12 | The Extended GARCH Models

Since the ARCH and GARCH models, $a_t = \sigma_t \varepsilon_t$ and $\varepsilon \sim i.i.d.(0,1)$, were introduced, there have been numerous GARCH variants proposed. Most of these models were motivated by the asymmetry property in volatilities .

Leverage effects Researchers have long found evidence that stock returns are negatively correlated with changes in returns volatility, i.e. volatility tends to rise in response to negative news and fall in response to positive news. Such asymmetry in response on shocks has become known as the *leverage effect*. GARCH models, however, assume that only the magnitude without algebraic sign determines future σ_t^2. If the distribution of ε_t is symmetric, the change in variance tomorrow is conditionally correlated with lagged σ_t^2 and lagged ε_t^2, and thus is invariant to ε_t being positive or negative.

With a skew distribution assumption of ε_t as the example in Handout 11, the effectiveness of capturing the asymmetry in volatility is still off. This is because the skewness in the distribution of ε_t is a constant and not updated with the most available information. The distribution of ε_t in the S&P example is the skewed Normal Inverse Gaussian. In Figure 11.7, there are more negative returns fall outside of the GARCH forecast bands than the positive ones, though the difference is not as severe as those in the literature.

The EGARCH Models

The most commonly known and often used model that accounts for the asymmetric property in volatility is the EGARCH, exponential GARCH model, proposed by Nelson (1991). Suppose that $a_t = \varepsilon_t \sigma_t$ and $\varepsilon_t \sim i.i.d.(0,1)$, the EGARCH(1,1) model is defined as,

$$\log(\sigma_t^2) = \omega + \beta \log(\sigma_{t-1}^2) + \alpha\, \varepsilon_{t-1} + \gamma \big\{ |\varepsilon_{t-1}| - E|\varepsilon_{t-1}| \big\}, \quad (12.1)$$

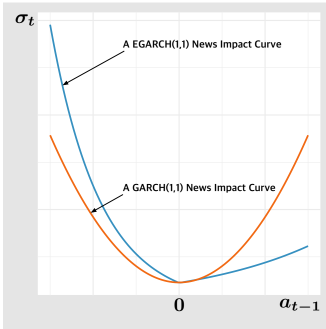
where ω, α, β and γ are constant parameters. The EGARCh model is asymmetric because $\varepsilon_{t-1} = a_{t-1}/\sigma_{t-1}$ (instead of a_t^2) has a coefficient α. This coefficient is typically negative so that positive return shocks generate less volatility than negative return shocks. Thus,

$$\begin{aligned} \sigma_t^2 &= A \cdot \exp\big\{(\alpha+\gamma)\,\varepsilon_{t-1}\big\}, \quad \varepsilon_{t-1} > 0 \;\; (\text{or } \; a_{t-1} > 0), \qquad (12.2) \\ \sigma_t^2 &= A \cdot \exp\big\{(\alpha-\gamma)\,\varepsilon_{t-1}\big\}, \quad \varepsilon_{t-1} < 0 \;\; (\text{or } \; a_{t-1} < 0), \end{aligned}$$

where $A = \sigma_{t-1}^{2\beta} \exp\big\{\omega - \gamma E|\varepsilon_{t-1}|\big\}$. When ε_t is standard norml, $E|\varepsilon_t| = \sqrt{2/\pi}$. When ε_t is standardized t, $t_\nu^{\text{std}}(0,1)$

$$E|\varepsilon_t| = \frac{2\sqrt{\nu-2}\,\Gamma((\nu+1)/2)}{\sqrt{\pi}\,(\nu-1)\Gamma(\nu/2)}\,.$$

News Impact Curve The news impact curve is obtained by replacing past σ_t in the model with unconditional σ. Holding the information earlier constant, we can examine the implied relation between ε_{t-1} and σ_t. News impact curves which measure the effect of a shock in the current period on the conditional variance in the subsequent period facilitate comparison between models.



The news impact curves show here are GARCH(1,1) and EGARCH(1,1) for $\alpha < 0$ but $\alpha + \gamma > 0$. The EGARCH model allows good news and bad news to have a different impact on volatility, while the standard GARCH model does not, and furthermore, the EGARCH model allows big news to have a greater impact on volatility than the standard GARCH model.

The EGARCH model represents a major shift from the ARCH and GARCH models. Rather than model the variance directly, as seen from (12.1), EGARCH model the log of the variance, there is no positivity restrictions on the parameters to ensure the conditional variance is positive.

An EGARCH(p,q) model a_t is defined as

$$\begin{aligned} a_t &= \varepsilon_t \sigma_t, \qquad \varepsilon_t \sim i.i.d.(0,1) \\ \log \sigma_t^2 &= \omega + \sum_{i=1}^{p} \big\{ \alpha_i \varepsilon_{t-i} + \gamma_i (|\varepsilon_{t-i}| - E|\varepsilon_{t-i}|) \big\} + \sum_{j=1}^{q} \beta_j \log \sigma_{t-j}^2 \,. \end{aligned}$$

Forecasting The 1-step ahead forecast of $\log \hat\sigma_{n+1}^2$ is simply replacing the parameters by the estimated ones and white noise by fitted residuals,

$$\log \hat\sigma_{n+1}^2 = \hat\omega + \sum_{i=1}^{p} \big\{ \hat\alpha_i \hat\varepsilon_{n-i} + \hat\gamma_i (|\hat\varepsilon_{n-i}| - \hat E|\varepsilon_{n-i}|) \big\} + \sum_{j=1}^{q} \hat\beta_j \log \hat\sigma_{n-j}^2 \,.$$

For the h-step ahead forecast, $h \geq 2$, the future values of ε_t and $(|\varepsilon_t| - E|\varepsilon_t|)$ are replaced by zeros and the future values of $\log \hat{\sigma}_t^2$ are replaced by the forecast. In the case of Gaussian EGARCH(1,1),

$$\log \hat{\sigma}_{n+1}^2 = \hat{\omega} + \hat{\alpha}\hat{\varepsilon}_n + \hat{\gamma}\Big(|\hat{\varepsilon}_n| - \sqrt{2/\pi}\Big) + \hat{\beta}\log \hat{\sigma}_n^2$$
$$\log \hat{\sigma}_{n+2}^2 = \hat{\omega} + \hat{\beta}\log \hat{\sigma}_{n+1}^2\,.$$

For those asset returns exhibiting leverage effects, the EGARCH models often provide better fit in conditional volatility than the standard GARCH models.

Models with a Threshold

The differentiated features of $a_{t-1} < 0$ and $a_{t-1} > 0$ in the EGARCH seen in (12.2) are embedded in its model specification. Another approach to handle leverage effects is making use of a threshold, an indicator function which explicitly includes a term only when $a_{t-1} < 0$. Two most common seen such models are the GJR- GARCH and TGARCH models.

The GJR-GARCH models The GJR-GARCH model was named after the authors who introduced it, Glosten, Jagannathan & Runkle (1993). A GJR-GARCH(p,q) is defined as $a_t = \varepsilon_t\sigma_t$, $\varepsilon_t \sim i.i.d.(0,1)$ and

$$\sigma_t^2 = \omega + \sum_{i=1}^{p}\left\{\alpha_i a_{t-i}^2 + \gamma_i a_{t-i}^2 \mathbb{I}_{\{a_{t-i}<0\}}\right\} + \sum_{j=1}^{q}\beta_j\sigma_{t-j}^2$$

where $\omega > 0$, $\alpha_i, \beta_i, \gamma_i \geq 0$ and $\mathbb{I}$ is the indicator function taking 1 when $a_{t-i} < 0$ and 0 otherwise. A positive a_{t-i} contributes $\alpha_i a_{t-i}^2$ to σ_t^2, whereas a negative a_{t-i} has a larger impact $(\alpha_i + \gamma_i)a_{t-i}^2$ with $\gamma_i > 0$. The model is the standard GARCH model if $\gamma_i = 0$ for all i.

The TGARCH models The threshold GARCH model by Zakoian (1994) makes one major change to the GJR-GARCH. Rather than modeling the variance directly using squared innovations, a TARCH model parameterizes the conditional standard deviation as a function of the lagged absolute value of the shocks. The TGARCH(p,q) is defined as $a_t = \varepsilon_t\sigma_t$, $\varepsilon_t \sim i.i.d.(0,1)$ and

$$\sigma_t = \omega + \sum_{i=1}^{p}\left\{\alpha_i|a_{t-i}| + \gamma_i|a_{t-i}|\mathbb{I}_{\{a_{t-i}<0\}}\right\} + \sum_{j=1}^{q}\beta_j\sigma_{t-j}\,.$$

The APARCH models The **A**symmetric **P**ower ARCH model is defined as $a_t = \varepsilon_t\sigma_t$, $\varepsilon_t \sim i.i.d.(0,1)$ and

$$\sigma_t^\delta = \omega + \sum_{i=1}^{p}\alpha_i\big(|a_{t-i}| - \gamma_i a_{t-i}\big)^\delta + \sum_{j=1}^{q}\beta_j\sigma_{t-j}^\delta\,,$$

where $\delta, \omega, \alpha_i, \beta_i > 0$ and $-1 < \gamma_i < 1$. The APGARCH can be veiwed as a general class of threshold models, various models arise from this class: the TGARCH model when $\delta = 1$; the GJR GARCH model when $\delta = 2$; The standard GARCH when $\delta = 2$ and $\gamma_j = 0$.

This feature of APARCH model can be useful when conducting hypothesis tests concerning the specification of conditional volatility models.

To fit an EGARCH or GJR-GARCH or APGARCH conditional volatility model, we only need to specify the `model` name `eGARCH`, or `"gjrGARCH"` or `"apARCH"` in the argument `variance.model` of the ugarchspec() function. For example an ARMA(1,0) + apARCH(1,1) with $\varepsilon_t \sim i.i.d.\, t_\nu^{\text{std}}(0,1)$,

```
spec = ugarchspec(mean.model = list(armaOrder = c(1,0)),
      variance.model = list(model = "apARCH",
      garchOrder = c(1,1)), distribution.model = "std")
```

To fit a TGARCH conditional volatility model, for example, a TGARCH (1,1) model, the specification is as follows.

```
variance.model = list(model = "fGARCH", submodel = "TGARCH",
               garchOrder = c(1,1))
```

Eg 12.1. Fitting an ARMA + extended GARCH to the S&P 500 daily returns. Continuation of Egs. 11.1, 11.2, 11.3 and 11.4. To fit any extended GARCH model, we would follow the same process as what was done for the standard GARCH model detailed in Handout 11.

Given that each model for σ_t is different, we should obtain a set of $\hat{\varepsilon}_t = \hat{a}_t/\hat{\sigma}_t$ for each extended model to search a suitable distribution. We have given a preliminary and the resulting $\hat{\varepsilon}_t$ to find the best fit distribution. It turns that the Normal Inverse Gaussian (NIG) distribution is the best fit all models. We omt the details.

The next step is to fit each GARCH model with AR(1) and NIG conditional distribution for order (p,q), $p,q = 1,2$. The `R` routine is identical to that o page 321 except 0 is not included in the candidate value of p. We will only show the AIC and BIC for each model.


```
cat("Informaion criteria of eGARCH:"); e.ic

## Informaion criteria of eGARCH:
##         garch11  garch12  garch21  garch22
## Akaike 2.615689 2.616240 2.612294 2.612797
## Bayes  2.630667 2.633091 2.631017 2.633392

cat("Informaion criteria of tGARCH:"); t.ic

## Informaion criteria of tGARCH:
##         garch11  garch12  garch21  garch22
## Akaike 2.606812 2.607278 2.603409 2.604008
## Bayes  2.621790 2.624128 2.622131 2.624603

cat("Informaion criteria of gjrGARCH:"); gjr.ic

## Informaion criteria of gjrGARCH:
##         garch11  garch12  garch21  garch22
## Akaike 2.620927 2.621482 2.620713 2.619470
## Bayes  2.635905 2.638332 2.639436 2.640065

cat("Informaion criteria of apARCH:"); ap.ic

## Informaion criteria of apARCH:
##         garch11  garch12  garch21  garch22
## Akaike 2.607292 2.607745 2.606360 2.606979
## Bayes  2.624142 2.626467 2.626955 2.629446
```

All AIC and BIC scores are lower than the lowest AIC and BIC of the standard GARCH model fit in Handout 11. The lowest BIC model is of order (1,1) for all extended GARCH models and AIC is of order (2,1) except the GJR-GARCH is of order (2,2). The optimal AIC and BIC among all GARCH models are the TGARCH(2,1) and TGARCH(1,1) respectively.

Our original AR(1) + GARCH(1,1) model shows only mild unbalance in numbers of outliers between the negative and positive ones.

We now compute the 1-step ahead rolling forecasts and their PIs of the two TGARCH models.

```
n.fore = dim(Yt)[1] - dim(Yn)[1];  ## number of forecasts to be validated
## ------ TGARCH(1,1) fit and forecasts ------
spec= ugarchspec(mean.model = list(armaOrder = c(1,0)),
              variance.model = list(model = "fGARCH", submodel = "TGARCH",
              garchOrder = c(1,1)),distribution.model = "nig")
fit.t11 = ugarchfit(Yt, spec = spec, out.sample = n.fore)
fore.t11 = ugarchforecast(fit.t11, n.roll = n.fore-1)
## ------ TGARCH(2,1) fit and forecasts ------
spec= ugarchspec(mean.model = list(armaOrder = c(1,0)),
              variance.model = list(model = "fGARCH", submodel = "TGARCH",
              garchOrder = c(2,1)), distribution.model = "nig")
fit.t21 = ugarchfit(Yt, spec = spec, out.sample = n.fore)
fore.t21 = ugarchforecast(fit.t21, n.roll = n.fore-1)
cat("AR(1) + TGARCH(1,1)"); rate(fore.t11);cat("\nAR(1) + TGARCH(2,1)");
rate(fore.t21);

## AR(1) + TGARCH(1,1)
##
##        One-Step Rolling Forecast
## ----------------------------------
##          coverage below PI beyond PI
## 95% PI    0.9569    0.0191    0.0241
## 90% PI    0.8967    0.0491    0.0542
##
## AR(1) + TGARCH(2,1)
##
##        One-Step Rolling Forecast
## ----------------------------------
##          coverage below PI beyond PI
## 95% PI    0.9579    0.0201    0.0221
## 90% PI    0.8997    0.0502    0.0502
```

The TGARCH(1,1) model over adjusts the negative shock effects, there are less negative than positive outliers. The TGARCH(2,1) model shows preform much better in capturing the negative random effects, the numbers of negative and positive outliers are closer in

the 95% coverage and they are equal in the 90% coverage. It has three more parameters than the original GARCH(1,1) model, $2p+q+1$ versus $p+q+1$ in modeling σ_t.

The output of TGARCH(2,1) fit below shows no remaining serial correlation and ARCH effects. One coefficient estimate `alph2` is not significant, but its companion coefficient `eta12` is.

```
showShort(fit.t21)

##
## *---------------------------------*
## *          GARCH Model Fit        *
## *---------------------------------*
## GARCH Model : fGARCH(2,1)
## fGARCH Sub-Model : TGARCH
## Mean Model : ARFIMA(1,0,0)
## Distribution : nig
##
## Optimal Parameters
## ------------------------------------
##         Estimate  Std. Error  t value Pr(>|t|)
## mu      0.025439    0.013820   1.8407 0.065659
## ar1    -0.072520    0.020878  -3.4736 0.000514
## omega   0.031497    0.008151   3.8644 0.000111
## alpha1  0.101071    0.012545   8.0567 0.000000
## alpha2  0.078702    0.044240   1.7790 0.075242
## beta1   0.852578    0.042115  20.2443 0.000000
## eta11   1.000000    0.121721   8.2155 0.000000
## eta12  -0.567188    0.250778  -2.2617 0.023715
## skew   -0.304417    0.035128  -8.6658 0.000000
## shape   1.894458    0.303037   6.2516 0.000000
##
## LogLikelihood : -4221.841
##
## Information Criteria
## ------------------------------------
##  Akaike  Bayes Shibata Hannan-Quinn
##  2.6034 2.6221  2.6034       2.6101
##
```

```
## Weighted Ljung-Box Test on Standardized Residuals
## ------------------------------------
##                          statistic p-value
## Lag[1]                      0.7372  0.3906
## Lag[2*(p+q)+(p+q)-1][2]     0.7373  0.8814
## Lag[4*(p+q)+(p+q)-1][5]     1.2496  0.9028
## d.o.f=1
## H0 : No serial correlation
##
## Weighted Ljung-Box Test on Standardized Squared Residuals
## ------------------------------------
##                           statistic p-value
## Lag[1]                        1.826  0.1766
## Lag[2*(p+q)+(p+q)-1][8]       3.436  0.6057
## Lag[4*(p+q)+(p+q)-1][14]      4.600  0.8172
## d.o.f=3
##
## Weighted ARCH LM Tests
## ------------------------------------
##             Statistic Shape Scale P-Value
## ARCH Lag[4]     1.186 0.500 2.000  0.2761
## ARCH Lag[6]     1.763 1.461 1.711  0.5451
## ARCH Lag[8]     1.911 2.368 1.583  0.7594
```

Figure12.1 shows the 95% prediction intervals based on the AR(1) + TGARCH(2,1) Noise and those of the AR(1) + *i.i.d.* NIG White Noise. The green dots are outliers of TGARCH(2,1) model, the red dots *i.i.d.* White Noise and blue dots sGARCH model.

The leverage effects of our S&P daily return data have been largely taken care of by the skew distribution of ε_t, the NIG, in the model. The difference between the positive and negative outliers is not large and they are randomly spread across the period as shown in the blue dots of Figure 12.1.

The TGARCH model gives higher value of σ_t for a large negative

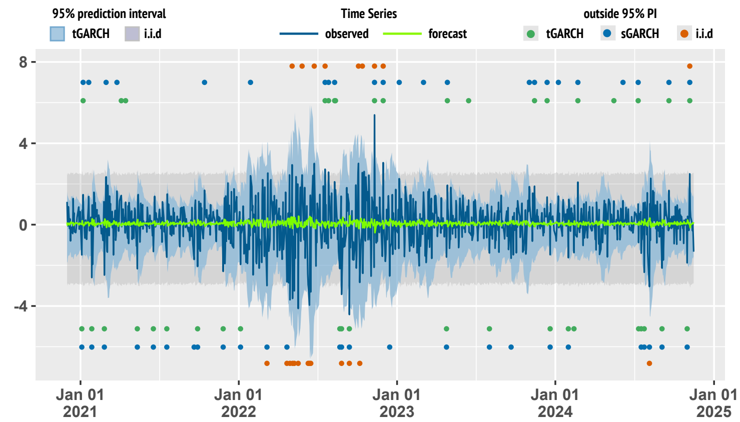


Figure 12.1: Rolling 1-step forecasts and 95% prediction intervals of S&P's daily returns from Dec 1, 2020 to Nov 15, 2024. The forecasts are based on AR(1) + *i.i.d.* NIG and AR(1) + TGARCH(2,1) models.

shock at $t-1$, as a result it has fewer outliers in higher volatility period because the PI tends to cover wider ranges. In Figure 12.1, we see very few green dots in the higher volatility period in 2022. The selected TGARCH(2,1) model is the least over-corrected extended model among all extended models selected by AIC and BIC when we studied all those models (not shown here).