation for $k = 2$ and $n = 4$ using Lexicographic order took 403 seconds, while it took less than one second with the above monomial order; similar extreme differences were noted in other cases. For larger cases, we also used the standard *Mathematica* `EliminationOrder` with the option `Sort -> True`, which allows the algorithm to reorder the variables.

Once the ideal of relations $\mathcal{R}$ of $\mathbb{R}[M_0]$ (and $\mathcal{R}^{\mathrm{SO}}$ of $\mathbb{R}[M_0^{\mathrm{SO}}]$) has been determined, we compute the Hilbert series of $\mathcal{R}$ and $\mathcal{Q}$ using *Macaulay2* [10]. Note that $\mathcal{R}$ contains the ideal $\mathcal{Q}$ by construction and the above observations. Hence, equality of these ideals can be verified by observing that their Hilbert series coincide. For smaller cases, we have also used the ideal membership and polynomial division features of *Macaulay2* to express each element of the Gröbner basis for $\mathcal{R}_{k,n}$ in terms of the generators of $\mathcal{Q}_{k,n}$ explicitly, though most of these expressions are too large to report here.

We check that each Hilbert series satisfies the functional equation

$$\mathrm{Hilb}(t^{-1}) = (-1)^d t^{-a} \, \mathrm{Hilb}(t) \tag{9}$$

where d is the Krull dimension and a is the a-invariant. As the corresponding quotient rings are Cohen-Macaulay integral domains by Corollary 3, this functional equation is equivalent to the rings being Gorenstein by [29, Theorem 4.4]. In fact, we observe in each case that $d = -a$, and hence that the quotient ring is graded Gorenstein; see [6, Section 3.7.3]. When $k = 2$, these computations were considered in [15, Section 8.3.1], and the relation between the coefficients of the Laurent expansion of the Hilbert series at $t = 1$ considered there and the graded Gorenstein condition is explained by [11, Corollary 1.8].

Note that the cases corresponding to $n = 1$ are orbifolds in a strict sense: the moment map is 0, and the group $\mathrm{O}_1 = \mathbb{Z}_2$ is finite. Hence in these cases, the fact that $\mathbb{R}[M_0]$ has rational singularities is a direct consequence of Boutot's Theorem [4], and the fact that $\mathbb{R}[M_0]$ is graded Gorenstein follows from Watanabe's Theorem [32, 33]. We have presented these cases for completeness, but they are particularly easy to compute. The corresponding SO_1-quotients are affine spaces as SO_1 is trivial.

To summarize the experimental results discussed below, we were able to compute the ideals $\mathcal{R}_{k,n}$ and $\mathcal{Q}_{k,n}$ for $(k,n) = (1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1)$, and $(4,2)$; for the case $(4,4)$, we were only able to compute $\mathcal{Q}_{k,n}$. We were able to compute $\mathcal{R}_{k,n}^{\mathrm{SO}}$ only for the cases $(k,n) = (2,2)$ and $(3,2)$. In each case, the ring of regular functions is graded Gorenstein, and $\mathcal{R}_{k,n} = \mathcal{Q}_{k,n}$ in each case where we were able to compute both. We also observe some interesting patterns, e.g. the

coefficients of t^2 in the numerator appears to be a square number when $k = n$ for both the O_n- and SO_n-quotients.

Many of these computations were completed in a few seconds on a laptop with a 2.4 GHz two-core processor (utilizing only one core for Gröbner basis computations) and 4 GB RAM. When longer computation times were encountered or more sophisticated equipment was required, it is noted below.

6.1. $k = 1$. When $k = 1$ and $n = 1$, the Gröbner basis for $\mathcal{R}_{1,1}$ consists of the single relation $x_{1,1}x_{2,2} - x_{1,2}^2$; this is equal to $Q_{1,1}$ so that $\mathcal{R}_{1,1} = \mathcal{Q}_{1,1}$ is obvious in this case. All other values of n yield the same result by Corollary 2. The Hilbert series of $\mathbb{R}[M_{0,1,n}]$ is given by

$$\mathrm{Hilb}_{\mathbb{R}[M_{0,1,n}]}(t) = \frac{1 + t^2}{(1 - t^2)^2},$$

which satisfies Equation (9) with $d = -a = 2$. Hence for each n, the symplectic quotient $M_{0,1,n}$ is $\mathbb{Z}^+$-graded regularly symplectomorphic to the orbifold $\mathbb{C}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts as multiplication by -1, which has already been observed in [3, 1].

In this case, there are no new SO_n-quotients, as the case $n = 1$ is affine space, and the cases $n \geq 2$ are all isomorphic to the corresponding O_n-quotients, described above, by Theorem 4.1

6.2. $k = 2$. When $k = 2$ and $n = 1$, we again have $\boldsymbol{J} = 0$. The Gröbner basis of $\mathcal{R}_{2,1}$ contains 20 elements, each easily recognized as a 2×2-minor of the Gram matrix X. The Hilbert series is

$$\mathrm{Hilb}_{\mathbb{R}[M_{0,2,1}]}(t) = \frac{1 + 6t^2 + t^4}{(1 - t^2)^4},$$

which satisfies Equation (9) with $d = -a = 2$ and coincides with the Hilbert series of the ideal $\mathcal{Q}_{2,1}$ so that $\mathcal{Q}_{2,1}$ is real radical.

When $k = 2$ and $n = 2$, the Gröbner basis of $\mathcal{R}_{2,2}$ contains the 9 elements

$$-x_{24}^2 x_{33} + 2x_{23}x_{24}x_{34} - x_{22}x_{34}^2 - x_{23}^2 x_{44} + x_{22}x_{33}x_{44}$$
$$= x_{24}Q_{23} - x_{23}Q_{24} + x_{22}Q_{34},$$
$$-x_{14}x_{23} + x_{13}x_{24} - x_{34}^2 + x_{33}x_{44} = Q_{34},$$
$$-x_{14}x_{24}x_{33} + 2x_{14}x_{23}x_{34} - x_{12}x_{34}^2 + x_{34}^3 - x_{13}x_{23}x_{44} + x_{12}x_{33}x_{44} - x_{33}x_{34}x_{44}$$
$$= x_{24}Q_{13} - x_{23}Q_{14} + (x_{12} - x_{34})Q_{34},$$
$$-x_{14}x_{22} + x_{12}x_{24} - x_{24}x_{34} + x_{23}x_{44} = Q_{24},$$
$$-x_{13}x_{22} + x_{12}x_{23} - x_{24}x_{33} + x_{23}x_{34} = Q_{23},$$
$$-x_{14}^2 x_{33} + 2x_{13}x_{14}x_{34} - x_{11}x_{34}^2 - x_{13}^2 x_{44} + x_{11}x_{33}x_{44}$$
$$= x_{14}Q_{13} - x_{13}Q_{14} + x_{11}Q_{34},$$
$$-x_{12}x_{14} + x_{11}x_{24} - x_{14}x_{34} + x_{13}x_{44} = Q_{14},$$
$$-x_{12}x_{13} + x_{11}x_{23} - x_{14}x_{33} + x_{13}x_{34} = Q_{13},$$
$$-x_{12}^2 + x_{11}x_{22} + x_{34}^2 - x_{33}x_{44} = Q_{12} - Q_{34}.$$

Hence, $\mathcal{R}_{2,2} = \mathcal{Q}_{2,2}$. This can also be seen by computing that for both $\mathcal{R}_{2,2}$ and $\mathcal{Q}_{2,2}$, the Hilbert series is

$$\mathrm{Hilb}_{\mathbb{R}[M_{0,2,2}]}(t) = \frac{1 + 4t^2 + 4t^4 + t^6}{(1 - t^2)^6},$$

which satisfies Equation (9) with $d = -a = 6$. In particular, the Laurent expansion of the Hilbert series at $t = 1$ begins

$$\text{Hilb}_{\mathbb{R}[M_{0,2,2}]}(t) = \frac{5}{32}(1-t)^{-6} + \frac{11}{128}(1-t)^{-4} + \frac{11}{128}(1-t)^{-3} + \cdots, \qquad (10)$$

which is used in the proof of Theorem 4.3. For each $n > 2$, $\mathbb{R}[M_{0,2,n}]$ is isomorphic to $\mathbb{R}[M_{0,2,2}]$ by Corollary 2.

The SO_n-symplectic quotient is affine space when $n = 1$ and isomorphic to the corresponding O_n quotients when $n \geq 3$, so the only SO_n-quotient not already described is $M_{0,2,2}^{\text{SO}}$. In this case, the Hilbert series is

$$\text{Hilb}_{\mathbb{R}[M_{0,2,2}^{\text{SO}}]}(t) = \frac{1 + 9t^2 + 9t^4 + t^6}{(1-t^2)^6},$$

which satisfies Equation (9) with $d = -a = 6$.

6.3. $k = 3$.
When $k = 3$ and $n = 1$, the Gröbner basis of $\mathcal{R}_{3,1}$ contains 105 elements, each easily recognized as a 2×2-minor of the Gram matrix X. The Hilbert series is

$$\text{Hilb}_{\mathbb{R}[M_{0,3,1}]}(t) = \frac{1 + 15t^2 + 15t^4 + t^6}{(1-t^2)^6},$$

which satisfies Equation (9) with $d = -a = 6$ and coincides with the Hilbert series of the ideal $\mathcal{Q}_{3,1}$, hence $\mathcal{Q}_{3,1}$ is real radical.

When $k = 3$ and $n = 2$, the Hilbert series is

$$\text{Hilb}_{\mathbb{R}[M_{0,3,2}]}(t) = \frac{1 + 11t^2 + 51t^4 + 51t^6 + 11t^8 + t^{10}}{(1-t^2)^{10}},$$

which satisfies Equation (9) with $d = -a = 10$ and coincides with the Hilbert series of the ideal $\mathcal{Q}_{3,2}$ so that $\mathcal{Q}_{3,2}$ is real radical.

When $k = 3$ and $n = 3$, the Hilbert series is

$$\text{Hilb}_{\mathbb{R}[M_{0,3,3}]}(t) = \frac{1 + 9t^2 + 30t^4 + 44t^6 + 30t^8 + 9t^{10} + t^{12}}{(1-t^2)^{12}},$$

which satisfies Equation (9) with $d = -a = 12$ and coincides with the Hilbert series of the ideal $\mathcal{Q}_{3,3}$ so that this ideal is real radical. In this case, $\mathcal{R}_{3,3}$ took about 96 minutes to compute on a laptop.

The SO_n-symplectic quotient is affine space when $n = 1$ and isomorphic to O_n-quotients when $n \geq 4$ by Theorem 4.1. When $n = 2$, the computation of $\mathcal{R}_{3,2}^{\text{SO}}$ takes about 7 minutes on a laptop, and is generated by 267 relations. The Hilbert series is

$$\text{Hilb}_{\mathbb{R}[M_{0,3,2}^{\text{SO}}]}(t) = \frac{1 + 25t^2 + 100t^4 + 100t^6 + 25t^8 + t^{10}}{(1-t^2)^{10}},$$

which satisfies Equation (9) with $d = -a = 10$. We were not able to compute $\mathcal{R}_{3,3}^{\text{SO}}$; using the option `Sort->True` and elimination order on *Mathematica* on a 244GB RAM machine, the computation had not completed in 50 hours.

6.4. $k = 4$.
When $n = 1$, the Gröbner basis of $\mathcal{R}_{4,1}$ is computed in under 2 minutes on a laptop and contains 336 elements, each easily recognized as a 2×2 minor of the Gram matrix X. The Hilbert series is

$$\text{Hilb}_{\mathbb{R}[M_{0,4,1}]}(t) = \frac{1 + 28t^2 + 70t^4 + 28t^6 + t^8}{(1-t^2)^8},$$

which satisfies Equation (9) with $d = -a = 8$. This coincides with the Hilbert series of $\mathcal{Q}_{4,1}$ so that $\mathcal{R}_{4,1} = \mathcal{Q}_{4,1}$, and $\mathcal{Q}_{4,1}$ is real radical.

When $n = 2$, the Gröbner basis of $\mathcal{R}_{4,2}$ contains 938 elements. It took just under 10 hours to compute on a system with 244GB RAM and only completed when we used the `Sort->True` option in the `GroebnerBasis` command along with elimination order. The Hilbert series is

$$\mathrm{Hilb}_{\mathbb{R}[M_{0,4,2}]}(t) = \frac{1 + 22t^2 + 225t^4 + 610t^6 + 610t^8 + 225t^{10} + 22t^{12} + t^{14}}{(1 - t^2)^{14}},$$

which satisfies Equation (9) with $d = -a = 14$. This coincides with the Hilbert series of $\mathcal{Q}_{4,2}$ so that $\mathcal{R}_{4,2} = \mathcal{Q}_{4,2}$, and $\mathcal{Q}_{4,2}$ is real radical.

The case $n = 3$ appears to be inaccessible; attempts to compute $\mathcal{R}_{4,3}$ all ran out of memory. Using the option `Sort->True` and elimination order on *Mathematica* on a 244GB RAM machine, the computation ran out of memory after about 8 hours.

For $n = 4$, $\mathcal{R}_{4,4}$ appears to be out of reach, but we were able to compute $\mathcal{Q}_{4,4}$. The Hilbert series of $\mathcal{Q}_{4,4}$ is

$$\begin{aligned}
\mathrm{Hilb}_{\mathcal{Q}_{4,4}}(t) = \Big(&1 + 16t^2 + 108t^4 + 395t^6 + 842t^8 + 1080t^{10} + 842t^{12} \\
&+ 395t^{14} + 108t^{16} + 16t^{18} + t^{20}\Big)/(1 - t^2)^{20}
\end{aligned}$$

which satisfies Equation (9) with $d = -a = 20$.

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