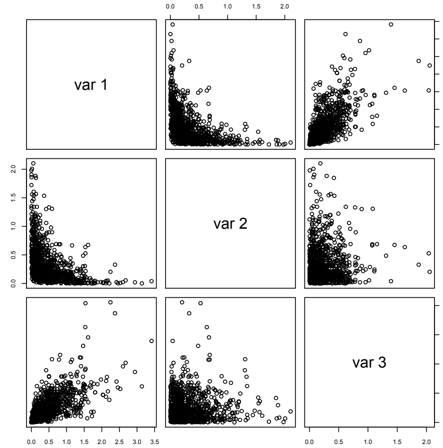
1. R-Lab§8.9.1. Solutions to Problems 1-2 are available on the book website.



Problem 3. The simulation is a multivariate exponential distribution generated from the Gaussian copula with the correlation matrix,

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & -0.60 & 0.75 \\ -0.60 & 1 & 0 \\ 0.75 & 0 & 1 \end{pmatrix}$$

(a) The marginal distributions are all exponential with rate parameters 2,3 and 4. The expected values are 1/2, 1/3 and 1/4 respectively.

The answers should be based on the code not on the plots. These are right skewed distributions, thus most of data points are at lower left.

(b) Yes. Components 2 and 3 are independent. Components 2 and 3 of the Gaussian are uncorrelated, this implies independence because they are normals (and only when they are normals). Any bi- or multi- variate distribution generated from independent copula has all its components independent.

2. §8.10 Exercises. Solution to Problem 1 is available on the book website.

Problem 2. The random variable $X \sim \text{Uniform}(0,1)$ and $Y = X^2$. Since X is positive, $Y = X^2$ is a strictly increasing function of X thus $\text{rank}(X) = \text{rank}(Y)$. Both Spearman's rho and Kendell's tau depend only on the ranks of the two random variables, Spearman's rho and Kendell's tau of X and Y are the same as those of X and X, which are equal to 1.

The person correlation depends on the number values of X and Y, and it is 1 if and only if X and Y always take the same values. The correlation is less than 1 because X^2 cannot be the same as X always.

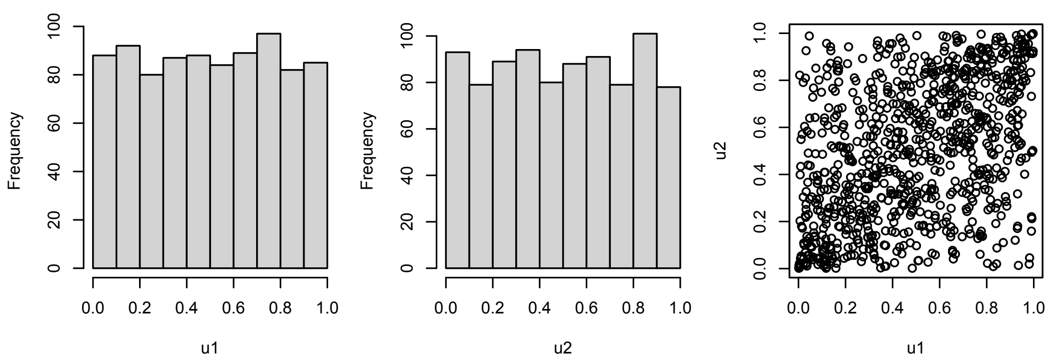
3.
```
> cat("Starting weeks:\n");head(yt,2);cat("\nEnding weeks:");tail(yt,2);
Starting weeks:                               Ending weeks:
                 AXP      INTC                                   AXP       INTC
2006-01-06 0.4184676  2.852950                2022-09-09  5.614172  0.7657983
2006-01-13 1.4323761 -1.996221                2022-09-16 -3.441531 -7.3179231

> n = dim(yt)[1];d = dim(yt)[2];cat("Sample size: n = ", n)
Sample size: n =  872
```

(a) The MLE's for fitting standard-t are shown in the output below. Specifically, the estimates of DF and their 95% CI indicate that the two t distributions do not share the same DF. From the three correlations, the two stock return series are associated including being correlated. These results imply a bivariate model should be considered but a bivariate-t is not suitable. Thus a copula bivariate model is a good candidate model.

```
> library(rugarch)
> est = se = matrix(ncol = 3, nrow = 2);
> rownames(est) = rownames(se) = syb
> colnames(est) = colnames(se) = c("m", "s","nu")
```

```
> for(i in 1:d){
+         mgd = fitdist("std", yt[,i])
+         est[i,] = mgd$pars
+         se[i,] = sqrt(diag(solve(mgd$hess)))
+ }
> cors = c(Person = cor(yt)[1,2], Kendall = cor(yt, method = "k")[1,2],
+          Spearman = cor(yt, method = "s")[1,2])
> nu_CI = est[,"nu"] + est[,"nu"]+qnorm(.975)*outer(se[,"nu"], c(-1,1))
> colnames(nu_CI) = c("lower_95%", "upper_95%")

> cat("* MLE of fitting standardized t *\n\nEstimates:");est;cat("\nStandard errors:\n");se;
cat("95% CI for DF");nu_CI;cat("correlations:"); cors

* MLE of fitting standardized t *

Estimates:                                95% CI for DF
          m     s    nu                        lower_95% upper_95%
AXP  0.2789 5.785 2.556                   AXP      4.597     5.628
INTC 0.1743 4.071 4.581                   INTC     7.695    10.630

Standard errors:                          correlations:
          m      s     nu                   Person  Kendall Spearman
AXP  0.1143 0.9072 0.2631                  0.45816  0.30687  0.43790
INTC 0.1187 0.1996 0.7487
```

(b) The probability transformation gives both series roughly a uniform distribution as expected. The scatter plot show the two series are positive associated.

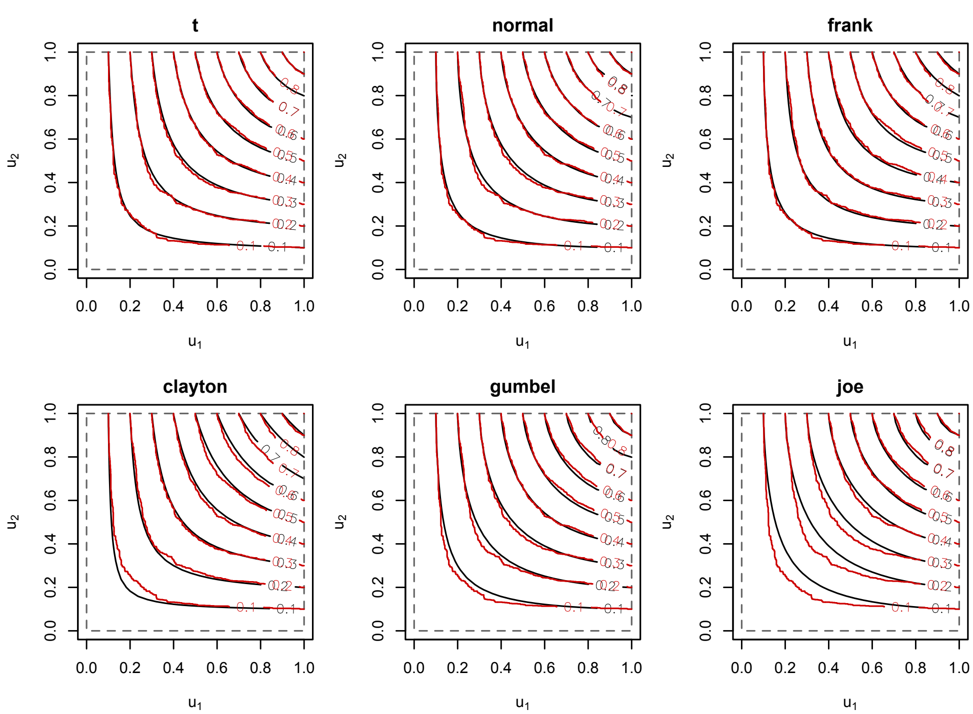



```
> ut = c()
> for(i in 1:d){
+ ut = cbind(ut,pdist("std", yt[,i], mu = est[i,"m"], sigma = est[i,"s"], shape = est[i,"nu"]))
+ }
> par(mfrow = c(1,3), pty = "s", lwd = 1.1, mar = c(4,4,2,1))
> hist(ut[,1],xlab = "u1", main = ""); hist(ut[,2], xlab = "u2", main = "");
> plot(ut, xlab = "u1", ylab = "u2")
```

(c) Both AIC and BIC select the t-copula being the best fit followed by Gumbel.

```
> library(copula)
> copNames = c("t", "normal", "frank","clayton", "gumbel", "joe")
> cops = vector("list",6);names(cops) = copNames
> for(i in 1:2){ ## 2 ellipticals
+         cops[[i]] = fitCopula(copula = ellipCopula(copNames[i],dim = 2), data = ut,
+                 method = "ml")
+ }
> for(i in 3:6){ ## 4 archimedeans
+         cops[[i]] = fitCopula(copula = archmCopula(copNames[i],dim = 2), data = ut,
+                 method = "ml")
+ }
> ic = matrix(nrow = 2, ncol = 6);rownames(ic) = c("aic","bic");colnames(ic) = copNames
> for(i in 1:6){
+         ic[1,i] = -2*cops[[i]]@loglik + 2*length(cops[[i]]@estimate)
+         ic[2,i] = -2*cops[[i]]@loglik + log(n)*length(cops[[i]]@estimate)
+ }
> ic
            t    normal     frank   clayton    gumbel       joe
aic -227.4861 -197.0858 -191.4413 -148.7538 -200.6815 -148.7808
bic -217.9445 -192.3150 -186.6705 -143.9830 -195.9107 -144.0100
```

(d) The PDF contours show that the t-copula fits best as the AIC and BIC suggest. The bottom three plots for Clayton, Gumbel and Joe do not fit as well comparing to the other three. However, according to the AIC and BIC, Gumbel is the 2nd best.



(e) I would choose t-copula based on the AIC, BIC and the PDF contours.

```
> cat("The estimates of t-copula:");c(rho = cops$t@estimate[1], DF = cops$t@estimate[2])
The estimates of t-copula:
      rho        DF
0.4614418 5.0380520
```

4.
```
> load("Midterm21.RData")
> head(yt,2); tail(yt,2)
                 BMY      MRK
2007-01-12 0.2670464 1.100065
2007-01-19 1.4372479 1.792284
                 BMY       MRK
2020-12-24 -3.170780 0.7640796
2020-12-31  2.223313 2.0502188
> syb
[1] "BMY" "MRK"
```

```
> uni.t.est
$BMY
            m         s        df
MLE 0.2620383 2.4600219 3.8845653
SE  0.1079559 0.1173096 0.6012481

$MRK
            m         s        df
MLE 0.2360303 2.3498820 3.9365648
SE  0.1033199 0.1065508 0.5803216
```

(a) The MLE's of DF's are 3.83 and 3.93 and their standard errors are .601 and .580. If we construct confidence intervals, they are practically the same one, implying not being significantly different between the two DF's.

(b) The MLE of correlation and DF are $\hat{\rho} = .4837$ and $\hat{\nu} = 3.89$, the MLE of the scale matrix $\mathbf{\Lambda}$ is given in the output.

```
> library(MASS); library(mnormt)
> df = seq(3.0, 4.5, .01) ## candidate values of DF
> loglik_p = c()
> for(i in 1:length(df)){
+   fit = cov.trob(yt, nu = df[i])
+   loglik_p[i] = sum(dmt(yt, mean = fit$center,
+                         S = fit$cov, df = df[i], log = T))
+ }
> nu = df[which.max(loglik_p)]
> bi_t =cov.trob(yt, nu = nu, cor = T)
> mu = bi_t$center; Lambda = bi_t$cov; rho = bi_t$cor[1,2]
> cat("The MLE of multivariate t:\n Mean vector");mu;
The MLE of multivariate t:
 Mean vector
      BMY       MRK
0.2539425 0.2355746
> cat("Lambda:");Lambda;cat("Degrees of freedom: nu = ", nu); cat("\ncorrelation: rho = ", rho)
Lambda:
         BMY      MRK
BMY 6.045934 2.812237
MRK 2.812237 5.589575
```

```
Degrees of freedom: nu =  3.89
correlation: rho =  0.4837609
```

The output `$cov` from `cov.trob()` is the scale matrix $\mathbf{\Lambda}$ which is defined for any degrees of freedom, it is NOT the covariance matrix $\mathbf{\Sigma}$, which is only defined for df > 2, similar to the standard t in the univariate case.

(c) The estimated marginal distributions are $t_{3.89}(.2539, 2.4588^2)$ and $t_{3.89}(.2356, 2.3642^2)$.

```
> lambda = sqrt(diag(Lambda))
> cat("Marginal distributions are t-distributions with df = ", nu, "and \n");
Marginal distributions are $t$-distributions with df =  3.89 and
> rbind(mean = mu, lambda = lambda)
             BMY       MRK
mean   0.2539425 0.2355746
lambda 2.4588481 2.3642283
```

5. (a) The tail dependence formula of a t-copula is given in eq (5.6), page 89 of Handout 5 and $\hat{\rho}$ and $\hat{\nu}$ are in part (b) of Q.1.

```
> tail_dep = 2*pt(-sqrt((nu + 1)*(1 - rho)/(1 + rho)), df = nu + 1)
> cat("Tail dependence of the t-Copula: lambda = ", paste(round(tail_dep,6)))
Tail dependence of the t-Copula: lambda =  0.250124
```

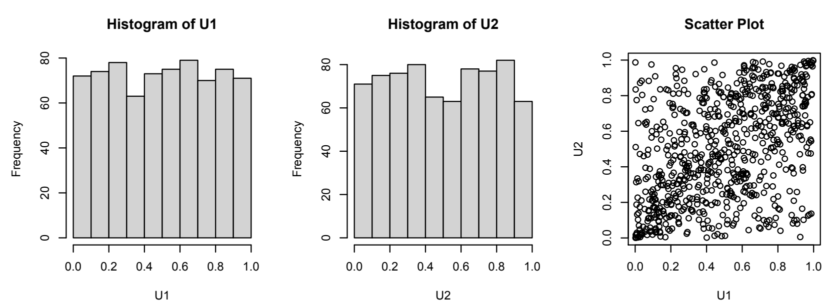
(b) The copula is probability transformation of the marginal CDFs. For a bivariate-t with marginals $t_\nu(\mu_i, \lambda_i^2)$, $i = 1, 2$,

$$U_{it} = F\Big(\frac{y_{it} - \mu_i}{\lambda_i} \,\Big|\, \nu\Big),$$

where F is the CDF of the t_ν distribution. The mle $\hat{\mu}_i$, $\hat{\lambda}_i$ and $\hat{\nu}$ are in part (c) of Q.1.

I have used classical t and the definition of location-scale families from Handout 3. If you use `rugarch`'s `pdist("std",...)` or `fGarch`'s `pstd()` for standard t, you have to convert $\hat{\lambda}_i$ to $\hat{\sigma}_i = \hat{\lambda}_i\sqrt{\hat{\nu}/(\hat{\nu}-2)}$.

```
> ut = sapply(1:2, function(u) pt((yt[,u]-mu[u])/lambda[u], df = nu))
```



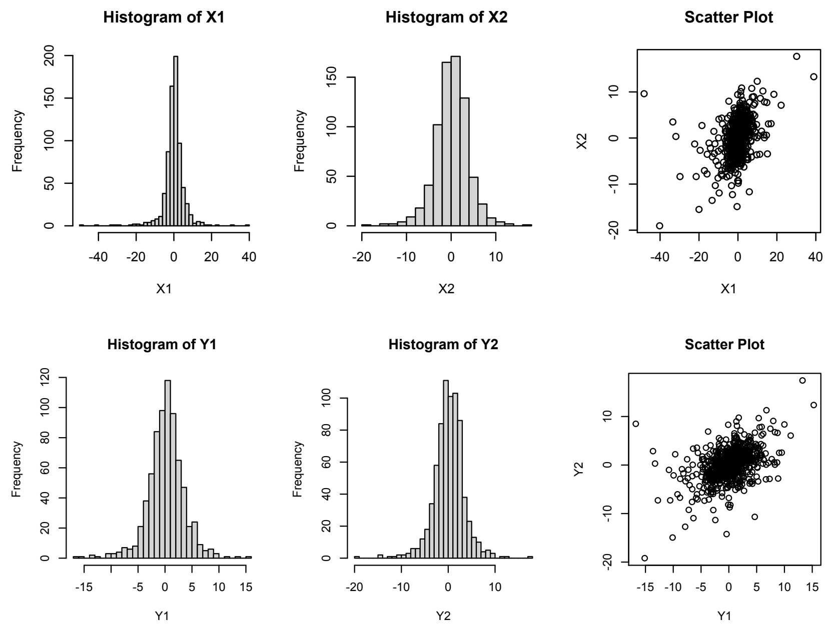
(c) If $\boldsymbol{X}_t$ is a bivariate meta-t with marginals $t_{\nu_i}(\mu_i, \lambda_i^2)$, then $\boldsymbol{X}_t$ are quantile transformation of U_{it}, $i = 1, 2$,

$$X_{it} = \mu + \lambda F^{-1}(u_i \,|\, \nu_i)$$

where F^{-1} is the quantile function of a t_ν , `qt()` in `R`.

The standard t is defined only for DF >2. We have to use the classical t to simulate in this question by a location-scale transformation of Student's t as the simulation for Question 1 of HW3. See the definitions in pages 38-39, Handout 3.

```
> mu.x = c(.25, .25); lambda.x = c(2.45,3); nu.x = c(2,5);
> xt = sapply(1:2, function(u) mu.x[u] + lambda.x[u]*qt(ut[,u], nu.x[u] ))
```



The plots of `yt` were not required. I plotted them for the sake of comparison.