Note:

1. The problems are from the textbook by G. H. Givens and J. A. Hoeting
2. There are total 2 questions: 7.1 and 7.2.
3. The full grade is 10 in this HW with point distributed as

7.1 (a)	1
(b)	2
(c)	1
(d)	1
(e)	1
7.2 (a)	2
(b)	2

7.1. The goal of this problem is to investigate the role of the proposal distribution in a Metropolis–Hastings algorithm designed to simulate from the posterior distribution of a parameter δ. In part (a), you are asked to simulate data from a distribution with δ known. For parts (b)–(d), assume δ is unknown with a Unif(0,1) prior distribution for δ. For parts (b)–(d), provide an appropriate plot and a table summarizing the output of the algorithm. To facilitate comparisons, use the same number of iterations, random seed, starting values, and burn-in period for all implementations of the algorithm.

a. Simulate 200 realizations from the mixture distribution in Equation (7.6) with $\delta = 0.7$. Draw a histogram of these data.

b. Implement an independence chain MCMC procedure to simulate from the posterior distribution of δ, using your data from part (a).

c. Implement a random walk chain with $\delta^* = \delta^{(t)} + \epsilon$ with $\epsilon \sim$Unif$(-1,1)$.

d. Reparameterize the problem letting $U = \log\{\delta/(1-\delta)\}$ and $U^* = u^{(t)} + \epsilon$. Implement a random walk chain in U-space as in Equation (7.8).

e. Compare the estimates and convergence behavior of the three algorithms.

7.2. Simulating from the mixture distribution in Equation (7.6) is straightforward [see part (a) of Problem 7.1]. However, using the Metropolis–Hastings algorithm to simulate realizations from this distribution is useful for exploring the role of the proposal distribution.

a. Implement a Metropolis–Hastings algorithm to simulate from Equation (7.6) with $\delta = 0.7$, using $N(x^{(t)}, 0.01^2)$ as the proposal distribution. For each of three starting values, $x^{(0)} = 0, 7$, and 15, run the chain for 10,000 iterations. Plot the sample path of the output from each chain. If only one of the sample paths was available, what would you conclude about the chain? For each of the simulations, create a histogram of the realizations with the true density superimposed on the histogram. Based on your output from all three chains, what can you say about the behavior of the chain?

b. Now change the proposal distribution to improve the convergence properties of the chain. Using the new proposal distribution, repeat part (a).