Introduction to Numerical Analysis

Day 3: Solving linear equations

Joaquin Cavieres

Geoinformatics, Bayreuth University

Outline

Linear equations

Linear equations

A linear equation can be expressed as follow:

$$u_1x_1 + u_2x_2, \ldots u_nx_n = b, \tag{1}$$

where $u_1, \ldots, u_n$ are the coefficients, $x_1, \ldots, x_n$ the variables and b is the response. For the above, we express a linear system of equations as:

$$\begin{aligned} u_{11}x_1 + u_{12}x_2, \ldots, u_{1n}x_n &= b_1 \\ u_{21}x_1 + u_{22}x_2, \ldots, u_{2n}x_n &= b_2 \\ &\vdots \\ u_{m1}x_1 + u_{m2}x_2, \ldots, u_{2mn}x_n &= b_m \end{aligned}$$

The system contains m linear equations associated with n coefficients, and has n variables.

The previous system can be rewrite as:

$$\boldsymbol{Ax} = \boldsymbol{b}, \tag{2}$$

where the $\boldsymbol{A}$ is of dimension $m \times n$ and the coefficients are u_{ij}, the vector $\boldsymbol{x}$ include the elements x_i and the vector $\boldsymbol{b}$ have the elements b_i.

The LU decomposition is a simplification of the Gaussian elimination. This type of decomposition is convenient when we want to calculate the inverse of a matrix or when we want to solve a linear system of equations.

LU decomposition

Definition

A matrix $\boldsymbol{A}$ of dimension $m \times n$ can be written as

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U} \tag{3}$$

where $\boldsymbol{L}$ is a lower triangular matrix $m \times m$ and $\boldsymbol{U}$ is an upper triangular matrix $m \times n$.

LU decomposition

Thus, if we have a matrix $\boldsymbol{A}$ of dimension 3×3, it can be expressed as:

$$\boldsymbol{A} = \boldsymbol{LU} \tag{4}$$

that is the same as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \tag{5}$$

LU decomposition

The above allows us to use the $\boldsymbol{LU}$ decomposition to solve:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{6}$$

since

$$\boldsymbol{L}(\boldsymbol{U}\boldsymbol{x}) = \boldsymbol{b} \tag{7}$$

The matrix $\boldsymbol{U}$ has m rows and the vector $\boldsymbol{x}$ has m elements, thus the result of their multiplication is also a vector of m elements.

LU decomposition

We can define a temporary vector $\boldsymbol{t}$ such that

$$\boldsymbol{t} = \boldsymbol{U}\boldsymbol{x} \tag{8}$$

and substituting $\boldsymbol{t}$ for $\boldsymbol{U}\boldsymbol{x}$ we have

$$\boldsymbol{L}\boldsymbol{t} = \boldsymbol{b} \tag{9}$$

Then we can solve for $\boldsymbol{x}$ by initially solving equation 9 for $\boldsymbol{t}$, then solving equation 8 for $\boldsymbol{x}$, yielding the expected result.

LU decomposition

See the example developed by hand and the `R` code for LU decomposition.

Cholesky decomposition

Cholesky decomposition

The Cholesky decomposition of a matrix provides an expression such that $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T$ where $\boldsymbol{L}^T$ is transpose of the matrix $\boldsymbol{L}$.

Cholesky decomposition is faster than the $\boldsymbol{LU}$ decomposition, but it is more limited since the Cholesky decomposition only can be used on symmetric positive definite matrices.

Cholesky decomposition

Symmetric positive definite matrices

- Symmetric matrices are matrices that are symmetric about the main diagonal, that is:

$$\forall\, i \text{ and } j,\;\; a_{ij} = a_{ji},\;\; \text{for a matrix } \boldsymbol{A}.$$

- Positive definite means that each of the pivot entries is positive.

Besides, for a positive definite matrix, the relationship $\boldsymbol{xAx} > 0$ for all vectors, $\boldsymbol{x}$.

Cholesky decomposition

So, for a matrix $\boldsymbol{A}$ of dimension $3 times 3$, the Cholesky decomposition can be expressed as:

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T \tag{10}$$

that is the same as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{23} \\ 0 & 0 & l_{33} \end{bmatrix} \tag{11}$$

Cholesky decomposition

See the example developed by hand and the `R` code for Cholesky decomposition.

See you next class!...

Howard, J. P. (2017). Computational Methods for Numerical Analysis with R. CRC Press.