Solving linear equations

Introduction to Numerical Analysis

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Solving linear equation by Cholesky decomposition

Consider the following equations: $25x + 15y - 5z = 35, 15x + 18y + 0z = 33, -5x + 0y + 11z = 6$. Using the Cholesky decomposition solve its corresponding linear system of equations.

Solution

We can express those equations a a linear system of equations as follow:

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix} \tag{1}$$

Thus, $\boldsymbol{A} = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$

Developing the exercise "by hand".

a) First step:

$$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{ij} l_{kj}}{l_{ii}}$$

b) Second step:

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

Start with the calculus:

I) $l_{11} = \sqrt{a_{11}} = \sqrt{25} = 5$

II) $l_{21} = \frac{a_{21}}{l_{11}} = 15/5 = 3$

III) $l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{18 - (3)^2} = 3$

IV) $l_{31} = \frac{a_{31}}{l_{11}} = -5/5 = -1$

V) $l_{32} = \frac{a_{32} - l_{31}l_{21}}{l_{22}} = \frac{0-(-1)*3}{3} = \frac{0-(-3)}{3} = 1$

VI) $l_{33} = \sqrt{a_{22} - l_{31}^2 - l_{32}^2} = \sqrt{11 - (-1)^2 - (1)^2} = \sqrt{11 - 2} = 3$

$\boldsymbol{L} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and its transpose is $\boldsymbol{L}^T = \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$.

As $\boldsymbol{Ax} = \boldsymbol{b}$, and $\boldsymbol{A} = \boldsymbol{LL}^T$, then we can do $\boldsymbol{LL}^T\boldsymbol{x} = \boldsymbol{b}$.
Let $\boldsymbol{L}^T\boldsymbol{x} = \boldsymbol{y}$, thus $\boldsymbol{Ly} = \boldsymbol{b}$.

$$\begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix} \tag{2}$$

that is the same as:

$$\begin{aligned} 5y_1 \qquad\qquad &= 35 \\ 3y_1 + 3y_2 \qquad &= 33 \\ -y_1 + \ \ y_2 + 3y_3 &= 6 \end{aligned} \tag{3}$$

Now, from the first equation of (3) of the linear system:

$$5y_1 = 35 \rightarrow \frac{35}{5} = 7$$

From second equation of (3),

$$3y_1 + 3y_2 = 33 \rightarrow 21 + 3y_2 = 33 \rightarrow 3y_2 = 33 - 21 \rightarrow 3y_2y_2 = 12/3 = 4$$

From third equation of (3),

$$-y_1 + y_2 + 3y_3 = 6 \rightarrow y_3 = 9/3 = 3$$

Now, $\boldsymbol{L}^T\boldsymbol{x} = \boldsymbol{y}$

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix} \tag{4}$$

$$\begin{aligned} 5x + 3y - \quad z &= 7 \\ 3y + \quad z &= 4 \\ 3z &= 3 \end{aligned}$$

Do the same and finally we have:

$x = 1, y = 1, z = 1.$

Solving linear equation by Cholesky decomposition with `R`

```
set.seed(10)
library(gear)
# create positive definite matrix A
A = crossprod(matrix(rnorm(25^2), nrow = 25))
# create vector x and matrix b
# x can be used to check the stability of the solution
x = matrix(rnorm(25))
b = A %*% x

# standard solve
x1 = solve(A, b)
all.equal(x, x1)
```

```
## [1] TRUE
```

```
# solve using cholesky decomposition
cholA = chol(A)
x2 = solve_chol(cholA, b)
all.equal(x, x2)
```

```
## [1] TRUE
```