STAT 608 HW 6

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11/15/24

1)

a)

$$\Sigma = \begin{bmatrix} 49 & 5 & 4 \\ 5 & 25 & 0 \\ 4 & 0 & 9 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & \frac{1}{7} & \frac{4}{21} \\ \frac{1}{7} & 1 & 0 \\ \frac{4}{21} & 0 & 1 \end{bmatrix}$$

b)

That formula can be generalized to:

$$Var(a^T X) = a^T \Sigma a$$

Where X is the matrix containing x_1, x_2, x_3, Σ is the covariance matrix and a is length three vector containing the coefficients on x_1, x_2, x_3.

```r
Sigma = matrix(c(49,5,4,5,25,0,4,0,9),nrow = 3, ncol = 3)
Sigma
```

```
     [,1] [,2] [,3]
[1,]   49    5    4
[2,]    5   25    0
[3,]    4    0    9
```

1.

$a = (2, -2, 0)$

```
a_1 <- c(2,-2,0)
as.numeric(t(a_1)%*%Sigma%*%a_1)
```

```
[1] 256
```

2.

$a = (2, -1, 5)$

```
a_2 <- c(2,-1,5)
as.numeric(t(a_2)%*%Sigma%*%a_2)
```

```
[1] 506
```

3.

$a = (-1, 1, 2)$

```
a_3 <- c(-1,1,2)
as.numeric(t(a_3)%*%Sigma%*%a_3)
```

```
[1] 84
```

4.

$a = (-\beta_1, -\beta_2, 1)$

$$\text{Var}(a_1X_1 + a_2X_2 + a_3X_3) = 49\beta_1^2 + 25\beta_2^2 + 9 + 10\beta_1\beta_2 - 8\beta_1$$

Let's say we have the multiple linear model:

$$y = \beta_1X_1 + \beta_2X_2 + e$$

If we solve for e_i we have:

$$e = y - \beta_1X_1 - \beta_2X_2$$

This is the identical linear combination from before except this time $X_3 = y$. In this setting we naturally want to find the minimum of $\mathrm{Var}(e)$, as it would give us the least square estimates of β_1, β_2.

c)

After taking derivatives with the respect of b_1 and b_2 and setting equal to zero we have the two normal equations:

$$98b_1 + 10b_2 = 8$$

$$10b_1 + 50b_2 = 0$$

d)

In matrix form we have:

$$\begin{bmatrix} 98 & 10 \\ 10 & 50 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

Which has solution, $b_1 = 1/12, b_2 = -\frac{1}{60}$.

2)

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e$$

We have design matrix X:

$$\begin{bmatrix} 1_n & x_1 & x_2 \end{bmatrix}$$

a)

When we compute $X^T X$ we have:

$$X^T X = \begin{bmatrix} 1_n \\ x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 1_n & x_1 & x_2 \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^n x_{1i} & \sum_{i=1}^n x_{2i} \\ \sum_{i=1}^n x_{1i} & \sum_{i=1}^n x_{1i}^2 & \sum_{i=1}^n x_{1i} x_{2i} \\ \sum_{i=1}^n x_{2i} & \sum_{i=1}^n x_{1i} x_{2i} & \sum_{i=1}^n x_{2i}^2 \end{bmatrix}$$

Knowing that the mean of x_1 and x_2 is zero we can say $\sum_{i=1}^n x_{1i} = \sum_{i=1}^n x_{2i} = 0$.

Knowing that the length of x_1 and x_2 is one we can say $\sum_{i=1}^n x_{1i}^2 = \sum_{i=1}^n x_{2i}^2 = 1$.

We can rewrite $\sum_{i=1}^{n} x_{1i}x_{2i}$ as $\sum_{i=1}^{n}(x_{1i} - \bar{x}_1)(x_{2i} - \bar{x}_2)$ since $\bar{x}_1 = \bar{x}_2 = 0$. That is the numerator of the sample correlation between x_1 and x_2.

The denominator of the sample correlation between x_1 and x_2 is $\sqrt{\sum_{i=1}^{n}(x_{1i} - \bar{x}_1)^2 \sum_{i=1}^{n}(x_{2i} - \bar{x}_2)^2}$. Using $\bar{x}_1 = \bar{x}_2 = 0$, we then have $\sqrt{\sum_{i=1}^{n} x_{1i}^2 \sum_{i=1}^{n} x_{2i}^2}$ both terms are equal to 1 since the length of x_1 and x_2 is one.

With that we have shown:

$$\sum_{i=1}^{n} x_{1i}x_{2i} = \frac{\sum_{i=1}^{n}(x_{1i} - \bar{x}_1)(x_{2i} - \bar{x}_2)}{\sqrt{\sum_{i=1}^{n}(x_{1i} - \bar{x}_1)^2 \sum_{i=1}^{n}(x_{2i} - \bar{x}_2)^2}} = \rho$$

So:

$$X^T X = \begin{bmatrix} n & 0 & 0 \\ 0 & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix}$$

b)

We can find the inverse by using row operations:

$$\begin{bmatrix} n & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \rho & \vdots & 0 & 1 & 0 \\ 0 & \rho & 1 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

We subtract ρ(row 2) from row 3:

$$\begin{bmatrix} n & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \rho & \vdots & 0 & 1 & 0 \\ 0 & 0 & 1-\rho^2 & \vdots & 0 & -\rho & 1 \end{bmatrix}$$

We divide row 3 by $(1 - \rho^2)$:

$$\begin{bmatrix} n & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & \rho & \vdots & 0 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & -\frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} \end{bmatrix}$$

We subtract ρ(row 3) from row 2:

$$\begin{bmatrix} n & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & 1+\frac{\rho^2}{1-\rho^2} & -\frac{\rho}{1-\rho^2} \\ 0 & 0 & 1 & \vdots & 0 & -\frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} \end{bmatrix}$$

We divide row 1 by n:

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 1/n & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & 1+\frac{\rho^2}{1-\rho^2} & -\frac{\rho}{1-\rho^2} \\ 0 & 0 & 1 & \vdots & 0 & -\frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} \end{bmatrix}$$

So:

$$(X^TX)^{-1} = \begin{bmatrix} 1/n & 0 & 0 \\ 0 & \frac{1}{1-\rho^2} & -\frac{\rho}{1-\rho^2} \\ 0 & -\frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} \end{bmatrix}$$

And with $c = \frac{1}{1-\rho^2}$ we have shown:

$$(X^TX)^{-1} = \begin{bmatrix} 1/n & 0 & 0 \\ 0 & c & -c\rho \\ 0 & -c\rho & c \end{bmatrix}$$

c)

We know:

$$Var(\beta) = \sigma^2(X^TX)^{-1}$$

Then:

$$Var(\widehat{\beta}_1 - \widehat{\beta}_2) = \sigma^2 \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1/n & 0 & 0 \\ 0 & c & -c\rho \\ 0 & -c\rho & c \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 2\sigma^2(c+c\rho) = 2\sigma^2(\frac{\rho+1}{1-\rho^2}) = 2\sigma^2(\frac{1}{1-\rho})$$

d)