Introduction to Numerical Analysis

Day 4: Montecarlo method

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Outline

Introduction

History

The method was named after the casino in Monte Carlo (Monaco) for being at that time "the capital of gambling", as roulette was considered a simple way to generate random numbers. The development of the Monte Carlo method began approximately in 1944 and then its use expanded enormously with the development of the computer.

History

The initial work stems from the research on the atomic bomb (World War II) at Los Alamos, U.S.A, and was aimed at simulating probabilistic problems applied to neutron diffusion. The birth of this method is credited to John von Neumann and Stanislaw Ulam who were involved in this research.

The Monte Carlo method is a numerical technique that is used to approximate complex mathematical expressions that are computationally expensive to evaluate exactly.

Introduction

Monte Carlo method

The main idea of the method is to evaluate integrals of the form:

$$\int_{X} h(x) f(x) dx$$

where f is a density function and from which we can generate an almost infinite number of random variables.

Monte Carlo method

From the above:

- Experiment with probabilistic results.
- Applying the Law of Large Numbers
- Apply the central limit theorem

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However, in numerical analysis the process must be deterministic, so in this topic numerical integration will be considered as such.

Monte Carlo method

The main problem to develop is as follows:

$$\mathbb{E}_f[h(X)] = \int_{\mathcal{X}} h(x)f(x)dx$$

where X is taking values in $\mathcal{X}$.

The Monte Carlo method generates samples (sampling) $X_1, ..., X_n$ from the density function f and approximates the integral by the following mathematical formula:

$$\bar{h}_n = \frac{1}{n}\sum_{j=1}^{n} h(x_j).$$

Monte Carlo method

The validation of the method can be done by the Convergence:

$$\bar{h}_n = \frac{1}{n}\sum_{j=1}^{n} h(x_j) \Longrightarrow \int_{\mathcal{X}} h(x)f(x)dx = \mathbb{E}_f[h(X)]$$

It is valid according the strong law of large numbers. Besides, when $h^2(X)$ has a finite expectation under f

$$\frac{\bar{h}_n - \mathbb{E}_f[h(X)]}{\sqrt{v_n}} \to \mathcal{N}(0,1)$$

is satisfied by the Central Limit Theorem, where the expression $v_n = \frac{1}{n^2}\sum_{j=1}^{n}[h(x_j) - \bar{h}_n]^2$.

Monte Carlo method

Example:

Let $X \sim f$ be a random variable and consider a real valued function g and the corresponding random variable $g(X)$. Then, the expectation of $g(X)$ is given by,

$$\mathbb{E}\left[g(X)\right] = \int_{\mathscr{D}} g(x) f(x) dx,$$

where $\mathscr{D}$ is the domain of X. Here, the expectation of X is $\mathbb{E}\left[g(X)\right]$ and the variance is $g(x) = (x - \mathbb{E}\left[X\right])^2$.

Monte Carlo method

If $\mu = \mathbb{E}\left[g(X)\right]$ is the expected value, then the Monte Carlo algorithm to compute an approximation to μ ($\hat{\mu}_{mc}$) is:

- Simulate n samples $X_1, X_2, \ldots, X_n$ from $f(x)$
- The approximation by the Monte Carlo method is:

$$\widehat{\mu_{mc}} = \frac{1}{n}\sum_{i=1}^{n} g(X_i).$$

See the examples in `R`.

Thanks!...

Robert, C. P., Casella, G., & Casella, G. (2010). Introducing monte carlo methods with r (Vol. 18). New York: Springer.