Introduction to Numerical Analysis

Day 1: Vector and matrices

Joaquin Cavieres

Geoinformatics, Bayreuth University

Outline

Vectors

Vectors

Definition

A vector $\boldsymbol{x}$ of dimension m (or order m) is a column of m numbers:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \tag{1}$$

The numbers x_i are the elements of $\boldsymbol{x}$, where $i = 1, \ldots, m$. The vector $\boldsymbol{x}$ also can be named as column vector.

Vectors

The transpose of $\boldsymbol{x}$ is expressed as a row vector:

$$\boldsymbol{x}^T = (x_1, \ldots, x_m)$$

Vectors are presumed to be column vectors unless specified otherwise.

- Addition and subtraction

$$\boldsymbol{x} + \boldsymbol{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_m + y_m \end{bmatrix}$$

It is not possible to add or subtract vectors which are not **conformable**, i.e., which do not have the same dimension.

Basic operations with vectors

- Scalar multiplication

$$\lambda \boldsymbol{x} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_m \end{bmatrix}$$

This operation is done element by element for any scalar (real number) λ

Vectors

General considerations:

- The results of addition and subtraction of two vectors are vectors of the same dimension.
- The results of adding to, subtracting from or multiplying a vector by a scalar are vectors of the same dimension.
- A vector with all elements 0 is denoted by $\mathtt{0}$.
- A vector with all elements 1 is and is denoted by $\boldsymbol{I}_m$,
- $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ (commutativity)
- $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$ (associativity)
- inner product $\Rightarrow \boldsymbol{x}^T\boldsymbol{y}$
- outer product $\Rightarrow \boldsymbol{x}\boldsymbol{y}^T$

Matrices

Matrices

Definition

A matrix of dimension $m \times n$ is a rectangular "structure" of scalar numbers:

$$\boldsymbol{X} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The matrix $\boldsymbol{X}$ has m rows and n columns. Commonly, we can say that "$\boldsymbol{X}$ is a matrix $m \times n$" or "of order $m \times n$". Also we can write to $\boldsymbol{X}$ as $\boldsymbol{X} = (x_{ij})$.

Matrices

Example

$$\boldsymbol{X} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$\boldsymbol{X}$ is a matrix of 2×3.

Note:

- There is a column vector of dimension $n \times 1$.
- There is a row vector of dimension $1 \times m$.

Matrices

Characteristics 1:

- The notation x_{ij} represents the components or elements of the matrix $\boldsymbol{X}$.
- A matrix is square if $m = n$, for example, the same number of rows and columns.
- The transpose of a matrix $m \times n$ is of dimension $n \times m$ and is denoted as $\boldsymbol{X}^T$.

Matrices

Characteristics 2:

- A square matrix $\boldsymbol{X}$ is symmetric if $\boldsymbol{X}^T = \boldsymbol{X}$.
- Two matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ are equals if they have the same dimension and each pair of their elements are equals (for example, $x_{ij} = y_{ij}$), for $i = 1, ..., m$ and $j = 1, ..., n$.
- A square matrix with all elements not on the diagonal equal to 0 is a diagonal matrix, for example, if $x_{ij} = 0 \ \forall \ i \neq j$ (y $x_{ii} \neq 0$ for at least one i).

Matrices

Characteristics 3:

- A diagonal matrix with all diagonal elements equals to 1 and all others not on the diagonal 0 is called $\boldsymbol{I}_n$. For example, if $x_{ii} = 1$, $i = 1, ..., n$ and $x_{ij} = 0 \;\; \forall \;\; i \neq j$ for $i, j = 1, ..., n$, then $\boldsymbol{X} = \boldsymbol{I}_n$. It is known as identity matrix

$$\boldsymbol{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- If $\boldsymbol{X}$ is a square matrix, then the diagonal diag($\boldsymbol{X}$) is the column vector with elements of $\boldsymbol{X}$.

Matrices

Characteristics 4:

- The trace of a square matrix is the sum of the diagonal elements of the matrix. For example, trace($\boldsymbol{X} = \text{tr}(x_{ij}) = \sum_{i=1}^{n} x_{ii}$.

  Observation: The $\text{tr}(\boldsymbol{I}_n) = n$.

Basic operations with matrices

- The addition and subtraction of two matrices with the same dimension are done element by element.

$$\boldsymbol{X} + \boldsymbol{Y} = (x_{ij}) + (y_{ij}) = (x_{ij} + y_{ij})$$

It is not possible to add or subtract matrices which do not have the same dimensions.

Basic operations with matrices

The scalar multiplication of a matrix is done element by element:

$$\lambda \boldsymbol{X} = \lambda(x_{ij}) = (\lambda x_{ij})$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} = 2 \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Matrix multiplication

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices, then we can multiply $\boldsymbol{A}$ and $\boldsymbol{B}$ only if the number of columns of $\boldsymbol{A}$ are equals to the number of rows of $\boldsymbol{B}$. This is:

> $\boldsymbol{A}$ is a matrix of $m \times n$ and $\boldsymbol{B}$ is a matrix of $n \times p$, the multiplication $\boldsymbol{AB}$ can be defined (but not the multiplication of $\boldsymbol{BA}$)

The result should be a matrix of $m \times p$ (the row number of the first matrix ($\boldsymbol{A}$) and the column number of the second matrix ($\boldsymbol{B}$))

Matrix multiplication

The element $(i, k) - th$ of $\boldsymbol{AB}$ is obtained by summing the products of the elements $i - th$ row of $\boldsymbol{A}$ with the elements of the $k - th$ columns of $\boldsymbol{B}$.

$$\boldsymbol{AB} = (\sum_{j=1}^{n} a_{ij} b_{jk})$$

Matrix multiplication

- If $\boldsymbol{C}$ is a matrix of dimension $m \times n$ and $\boldsymbol{D}$ of $p \times q$, then the product $\boldsymbol{CD}$ only can be defined if $n = p$, in that case the matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ are conformables. If $\boldsymbol{CD}$ is not defined then the matrices are non conformable.

- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors of dimension m and n respectively ($m \times 1$ for $\boldsymbol{x}$ and $n \times 1$ for $\boldsymbol{y}$), then $\boldsymbol{x}$ and $\boldsymbol{y}^T$ are conformable if the product $\boldsymbol{xy}^T$ is defined and is a matrix $m \times n$ with (i, j) elements $x_i y_j$, for $i = 1, ..., m$ y $j = 1, ..., n$. This is called outer product of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Example 1

If $\boldsymbol{W} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{Z} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, then $\boldsymbol{W}$ is of dimension 2×2 and $\boldsymbol{Z}$ is of dimension 2×2. This means that $\boldsymbol{WZ}$ is $2 \times 2 \times 2 \times 2 \times \equiv 2 \times 2$ and $\boldsymbol{ZW}$ is $2 \times 2 \times 2 \times 2 \times \equiv 2 \times 2$.

$$\boldsymbol{WZ} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1*5+2*7 & 1*6+2*8 \\ 3*5+4*7 & 3*6+4*8 \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

and

$$\boldsymbol{ZW} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Thus $\boldsymbol{WZ} \neq \boldsymbol{ZW}$.

Example 2

If $\boldsymbol{U} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $\boldsymbol{V} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $\boldsymbol{U}$ is of dimension 2×3 and $\boldsymbol{V}$ is of dimension 3×2. That means $\boldsymbol{UV}$ is $2 \times 3 \times 3 \times 2 \times \equiv 2 \times 2$ and $\boldsymbol{VU}$ is $3 \times 2 \times 2 \times 3 \times \equiv 3 \times 3$.

$$\boldsymbol{UV} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1*1+2*3+3*5 & 1*2+2*4+3*6 \\ 4*1+5*3+6*5 & 4*2+5*4+6*6 \end{bmatrix} = \begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}$$

and

$$\boldsymbol{VU} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1*1+2*4 & 1*2+2*5 & 1*3+2*6 \\ 3*1+4*4 & 3*2+4*5 & 3*3+4*6 \\ 5*1+6*4 & 5*2+6*5 & 5*3+6*6 \end{bmatrix} = \begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{bmatrix}$$

Thus $\boldsymbol{UV} \neq \boldsymbol{VU}$.

Example 3

If $\boldsymbol{W} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{V} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $\boldsymbol{W}$ is of dimension 2×2 and $\boldsymbol{V}$ is of dimension 3×2.

- ¿$\boldsymbol{WV}$?

Example 3

If $\boldsymbol{W} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{V} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $\boldsymbol{W}$ is of dimension 2×2 and $\boldsymbol{V}$ is of dimension 3×2.

- ¿$\boldsymbol{WV}$?
- ¿$\boldsymbol{W}$ multiplied by $\boldsymbol{V}^T$?

Example 3

If $\boldsymbol{W} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{V} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $\boldsymbol{W}$ is of dimension 2×2 and $\boldsymbol{V}$ is of dimension 3×2.

- ¿$\boldsymbol{WV}$?
- ¿$\boldsymbol{W}$ multiplied by $\boldsymbol{V}^T$?

Solution

- It is not possible such as the number of columns of $\boldsymbol{W}$ is different to the number of rows of $\boldsymbol{V}$.

Example 3

If $\boldsymbol{W} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\boldsymbol{V} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, then $\boldsymbol{W}$ is of dimension 2×2 and $\boldsymbol{V}$ is of dimension 3×2.

- ¿$\boldsymbol{WV}$?
- ¿$\boldsymbol{W}$ multiplied by $\boldsymbol{V}^T$?

Solution

- It is not possible such as the number of columns of $\boldsymbol{W}$ is different to the number of rows of $\boldsymbol{V}$.
- That means that $\boldsymbol{WV}$ is $2 \times 2 \times 2 \times 3 \equiv 2 \times 3$. Thus, it is possible to do the multiplication.

Practical exercises in R

Build the following matrices in R:

$$
\begin{aligned}
\mathtt{W} &= \mathtt{matrix}(\mathtt{c}(1,2,3,4),2,2,\mathtt{byrow}=\mathtt{T})\\
\mathtt{Z} &= \mathtt{matrix}(\mathtt{c}(5,6,7,8),2,2,\mathtt{byrow}=\mathtt{T})\\
\mathtt{A} &= \mathtt{matrix}(\mathtt{c}(2,2,3,5),2,2,\mathtt{byrow}=\mathtt{T})\\
\mathtt{U} &= \mathtt{matrix}(\mathtt{c}(1,2,3,4,5,6),2,3,\mathtt{byrow}=\mathtt{T})\\
\mathtt{V} &= \mathtt{matrix}(\mathtt{c}(1,2,3,4,5,6),3,2,\mathtt{byrow}=\mathtt{T})
\end{aligned}
$$

Exercise in R

- Build two matrices, $\boldsymbol{A}_{m\times n}$ and $\boldsymbol{B}_{p\times m}$, where $m = 3$, $n = 3$ y $p = 3$. Invent the values and propose an expression of multiplication in R.
- Do the same with $p = 2$, ¿What is the result?

See you next class!...

Howard, J. P. (2017). Computational Methods for Numerical Analysis with R. CRC Press.