The Poisson and Exponential Distributions

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1. Introduction

The Poisson distribution is a discrete distribution with probability mass function

$$P(x) = \frac{\mathrm{e}^{-\mu}\mu^x}{x!},$$

where $x = 0, 1, 2, \ldots$, the mean of the distribution is denoted by μ, and e is the exponential. The variance of this distribution is also equal to μ.

The exponential distribution is a continuous distribution with probability density function

$$f(t) = \lambda\mathrm{e}^{-\lambda t},$$

where $t \geq 0$ and the parameter $\lambda > 0$. The mean and standard deviation of this distribution are both equal to $1/\lambda$.

The cumulative exponential distribution is

$$F(t) = \int_0^\infty \lambda\mathrm{e}^{-\lambda t}\,\mathrm{d}t = 1 - \mathrm{e}^{-\lambda t}. \tag{1}$$

2. Relation between the Poisson and exponential distributions

An interesting feature of these two distributions is that, if the Poisson provides an appropriate description of the number of occurrences per interval of time, then the exponential will provide a description of the length of time between occurrences. To understand this, consider that, in a Poisson process, if events occur on average at the rate of λ per unit of time, then there will be on average λt occurrences per t units of time. The Poisson distribution describing this process is therefore $P(x) = \mathrm{e}^{-\lambda t}(\lambda t)^x/x!$, from which $P(x = 0) = \mathrm{e}^{-\lambda t}$ is the probability of no occurrences in t units of time.

Another interpretation of $P(x = 0) = \mathrm{e}^{-\lambda t}$ is that this is the probability that the time, T, to the first occurrence is greater than t, i.e.

$$P(T > t) = P(x = 0 \mid \mu = \lambda t) = \mathrm{e}^{-\lambda t}.$$

Conversely, the probability that an event does occur during t units of time is given by

$$P(T \leq t) = 1 - P(x = 0 \mid \mu = \lambda t) = 1 - \mathrm{e}^{-\lambda t}.$$

Note that this is the cumulative exponential distribution which, when differentiated with respect to t, produces the probability density function of the exponential distribution $f(t) = \lambda\mathrm{e}^{-\lambda t}$.