MCMC #4 (M-H sampler)

Sunday, April 12, 2020 11:04 PM

Metropolis - Hastings Sampler

Start with an initial X_0, obtain the state of the chain X_{t+1}, $t \geq 0$ by sampling a candidate Y from a proposal density $q(\cdot \mid X_t)$.

$q(\cdot \mid X_t)$ as prob. density on S satisfying regularity conditions and depends only on the previos state X_t.

If $q(\cdot \mid X_t)$ has positive density at the same support of π, then the conditions satifies.

Example of candidate density

must be Known

$$q(\cdot \mid X_t) \sim N(X_t, \sigma^2)$$

or $$q(\cdot \mid X_t) \sim Unif(X_t - a, X_t + a)$$

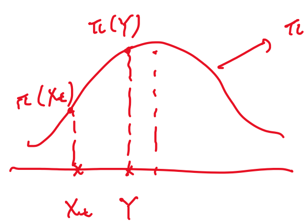
example Domain of π is $(0, 1)$

If $X_t = 0.7$, the possible candidate $Y \sim Unif(0.7-0.5,\ 0.7+0.5)$

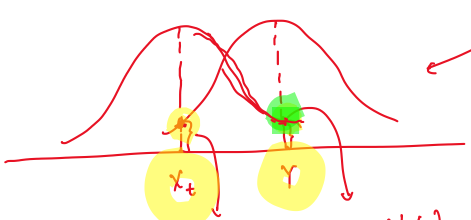
$= Unif(0.2, 1.2) \cap [0, 1]$

Then, we accept the candidate point Y with probability

$$\alpha(X_t, Y) = \min\left\{1, \frac{\pi(Y) \cdot q(X_t \mid Y)}{\pi(X_t) \cdot q(Y \mid X_t)}\right\}$$



If the point Y is not accepted, $X_{t+1} = X_t$ "stay"



$q(X_t \mid Y) = q(Y \mid X_t)$ if $q(\cdot)$ is Symmetric and can be ignored $\Rightarrow$ $\boxed{\text{Matropolis Sampler.}}$

Procedure

1. Set an initial X_0 at $t = 0$
2. Sample a candidate Y from $q(\cdot \mid X_t)$ ← previous state
3. Generate $u \sim U(0, 1)$

4. If $u \leq \alpha(X_t, Y)$, set $X_{t+1} = Y$ ("accept")
   else set $X_{t+1} = X_t$ ("stay")

5. Set $t = t+1$ and return to step 2.

Example 14.2

Want to generate r.s. from pop. with pdf

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

"Cauchy dist."

Use candidate $Y \sim N(\underbrace{X_t}_{\text{prev. state}}, \underbrace{\sigma}_{\text{assume to be given as } \sigma=2})$

<go over the Example>

$\boxed{\text{In-class Assignment}}$

Want to generate r.s. from Gamma(2,3). Use an initial $X_0 = 2$.

(a) Use $N(X_t, \alpha\beta^2)$ as the candidate density. Try different σ^2 and discuss how it affects the mixing rate.
$\sigma^2 = 18$

(b) $Exp(X_t)$ as the candidate density, and compare the chain
prev state

1. Generate r.s. of $n = 3000$ using M-H.
   Burn-in the first 5% ~ 10%
2. Plot the MC after the burn-in. Calculate the mean and the standard deviation of the chain and compare with the true values.
3. For your r.s. in 1, provide the normal Kernel density est.
4. Plot histogram, Kernel density, and the true density in a same graph.