Homework 5: MC Integration

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Problem 6.7

Consider pricing a European call option on an underlying stock withcurrent price $S^{(0)} = 50$, strike price $K = 52$, and volatility $\sigma = 0.5$. Suppose that there are $N = 30$ days to maturity and that the risk-free rate of return is $r = 0.05$.

a. Confirm that the fair price for this option is 2.10 when the payoff is based on $S^{(30)}$ [i.e. a standard option with payoff as in (6.74)]

$$\begin{aligned}
E[C] &= exp(-\frac{rT}{365})E[max(0, S^{(0)} - K)] \\
&= exp(-\frac{rT}{365})S^{(0)}exp((r - \frac{\sigma^2}{2})\frac{T}{365}) \\
&((1 - \Phi(c - \sqrt{\frac{T}{365}}))exp(\frac{\sigma^2 T}{365 \times 2}) - (1 - \Phi(c))\frac{K}{S^{(0)}}exp(-(\frac{r - \sigma^2}{2}\frac{T}{365})) \\
&\approx 2.101198
\end{aligned} \tag{1}$$

```r
S0    <- 50
K     <- 52
sigma <- 0.5
N     <- 30
r     <- 0.05
T     <- 30
n <- 1000
m <- 100

estimatedMu <- NULL
for(j in 1:m){
  ST <- S0exp((r-(sigma^2)/2)T/365 + sigmarnorm(n)sqrt(T/365))
  C  <- NULL
  for(i in 1:n){
    C[i] <- exp(-rT/365)max(c(0,ST[i] - K))
  }
  estimatedMu[j] <- mean(C)
}
```

The fair price estimated by MC is `2.106494`, which is pretty close to `2.10`.

b. Consider the analogous Asian option (same $S^{(0)}$, K, σ, N, and r) with payoff based on the arithmetic mean stock price during the holding period, as in (6.77). Using simple Monte Carlo, estimate the fair price for this option.

```
estimatedMu <- NULL
for(j in 1:m){
  #calculate MC estimate of A and theta
  A <- NULL
  for(i in 1:n){
    ST    <- NULL
    ST[1] <- S0
    for(k in 2:T){
      ST[k] <- ST[k-1]*exp(((r-(sigma^2)/2)/365) +
                              sigma*rnorm(1)/sqrt(365))
    }
    A[i] <- exp(-rT/365)max(c(0, mean(ST) - K))
  }
  estimatedMu[j] <- mean(A)
}
```

The estimated fair price is `0.8562979` after average. And its variance is `0.003393549`

c. Improve upon the estimate in (b) using the control variate strategy described in Example 6.13

```
estimatedMuMC  <- NULL
estimatedTheta <- NULL
for(j in 1:m){
  # calculate MC estimate of A and theta
  A     <- NULL
  theta <- NULL
  for(i in 1:n){
    ST    <- NULL
    ST[1] <- S0
    for(k in 2:T){
      ST[k] <- ST[k-1]*exp(((r-(sigma^2)/2)/365) +
                              sigma*rnorm(1)/sqrt(365))
    }
    A[i]     <- exp(-r*T/365)*max(c(0, mean(ST) - K))
    theta[i] <- exp(-r*T/365)*max(c(0,exp(mean(log(ST))) - K))
  }
  estimatedMuMC[j]  <- mean(A)
  estimatedTheta[j] <- mean(theta)
```

```
}

# Analytic solution for theta
N  <- T
c3 <- 1 + 1/N
c2 <- sigma*((c3*T/1095)*(1 + 1/(2*N)))^.5
c1 <- (1/c2)*(log(S0/K) + (c3*T/730)*(r - (sigma^2)/2) +
              (c3*(sigma^2)*T/1095)*(1 + 1/(2*N)))
theta <- S0*pnorm(c1)*exp(-T*(r + c3*(sigma^2)/6)*(1 - 1/N)/730) -
         K*pnorm(c1-c2)*exp(-r*T/365)

# Control variate
estimatedMuCV <- estimatedMuMC - 1 * (estimatedTheta-theta)
```

The estimated fair price is `0.9440792` and its variance is `7.143709e-06`, which is way much better than the result in (b).

d. Try an antithetic approach to estimate the fair price for the option described in part (b).

```
estimatedMu1 <- NULL
estimatedMu2 <- NULL
for(j in 1:m){
  # calculate MC estimate of A and theta
  A1 <- NULL
  A2 <- NULL
  for(i in 1:n/2){
    ST1    <- NULL
    ST2    <- NULL
    ST1[1] <- S0
    ST2[1] <- S0
    for(k in 2:T){
      sample <- rnorm(1)
      ST1[k] <- ST1[k-1]*exp(((r-(sigma^2)/2)/365) +
                                 sigma*sample/sqrt(365))
      ST2[k] <- ST2[k-1]*exp(((r-(sigma^2)/2)/365) +
                                 sigma*(-sample)/sqrt(365))
    }
    A1[i] <- exp(-r*T/365)*max(c(0, mean(ST1) - K))
    A2[i] <- exp(-r*T/365)*max(c(0, mean(ST2) - K))
  }
  estimatedMu1[j] <- mean(A1)
  estimatedMu2[j] <- mean(A2)
}

estimatedMuA <- (estimatedMu1 + estimatedMu2)/2
```

The estimated fair price is `0.8709188` and its variance is `0.002547244`, which is slightly better than the result in (b), but worse than (c).

e. Compare the sampling distributions of the estimators in (b), (c), (d).

Please see the details in (b), (c), (d).

Problem 6.8

Consider the model given by $X\ Lognormal(0, 1)$ and $logY = 9 + logX + \epsilon$, where $\epsilon\ N(0, 1)$. We wish to estimate $E\{Y/X\}$. Compare the performance of the standard Monte Carlo estimator and the Rao-Blackwellized estimator.

```
n  <- 1000000
x  <- exp(rnorm(n, 0, 1))
y1 <- exp(9+3*log(x)+rnorm(n, 0, 1))
z1 <- y1/x
mean(z1)
var(z1)

# since E[Y/X | X] = E[exp(9+log(X))*exp(Normal(0, 1)/ X] = exp(9+log(X))*exp(1/2)/X

x  <- exp(rnorm(n, 0, 1))
y2 <- exp(9+3*log(x))*exp(1/2)/x
y2 <- exp(9)*x^2*exp(1/2)
z2 <- y2
mean(z2)
var(z2)

# var(z2) < var(z1)
```

The estimated values are `98972.44` and `97806.38` respectively. However, the variance of Rao-Blackwellized estimator (`399722278386`) is smaller than the Monte Carlo one (`2.186901e+12`).