Introduction to Numerical Analysis

Day 2: Singular Value Decomposition (SVD)

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Outline

Singular values

Let $\boldsymbol{A}$ be an $m \times n$ matrix. If we consider $\boldsymbol{A}\boldsymbol{A}^T$ is a symmetric matrix of dimension $n \times n$, its eigenvalues are real. So,

If λ is an eigenvalue of $\boldsymbol{A}\boldsymbol{A}^T$, then $\lambda \geq 0$.

Singular values

As $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\boldsymbol{A}\boldsymbol{A}^T$, we can order these such that $\lambda_1 \geq \lambda_2 \geq, \ldots \lambda_n \geq 0$. On the other hand, doing $\sigma_i = \sqrt{\lambda_i}$, so that $\sigma_1 \geq \sigma_2 \geq, \ldots \sigma_n \geq 0$. Thus:

Definition

The numbers $\sigma_i = \sqrt{\lambda_i}$, so that $\sigma_1 \geq \sigma_2 \geq, \ldots \sigma_n, \geq 0$ defined above are called the singular values of $\boldsymbol{A}$.

SVD decomposition

Let $\boldsymbol{A}$ is a matrix of dimension $m \times n$ with singular values $\sigma_1 \geq \sigma_2 \geq, \ldots \sigma_n, \geq 0$, and r denote the number of nonzero singular values of $\boldsymbol{A}$ (the rank of $\boldsymbol{A}$).

Definition

The SVD of the matrix $\boldsymbol{A}$ can be expressed as:

$$\boldsymbol{A} = \boldsymbol{U}\Sigma\boldsymbol{V}^{\boldsymbol{T}}$$

where:

- $\boldsymbol{U}$ is an $m \times m$ orthogonal matrix.
- Σ is an $m \times n$ matrix whose i-*th* diagonal entry equals the i-*th* singular value σ_i for $i = 1, \ldots, r$. All the other entries of Σ are zero.
- $\boldsymbol{U}$ is an $n \times n$ orthogonal matrix.

Example

Consider a matrix $\boldsymbol{A}$ of dimension 3×2 and compute the SVD.

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \tag{1}$$

Step 1: Calculate AA^T

$$\boldsymbol{AA^T} = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \tag{2}$$

It is a square symmetric matrix through which we can find the eigenvalues.

Step 2: Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{A}\boldsymbol{A}^T$

You can find eigenvalues and eigenvectors by treating a matrix as a system of linear equations. For example, for the next matrix:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{3}$$

we can apply the following formula:

$$\boldsymbol{A}\vec{v} = \lambda\vec{v} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

The equation (4) is the same as:

$$
\begin{aligned}
2x_1 + x_2 &= \lambda x_1 && (5)\\
x_1 + 2x_2 &= \lambda x_2 && (6)
\end{aligned}
$$

which can be rearranged as:

$$
\begin{aligned}
(2-\lambda)x_1 + x_2 &= 0 && (7)\\
x_1 + (2-\lambda)x_2 &= 0 && (8)
\end{aligned}
$$

The linear system does not to have a nonzero vector $(x_1, x_2)^T$, that is the determinant of the coefficient matrix must be 0.

$$\begin{vmatrix} (2-\lambda) & 1 \\ 1 & (2-\lambda) \end{vmatrix} = 0$$

so,

$$\begin{aligned} (2-\lambda)(2-\lambda) - 1 * 1 =& 0 \\ \lambda^2 - 4\lambda + 3 =& 0 \\ (\lambda - 3)(\lambda - 1) =& 0 \end{aligned}$$

Two values of λ satisfy the last equation; $\lambda_1 = 3$ and $\lambda_2 = 1$.

Step 3: Find the eigenvectors

We can start with $\lambda = 3$ to find its eigenvector, so we will use the equation (5):

$$(2 - \lambda)x_1 + x_2 = 0$$

substituting we get

$$(2 - 3)x_1 + x_2 = 0$$

that is:

$$x_1 = x_2$$

Since could be an infinite values for x_1 which satisfy this equation; the only restriction that not all the components in an eigenvector can equal zero. Thus, if $x_1 = 1$, then $x_2 = 1$ and this condition is satisfied. Finally, the eigenvector corresponding to $\lambda = 3$ is $(1, 1)^T$.

Do the same for the eigenvector corespondent to $\lambda = 1$

Continuing with our example:

$$\begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

That is the same as:

$$\begin{aligned} 11x_1 + x_2 &= \lambda x_1 \\ x_1 + 11x_2 &= \lambda x_2 \end{aligned}$$

which can be rearranged as:

$$\begin{aligned} (11 - \lambda)x_1 + x_2 &= 0 \\ x_1 + (11 - \lambda)x_2 &= 0 \end{aligned}$$

Calculating the det:

$$\begin{vmatrix} (11-\lambda) & 1 \\ 1 & (11-\lambda) \end{vmatrix} = 0$$

so,

$$\begin{aligned} (11-\lambda)(11-\lambda) - 1 * 1 &= 0 \\ (\lambda - 10)(\lambda - 12) &= 0 \end{aligned}$$

Thus $\lambda = 10$ and $\lambda = 12$

Using the lambda's calculates in the original equations gives us our eigenvectors. For example, $\lambda = 10$:

$$\begin{aligned}(11-10)x_1 + x_2 =& 0\\ x_1 =& -x_2\end{aligned}$$

we can consider $x_1 = 1$ and $x_2 = -1$, so our eigenvector is $(1, -1)^T$. For $\lambda = 12$:

$$\begin{aligned}(11-12)x_1 + x_2 =& 0\\ x_1 =& x_2\end{aligned}$$

and our eigenvector is $(1, 1)^T$.

See you next class!...

Howard, J. P. (2017). Computational Methods for Numerical Analysis with R. CRC Press.