Introduction to Numerical Analysis

Day 2: Linear algebra and matrix decomposition

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Outline

1. Vector and matrix operations
2. Gaussian elimination

Vectors

Vector and matrix operations

To start, we will assume there are two vectors of length m, the vector $\boldsymbol{u}$ and the vector $\boldsymbol{v}$, such that:

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

If we add a scalar to a vector, then the number is added to each element,

$$\boldsymbol{u} + y = \begin{bmatrix} u_1 + y \\ u_2 + y \\ \vdots \\ u_m + y \end{bmatrix}$$

Vector and matrix operations

And if we add a vector to a another vector, we create a new vector that is the element-wise sum of the two vectors. That is,

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_1 + v_2 \\ \vdots \\ u_1 + v_m \end{bmatrix}$$

Matrices are similar, but encounter somewhat different semantics than the vector because of their multiple dimensions. For example,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

Vector and matrix operations

Remember that adding $\boldsymbol{A}$ and $\boldsymbol{B}$ adds the two matrices, element-wise, if and only if the matrices are the same size, thus:

$$\boldsymbol{A}+\boldsymbol{B}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m+1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

Vector and matrix operations

See the example in `R`

Vector and matrix operations

For vectors, the effect of matrix multiplication is known as the dot product. This is mathematically equivalent to the row-by-column multiplication product for matrices when the inner dimension of both matrices is one unit.

We call this the **dot product** or the **inner product**.

Vector and matrix operations

The inner product (dot product) can only be executed on two vectors of the same length, that is:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i}^{n} u_i * v_i \tag{1}$$

The result of the dot product is a scalar (a real number in a matrix of one row and one column)

Gaussian elimination

Gaussian elimination

In the last section, we explored a suite of tools for manipulating and transforming matrices. Now we can use these tools now to solve matrix equations.

Row echelon form

The first step toward solving mathematical equations is reducing a matrix to row echelon form using Gaussian elimination. A matrix is in row echelon form if the matrix meets two conditions.

First, rows with only values of zero must be below rows with any nonzero values.

Second, the first nonzero entry of any other row must be to the right of the row above it.

Row echelon form

The above gives matrices that look something like

$$\boldsymbol{A} = \begin{bmatrix} d_1 & a_{12} & \dots & a_{1n} \\ 0 & d_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m \end{bmatrix} \qquad \text{or} \qquad \boldsymbol{B} = \begin{bmatrix} d_1 & b_{12} & b_{12} & \dots & b_{1n} \\ 0 & 0 & d_2 & \dots & b_{2n} \\ 0 & 0 & 0 & \dots & b_{3n} \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{mn} \end{bmatrix}$$

The matrix will have an increasing number of zeroed entries in the lower rows.

Row echelon form

The algorithm for Gaussian elimination captures the approach when performing Gaussian elimination by hand.

See the example developed by "hand" and the example in `R`.

See you next class!...

Howard, J. P. (2017). Computational Methods for Numerical Analysis with R. CRC Press.