Existence of Attracting Periodic Orbits for the Newton Method

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Abstract

We describe a method for constructing polynomials whose corresponding Newton transform has an attracting periodic orbit. Using these results we prove that, for any integer $n \geq 2$, there is an open subset of the set of complex analytic functions whose Newton transform has an attracting periodic orbit of period n.

1 Introduction

Newton's method may be used to approximate both real and complex solutions of the equation $f(x) = 0$; these solutions will be called zeroes or roots of f. Let $x_0 \in \mathbb{C}$ be an initial guess for a zero of f. Compute

$$x_{n+1} = N_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \, , \quad \text{with} \quad n = 0, 1, 2, \cdots \, ,$$

which defines a discrete dynamical system called the Newton method of f. The N_f of above is called the Newton transform of f. If we choose the initial guess sufficiently near a zero x^* of f, then the sequence $(x_n)_{n=0,1,2,\cdots}$ converges to x^*. The convergence of $(x_n)_{n=0,1,2,\cdots}$ to x^* is quadratic if x^* is a simple zero of f, and linear if x^* is a

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root of multiplicity greater than or equal to two (that is, there is $m \geq 2$ such that $f'(x^*) = \cdots = f^{(m-1)}(x^*) = 0$ and $f^{(m)}(x^*) \neq 0$). On the other hand, if we do not choose the initial guess x_0 near a root of f, then the sequence $(x_n)_{n=0,1,\dots}$ may or may not converge to a root of f. For example, if x_0 belongs to a periodic orbit, say of period $k \geq 2$, of N_f (that is, $N_f^k(x_0) = x_0$ and $N_f^j(x_0) \neq x_0$, for $j = 1, \dots, k-1$), then the sequence $(x_n)_{n=1,\dots}$ does not converge to a root of f, since any root x^* of $f(x) = 0$ is a fixed point of N_f. We note that the roots of f are attracting fixed points of N_f. In fact, if x^* is a simple root of f (that is, $f'(x^*) \neq 0$), then $N_f'(x^*) = 0$; and if x^* is a multiple root of multiplicity $m \geq 2$, then $N_f'(x^*) = (m-1)/m$.
The Newton method appears repeatedly in the literature of numerical analysis and in that of discrete dynamical systems (see, for example, [3] and [2]). A historical study of iteration of rational functions, of which the Newton transform of a polynomial is a particular case, may be found in [1].
We give a method for constructing polynomials whose Newton transform has an attracting periodic orbit of period greater than or equal to two. Thus, the set of exceptional points contains an open set.

2 Basic Features of the Newton Method

We give a revision of the basic features of the Newton transform.
Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. The Newton transform associated to f is

$$N_f(x) = x - \frac{f(x)}{f'(x)} , \quad \text{if } f'(x) \neq 0 .$$

Since $N_f'(x) = (f(x)f''(x))/(f'(x))^2$, the critical points of N_f are the roots of f and of $f''(x) = 0$. We say x_0 is a simple root of f if $f(x_0) = 0$ and $f'(x_0) \neq 0$. In this case, it is easy to see that $N_f(x_0) = x_0$ if and only if $f(x_0) = 0$; that is, the simple roots of f are fixed points of N_f. On the other hand, if x_0 is a multiple root of f of multiplicity $m \geq 2$, then we also have that $N_f(x_0) = x_0$. Thus if x_0 is a simple root of f, then x_0 is a super–attractive fixed point of N_f, that is, $N_f'(x_0) = 0$; and if x_0 is a multiple root of f of multiplicity $m \geq 2$, then $N_f'(x_0) = \dfrac{m-1}{m}$. Therefore, the convergence of the iterated $N_f^n(x) = N_f \circ \cdots \circ N_f(x)$ (n times) is at least quadratic in a neighborhood of a simple root x_0 and linear in a neighborhood of a multiple root of f.

3 Results

Proposition 1 *Let $f : \mathbb{C} \to \mathbb{C}$ be a complex analytic function, and let $\mathcal{O} = \{x_1, \ldots, x_n\} \subset \mathbb{C}$, with $x_i \neq x_j$ for all $i, j \in \{1, \ldots, n\}$. Then $\mathcal{O}$ is a periodic orbit of Newton transform N_f if and only if f satisfies*

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}\,,\ i = 1, \ldots, n-1\,, \quad \textit{and} \quad f'(x_n) = \frac{f(x_n)}{x_n - x_1}\,. \tag{1}$$

Proof. Assume $\mathcal{O}$ is a periodic orbit of N_f. Let $x_i \in \mathcal{O}$. Then $N_f(x_i) = x_{i+1}$, for $i = 1, \ldots, n-1$, and $N_f(x_n) = x_1$. Since $N_f(x_i) = x_i - \dfrac{f(x_i)}{f'(x_i)}$, and since $N_f(x_i) = x_{i+1}$, we have that $f'(x_i) = \dfrac{f(x_i)}{x_i - x_{i+1}}$, for $i = 1, \ldots, n-1$. Finally, since $N_f(x_n) = x_1$, and since $N_f(x_n) = x_n - \dfrac{f(x_n)}{f'(x_n)}$, we have that $f'(x_n) = \dfrac{f(x_n)}{x_n - x_1}$.
Conversely, suppose f satisfies condition (1). Then, for $i = 1, \ldots, n-1$, we have $N_f(x_i) = x_i - \dfrac{f(x_i)}{\frac{f(x_i)}{x_i - x_{i+1}}} = x_{i+1}$. In particular, $N_f(x_{n-1}) = x_n$ and $N_f(x_n) = x_n - \dfrac{f(x_n)}{\frac{f(x_n)}{x_n - x_1}} = x_1$. Consequently, $\mathcal{O} = \{x_1, \ldots, x_n\}$ is a periodic orbit of period n of N_f which completes the proof.

Proposition 2 *For any positive integer $n \geq 2$, there is a polynomial f of degree less than or equal to $2n-1$ for which its Newton transform has a periodic orbit of period n.*

Proof. Let $x_1, x_2, \ldots, x_n$ be given numbers, with $x_i \neq x_j$ if $i \neq j$ for all $i, j \in \{1, \ldots, n\}$, and let $y_1, y_2, \ldots, y_n$, with $y_j \neq 0$ for all $j = 1, \ldots, n$. Assume there is a polynomial f such that

$$\begin{cases} f(x_i) = y_i\,,\ i = 1, \ldots, n\,, \\[2ex] f'(x_i) = \dfrac{y_i}{x_i - x_{i+1}}\,,\ i = 1, \ldots, n-1\,, \quad \text{and} \\[2ex] f'(x_n) = \dfrac{y_n}{x_n - x_1}\,. \end{cases} \tag{2}$$

By Proposition 1, $\mathcal{O} = \{x_1, x_2, \ldots, x_n\}$ is a periodic orbit of period n of N_f.

We now show that such a polynomial f exists. For this, we use the Hermite interpolation method, which allows us to construct a polynomial of degree $2n-1$ that satisfies conditions (2).
We begin the construction by writing f as

$$f(x) = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_{2n} f_{2n}(x)$$

where the functions f_i are polynomials of degree $i-1$, for each $i = 1, \ldots, 2n$, and defined inductively as follows:

$$\begin{array}{rcl}
f_1(x) & = & 1 \\
f_2(x) & = & f_1(x) \cdot (x - x_1) = x - x_1 \\
f_3(x) & = & f_2(x) \cdot (x - x_1) = (x - x_1)^2 \\
f_4(x) & = & f_3(x) \cdot (x - x_2) = (x - x_1)^2 \cdot (x - x_2) \\
f_5(x) & = & f_4(x) \cdot (x - x_2) = (x - x_1)^2 \cdot (x - x_2)^2 \\
f_6(x) & = & f_5(x) \cdot (x - x_3) = (x - x_1)^2 \cdot (x - x_2)^2 \cdot (x - x_3) \\
 & \vdots & \\
f_{2i-1}(x) & = & f_{2i-2}(x) \cdot (x - x_{i-1}) \\
f_{2i}(x) & = & f_{2i-1}(x) \cdot (x - x_i) \\
 & \vdots & \\
f_{2n-1}(x) & = & f_{2n-2}(x) \cdot (x - x_{n-1}) \\
f_{2n}(x) & = & f_{2n-1}(x) \cdot (x - x_n) \,.
\end{array}$$

Note that $f_{2i-1}(x_i) = f_{2i-2}(x_i) \cdot (x_i - x_{i-1}) \neq 0$ and that $f_{2i}(x_i) = f_{2i-1}(x_i) \cdot (x_i - x_i) = 0$; that is, $f_j(x_i) \neq 0$, for $j \leq 2i-1$, and $f_j(x_i) = 0$, for $j \geq 2i$.
On the other hand, we have that $f'_{2i}(x) = f'_{2i-1}(x) \cdot (x - x_i) + f_{2i-1}(x)$. Thus $f'_{2i}(x_i) = f_{2i-1}(x_i) \neq 0$ and $f'_{2i+1}(x) = f'_{2i}(x) \cdot (x - x_i) + f_{2i}(x)$. Therefore $f'_{2i+1}(x_i) = f_{2i}(x_i) = 0$ and hence $f'_j(x_i) \neq 0$, for $j \leq 2i$, and $f'_j(x_i) = 0$, for $j \geq 2i+1$.
To determine the polynomial f we must find suitable coefficients a_i, for $i = 1, \ldots, 2n$. For this we must solve a linear system of $2n$ equations with $2n$ unknowns. The associated matrix is a lower triangular matrix whose rows are, for $i = 1, \ldots, n$,

$$\begin{array}{lcccccccccc}
A_{2i-1} & = & f_1(x_i) & f_2(x_i) & f_3(x_i) & \ldots & f_{2i-1}(x_i) & 0 & 0 & 0 \ \ldots & 0 \\
A_{2i} & = & f'_1(x_i) & f'_2(x_i) & f'_3(x_i) & \ldots & f'_{2i-1}(x_i) & f'_{2i}(x_i) & 0 & 0 \ \ldots & 0 \\
A_{2i+1} & = & f_1(x_{i+1}) & f_2(x_{i+1}) & f_3(x_{i+1}) & \ldots & f_{2i-1}(x_{i+1}) & f_{2i}(x_{i+1}) & f_{2i+1}(x_{i+1}) & 0 \ \ldots & 0 \,.
\end{array}$$

Thus the system of equations may be written in the form $Ax = b$ where

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_{2n} \end{pmatrix}_{2n\times 2n}, \quad x = \begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix}_{2n\times 1}, \quad \text{and} \quad b = \begin{pmatrix} y_1 \\ \dfrac{y_1}{x_1 - x_2} \\ \vdots \\ y_n \\ \dfrac{y_n}{x_n - x_1} \end{pmatrix}_{2n\times 1}.$$

We note that the linear system $Ax = b$ has a solution, since the determinant of the matrix A is nonzero. To prove the last assertion, we note that the components of the diagonal of A are $f_{2i-1}(x_i)$ and $f'_{2i}(x_i)$, which are nonzero for $i = 1, \ldots, n$. The proof is now complete.

Example: We construct a polynomial whose Newton transform has a periodic orbit of period three.
Let

$$\begin{cases} x_1 = 0\,, & y_1 = 1 \\ x_2 = 1\,, & y_2 = -1 \\ x_3 = 2\,, & y_3 = 1\,. \end{cases}$$

Then $f(0) = 1$, $f(1) = -1$ and $f(2) = 1$, and $f'(0) = -1$, $f'(1) = 1$ and $f'(2) = 1/2$. We now construct the polynomials $f_i(x)$, for $i = 1, \ldots, 6$,

$$\begin{aligned} f_1(x) &= 1 \\ f_2(x) &= f_1(x) \cdot (x - x_1) = x \\ f_3 x) &= f_2(x) \cdot (x - x_1) = x^2 \\ f_4(x) &= f_3(x) \cdot (x - x_2) = x^2 \cdot (x-1) \\ f_5(x) &= f_4(x) \cdot (x - x_2) = x^2 \cdot (x-1)^2 \\ f_6(x) &= f_5(x) \cdot (x - x_3) = x^2 \cdot (x-1)^2 \cdot (x-2)\,. \end{aligned}$$

Hence $f(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^2 (x-1) + a_5 x^2 (x-1)^2 + a_6 x^2 (x-1)^2 (x-2)$ and the rows of the matrix are

$$
\begin{array}{rclllllll}
A_1 &=& f_1(x_1) & 0 & 0 & 0 & 0 & 0 \\
A_2 &=& f_1'(x_1) & f_2'(x_1) & 0 & 0 & 0 & 0 \\
A_3 &=& f_1(x_2) & f_2(x_2) & f_3(x_2) & 0 & 0 & 0 \\
A_4 &=& f_1'(x_2) & f_2'(x_2) & f_3'(x_2) & f_4'(x_2) & 0 & 0 \\
A_5 &=& f_1(x_3) & f_2(x_3) & f_3(x_3) & f_4(x_3) & f_5(x_3) & 0 \\
A_6 &=& f_1'(x_3) & f_2'(x_3) & f_3'(x_3) & f_4'(x_3) & f_5'(x_3) & f_6'(x_3)\,.
\end{array}
$$

The linear system $Ax = b$ is

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 \\
1 & 2 & 4 & 4 & 4 & 0 \\
0 & 1 & 4 & 8 & 12 & 4
\end{pmatrix}
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}
=
\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ \frac{1}{2} \end{pmatrix} .
$$

From this system we easily obtain that $a_1 = 1$, $a_2 = -1$, $a_3 = -1$, $a_4 = 4$, $a_5 = -5/2$, and $a_6 = 7/8$. Hence the polynomial $f(x)$ is given by $f(x) = 1 - x - x^2 + 4x^2(x-1) - \frac{5}{2}x^2(x-1)^2 + \frac{7}{8}x^2(x-1)^2(x-2)$.

Proposition 3 *Let $f(x)$ be a polynomial whose Newton transform has a periodic orbit of period n, say $\mathcal{O} = \{x_1, x_2, \ldots, x_n\}$. If $f''(x_i) = 0$, for some $x_i \in \mathcal{O}$, then $\mathcal{O}$ is an attracting periodic orbit.*

Proof. By Proposition 1, f satisfies

$$
\begin{cases}
f(x_i) = y_i\,, \ \ i = 1, \ldots, n\,, \\[2ex]
f'(x_i) = \dfrac{y_i}{x_i - x_{i+1}}\,, \ \ i = 1, \ldots, n-1, \quad \text{and} \\[2ex]
f'(x_n) = \dfrac{y_n}{x_n - x_1}\,.
\end{cases}
$$

Without loss of generality, suppose that $f''(x_1) = 0$. We have

$$
(N_f^n)'(x_1) = N_f'(N_f^{n-1}(x_1)) \cdots N_f'(N_f(x_1)) \cdot N_f'(x_1) = N_f'(x_n) \cdot N_f'(x_{n-1}) \cdots N_f'(x_2) \cdot N_f'(x_1)\,.
$$

Now since $N_f'(x) = \dfrac{f(x)f''(x)}{f'(x)}$, we obtain

$$(N_f^n)'(x_1) \quad = \quad \frac{f(x_n)\cdot f''(x_n)}{(f'(x_n))^2}\cdot\frac{f(x_{n-1})\cdot f''(x_{n-1})}{(f'(x_{n-1}))^2}\cdots\frac{f(x_1)\cdot f''(x_1)}{(f'(x_1))^2}\,.$$

Thus, since $f''(x_1)=0$, we have that $(N_f^n)'(x_1)=0$ and the result now follows.

Theorem 1 *Let $n \geq 2$ be an integer. Then there is a polynomial of degree less than or equal to $2n$ whose Newton transform has an attracting periodic orbit of period n.*

Proof. Given $\mathcal{O}=\{x_1,\ldots,x_n\}$, with $x_i \neq x_j$ for all $i,j \in \{i,\ldots,j\}$, and $\{y_1,\ldots,y_n\}$, with $y_k \neq 0$ for all $k=1,\ldots,n$. Proposition 2 permit us to construct a polynomial f such that $\mathcal{O}$ is a periodic orbit of period n for N_f. Recall that

$$f(x) = a_1 f(x) + \cdots + a_{2n} f_{2n}(x)\,.$$

for suitable values of $a,\ldots,a_{2n}$. We now define a new polynomial $\tilde{f}$ by

$$\tilde{f}(x) = f(x) + a_{2n+1} f_{2n+1}(x)$$

where the coefficient a_{2n+1} is a parameter to be determined.
The new condition does not alter the periodic orbit $\mathcal{O}=\{x_1,\ldots,x_n\}$, since we have that $f_{2n+1}(x_i)=0$ for each $i=1,\ldots,n$. Finally, to determine a_{2n+1} we use the condition $f''(x_1)=0$ and solve the equation for a_{2n+1}. By Proposition 3 the periodic orbit $\mathcal{O}$ is attracting and the proof is now complete.

Example. By the above example, $f(x) = 1-x-x^2+4x^2(x-1)-\frac{5}{2}x^2(x-1)^2+\frac{7}{8}x^2(x-1)^2(x-2)$, hence $\tilde{f}(x) = f(x)+ax^2(x-1)^2(x-2)^2$. We next solve $\tilde{f}''(0)=0$ and obtain $a=37/16$. The new polynomial $\tilde{f}$ is such that its Newton transform $N_{\tilde{f}}$ has the attracting periodic orbit $\mathcal{O}=\{0,1,2,\}$ of period 3.
Since the hyperbolic periodic orbits of an analytic map are stable, we now have the following theorem.

Theorem 2 *There is an open set $\mathcal{U}$ of the set of analytic functions $f:\mathbb{C}\to\mathbb{C}$ such that, for any $f\in\mathcal{U}$, its Newton transform N_f has an attracting periodic orbit of period greater than or equal to two.*

References

[1] D. S. Alexander, *A history of complex dynamics: from Schröder to Fatou and Julia*, Vieweg, 1994.

[2] R. Holmgren, *A First Course in Discrete Dynamical Systems*, Springer–Verlag, 1994.

[3] H. Peitgen (ed.), *Newton, Method and Dynamical Systems*, Kluwer Academic Publishers, (1989), 54–58.