Gibbs Sampler

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The Gibbs sampler

Specially adapted for multidimmensional target distribution.

The goal is to construct MC whose stationary distribution equals to the target π.

Idea : Sequential sampling from marginal distribution.

Bivariate Case

$$\underline{X} = (X_1, X_2) \sim f(x_1, x_2) = \pi$$

We want to generate a MC from the target $f(x_1, x_2)$

$$\underline{X}_t = (X_{t,1}, X_{t,2})$$

Suppose that $X_1 | X_2 \sim f(x_1 | x_2)$ are easily sampled.
and $X_2 | X_1 \sim f(x_2 | x_1)$

Procedure

1. Set an initial point $\underline{X}_0 = (x_{0,1}, x_{0,2})$ at $t=0$
2. Generate $X_{t+1,1}$ from $f(X_{t,1} | X_{t,2} = x_{t,2})$ → prev. value for $X_{t,2}$
3. " $X_{t+1,2}$ from $f(X_{t,2} | X_{t+1,1} = x_{t+1,1})$ → from Step 2
4. Set $t = t+1$

Repeat step 2 ~ 4.

The Gibbs sampler always accept the generated value.

Example 14.6 Bivariate (p594)

$$f(x, y) \propto \binom{n}{x} y^{x+\alpha-1} \cdot (1-y)^{n-x+\beta-1} \quad . \quad x = 0, 1, \ldots, n$$

$0 \le y \le 1$

Want to generate sample from $\underbrace{f(x)}_{\text{marginal}}$

Conditional distribution

$$f(x|y) \propto \binom{n}{x} y^x (1-y)^{n-x} \cdot \underbrace{y^{\alpha-1} \cdot (1-y)^{\beta-1}}_{\text{constant given } y, \alpha, \beta}$$

$$\propto \binom{n}{x} y^x (1-y)^{n-x} \sim \text{Binomial}(n, y)$$

$$f(y|x) \propto \binom{n}{x} y^{x+\alpha-1} (1-y)^{n-x+\beta-1} \sim \text{Beta}(\underbrace{x}_{\text{given}}+\alpha, n-x+\beta)$$

Procedure

1. Initial (x_0, y_0) guess

   x_0 r.s. from Binomial$(n, 0.5)$

   y_0 r.s from Beta$(x_0+\alpha, n-x_0+\beta)$

2. X_t r.s. from Binomial(n, y_{t-1})

   Y_t r.s from Beta$(X_t+\alpha, n-X_t+\beta)$

3. Continue.

Example 14.7 (p597)

Random sampling from Bivariate Normal distribution

Background

$$Z_1, Z_2 \overset{iid}{\sim} N(0,1) \Rightarrow f(z_1, z_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)}$$

Let $X_1 = \sigma_1 Z_1 + \mu_1$ and

$$X_2 = \sigma_2\left[\rho Z_1 + (1-\rho^2)^{1/2} \cdot Z_2\right] + \mu_2$$

Then

$$f(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_1\cdot\sigma_2}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

Here $E(X_1) = \mu_1$, $Var(X_1) = \sigma_1^2$

$E(X_2) = \mu_2$, $Var(X_2) = \sigma_2^2$

correlation $\rho(X_1, X_2) = \rho$

Then X_1 and X_2 have a bivariate Normal distribution

$$N\left(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}, \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}\right)$$

$$\Sigma$$

Conditional distribution

$X_2 \mid X_1 = x_1$

$$Z_1 = \frac{X_1-\mu_1}{\sigma_1} \quad \text{and} \quad Z_2 = \frac{X_2-\mu_2}{\sigma_2}$$

$$X_2 = \sigma_2\left[\rho\left(\frac{X_1-\mu_1}{\sigma_1}\right) + \sqrt{(1-\rho^2)}\left(\frac{X_2-\mu_2}{\sigma_2}\right)\right] + \mu_2$$

$$E[X_2 \mid X_1 = x_1] = \rho\cdot\sigma_2\left(\frac{x_1-\mu_1}{\sigma_1}\right) + \mu_2 \quad ,$$

$$Var[X_2 \mid X_1 = x_1] = \sigma_2^2(1-\rho^2)\,V\left(\cancel{\frac{x_2-\mu_2}{\sigma_2}}\right)$$

$$= \sigma_2^2(1-\rho^2)$$

$$X_2 \mid X_1 = x_1 \sim N\left(\mu_2 + \rho\cdot\frac{\sigma_2}{\sigma_1}(x_1-\mu_1),\ \sigma_2^2(1-\rho^2)\right)$$

Noting that $f(x_1, x_2)$ is symmetric in Z_1 and Z_2

$$X_1 \mid X_2 = x_2 \sim N\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2-\mu_2),\ \sigma_1^2(1-\rho^2)\right)$$

mean variance

Procedure

1. Take r.s. $Z_{1,0} \sim N(0,1)$, $Z_{2,0} \sim N(0,1)$

   Then $X_{1,0} = \mu_1 + \sigma_1 Z_{1,0}$

   $$X_{2,0} = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(X_{1,0} - \mu_1) + \sigma_2 \cdot \sqrt{1-\rho^2} \cdot Z_{2,0}$$

2. $$X_{1,1} = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}(X_{2,0} - \mu_2) + \sigma_1 \cdot \sqrt{1-\rho^2} \cdot Z_{1,1}$$

   $$X_{2,1} = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(X_{1,1} - \mu_1) + \sigma_2 \sqrt{1-\rho^2} \cdot Z_{2,1}$$

   $$\vdots$$

   $$X_{1,t+1} = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}(X_{2,t} - \mu_2) + \sigma_1 \cdot \sqrt{1-\rho^2} \cdot Z_{1,t+1}$$

   $$X_{2,t+1} = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(X_{1,t+1} - \mu_1) + \sigma_2 \cdot \sqrt{1-\rho^2} \cdot Z_{2,t+1}$$

3. Repeat.

$\boxed{\text{In-class}}$

of size 5000 (burn-in 20%)

Generate MC from Bivariate normal distribution

with mean $\underset{\sim}{\mu} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 1.5 & 0.6 \\ 0.6 & 1 \end{bmatrix}$

Plot each of X_1 and X_2 MC and histograms

Plot scatterplot of X_1 vs X_2 MC.

$\mu_1 = 3$

$\mu_2 = 2$

$\sigma_1 = \sqrt{1.5}$

$\sigma_2 = 1$

$\rho = \dfrac{0.6}{\sqrt{1.5} \cdot \sqrt{1}}$

Bivariate Mixture

Bivariate Mixture

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim p \cdot N_2(\underline{\mu}, \Sigma_1) + (1-p) \cdot N_2(\underline{v}, \Sigma_2)$$

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \Sigma_1 = \begin{bmatrix} a_1 & c_1 \\ c_1 & b_1 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} a_2 & c_2 \\ c_2 & b_2 \end{bmatrix}$$

It can be shown that

determinant

$$X \mid Y = y \sim w_y \cdot N\left(\mu_1 + \frac{(y-\mu_2)\cdot c_1}{b_1}, \frac{|\Sigma_1|}{b_1}\right) + (1-w_y)\cdot N\left(v_1 + \frac{(y-v_2)\cdot c_2}{b_2}, \frac{|\Sigma_2|}{b_2}\right)$$

$$Y \mid X = x \sim w_x \cdot N\left(\mu_2 + \frac{(x-\mu_1)c_1}{a_1}, \frac{|\Sigma_1|}{a_1}\right) + (1-w_x)\cdot N\left(v_2 + \frac{(x-v_1)c_2}{a_2}, \frac{|\Sigma_2|}{a_2}\right)$$

where

$$w_x = \frac{p \cdot a_1^{-1/2} \cdot \exp\left(-(x-\mu_1)^2/(2a_1)\right)}{p \cdot a_1^{-1/2} \exp\left(-(x-\mu_1)^2/(2a_1)\right) + (1-p)\cdot a_2^{-1/2} \exp\left(-(x-v_1)^2/(2a_2)\right)}$$

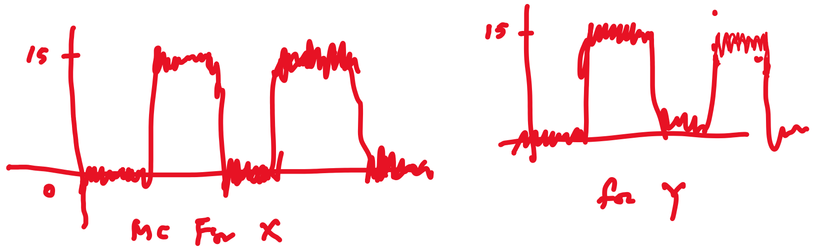
$$w_y = \frac{p \cdot b_1^{-1/2} \cdot \exp\left(-(y-\mu_2)^2/(2b_1)\right)}{p \cdot b_1^{-1/2} \cdot \exp\left(-(y-\mu_2)^2/(2b_1)\right) + (1-p)\cdot b_2^{-1/2} \cdot \exp\left(-(y-v_2)^2/(2b_2)\right)}$$

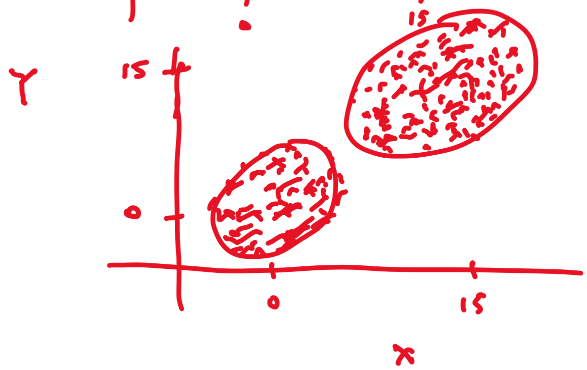
In-class Use Gibbs sampler to generate bivariate MC, $\begin{bmatrix} X \\ Y \end{bmatrix}$, of size 5000 from bivariate mixture normal with

$$\underline{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} 15 \\ 15 \end{pmatrix}, \quad p = 0.3, \quad \Sigma_1 = \Sigma_2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

After burn 10%, plot each MC and histogram.

Give the scatterplot of X vs Y MCs.





$\boxed{\text{In-class 2}}$ Given

$$f(x|y) = y\cdot e^{-xy} \sim Exp\left(\frac{1}{y}\right), \quad 0 < x < B \quad 10$$

$$f(Y|x) = x\, e^{-xy}, \quad 0 < y < B$$

$$\sim Exp\left(\frac{1}{x}\right)$$

Use Gibbs sampler to generate MC who invariance distribution is the marginal distribution $f(x)$

Procedure

1. Initial (x_0, y_0)

   X_0 r.s from $Exp(1)$

   Y_0 " "

2. X_1 r.s. from $Exp\left(\frac{1}{y_0}\right)$

   Y_1 r.s. from $Exp\left(\frac{1}{x_1}\right)$

3. X_2. r.s. from $\text{Exp}\left(\frac{1}{y_1}\right)$

   y_2 " " $\text{Exp}\left(\frac{1}{x_2}\right)$

   ⋮

   X_t r.s from $\text{Exp}\left(\frac{1}{y_{t-1}}\right)$

   y_t " " $\text{Exp}\left(\frac{1}{x_t}\right)$

Continue

a) Estimate $E(X)$, $V(X)$, skeness (X), Kurtosis (X)

b) Histogram of X, plot of MC vs t

c) Find $\hat{f}(x)$ at $X = 0.1, 1.8, 3, 9$