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STAT510 Homework 2

Required Python Modules:

In [1]:
```
#PANDAS
import pandas as pd
from pandas import DataFrame, Series

#NUMPY
import numpy as np

#SCIPY t and F distributions
from scipy.stats import t
from scipy.stats import f

#STATMODELS
import statsmodels.formula.api as sm
#import statsmodels.api as sm

#SEABORN plotting
import seaborn as sns

#MATPLOTLIB plotting
import matplotlib.pyplot as plt
%matplotlib inline
```

Problem 1.3

Although the statement of the participant if there is a precise deterministic relation between plastic hardnes and elapsed time is technically correct, the veracity of the claim is highly unlikely. The participant's claim implies that there is an invariant relationship between the variables. In reality, the more plausible scenario is that the resulting plastic hardness varies for a given measure of elapsed time. If plastic hardness exhibits variability at each level of elapsed time, then regression analysis is precisely the appropriate approach for the analysis.

Furthermore, the participant's claim also implies that plastic hardness can be measured with infinite precission. At some small scale variability in measurement is bound to be present.

Problem 1.22

First load the data set for the plastic hardness data

In [2]:
```
filename = '~/Documents/LinearRegression/STAT510/Kutner/CH1DS/CH01PR22.txt
'

df = pd.read_table(filename, delim_whitespace=True, names=['hardness','hou
rs'])
```

In [3]:
```
df.head()
```

Out[3]:

	hardness	hours
0	199	16
1	205	16
2	196	16
3	200	16
4	218	24

Next we fit the model

In [4]:
```
model = sm.ols(formula="hardness ~ hours", data=df).fit()

b0 = model.params[0]
b1 = model.params[1]
n = len(df)
```

Part a.

The table below summarizes the results of fitting the model to plastic hardness data:

In [5]:
```
print model.summary()
```

```
                            OLS Regression Results                            

==============================================================================
====
Dep. Variable:               hardness   R-squared:                       0
.973
Model:                            OLS   Adj. R-squared:                  0
.971
Method:                 Least Squares   F-statistic:                     5
06.5
Date:                Tue, 22 Sep 2015   Prob (F-statistic):           2.16
e-12
Time:                        14:18:06   Log-Likelihood:                -40
.414
No. Observations:                  16   AIC:                             8
4.83
Df Residuals:                      14   BIC:                             8
6.37
Df Model:                           1                                         

Covariance Type:            nonrobust                                         

==============================================================================
====
                 coef    std err          t      P>|t|      [95.0% Conf. I
nt.]
------------------------------------------------------------------------------
----
Intercept    168.6000      2.657     63.454      0.000       162.901   174
.299
hours          2.0344      0.090     22.506      0.000         1.840     2
.228
==============================================================================
====
Omnibus:                        0.955   Durbin-Watson:                   2
.466
Prob(Omnibus):                  0.620   Jarque-Bera (JB):                0
.711
Skew:                           0.068   Prob(JB):                        0
.701
Kurtosis:                       1.976   Cond. No.                         
96.7
==============================================================================
====

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is cor
rectly specified.
```

```
/home/jcapitz/anaconda/lib/python2.7/site-packages/scipy/stats/stats.py:12
77: UserWarning: kurtosistest only valid for n>=20 ... continuing anyway,
n=16
  "anyway, n=%i" % int(n))
```

The estimated regression function is:

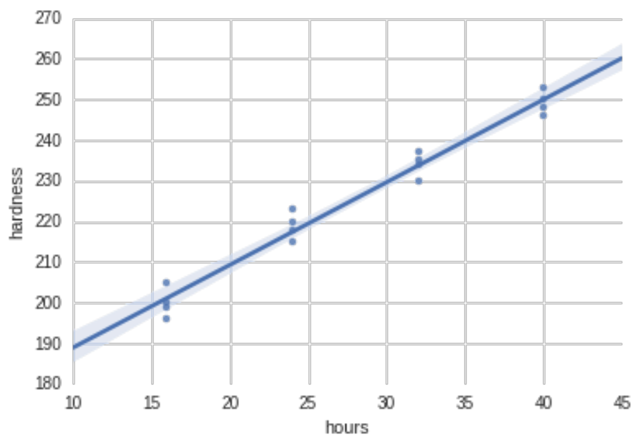
$$\hat{y} = 168.6 + 2.0344x$$

Where $\hat{y}$ represents the estimated hardness and x represents the elapsed time in hours.

Plot of the estimated regression function and the data:

The plot indicates that the data fits reasonably well the linear model represented by the regression function.

In [6]:
```
ax = sns.regplot(x="hours", y="hardness", data=df)
```



Part b.

To obtain a point estimate of the mean hardness when X = 40 hours, we simply let X = 40 be the input into the estimated regression function:

In [7]:
```
x = DataFrame({'hours':40}, index = range(1))

print 'The point estimate of the mean hardness when X = 40 is %0.2f Brinel
l units' %model.predict(x)[0]
```

```
The point estimate of the mean hardness when X = 40 is 249.97 Brinell unit
s
```

Part c.

The point estimate of the change in mean hadness when X is increased by one hour is simply the estimated coefficient for the X variable (hours) from the estimated regression function:

In [8]:
```
print '''The point estimate of the change in mean hardness when X is incre
ased by one unit is %0.3f Brinell units per hour.
''' %model.params['hours']
```

```
The point estimate of the change in mean hardness when X is increased by o
ne unit is 2.034 Brinell units per hour.
```

Problem 1.26

Part a.

Obtain resisuals and check wether the sum adds to 0:

In [9]:
```
model.resid
```

Out[9]:
```
0    -2.150
1     3.850
2    -5.150
3    -1.150
4     0.575
5     2.575
6    -2.425
7     5.575
8     3.300
9     0.300
10    1.300
11   -3.700
12    0.025
13   -1.975
14    3.025
15   -3.975
dtype: float64
```

Below we can see that the sum of the residuals is zero:

In [10]:
```
print 'The sum of the errors is %0.3f' %sum(model.resid)
```

```
The sum of the errors is 0.000
```

Part b.

A point estimate of the variance is given by:

$$\hat{\sigma}^2 = \frac{\sum\left(y_i - \hat{y}_i\right)^2}{n-2}$$

The point estimate of the variance is calculated below:

In [11]:
```
variance = sum(model.resid**2)/(n-2)

print 'The variance is %0.2f.' %variance
```

```
The variance is 10.46.
```

The estimate for sigma is given by:

$$\hat{\sigma} = \sqrt{\frac{\sum \left(y_i - \hat{y}_i\right)^2}{n-2}}$$

Sigma is expressed in Brinell units and is calculated below:

In [12]:
```
stdev = np.sqrt(variance)

print 'The standard deviation is %.2f' %stdev
```

```
The standard deviation is 3.23
```

Problem 1.31

No it would not. The measurements of hardness taken of the same item at different time intervals is a time series and each observation is not necessarily independent of one another. Each succesive measure of hardness will depend on the previous level of hardness. This implies that the error terms will be serially correlated.

Question 2

To show $\sum K_i^2 = \frac{1}{S_{xx}}$ we recognize that:

$$K_i^2 = \frac{(x_i - \bar{x})^2}{(S_{xx})^2}$$

then:

$$\begin{aligned} \sum K_i^2 &= \sum \frac{(x_i - \bar{x})^2}{(S_{xx})^2} \\ &= \frac{1}{(S_{xx})^2} \sum (x_i - \bar{x})^2 \\ &= \frac{S_{xx}}{(S_{xx})^2} \\ &= \frac{1}{S_{xx}} \end{aligned}$$

Question 3

To find the variance of b_0 we substitute $b_0 = \bar{Y} - b_1\bar{X}$ and workout the variance:

$$
\begin{aligned}
V(b_0) &= V\left(\bar{Y} - b_1\bar{X}\right) \\
&= V\left(\bar{Y}\right) - 2\bar{X}Cov\left(\bar{Y}, b_1\right) + \bar{X}^2 V(b_1) \\
&= \frac{\sigma^2}{n} - 2\bar{X}Cov\left(\bar{Y}, b_1\right) + \bar{X}^2 \frac{\sigma^2}{S_{xx}}
\end{aligned}
$$

$$\text{since } Cov\left(\bar{Y}, b_1\right) = 0 \text{ by independence of the } Y_i \text{ then:}$$

$$
\begin{aligned}
V(b_0) &= \frac{\sigma^2}{n} + \bar{X}^2 \frac{\sigma^2}{S_{xx}} \\
&= \sigma^2 \left[\frac{1}{n} + \frac{\bar{X}^2}{S_{xx}}\right]
\end{aligned}
$$

Problem 2.7

We can calculate the standard error of the b_1 coefficient or read it of the summary table.

$$SE(b_1) = \frac{\hat{\sigma}}{\sqrt{\sum (x_i - \bar{x})^2}}$$

In [13]:
```
stderr_b1 = np.sqrt(variance/sum((df['hours']-np.mean(df['hours']))**2))

print 'The standard error is %.3f.' %stderr_b1
```

```
The standard error is 0.090.
```

Part a.

Now we obtain the critical value $t_{\alpha/2,n-2}$ by finding the inverse $\alpha/2 = CDF = 0.005$ value from the t-distribution with 14 degrees of freedom:

In [14]:
```
tcrit = abs(t.ppf(0.005, n-2, loc=0, scale=1))

print 'The t critical value is %0.3f' %tcrit
```

```
The t critical value is 2.977
```

Now we can construct an 99% interval for b_1. This interval is given by:

$$b_1 \pm t_{0.005,14} SE(b_1)$$

In [15]:
```
CI_99 = [b1-tcrit*stderr_b1, b1+tcrit*stderr_b1]

print 'The 99 percent confidence interval is (%.3f, %.3f)' %(CI_99[0],CI_9
9[1])
```

```
The 99 percent confidence interval is (1.765, 2.303)
```

Part b.

The hypothesis test is as follows:

$H_o : \beta_1 = 2$

$H_a : \beta_1 \neq 2$

The decision rule is that if $|t_{stat}| > |t_{0.005,14}|$, we would reject the null hypothesis H_o and fail to reject H_o otherwise. We calculate t_{stat} below:

$$t_{stat} = \frac{b_1 - \beta_1^{null}}{SE(b_1)} \approx \frac{2.03 - 2}{0.09} \approx 0.38$$

A better approximation is computed below:

In [16]:
```
beta_null = 2

tstat = (b1-beta_null)/stderr_b1

print 'The t statistic value is %0.3f' %tstat
```

```
The t statistic value is 0.380
```

Since $|t_{stat}| < |t_{0.005,14}| = 2.977$ we fail to reject H_o.

The p-value is given below:

In [17]:
```
pval = 2*(1-t.cdf(tstat,n-2))

print 'The p-value is %.3f' %pval
```

```
The p-value is 0.709
```

Part c.

Since the standard is being exceeded by 0.3 Brinell, that means $\beta_1 - \beta_1^{null} = 0.3$, then the actual parameter $\beta_1 = 2.3$. To calculate the power of the test when $\beta_1 = 2.3$, we need to calculate $1 - P\left(t_{stat} \notin RR \mid \beta_1 = 2.3\right)$, where RR is the rejection region of the test from part b). This is the power of the test.

From part b) we know the rejection region is:

$$RR = \left\{ \left| \frac{b_1 - \beta_1^{null}}{SE\left(b_1\right)} \right| > t_{crit} \right\}$$
$$= \left\{ b_1 > \beta_1^{null} + t_{crit} SE\left(b_1\right) \text{ or } b_1 < \beta_1^{null} - t_{crit} SE\left(b_1\right) \right\}$$

Therefore:

$$P\left(t_{stat} \notin RR \mid \beta_1\right) = P\left(\beta_1^{null} - t_{crit} SE\left(b_1\right) < b_1 < \beta_1^{null} + t_{crit} SE\left(b_1\right) \;\middle|\; \beta_1\right)$$

and:

$$Power(\beta_1) = 1 - P\left(\beta_1^{null} - t_{crit} SE\left(b_1\right) < b_1 < \beta_1^{null} + t_{crit} SE\left(b_1\right) \;\middle|\; \beta_1\right)$$
$$= 1 - P\left(\frac{\beta_1^{null} - \beta_1 - t_{crit} SE\left(b_1\right)}{\sigma\{b_1\}} < t < \frac{\beta_1^{null} - \beta_1 + t_{crit} SE\left(b_1\right)}{\sigma\{b_1\}} \;\middle|\; \beta_1\right)$$

Below we perform the power calculations for $\alpha = 0.01$ and $t_{crit} = t_{0.005,14} = 2.9768427$:

In [18]:
```
beta1 = 2.3

sigma_b1 = 0.1

tleft = (beta_null - beta1 - tcrit * stderr_b1)/sigma_b1

tright = (beta_null - beta1 + tcrit * stderr_b1)/sigma_b1

pType_2err = t.cdf(tright,n-2) - t.cdf(tleft,n-2)

power = 1 - pType_2err

print 'The power of the test is %.3f' %power
```

```
The power of the test is 0.619
```

Hence $Power(\beta_1 = 2.3) = 0.619$ for $\alpha = 0.01$.

Problem 2.16

Part a.

First calculate $\bar{X}$ and S_{xx}:

In [19]:
```
X = df.loc[:,'hours']

Xh = DataFrame({'hours':30}, index = range(1))

Xbar = np.mean(X)

Sxx = np.sum((X - Xbar)**2)
```

In [20]:
```
print Xbar
```

```
28.0
```

In [21]:
```
print Sxx
```

```
1280.0
```

$\hat{Y}$ is given from the model:

In [22]:
```
yhat = model.predict(Xh)[0]

print yhat
```

```
229.63125
```

Then we find the MSE of the model and calculate the standard error of $\hat{Y}$:

In [23]:
```
MSE = model.mse_resid

print MSE
```

```
10.4589285714
```

The standard error of $\hat{Y}$ is approximated by:

$$S\left(\hat{Y}\right) = \sqrt{MSE\left[\frac{1}{n} + \frac{\left(X_h - \bar{X}\right)^2}{S_{xx}}\right]}$$

which we calculate below:

In [24]:
```
var_yhat = MSE * ((1/ n) + ((Xh - Xbar)**2)/Sxx).loc[0,'hours']

stderr_yhat = np.sqrt(var_yhat)

print stderr_yhat
```

```
0.180787587477
```

The 98% confidence interval is given by $\hat{Y} \pm t_{0.01,14}\ \ S\left(\hat{Y}\right)$ and is calculated below:

In [25]:
```
tcrit = abs(t.ppf(0.01, n-2, loc=0, scale=1))

CI_98 = [yhat-tcrit*stderr_yhat, yhat+tcrit*stderr_yhat]

print 'The 98 percent confidence interval is (%0.2f, %0.2f)' %(CI_98[0],CI
_98[1])
```

```
The 98 percent confidence interval is (229.16, 230.11)
```

This interval tells us that the true mean hardness value in Brinell units after 30 hours have elapsed, will be contained in the interval $(229.16,\ 230.11)$ 98% of the time.

Part b.

Here the calculations are similar to part a. above, except that for the prediction interval we use the following standard error:

$$S\,(predict) = \sqrt{MSE\left[1 + \frac{1}{n} + \frac{\left(X_h - \bar{X}\right)^2}{S_{xx}}\right]}$$

which we calculate below:

In [26]:
```
var_pred = MSE * (1 + (1/ n) + ((Xh - Xbar)**2)/Sxx).loc[0,'hours']

stderr_pred = np.sqrt(var_pred)

print stderr_pred
```

```
3.23907590575
```

The 98% confidence interval is given by $\hat{Y}_{predict} \pm t_{0.01,14}\ \ S\,(predict)$ and is calculated below:

In [27]:
```
CI_98_pred = [yhat-tcrit*stderr_pred, yhat+tcrit*stderr_pred]

print 'The 98 percent prediction interval is (%0.2f, %0.2f).' %(CI_98_pred
[0],CI_98_pred[1])
```

```
The 98 percent prediction interval is (221.13, 238.13).
```

This interval tells us that the true hardness value in Brinell units for a new molded test item after 30 hours have elapsed, will be contained in the interval $(221.13,\ 238.13)$ 98% of the time. Notice that this is a much wider interval relative to the one in part a., reflecting the greater variability of a prediction which of course deviates from the mean response $\hat{Y}$.

Problem 2.26

Part a.

With a few comands we can set up the ANOVA table shown below:

In [28]:
```
import statsmodels.api as sms

table = sms.stats.anova_lm(model, typ=2)

table.insert(2, 'mean_sq', table['sum_sq']/table['df'])

table = table.set_value('total', ['sum_sq', 'df'], [np.sum(table['sum_sq']
),np.sum(table['df'])])

table.index = ['Model', 'Residuals','Total']

table
```

Out[28]:

	sum_sq	df	mean_sq	F	PR(>F)
Model	5297.5125	1	5297.512500	506.506232	2.158814e-12
Residuals	146.4250	14	10.458929	NaN	NaN
Total	5443.9375	15	NaN	NaN	NaN

Part b.

We can conduct a F test of the following null and alternative hypothesis:

$H_0 : \ \beta_1 = 0$

$H_a : \ \beta_1 \neq 0$

The decision rule is:

If $F \leq F_{(0.99,\ 14)}$, then we fail to reject H_0.

If $F > F_{(0.99,\ 14)}$, then we reject H_0.

In [29]:
```
fcrit = f.ppf(0.99, 1, n-2)

print 'The F critical value is %0.2f' %fcrit
```

```
The F critical value is 8.86
```

From the ANOVA table we have $F = 506.51$. Since $F \gg F_{(0.99,\ 14)}$, we reject H_0 and conclude that there exist a linear association between plastic hardness and the elapsed time.