9 Appendix

The following section presents the complete proofs for the Theorems whose details were omitted in the paper. For convenience, we present the equations associated with the production rules included in this section as a courtesy to the reader.

For **production rule (e)**, we have:

$$<Blocks> \rightarrow <SB> <Blocks>$$

The derivation of production rule (e) creates a subgraph $G_i^{BLKS'}$ concatenating a previously created aggregate blocks basic block subgraph G_i^{SB} in series with a previously created subgraph G_i^{BLKS}. The associated WCET cost function is given by:

$$\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ}) = min_{r,s}\{(\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) + \max_{\delta_i^m,\delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] + \Phi_i^{SB}(\zeta_{pred_s}, \zeta_{succ})\} \quad (21)$$

where valid solution combinations are subject to the following constraints:

$$(\zeta_{succ_r} + \max_{\delta_i^m,\delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] + \zeta_{pred_s}) \leq Q_i \quad (22)$$

$$\delta_i^m \in \rho_i^{succ}(G_i^{BLKS}, \zeta_{pred}, \zeta_{succ_r}) \quad (23)$$

$$\delta_i^n \in \rho_i^{pred}(G_i^{SB}, \zeta_{pred_s}, \zeta_{succ}) \quad (24)$$

The associated preemption point function is given by:

$$\rho_i^{BLKS'}(\zeta_{pred}, \zeta_{succ}) = \rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) \cup \rho_i^{SB}(\zeta_{pred_s}, \zeta_{succ}) \quad (25)$$

where ζ_{succ_r}, and ζ_{pred_s} represent the values where the function $\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ})$ is minimized.

▶ **Theorem 2.** *Given Φ_i and ρ_i functions for each substructure of $BLKS$ where each $\rho_i^A(\zeta_{pred}, \zeta_{succ})$ represents a feasible solution for substructure A given preemptions ζ_{pred} before, ζ_{succ} after, and Φ_i^A is a safe bound on the total WCET and preemption cost of that solution. Applying production (e) over a feasible G_i, G_i^{BLKS} and Q_i results in a feasible solution ρ_i^{BLKS} and a safe bound Φ_i^{BLKS} given by Equations 21, 22-24, and 25 respectively.*

Proof. The proof is by direct argument. We need to prove that our solution ensures that the task level Q_i constraint is not violated and the cost function $\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ})$ results in a safe upper bound. To prove the Q_i constraint is not violated, we must show 1) the non-preemptive execution time of the combined solutions does not exceed Q_i at each solution interface, and 2) the non-preemptive execution time of the combined solution at the new predecessor and successor interfaces does not exceed Q_i. Let $\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ_s})$ with ζ_{pred},$\zeta_{succ_s} \in [0 \ldots Q_i]$ represent a safe upper bound cost solution for subgraph G_i^{BLKS} for basic block δ_i^j, with its corresponding set of selected preemption points denoted by $\rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ_s})$ be a limited preemption execution safe upper bound cost solution for basic block δ_i^j. We make an identical statement for subgraph G_i^{SB} for basic block δ_i^k, whose cost function is denoted $\Phi_i^{SB}(\zeta_{pred_u}, \zeta_{succ})$, and whose set of selected preemption points are denoted $\rho_i^{SB}(\zeta_{pred_u}, \zeta_{succ})$. Since we have a safe upper bound cost solution for each of the combined subgraphs, we can conclude that $\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ})$ computed in Equation 21 represents a safe upper bound cost solution for the concatenated series subgraphs $G_i^{BLKS} \cup G_i^{SB}$

starting at basic block δ_i^j, and ending at basic block δ_i^k with its corresponding selected preemption points denoted by $\rho_i^{BLKS'}(\zeta_{pred}, \zeta_{succ})$ and computed in Equation 25. Condition 1 is met in accordance with Equation 22 whose purpose is to ensure the non-preemptive execution time of the combined solutions does not exceed Q_i at each solution interface. Condition 2 is met per the definition of the parameters ζ_{pred}, and ζ_{succ} respectively, whose range is given by $[0 \dots Q_i - 1]$. Thus, the problem finds a feasible safe upper bound cost preemption points solution when applying production (e). ◀

For **production rule (f)**, we have:

<Blocks> → <CB> <Blocks>

The derivation of production rule (f) creates a subgraph $G_i^{BLKS'}$ concatenating a previously created conditional block subgraph G_i^{CB} in series with a previously created aggregate blocks subgraph G_i^{BLKS}. The associated WCET cost function is given by:

$$\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ}) \;=\; min_{r,s}\{(\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) \;+\; max_{\delta_i^m, \delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] \;+ \\ \Phi_i^{CB}(\zeta_{pred_s}, \zeta_{succ})\} \tag{26}$$

The associated preemption point function is given by:

$$\rho_i^{BLKS'}(\zeta_{pred}, \zeta_{succ}) \;= \rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) \;\cup\; \rho_i^{CB}(\zeta_{pred_s}, \zeta_{succ}) \tag{27}$$

where ζ_{succ_r}, and ζ_{pred_s} represent the values where the function $\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ})$ is minimized and valid solution combinations are subject to the following constraints:

$$(\zeta_{succ_r} \;+\; max_{\delta_i^m, \delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] \;+\; \zeta_{pred_s}) \le Q_i \tag{28}$$

$$\delta_i^m \in \rho_i^{succ}(G_i^{BLKS}, \zeta_{pred}, \zeta_{succ_r}) \tag{29}$$

$$\delta_i^n \in \rho_i^{pred}(G_i^{CB}, \zeta_{pred_s}, \zeta_{succ}) \tag{30}$$

▶ **Theorem 3.** *Given Φ_i and ρ_i functions for each substructure of $BLKS$ where each $\rho_i^A(\zeta_{pred}, \zeta_{succ})$ represents a feasible solution for substructure A given preemptions ζ_{pred} before, ζ_{succ} after, and Φ_i^A is a safe upper bound on the total WCET and preemption cost of that solution. Applying production (e) over a feasible G_i, G_i^{BLKS} and Q_i results in a feasible solution ρ_i^{BLKS} and a safe upper bound Φ_i^{BLKS} given by Equations 26, 27, and 28-30 respectively.*

Proof. The proof is by direct argument. We need to prove that our solution ensures that the task level Q_i constraint is not violated and the cost function $\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ})$ results in a safe upper bound. To prove the Q_i constraint is not violated, we must show 1) the non-preemptive execution time of the combined solutions does not exceed Q_i at each solution interface, and 2) the non-preemptive execution time of the combined solution at the new predecessor and successor interfaces does not exceed Q_i. Let $\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ_s})$ with ζ_{pred},$\zeta_{succ_s} \in [0 \dots Q_i]$ represent a safe upper bound cost solution for subgraph G_i^{BLKS} for basic block δ_i^j, with its corresponding set of selected preemption points denoted by $\rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ_s})$ be a limited preemption execution safe upper bound cost solution for basic block δ_i^j. We make an identical statement for subgraph G_i^{CB} for basic block δ_i^k, whose cost function is denoted $\Phi_i^{CB}(\zeta_{pred_u}, \zeta_{succ})$, and whose set of selected preemption points are denoted $\rho_i^{CB}(\zeta_{pred_u}, \zeta_{succ})$. Since we have a safe upper bound cost solution for each of the combined subgraphs, we can conclude that $\Phi_i^{BLKS'}(\zeta_{pred}, \zeta_{succ})$ computed in Equation 26 represents a safe upper bound cost solution for the concatenated series subgraphs $G_i^{BLKS} \;\cup\; G_i^{CB}$

starting at basic block δ_i^j, and ending at basic block δ_i^k with its corresponding selected preemption points denoted by $\rho_i^{BLKS'}(\zeta_{pred}, \zeta_{succ})$ and computed in Equation 27. Condition 1 is met in accordance with Equation 28 whose purpose is to ensure the non-preemptive execution time of the combined solutions does not exceed Q_i at each solution interface. Condition 2 is met per the definition of the parameters ζ_{pred}, and ζ_{succ} respectively, whose range is given by $[0 \dots Q_i - 1]$. Thus, the problem finds a feasible safe upper bound cost preemption points solution when applying production (f). ◀

For **production rule (j)**, we have:

<FunctionCall> $\rightarrow$ [*<Blocks>* *<FunctionName>*]

The derivation of production rule (j) creates a subgraph G_i^{FCALL} that is equivalent to the subgraph G_i^{BLKS}. The associated WCET cost function is given by:

$$\Phi_i^{FCALL}(\zeta_{pred}, \zeta_{succ}) = min_{r,s}\{(\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) + max_{\delta_i^m, \delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] + \Phi_i^{FUNC}(\zeta_{pred_s}, \zeta_{succ})\} \tag{38}$$

The associated preemption point function is given by:

$$\rho_i^{FCALL}(\zeta_{pred}, \zeta_{succ}) = \rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ_r}) \cup \rho_i^{FUNC}(\zeta_{pred_s}, \zeta_{succ}) \tag{39}$$

where ζ_{succ_r}, and ζ_{pred_s} represent the values where the function $\Phi_i^{FCALL}(\zeta_{pred}, \zeta_{succ})$ is minimized and valid solution combinations are subject to the following constraints:

$$(\zeta_{succ_r} + max_{\delta_i^m, \delta_i^n}[\xi_i(\delta_i^m, \delta_i^n)] + \zeta_{pred_s}) \leq Q_i \tag{40}$$

$$\delta_i^m \in \rho_i^{succ}(G_i^{BLKS}, \zeta_{pred}, \zeta_{succ_r}) \tag{41}$$

$$\delta_i^n \in \rho_i^{pred}(G_i^{FUNC}, \zeta_{pred_s}, \zeta_{succ}) \tag{42}$$

▶ **Theorem 4.** *Given Φ_i and ρ_i functions for each substructure of $FCALL$ where each $\rho_i^A(\zeta_{pred}, \zeta_{succ})$ represents a feasible solution for substructure A given preemptions ζ_{pred} before, ζ_{succ} after, and Φ_i^A is a safe upper bound on the total WCET and preemption cost of that solution. Applying production (e) over a feasible G_i, G_i^{FCALL} and Q_i results in a feasible solution ρ_i^{FCALL} and a safe upper bound Φ_i^{FCALL} given by Equations 38, 39, and 40-42 respectively.*

Proof. The proof is by direct argument. We need to prove that our solution ensures that the task level Q_i constraint is not violated and the cost function $\Phi_i^{FCALL}(\zeta_{pred}, \zeta_{succ})$ results in a safe upper bound. To prove the Q_i constraint is not violated, we must show 1) the non-preemptive execution time of the combined solutions does not exceed Q_i at each solution interface, and 2) the non-preemptive execution time of the combined solution at the new predecessor and successor interfaces does not exceed Q_i. Let $\Phi_i^{BLKS}(\zeta_{pred}, \zeta_{succ_s})$ with ζ_{pred},$\zeta_{succ_s} \in [0 \dots Q_i]$ represent a safe upper bound cost solution for subgraph G_i^{BLKS} for basic block δ_i^j, with its corresponding set of selected preemption points denoted by $\rho_i^{BLKS}(\zeta_{pred}, \zeta_{succ_s})$ be a limited preemption execution safe upper bound cost solution for basic block δ_i^j. We make an identical statement for subgraph G_i^{FUNC} for basic block δ_i^k, whose cost function is denoted $\Phi_i^{FUNC}(\zeta_{pred_u}, \zeta_{succ})$, and whose set of selected preemption points are denoted $\rho_i^{FUNC}(\zeta_{pred_u}, \zeta_{succ})$. Since we have a safe upper bound cost solution for each of the combined subgraphs, we can conclude that $\Phi_i^{FCALL}(\zeta_{pred}, \zeta_{succ})$ computed in Equation 38 represents a safe upper bound cost solution for the concatenated series subgraphs $G_i^{BLKS} \cup G_i^{FUNC}$ starting at basic block δ_i^j, and ending at basic block δ_i^k with its corresponding selected preemption points denoted by $\rho_i^{FCALL}(\zeta_{pred}, \zeta_{succ})$ and computed in Equation 39.

Condition 1 is met in accordance with Equation 40 whose purpose is to ensure the non-preemptive execution time of the combined solutions does not exceed Q_i at each solution interface. Condition 2 is met per the definition of the parameters ζ_{pred}, and ζ_{succ} respectively, whose range is given by $[0 \ldots Q_i - 1]$. Thus, the problem finds a feasible safe upper bound cost preemption points solution when applying production (j). ◀