STAT 608 HW 7

Jack Cunningham (jgavc@tamu.edu)

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1)

We have the logistic regression model with one predictor:

$$\log(\frac{\theta(x)}{1-\theta(x)}) = \beta_0 + \beta_1 x$$

a)

We apply e^x to both sides:

$$\frac{\theta(x)}{1-\theta(x)} = e^{\beta_0+\beta_1 x}$$

Multiply both sides by $1-\theta(x)$:

$$\theta(x) = (1-\theta(x))e^{\beta_0+\beta_1 x}$$

$$\theta(x) = e^{\beta_0+\beta_1 x} - e^{\beta_0+\beta_1 x}\theta(x)$$

$$\theta(x) + e^{\beta_0+\beta_1 x}\theta(x) = e^{\beta_0+\beta_1 x}$$

On left side we factor out $\theta(x)$:

$$\theta(x)(1+e^{\beta_0+\beta_1 x}) = e^{\beta_0+\beta_1 x}$$

We are left with the desired result:

$$\theta(x) = \frac{e^{\beta_0+\beta_1 x}}{1+e^{\beta_0+\beta_1 x}}$$

b)

Starting from our result in a, we multiply by $e^{-(\beta_0+\beta_1 x)}/e^{-(\beta_0+\beta_1 x)}$:

$$\theta(x) = (\frac{e^{\beta_0+\beta_1 x}}{1+e^{\beta_0+\beta_1 x}})\frac{e^{-(\beta_0+\beta_1 x)}}{e^{-(\beta_0+\beta_1 x)}}$$

With $e^{-(\beta_0+\beta_1 x)}e^{\beta_0+\beta_1 x} = 1$, we distribute through and are left with the desired result:

$$\theta(x) = \frac{1}{1+e^{-(\beta_0+\beta_x)}}$$

2)

We can use the Bernoulli distribution for the conditional distribution of $X|Y=1$ and $X|Y=0$. So:

$$P(X=x|Y=0) = \pi_0^x(1-\pi_0)^{1-x}$$

$$P(X=x|Y=1) = \pi_1^x(1-\pi_1)^{1-x}$$

This will help with the next two parts.

a)

On page 283 it is derived that when X is a discrete random variable we have:

$$\log\left(\frac{\theta(x)}{1-\theta(x)}\right) = \log\left(\frac{P(Y=1)}{P(Y=0)}\right) + \log\left(\frac{P(X=x|Y=1)}{P(X=x|Y=0)}\right)$$

Using our conditional distributions from earlier we have:

$$\log\left(\frac{\theta(x)}{1-\theta(x)}\right) = \log\left(\frac{P(Y=1)}{P(Y=0)}\right) + \log\left(\frac{\pi_1^x(1-\pi_1)^{1-x}}{\pi_0^x(1-\pi_0)^{1-x}}\right)$$