Introduction to Numerical Analysis

Day 2: Linear algebra and matrix decomposition

Joaquin Cavieres

Geoinformatics, Bayreuth University

Outline

Matrix decomposition

Matrix decomposition

A **matrix decomposition** or **matrix factorization** is a factorization of a matrix into a product of matrices and there are different matrix decompositions.

LU decomposition

LU decomposition

The LU decomposition is a simplification of the Gaussian elimination. This type of decomposition is convenient when we want to calculate the inverse of a matrix or when we want to solve a linear system of equations.

LU decomposition

Definition

A matrix $\boldsymbol{A}$ of dimension $m \times n$ can be written as

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U} \tag{1}$$

where $\boldsymbol{L}$ is a lower triangular matrix $m \times m$ and $\boldsymbol{U}$ is an upper triangular matrix $m \times n$.

LU decomposition

Thus, if we have a matrix $\boldsymbol{A}$ of dimension 3×3, it can be expressed as:

$$\boldsymbol{A} = \boldsymbol{LU} \tag{2}$$

that is the same as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \tag{3}$$

LU decomposition

See the example developed by hand and the `R` code for LU decomposition.

Cholesky decomposition

Cholesky decomposition

The Cholesky decomposition of a matrix provides an expression such that $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T$ where $\boldsymbol{L}^T$ is transpose of the matrix $\boldsymbol{L}$.

Cholesky decomposition is faster than the $\boldsymbol{LU}$ decomposition, but it is more limited since the Cholesky decomposition only can be used on symmetric positive definite matrices.

Cholesky decomposition

Symmetric positive definite matrices

- Symmetric matrices are matrices that are symmetric about the main diagonal, that is:

$$\forall\ i \ \text{ and } \ j, \ \ a_{ij} = a_{ji}, \ \text{ for a matrix } \ \boldsymbol{A}.$$

- Positive definite means that each of the pivot entries is positive.

Besides, for a positive definite matrix, the relationship $\boldsymbol{xAx} > 0$ for all vectors, $\boldsymbol{x}$.

Cholesky decomposition

So, for a matrix $\boldsymbol{A}$ of dimension $3 times 3$, the Cholesky decomposition can be expressed as:

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T \tag{4}$$

that is the same as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{23} \\ 0 & 0 & l_{33} \end{bmatrix} \tag{5}$$

Cholesky decomposition

See the example developed by hand and the `R` code for Cholesky decomposition.

See you next class!...

Howard, J. P. (2017). Computational Methods for Numerical Analysis with R. CRC Press.