## IX. Appendix

The following section presents the remaining production rules for conditional structures.

### A. Conditional Structures

For **blocks production rule (c)**, we have:

$$\textit{<Blocks>} \rightarrow \ [\ \textit{<SB>}\ ]$$

The derivation of production rule (c) creates a subgraph $G_i^{BLKS}$ that is equivalent to the subgraph $G_i^{SB}$. The associated WCET cost function is given by:

$$\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) \ = \ \Phi_i^{SB}(\zeta_{pred}, \zeta_{succ}) \tag{14}$$

The associated set of selected preemption points function is given by:

$$\rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) = \rho_i^{SB}(\zeta_{pred}, \zeta_{succ}) \tag{15}$$

For **blocks production rule (d)**, we have:

$$\textit{<Blocks>} \rightarrow \ [\ \textit{<CB>}\ ]$$

The derivation of production rule (d) creates a subgraph $G_i^{BLKS}$ that is equivalent to the subgraph $G_i^{CB}$. The associated WCET cost function is given by:

$$\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) \ = \ \Phi_i^{CB}(\zeta_{pred}, \zeta_{succ}) \tag{16}$$

The associated set of selected preemption points function is given by:

$$\rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) = \rho_i^{CB}(\zeta_{pred}, \zeta_{succ}) \tag{17}$$

For **aggregate blocks production rule (e)**, we have:

$$\textit{<Blocks>} \rightarrow \textit{<SB>}\ \textit{<Blocks>}$$

The derivation of production rule (e) creates a subgraph $G_i^{BLKS'}$ concatenating a previously created aggregate blocks basic block subgraph $G_i^{SB}$ in series with a previously created subgraph $G_i^{BLKS}$. The associated WCET cost function is given by:

$$\begin{aligned}\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ}) \ = \ min_{r,s}\{(\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) \ + \\ \max_{\delta_i^m, \delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] \ + \\ \Phi_i^{SB}(\zeta_{pred_s}, \zeta_{succ})\}\end{aligned} \tag{18}$$

where $\zeta_{succ_r}$, and $\zeta_{pred_s}$ represent the values where the function $\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ})$ is minimized and valid solutions are subject to the following constraints:

$$\begin{aligned}(\zeta_{succ_r} \ + \ \max_{\delta_i^m, \delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] \ + \ \zeta_{pred_s}) \leq Q_i \\ \delta_i^m \in \rho_i^{succ}(G_i^{BLKS}, \zeta_{pred}, \zeta_{succ_r}) \\ \delta_i^n \in \rho_i^{pred}(G_i^{SB}, \zeta_{pred_s}, \zeta_{succ})\end{aligned} \tag{19}$$

The associated preemption point function is given by:

$$\begin{aligned}\rho_i^{BLKS'}(\zeta_{pred}, \zeta_{succ}) \ = \rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) \cup \\ \rho_i^{SB}(\zeta_{pred_s}, \zeta_{succ})\end{aligned} \tag{20}$$

**Theorem 2.** *Given $\Phi_i$ and $\rho_i$ functions for each substructure of $BLKS$ where each $\rho_i^A(\zeta_{pred}, \zeta_{succ})$ represents a feasible solution for substructure $A$ given preemptions $\zeta_{pred}$ before, $\zeta_{succ}$ after, and $\Phi_i^A$ is a safe upper bound on the total WCET and preemption cost of that solution. Applying production (e) over a feasible $G_i$, $G_i^{BLKS}$ and $Q_i$ results in a feasible solution $\rho_i^{BLKS}$ and a safe upper bound $\Phi_i^{BLKS}$ given by Equations 18, 19, and 20 respectively.*

*Proof:* The proof is by direct argument. The proof structure and line of reasoning are identical to the proof for Theorem 1, hence we omit the details here. ■

For **aggregate blocks production rule (f)**, we have:

$$\textit{<Blocks>} \rightarrow \textit{<CB>}\ \textit{<Blocks>}$$

The derivation of production rule (f) creates a subgraph $G_i^{BLKS'}$ concatenating a previously created conditional block subgraph $G_i^{CB}$ in series with a previously created aggregate blocks subgraph $G_i^{BLKS}$. Production rule (f) exhibits the maximum time complexity for our algorithm executing in $O(N_i log(N_i) Q_i^4)$ time. Each *<SB>* contains $Q_i$ solutions with each *<Blocks>* and *<CB>* structure containing $Q_i^2$ solutions. The associated WCET cost function is given by:

$$\begin{aligned}\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ}) \ = \ min_{r,s}\{(\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) \ + \\ max_{\delta_i^m, \delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] \ + \\ \Phi_i^{CB}(\zeta_{pred_s}, \zeta_{succ})\}\end{aligned} \tag{21}$$

where $\zeta_{succ_r}$, and $\zeta_{pred_s}$ represent the values where the function $\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ})$ is minimized and valid solution combinations are subject to the following constraints:

$$\begin{aligned}(\zeta_{succ_r} \ + \ max_{\delta_i^m, \delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] \ + \ \zeta_{pred_s}) \leq Q_i \\ \delta_i^m \in \rho_i^{succ}(G_i^{BLKS}, \zeta_{pred}, \zeta_{succ_r}) \\ \delta_i^n \in \rho_i^{pred}(G_i^{CB}, \zeta_{pred_s}, \zeta_{succ})\end{aligned} \tag{22}$$

The associated preemption point function is given by:

$$\begin{aligned}\rho_i^{BLKS'}(\zeta_{pred}, \zeta_{succ}) \ = \rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) \cup \\ \rho_i^{CB}(\zeta_{pred_s}, \zeta_{succ})\end{aligned} \tag{23}$$

**Theorem 3.** *Given $\Phi_i$ and $\rho_i$ functions for each substructure of $BLKS$ where each $\rho_i^A(\zeta_{pred}, \zeta_{succ})$ represents a feasible solution for substructure $A$ given preemptions $\zeta_{pred}$ before, $\zeta_{succ}$ after, and $\Phi_i^A$ is a safe upper bound on the total WCET and preemption cost of that solution. Applying production (e) over a feasible $G_i$, $G_i^{BLKS}$ and $Q_i$ results in a feasible solution $\rho_i^{BLKS}$ and a safe upper bound $\Phi_i^{BLKS}$ given by Equations 21, 22, and 23 respectively.*

*Proof:* The proof is by direct argument. The proof structure and line of reasoning are identical to the proof for Theorem 1, hence we omit the details here. ∎

The grammar we have presented thus far are focused on the production rules for conditional structures. Non-unrolled loops and functions are structured programming constructs that are also prevalent in real-time code. We present the production rules supporting these structured programming elements in the following subsections.

### B. Non Unrolled Loops

For **loops production rule (g)**, we have:

*<Loop>* → [ *<Blocks>* *<MaxIter>* ]

The derivation of production rule (g) creates a subgraph $G_i^{LOOP}$ that is equivalent to the subgraph $G_i^{BLKS}$. The associated WCET cost function is given by:

$$\Phi_i^{LOOP}(\zeta_{pred}, \zeta_{succ}) \;=\; \Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) \;\times\; MaxIter \tag{24}$$

where valid solution combinations are subject to the following constraints:

$$(\zeta_{succ} \;+\; \zeta_{pred}) \leq Q_i \tag{25}$$

The associated set of selected preemption points function is given by:

$$\rho_i^{LOOP}(\zeta_{pred}, \zeta_{succ}) = \rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) \tag{26}$$

For **blocks production rule (h)**, we have:

$$\textit{<Blocks>} \rightarrow \;[\; \textit{<LOOP>} \;] \tag{27}$$

The derivation of production rule (h) creates a subgraph $G_i^{BLKS}$ that is equivalent to the subgraph $G_i^{LOOP}$. The associated WCET cost function is given by:

$$\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) \;=\; \Phi_i^{LOOP}(\zeta_{pred}, \zeta_{succ}) \tag{28}$$

The associated set of selected preemption points function is given by:

$$\rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) = \rho_i^{LOOP}(\zeta_{pred}, \zeta_{succ}) \tag{29}$$

### C. Inline Functions

Functions are split into two grammar elements, namely, function definition and function invocation. The conditional PPP algorithm generates solutions for the function definition blocks consistently with the main task function. The generated function definition preemption solutions are combined with the function invocation preemption solutions at each graph location where the function is called.

For **function definition production rule (i)**, we have:

*<Function>* → [ *<Blocks>* *<FunctionName>* ]

The derivation of production rule (i) creates a subgraph $G_i^{FUNC}$ that is equivalent to the subgraph $G_i^{BLKS}$. The associated WCET cost function is given by:

$$\Phi_i^{FUNC}(\zeta_{pred}, \zeta_{succ}) \;=\; \Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) \tag{30}$$

The associated set of selected preemption points function is given by:

$$\rho_i^{FUNC}(\zeta_{pred}, \zeta_{succ}) = \rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) \tag{31}$$

For **function call production rule (j)**, we have:

*<FunctionCall>* → [ *<Blocks>* *<FunctionName>* ]

The derivation of production rule (j) creates a subgraph $G_i^{FCALL}$ that is equivalent to the subgraph $G_i^{BLKS}$. The associated WCET cost function is given by:

$$\begin{aligned}\Phi_i^{FCALL}(\zeta_{pred}, \zeta_{succ}) \;=\; min_{r,s}\{(\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) \;+ \\ max_{\delta_i^m, \delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] \;+ \\ \Phi_i^{FUNC}(\zeta_{pred_s}, \zeta_{succ})\}\end{aligned} \tag{32}$$

where $\zeta_{succ_r}$, and $\zeta_{pred_s}$ represent the values where the function $\Phi_i^{FCALL}(\zeta_{pred}, \zeta_{succ})$ is minimized and valid solution combinations are subject to the following constraints:

$$\begin{aligned}(\zeta_{succ_r} \;+\; max_{\delta_i^m, \delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] \;+\; \zeta_{pred_s}) \leq Q_i \\ \delta_i^m \in \rho_i^{succ}(G_i^{BLKS}, \zeta_{pred}, \zeta_{succ_r}) \\ \delta_i^n \in \rho_i^{pred}(G_i^{FUNC}, \zeta_{pred_s}, \zeta_{succ})\end{aligned} \tag{33}$$

The associated preemption point function is given by:

$$\begin{aligned}\rho_i^{FCALL}(\zeta_{pred}, \zeta_{succ}) \;= \rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) \cup \\ \rho_i^{FUNC}(\zeta_{pred_s}, \zeta_{succ})\end{aligned} \tag{34}$$

**Theorem 4.** *Given $\Phi_i$ and $\rho_i$ functions for each substructure of $FCALL$ where each $\rho_i^A(\zeta_{pred}, \zeta_{succ})$ represents a feasible solution for substructure $A$ given preemptions $\zeta_{pred}$ before, $\zeta_{succ}$ after, and $\Phi_i^A$ is a safe upper bound on the total WCET and preemption cost of that solution. Applying production (e) over a feasible $G_i$, $G_i^{FCALL}$ and $Q_i$ results in a feasible solution $\rho_i^{FCALL}$ and a safe upper bound $\Phi_i^{FCALL}$ given by Equations 32, 33, and 34 respectively.*

*Proof:* The proof is by direct argument. The proof structure and line of reasoning are identical to the proof for Theorem 1, hence we omit the details here. ∎

For **blocks production rule (k)**, we have:

*<Blocks>* → [ *<FunctionCall>* ]

The derivation of production rule (k) creates a subgraph $G_i^{BLKS}$ that is equivalent to the subgraph $G_i^{FCALL}$. The associated WCET cost function is given by:

$$\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) \;=\; \Phi_i^{FCALL}(\zeta_{pred}, \zeta_{succ}) \tag{35}$$

The associated set of selected preemption points function is given by:

$$\rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ}) = \rho_i^{FCALL}(\zeta_{pred}, \zeta_{succ}) \tag{36}$$