Introduction to Numerical Analysis

Day 4: Interpolation

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Outline

Introduction

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Generally, in statistical or mathematical problems, we want to evaluate a function at one or more points. However, this is not so simple since some inconveniences arise such as:

- Computational time of execution (expensive)
- Evaluating complex functions
- We only have one value for a function on a finite set of points

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These "simple" functions are almost always chosen from polynomial, trigonometric, rational, etc.

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A function $f : \mathbb{R} \to \mathbb{R}$ can be effectively evaluated in any data point ?

- Power functions: $f(x) = x^n, n \in \mathbb{N}$
- Polynomial functions:

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \ldots + \beta_n x^n$$

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but, we can evaluate other functions as well?$\rightarrow$ by a polynomial function, for example: Lagrange interpolation.

Interpolation

Definition

The interpolation of a function f through a other function g, consists of, given the following points:

- $n+1$ different points $x_0, ..., x_n$
- $n+1$ values on that points, $f(x_0) = \omega_0, f(x_1) = \omega_1, ..., f(x_n) = \omega_n$,

find a function g such that $g(x_i) = \omega_i$ for $i = 0, 1, ..., n$.

The points $x_0, ..., x_n$ are called "knots" and the function g is called interpolant of f in the points $x_0, ..., x_n$.

Interpolation

We will only consider:

- Polynomial interpolation:

$$g(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n = \sum_{k=0}^{n} a_k x^k$$

- Piecewise polynomial interpolation

$$g(x) = \begin{cases} p_1(x) & \text{if } \ x \in (x_0^*, x_1^*) \\ p_2(x) & \text{if } \ x \in (x_1^*, x_2^*) \\ \dots & \\ p_m(x) & \text{if } \ x \in (x_{m-1}^*, x_m^*) \end{cases}$$

where x_0^*, x_m^* is a partition of the interval that contains the knots of the interpolation (x_0, x_n) and $p_i(x)$ are the polynomials.

Lineal interpolation

In this simple case we are going to represent two points, for example the growth of a child, the amount of sugar in a soft drink or the number of computers in a house during a year. For this type of real, making two measurements is more feasible than making continuous measurements.

Linear interpolation

Here x is where we have our measurement and the observed value is y. Thus, by a basic compute:

$$y = mx + b$$

So, to find m (slope) we do:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and the intercept $b = y_2 - m * x_2$

See example 1 in `R`

Higher-order Polynomial Interpolation

Given two data points, a line of a polynomial interpolation will always pass exactly through the two points, provided the two values of x are different. If the two values of x are the same, the slope is undefined because it is infinite and the line is vertical.

1For n data points, a polynomial of degree $n-1$ is necessary and sufficient to fit the data points.

This calculation yield in the function, $g(x)$, which is the polynomial approximation of $f(x)$, the source function for the data points.

Higher-order Polynomial Interpolation

Given a set of observations (x_i, y_i), the interpolating function $g(x)$ must meet the requirement,

$$g(x_i) = y_i, \quad \forall\, i \tag{1}$$

Thus, an interpolating function is a polynomial of the form:

$$g(x) = \beta_n x^n + \beta_{n-1} x^{n-1} + \ldots + \beta_1 x + \beta_0 \tag{2}$$

and, in matrix form:

$$\begin{bmatrix} x_1^n & x_1^{n-1} & \ldots & x_1 + 1 \\ x_2^n & x_2^{n-1} & \ldots & x_2 + 1 \\ \vdots & \vdots & & \vdots \\ x_n^n & x_n^{n-1} & \ldots & x_n + 1 \end{bmatrix} = \begin{bmatrix} \beta_n \\ \beta_{n-1} \\ \vdots \\ \beta_0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{3}$$

So, the expression in (3) can be reduced to:

$$\boldsymbol{X}\beta = \boldsymbol{y}, \tag{4}$$

where the matrix $\boldsymbol{X}$ is known as Vandermonde matrix. Solving the equation for β returns a vector of values for the coefficients of the polynomial (See example 2 in `R`).

Piecewise Interpolation

While the higher-degreed polynomial is guaranteed to pass through all of the points given, it may fluctuate wildly between two given points, a pattern known as Runge's phenomenon.

In Piecewise interpolation, we observe that at different points along a curve, a function value may be better approximated using two or more interpolations.

Piecewise Interpolation

We will want to use lower-degreed polynomials for each part of a curve because we are able to efficiently analyze those polynomials to approximately analyze the underlying data.

Piecewise Interpolation

- Order the data points such that $x_1 < x_w, \ldots < x_k$
- Define the basis

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{when } x_{i-1} < x \leq x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{when } x_i < x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

for $i = 1, \ldots, k-1$ with boundary basis $\phi_1(x)$ and $\phi_k(x)$

- Piecewise interpolation:

$$f(x) = \sum_{i=1}^{k} y_i \phi_i(x)$$

This is a continuous, but non-smooth function.

Piecewise Linear Interpolation

A piecewise interpolant is of greater value if the resulting function is continuous. A continuous function has smoother transformations from region to region, and that feels more natural (See example 3 in `R`).

Cubic spline

Cubic spline interpolation is the process of constructing a spline $f : [x_1, x_{n+1}] \rightarrow \mathbb{R}$ which consists of n polynomials of degree three, referred to as to f_1 to f_n, and a spline is a function defined by piecewise polynomials.

Cubic spline

The function has the following structure:

$$f(x) = \begin{cases} a_1x^3 + b_1x^2 + c_1x + d_1 & \text{if } x \in [x_1, x_2] \\ a_2x^3 + b_2x^2 + c_2x + d_2 & \text{if } x \in (x_2, x_3] \\ \dots & \\ a_nx^3 + b_nx^2 + c_nx + d_n & \text{if } x \in (x_n, x_{n+1}] \end{cases} . \tag{6}$$

All the polynomials only are valid within an interval; they compose the interpolation function.

See example 4 in `R`.

See you next class!...

Howard, J. P. (2017). Computational Methods for Numerical Analysis with R. CRC Press.