STAT510 - Homework Assignment 1

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Question 1

The pdf of the random variable Y is:

$$f(y;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$$

Since Y is iid, we obtain the likelihood as follows:

$$L(\mu) = \prod_{i=1}^{n} \left(2\pi\sigma^2\right)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2}$$

$$= \left(2\pi\sigma^2\right)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

To find the MLE of μ, we proceed to maximize the log of $L(\mu)$ with respect to μ. This is equivalent to maximizing $L(\mu)$ with respect to μ. To accomplish this we can use the first and second derivatives tests from calculus to find the MLE:

$$\ln L(\mu) = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i-\mu\right)^2$$

$$\frac{d\ln L(\mu)}{d\mu} = \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} y_i - n\mu\right) \overset{set}{=} 0$$

$$\hat{\mu}_{mle} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

The second derivative test confirms that $\hat{\mu}_{mle}$ is indeed a maximum:

$$\frac{d^2\ln L(\mu)}{d\mu^2} = -\frac{n}{\sigma^2} < 0$$

Question 2

To find $E\left(\bar{Y}\right)$, we use the fact that the random variable Y is distributed iid normal. Therefore we know that:

$$\frac{\bar{Y}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

Then:

$$E\left(\frac{\bar{Y}-\mu}{\frac{\sigma}{\sqrt{n}}}\right) = 0$$

$$E\left(\bar{Y}-\mu\right) = 0$$

$$E\left(\bar{Y}\right) = \mu$$

For the variance, we use the definition:

$$V\left(\bar{Y}\right) = E\left(\bar{Y}-E\left(\bar{Y}\right)\right)^2$$

and given that Y is distributed iid normal we know:

$$E\left(\frac{\bar{Y}-\mu}{\frac{\sigma}{\sqrt{n}}}-0\right)^2 = 1$$

then:

$$E\left(\bar{Y}-\mu\right)^2 = \frac{\sigma^2}{n}$$

$$E\left(\bar{Y}-E\left(\bar{Y}\right)\right)^2 = \frac{\sigma^2}{n}$$

$$V\left(\bar{Y}\right) = \frac{\sigma^2}{n}$$

Question 3

We can use linear algebra to solve the system of normal equations:

$$\begin{pmatrix} n & \sum_i x_i & \sum_i y_i \\ \sum_i x_i & \sum_i x_i^2 & \sum_i x_i y_i \end{pmatrix}$$

Dividing the first row by n:

$$\begin{pmatrix} 1 & \bar{x} & \bar{y} \\ \sum_i X_i & \sum_i x_i^2 & \sum_i x_i y_i \end{pmatrix}$$

Multiply the first row by $\sum_i x_i$ and subtract it from the second row to get the second pivot:

$$\begin{pmatrix} 1 & \bar{x} & \bar{y} \\ 0 & \sum_i x_i^2 - \bar{x}\sum_i x_i & \sum_i x_i y_i - \bar{y}\sum_i x_i \end{pmatrix}$$

The above result implies that:

$$b_1 = \frac{\sum_i x_i y_i - \bar{y}\sum_i x_i}{\sum_i x_i^2 - \bar{x}\sum_i x_i}$$

To show that:

$$b_1 = \frac{\sum_i \left(x_i - \bar{x}\right)\left(y_i - \bar{x}\right)}{\sum_i \left(x_i - \bar{x}\right)^2}$$

expand $\sum_i \left(x_i - \bar{x}\right)\left(y_i - \bar{x}\right)$:

$$\begin{aligned} &= \sum_i \left(x_i y_i - \bar{y}x_i - \bar{x}y_i + \bar{x}\bar{y}\right) \\ &= \sum_i x_i y_i - 2\bar{y}\sum_i x_i + n\bar{x}\bar{y} \\ &= \sum_i x_i y_i - 2\bar{y}\sum_i x_i + \bar{y}\sum_i x_i \\ &= \sum_i x_i y_i - \bar{y}\sum_i x_i \end{aligned} \tag{1}$$

Now expand $\sum_i \left(x_i - \bar{x}\right)^2$:

$$\begin{aligned} &= \sum_i \left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right) \\ &= \sum_i x_i^2 - 2\bar{x}\sum_i x_i + n\bar{x}^2 \\ &= \sum_i x_i^2 - 2\bar{x}\sum_i x_i + \bar{x}\sum_i x_i \\ &= \sum_i x_i^2 - \bar{x}\sum_i x_i \end{aligned} \tag{2}$$

Based on result (1) and result (2), we have shown that:

$$\frac{\sum_i \left(x_i - \bar{x}\right)\left(y_i - \bar{x}\right)}{\sum_i \left(x_i - \bar{x}\right)^2} = \frac{\sum_i x_i y_i - \bar{y}\sum_i x_i}{\sum_i x_i^2 - \bar{x}\sum_i x_i} = b_1$$