HW 7

2. Proofs:

a. Let A be an orthogonal matrix $(A^{-1}) = (A^T)$

$\det(A) = \det(A^T)$

$A(A^{-1}) = I$

$\det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = \cancel{\det(A^T)\det(A^{-1})}$

$\det(I) = \det(A^T)\det(A^{-1}) = \det(A^T)^2$

$\det(I) = 1$

$\det(I) = \det(A^T)^2 = 1$

$\det(A^T) = \boxed{\pm\sqrt{1} = \det(A)}$

QED.

b. Let A be an $n \times n$ square matrix. We will prove that $|\det(A)| = \prod_{i=1}^{n} \sigma_i$

$A = U\Sigma V^T$, U is $n \times r$, Σ is $r \times r$, V is $r \times n$

$\det(A) = \det(U\Sigma V^T) = \det(U)\det(\Sigma)\det(V^T)$

We proved in part a that the determinant of an orthogonal matrix equals ± 1

U and V are orthogonal matrices by the definition of SVD, hence:

$\det(A) = \pm 1 \cdot \det(\Sigma) \cdot \pm 1$

Given from 132 notes, ~~the~~ the determinant of ~~an orthogonal~~ a diagonal matrix is the product of the diagonal values.

So, $\det(\Sigma) = \prod_{i=1}^{n} \Sigma_{i,i}$. Since Σ is the singular matrix, this equals $\prod_{i=1}^{n} \sigma_i$

Inserting this, we get:

$\det(A) = \pm 1 \cdot \prod_{i=1}^{n} \sigma_i \cdot \pm 1$,

So $|\det(A)| = \left|\pm 1 \prod_{i=1}^{n} \sigma_i \cdot \pm 1\right| = \left|\prod_{i=1}^{n} \sigma_i\right|$

QED.

HW 8

3. SVD

$A = U\Sigma V^T$

a) ~~Show that $A^T u_j = \sigma_j v_j$, $\forall j$ $1 \le j \le$ rank(A)~~

~~$V = v_1 ... v_j$ (orthonormal basis for A, proved in 132, L25)~~

~~So for $(Av_i)^T(Av_j) = 0$ where $i \neq j$~~

~~$A = U\Sigma V^T$, $V^T = V^{-1}$~~

~~$AV = U\Sigma$ $A = n \times n$, $V = n \times r$, $U = r \times n$, $\Sigma = r \times r$,~~ where r is the matrix rank A

~~So for all columns of AV 1, r,~~

b.) Given $A = U\Sigma V^T$

U and V are orthogonal, so $U^{-1} = U^T$ and $V^{-1} = V^T$

$$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & & 0 \\ & 1/\sigma_2 & \\ 0 & & \ddots \end{bmatrix}$$

$$A^{-1} = V\Sigma^{-1}U^T$$

c)

1. Let $A^{n \times m}$ be a real matrix (given)
2. Let $P \in \mathbb{R}^{n \times n}$ be an orthogonal matrix (given)
3. $P^T P = I$ (definition of orthogonal matrix)
4. For $A = U\Sigma V^T$, $\Sigma^{r \times r} =$ eigenvalues of A down the diagonal
   $U =$ eigenvectors of A, normalized
   $V =$ eigenvectors of A
5. eig (A) = eigenvalue decomposition of $A^T A$

~~6. eig (AP) = eigenvalue decomposition of $(AP)^T AP$~~

~~7. $(AP)^T AP$~~

6. $(PA)^T PA = A^T P^T P A$ (def. transpose)
7. $A^T P^T P A = A^T I A = A^T A$ (product of $P^T P$ when P is orthogonal)
8. eig (PA) = eig. decomposition of $(PA)^T PA =$ " , of $A^T A$
9. eig (A) = eig (PA), (7, 8)

HW 8

3. SVD

a. $A = U\Sigma V^T$

1. $A = U\Sigma V^T$ (given)
2. $AV = U\Sigma$ (multiply both sides by V, ($V^T = V^{-1}$ by orthogonal matrix definition))
3. $Av_i = U_i \sigma_i$ (decomposition of matrices into columns)
4. $u_j = Av_i / \|Av_i\| = \frac{1}{\sigma_i} Av_i$ (definition from 132 notes)
5. ~~$A^T = V\Sigma^T U^T$~~ $A^T = (U\Sigma V^T)^T$
6. $A^T = V\Sigma^T U^T$ (definition of transpose)
7. $A^T = V\Sigma U^{-1}$ (definition of orthogonal matrix)
8. $A^T U = V\Sigma$ (Multiply both sides by U)
9. $A^T u_j = V_j \sigma_j$ (decomposition of matrices into columns)
10. $A^T u_j$ is $m \times n$ times $n \times n$, $V_j \sigma_j$ is $n \times r \times r \times r$, so the range of j is r
11. $r = \text{rank}(A)$ (definition of singular value dimensions.
12. Thus for all j in range 0 through rank(A), $A^T u_j = V_j \sigma_j$

QED.

3c. (continued)

10. The singular values of a matrix are the square roots of the eigenvalues of that matrix

11. singular values for A = Singular values for PA because eig(A) = eig(PA), the values are the same so the square roots will be too,