The Pigeon Hole Principle applied to Sequences

A paper originally published by A. Seidenberg in 1959
A simple proof of a Theorem of Erdos and Szekeres
modified by Mark E. Lehr for CSC 7 Discrete Mathematics

May 2021

1 Introduction

One of your classmates "Wesley Duong" asked a question about a problem from Section 9.4 H.38. Even though the book indicates the solution or hints at how to proceed, it was intriguing enough to warrant an in-depth investigation. I created a program to examine the possible results and at the same time show a technique called dynamic programming which solves the search problem by expanding the solution one element at a time. Similar to the bloom filter/hashing/recursive expectation previously shown.

Principle 1.1. *In mathematics, the pigeonhole principle states that if ν items are put into δ containers, $\nu > \delta$, then at least one container must hold more than one item. It is commonly called Dirichlet's box principle or Dirichlet's drawer principle.*

The principle has several generalizations and can be stated in various ways. In a more quantified version: for natural numbers δ and μ, if $\nu = \delta * \mu + 1$ objects are distributed among δ sets, then the pigeonhole principle asserts that at least one of the sets will contain at least $\mu + 1$ objects.

Though the most straightforward application is to finite sets, it is also used with infinite sets that cannot be put into one-to-one correspondence. To do so requires the formal statement of the pigeonhole principle, which is "there does not exist an injective function whose co-domain is smaller than its domain".

Theorem 1.1. *Given a sequence $S : (s)_{i=1}^{\nu} = \{s_1, s_2, \ldots, s_\nu\}, s_i \in \mathbb{R}, s_i \neq s_j$ with $\nu > \delta * \mu$, then the number of terms of every decreasing sub-sequence is at most δ, then there exists an increasing sub-sequence of more than μ terms.*

Proof. To each s_i assign a pair of numbers (δ_i, μ_i), where δ_i is the largest number of terms of a decreasing sub-sequence beginning with s_i and μ_i the largest number of terms of an increasing sub-sequence beginning with s_i. To two distinct terms $s_i, s_j, i < j$, correspond distinct pairs. For if $s_i < s_j$ then $\mu_i \geq 1+\mu_j$

and if $s_i > s_j$ then $\delta_i \geq 1 + \delta_j$. We have $\nu > \delta * \mu$ distinct pairs (δ_i, μ_i). For all of them $1 \leq \delta_i \leq \delta$. If all μ_i were less than or equal to μ, then at most $\delta * \mu$ of the above pairs would be distinct. Thus at least one μ_i is greater than μ. ■

And a consequence of Theorem 1.1 is the statement in the next 2 corollaries.

Corollary 1.1.1. *If the longest decreasing sub-sequence has δ terms, then the longest increasing sub-sequence has at least ν/δ terms. $\mu \geq \nu/\delta$*

Corollary 1.1.2. *If $\nu > n^2$, then S has a monotone sub-sequence of more than n terms.*

Remark. A well known example of a sequence of $\nu = \delta * \mu$ terms

$$
\begin{gathered}
\delta, \delta - 1, \ldots, 1, \\
2\delta, 2\delta - 1, \ldots, \delta + 1, \\
3\delta, 3\delta - 1, \ldots, 2\delta + 1, \\
\vdots \\
\mu\delta, \mu\delta - 1, \ldots, (\mu - 1)\delta + 1
\end{gathered}
$$

which has μ decreasing sub-sequences of length δ and δ increasing sub-sequences of length μ. This shows that the bound given by the theorem is exact. In the code that illustrates this problem, the above sequence is generated, then reversed, and finally shuffled so that all possible sequences can be explored and the sub-sequences can thus be quantified.