Math 118A Theorems and Definitions

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Chapter 1: Topology in Metric Space

Definition: Let X be a set, a metric $d : X \times X \to \mathbb{R}$ is a map satisfying the following properties:

1. Non-negativity: $d(x, y) > 0$ if $x, y \in X$ and $x \neq y$. $d(x, x) = 0$ for all $x \in X$.
2. Symmetry: $d(x, y) = d(y, x)$.
3. Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then, (X, d) is a metric space.

Example:

1. Euclidean metric space: Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$.
   - $d(x, y) > 0$ for all $x, y \in \mathbb{R}$ and $d(x, x) = |x - x| = 0$
   - $d(x, y) = |x - y| = |y - x| = d(y, x)$
   - $d(x, z) = |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$
2. Let X be an infinite set. Define $d : X \times X \to \mathbb{R}$ by $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Then (X, d) is a metric space.

Definition: Let (X, d) be a metric space, let $a \in X$ and $r > 0$. Define the ball centered at a with radius $r > 0$ to be

$$B(a, r) = \{x \in X : d(x, a) < r\}$$

$B(a, r)$ is also said to be a neighborhood of a.

Example: $X = \mathbb{R}$ and $d(x, y) = |x - y|$, so

$$\begin{aligned} B(a, r) &= \{x \in \mathbb{R}, |x - a| < r\} \\ &= \{x \in \mathbb{R}, -r \leq x - a \leq r\} \\ &= \{x \in \mathbb{R}, -r + a \leq x \leq r + a\} \\ &= (a - r, a + r) \end{aligned}$$

Definition: Let (X, d) be a metric space,

1. Set $O \subseteq X$ is said to be open if for every $a \in O$, there is a neighborhood $B(a, r)$ of a such that $B(a, r) \subseteq O$.

2. $F \subseteq X$ is said to be closed if $F^{\complement}$ is open.

3. A set $E \subseteq X$ is said to be bounded if there is $a \in E$ and $M > 0$ *s.t.* $d(x,a) \leq M$ for all $x \in E$.

4. A point $a \in X$ is said to be a limit point of E if every neighborhood of a intersects E in a point different from a.

5. Let $E \subseteq X$, define E' to be the set of limit points of E, then the closure of E is

$$\overline{E} = E \cup E'$$

Example: $X = \mathbb{R}$ and $d(x,y) = |x-y|$, so

$$\begin{aligned} B(a,r) &= \{x \in \mathbb{R}, |x-a| < r\} \\ &= \{x \in \mathbb{R}, -r \leq x-a \leq r\} \\ &= \{x \in \mathbb{R}, -r+a \leq x \leq r+a\} \\ &= (a-r, a+r) \end{aligned}$$

Theorem: Let (X,d) be a metric space,

1. Let $\mathcal{F}$ be a collection of open subsets of X, then $\bigcup_{E\in\mathcal{F}}$ is open.

2. Let $E_1, E_2, \ldots, E_N$ be open sets in X. Then $E_1 \cap E_2 \cap \cdots \cap E_N$ is open.

3. Let $\mathcal{F}$ be a collection of closed sets in X, then $\bigcap_{E\in\mathcal{F}}$ is closed.

4. Let $E_1, E_2, \ldots, E_N$ be closed sets in X. Then $E_1 \cup E_2 \cup \cdots \cup E_N$ is closed.

Theorem: Let (X,d) be a metric space, let $a \in X$ and $r > 0$,

1. $B(a,r)$ is open.

2. $\overline{B(a,r)} \subseteq \{x \in X : d(a,r) \leq r\}$.

Theorem: Let (X,d) be a metric space, let $E \subseteq X$,

1. $\overline{E}$ is closed.

2. E is closed $\Longleftrightarrow E = \overline{E}$

3. $\overline{E}$ is the smallest set containing E, in other words, if F is a closed set, *s.t.* $E \subseteq F$, then $\overline{E} \subseteq F$.

Definition: Let (X,d) be a metric space, let $E \subseteq X$, a family S of open sets is said to be an open cover of E if $E \subseteq \bigcup_{B\in S} B$. A subcover of E is a subfamily of S that is also an open cover of E. E is said to be compact if every open cover of E has a finite subcover.

Example:

$$(\mathbb{R}, |x-y|)$$

- The interval $(0,1)$ is not compact. Look at:

$$S = \left\{ \left(\frac{1}{n}, 1\right) : n \in \mathbb{N} \right\}$$

Note that S is an open cover of $(0,1)$ but it does not have a finite subcover, we cannot have finite

$$\left(\frac{1}{n_1},1\right)\cup\left(\frac{1}{n_2},1\right)\cup\cdots\cup\left(\frac{1}{n_k},1\right)$$

because eventually there will be some part of $(0,1)$ that cannot be covered.

Heine-Borel Theorem: Let $X=\mathbb{R}^d$ and $d(x,y)=|x-y|$. Let $E\subseteq X$, then E is compact if and only if E is closed and bounded.

Theorem: Let (X,d) be a metric space, if E is a compact subset of X, then E is closed and bounded.

Chapter 2: Sequences and Series

0.1 Sequences in Metric Space

Definition: Let (X,d) be a metric space, let (a_n) be a sequence in X. We say that (a_n) converges to $a\in X$, if for all $\varepsilon>0$, there exists $N\in\mathbb{N}$ $s.t.$

$$n\geq N\Longrightarrow d(a_n,a)<\varepsilon.$$

Theorem: Let (X,d) be a metric space, let (a_n) be a sequence in X.

1. Let $a\in X$, then (a_n) converges to $a\iff$ every neighborhood of a contains all but finitely many terms of a.
2. If $a_1,a_2\in X$ and (a_n) converges to a_1 and a_2, then $a_1=a_2$. [Show that $d(a_1,a_2)=0\iff d(a_1,a_2)<\varepsilon$ for any $\varepsilon>0$.]
3. If (a_n) converges to a, then (a_n) is bounded.
4. If $E\subseteq X$ and a is a limit point of E, then there is a sequence (a_n) in E $s.t.$ $a_n\xrightarrow{n\to\infty}a$.

Theorem: Let (a_n) and (b_n) be sequences of real numbers converge to $a\in\mathbb{R}$ and $b\in\mathbb{R}$ respectively.

1. (a_n+b_n) converges to $a+b$.
2. (a_nb_n) converges to ab.
3. $\left(\frac{1}{a_n}\right)$ converges to $\frac{1}{a}$ provided that $a_n\neq 0$ for all $n\in\mathbb{N}$ and $a\neq 0$.

Theorem: Let (a_n) be a sequences in $\mathbb{R}$ $s.t.$ (a_n) is increasing and bounded from above, then (a_n) converges to

$$a=\sup_{n\in\mathbb{N}}a_n\in\mathbb{R}.$$

Corollary: Let (a_n) be a sequences in $\mathbb{R}$ $s.t.$ (a_n) is decreasing and bounded from below, then (a_n) converges to

$$a=\inf_{n\in\mathbb{N}}a_n\in\mathbb{R}.$$

0.2 Subsequences in Metric Space

Definition: Let (X, d) be a metric space, let $(a_n) \subseteq X$. Consider a sequence of positive integers *s.t.* $n_1 < n_2 < \ldots$. Then (a_{n_k}) is called a subsequence of (a_n). If (a_{n_k}) converges, its limit is called a subsequential limit

$$\begin{array}{llllll} (a_n): a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots & & & & & \\ (a_{2n}): & a_2, & a_4, & a_6, & a_8, \ldots & n_k \geq k \\ & a_{n_1}, & a_{n_2}, & a_{n_3}, & a_{n_4}, \ldots & \end{array}$$

Theorem: Let (X, d) be metric space, let $(a_n) \subseteq X$, then $a_n \xrightarrow{n\to\infty} a \in X \iff$ every subsequence of (a_n) converge to a.

Theorem: Let (X, d) be a compact metric space, let $(a_n) \subseteq X$. Then (a_n) has a subsequence that converges to some $a \in X$.

Corollary (Bolzano - Weierstrass): Every bounded sequence in $\mathbb{R}^d$ has a convergent subsequence.

Theorem: Let (X, d) be metric space, let $(a_n) \subseteq X$. Denote by S the set of subsequential limits of (a_n), then S is closed.

Definition: Let $(a_n) \subseteq \mathbb{R}$, we say that (a_n) diverges to ∞ if for all $M > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \Longrightarrow a_n > M$$

and diverges to $-\infty$ if for all $M < 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \Longrightarrow a_n < M$$

Definition: Let $(a_n) \subseteq \mathbb{R}$, let S be the set of subsequential limits. Define $\limsup a_n$ to be the supremum of S and $\liminf a_n$ to be the infimum of S.

Theorem: Let $(a_n) \subseteq \mathbb{R}$, let S be the set of subsequential limits. Then

1. $\limsup a_n \in S$ and $\liminf a_n \in S$.
2. (a_n) converges to $a \in \mathbb{R}$ if and only if $S = \{a\}$.

Lemma: Let $(a_n) \subseteq \mathbb{R}$,

1. If (a_n) is bounded, (a_n) has a subsequence converging to $a \in \mathbb{R}$. (BW)
2. If (a_n) is unbounded above, (a_n) has a subsequence diverging to ∞.
3. If (a_n) is unbounded below, (a_n) has a subsequence diverging to $-\infty$.

0.3 Cauchy Sequences in Metric Spaces

Definition: Let (X, d) be a metric space, and $(a_n) \subseteq X$, (a_n) is said to be Cauchy if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ *s.t.* $m, n \geq N \Longrightarrow d(a_m, a_n) < \varepsilon$.

Theorem: Let (X, d) be a metric space, and $(a_n) \subseteq X$.

1. If (a_n) converges, then (a_n) is Cauchy.

2. If X is compact and (a_n) is Cauchy, then (a_n) converges to some $a \in X$.

3. If $X \in \mathbb{R}^n$, then (a_n) converges $\iff (a_n)$ is Cauchy.

Definition: A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to some $a \in X$.

Example: Let $X = (0, 1)$ and $d(x, y) = |x - y|$. Note that $\frac{1}{n}$ is Cauchy but does not converge to an element in X. It converges to 0, but $0 \notin X$.

Theorem: Let (X, d) be a complete metric space, and E is a closed subset of X, then (E, d) is complete.

0.4 Series

Definition: Let (a_n) be a sequence in $\mathbb{R}$. Define the sequence of partial sums $S_n = \sum\limits_{k=1}^{n} a_k$, if $S_n \to S$, we say that the series converges and we write $S = \sum\limits_{n=1}^{\infty} a_n$.

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots
\end{aligned}$$

Lemma: Let $(a_n) \subseteq \mathbb{R}$, then (a_n) is Cauchy $\iff \exists N_0 \in \mathbb{N}$ such that $m, n \geq N_0 \implies |a_m - a_{n-1}| \leq \varepsilon$.

Cauchy Criterion: Let $(a_n) \subseteq \mathbb{R}$, then $\sum\limits_{n=1}^{\infty} a_n$ converges $\iff$ for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $m \geq n \geq N \implies \left|\sum\limits_{k=n}^{\infty} a_k\right| < \varepsilon$

Corollary: Let $(a_n) \subseteq \mathbb{R}$, we say that $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely if $\left|\sum\limits_{n=1}^{\infty} a_n\right|$ converges. If $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

Theorem: Let $(a_n) \subseteq \mathbb{R}$, if $\sum\limits_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Comparison Test: Let $(a_n) \in \mathbb{R}$,

1. Suppose $|a_n| \leq c_n$ for all $n \in \mathbb{N}$, and $\sum\limits_{n=1}^{\infty} c_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

2. Suppose $a_n \geq d_n \geq 0$ for all $n \in \mathbb{N}$, and $\sum\limits_{n=1}^{\infty} d_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Theorem:

1. If $0 \leq x \leq 1$, then $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$, if $x \geq 1$, the series diverges.

2. If $p > 1$, $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges, if $p \leq 1$, $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Lemma: Let $(a_n) \subseteq \mathbb{R}$, let $S = \limsup a_n$, let $\alpha > S$, then there exists $N \in \mathbb{N}$ such that $n \geq N \Longrightarrow a_n \leq \alpha$.

Root Test: Let $(a_n) \in \mathbb{R}$, let $\alpha = \limsup |a_n|^{\frac{1}{n}}$,

1. If $\alpha < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

2. If $\alpha > 1$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

3. When $\alpha = 1$, the result is inconclusive.

Ratio Test: Let $(a_n) \in \mathbb{R}$, let $\beta = \limsup \frac{|a_{n+1}|}{|a_n|}$,

1. If $\beta < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

2. If $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for all integers n, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Definition: Let $(a_n) \subseteq \mathbb{R}$, the series $\sum\limits_{n=0}^{\infty} a_n x^n$ is a power series, it is a function.

Theorem: Let $(a_n) \subseteq \mathbb{R}$, let $R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$, then $\sum\limits_{n=0}^{\infty} a_n x^n$ converges if $|x| < R$ and diverges if $|x| > R$.

Chapter 3: Continuity

01. Continuity in Metric Spaces

Definition: Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subset X$. A function $f : E \to Y$ is said to be continuous at (fixed) $x_0 \in E$ if for each $\varepsilon > 0$, there is $\delta > 0$ such that $\underset{d_X(x,x_0)<\delta}{x} \in E \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon$.

Theorem: Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subseteq X$, let function $f : E \to Y$ be continuous at $a \in E$. Then for every sequence $(a_n) \subseteq E$ converging to $a \in E$, we have $\lim\limits_{n\to\infty} d_Y(f(a_n), f(a)) = 0$.

Theorem: Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subseteq X$. A function $f : E \to Y$ is continuous on $E \Longleftrightarrow f^{-1}(O)$ is open in E for every open set O in Y.

Theorem: Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subseteq X$, let $f : E \to Y$ be a continuous function. If E is compact, then $f(E)$ is compact.

Corollary: Let (X, d) be a metric space, let $E \subseteq X$ be an nonempty compact subset of X. Let $f : E \to \mathbb{R}$ be continuous. Then f attains its max and min of E.

02. Uniform Continuity in Metric Spaces

Definition: Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subset X$, a function $f : E \to Y$ is said to be uniformly continuous on E if for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$\underset{x,y \,\in\, E}{d_X(x, y)} < \delta \Longrightarrow d_Y(f(x), f(y)) < \varepsilon.$$

(for every $x, y \in E$).

Theorem: Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subset X$ and $f : E \to Y$ be uniformly continuous. If (a_n) is Cauchy in E, then $(f(a_n))$ is Cauchy in Y. It is useful to prove a function isn't continuous.

Theorem: Let (X, d_X) and (Y, d_Y) be metric spaces, let $E \subset X$ be compact. Let $f : E \to Y$ be a function. Then, f is continuous on $E \Longleftrightarrow f$ is uniformly continuous on E.

03. Continuity and Connectedness

Definition: Let (X, d) be a metric space, let $A, B \subset X$. A and B are said to be separated if $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$. A set E is said to be connected if E is not the union of 2 non-empty separated sets.

Theorem: Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : x \to Y$ be continuous. If $E \subseteq X$ is connected, then $f(E)$ is connected.