6 | Portfolio Selection

One assumption in investment is the risk-return tradeoff, a higher level of risk implies a higher potential return. Then one of the important problems is how to quantify the tradeoff between risk and returns. Statistics has become the foundation of this quantification since Markowitz's seminal paper "*Portfolio selection*" (1952), in which he pioneered *Modern Portfolio Theory*.

Modern Portfolio Theory

The Markowitz modern portfolio theory (MPT) characterizes the return and risk of investments by expectation and standard deviation. MPT is thus also termed mean-variance portfolio theory.

Investors in general would like to maximize the expected return and minimize the risk. Simply put, portfolio theory is about finding the balance between maximizing the return and minimizing the risk. One key concept is to reduce risk by diversifying the portfolio. According to Markowitz's theory, there is an optimal portfolio which offers the highest expected return for a specific level of risk or the lowest risk for a given level of expected return.

One Risky Asset and One Risk-Free Asset

The simplest example is one risky asset, which could be a portfolio, for example, a mutual fund. Assume that the expected return is

0.15 and the standard deviation of the return is 0.25. Assume there is a risk-free asset, usually an US Treasury bond, say a 90-day T-Bill with a return 0.06.

An investment portfolio is constructed over *a holding period*, which could be a day or month or several years. We will assume only one holding period for now. Assume that a fraction w of wealth is invested in the risky asset and the remaining fraction $1-w$ is invested in the risk-free asset. Then the expected return is

$$E(R) = w(0.15) + (1-w)(0.06) = 0.06 + 0.09w$$

and the variance of the return is

$$\sigma_R^2 = w^2(.25)^2 + (1-w)^2(0)^2 = w^2(.25)^2 \implies \sigma_R = 0.25\omega\,.$$

To decide what proportion w of one's wealth to invest in the risky asset, one chooses either the expected return $E(R)$ one wants or the level of risk σ_R with which one is willing to take. Once either $E(R)$ or σ_R is chosen, w can be determined.

An alternative measure of risk is actual money loss.

Suppose that a firm is to invest \$1,000,000 and has capital reserves that could cover a loss of \$150,000 but no more. Thus if there is a loss, it cannot exceed 15% or $R > -0.15$. Suppose that R is normally distributed, the only way to guarantee that $R > -0.15$ with probability 1 is to invest entirely in the risk-free asset. The firm might instead require only $P(R < -0.15)$ be small, say 0.01. Therefore, w

should be such that

$$\Phi\left(\frac{-0.15-\mu_R}{\sigma_R}\right) = 0.01 \implies \frac{-0.15-(0.06+0.09\,w)}{0.25w} = \Phi^{-1}(0.01)$$

The solution is $w = 0.4272$.

The quantity \$150,000 is called Value at Risk (VaR) and $0.99 = 1-0.01$ the confidence coefficient. This is an example of finding the portfolio with VaR $=$ \$150,000 and confidence coefficient 0.99. A more realistic assumption is that the return R has a heavy tailed distribution instead of normal.

If the expected returns on the risky and risk-free assets are μ_1 and μ_f and the standard deviation of the risky asset is σ_1, then the expect return on the portfolio is $w\mu_1 + (1-w)\mu_f$ and the standard deviation is $\sigma|w|$.

The value of μ_f is known since Treasury bond rates are published. The μ_1 and σ_1 of the risky asset need to be estimated base on the past returns. We will estimate them with their marginal distributions. In the later chapters, time dependent models, the conditional mean models, the conditional standard deviation models and the combination of them can be considered to estimate the expected returns and risks. Examples of such models are the ARMA + GARCH models.

Two Risky Assets

Consider a portfolio of two risky assets having returns R_1 and R_2. Then the return on the portfolio is $R_p = wR_1 + (1-w)R_2$. The expected return and the variance on the portfolio are, respectively,

$$
\begin{aligned}
E(R_p) &= w\mu_1 + (1-w)\mu_2 \\
\sigma^2_{R_p} &= w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\rho_{12}\sigma_1\sigma_2\,,
\end{aligned}
$$

where $\rho_{12} = \text{Corr}(R_1, R_2)$.

Eg 6.1. $\mu_1 = 0.14$, $\mu_2 = 0.08$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$ and $\rho = 0$. Then $E(R_p) = 0.08 + 0.06w$ and

$$\sigma^2_{R_p} = (0.2)^2 w^2 + (0.15)^2(1-w)^2.$$

One can easily show, by the elementary calculus, that the portfolio with the minimum risk is at $w = 0.36$. For this portfolio $E(R_p) = 0.1016$ and $\sigma_{R_p} = 0.12$.

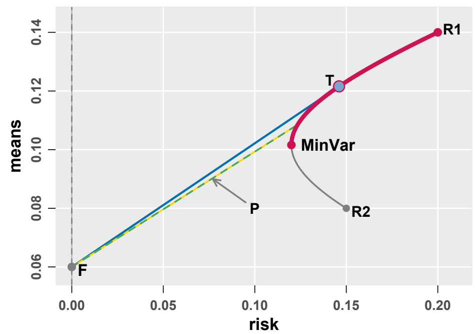


Figure 6.1: Expected return versus risk for Eg. 6.1. F = risk-free asset, T = tangency portfolio. MinVar is the minimum variance portfolio. The efficient frontier is the red curve. All points connecting R2 and R1 are attainable with $0 \le w \le 1$, but the black curve are suboptimal. P is typical portfolio on the efficient frontier.

Efficient frontier The parabolic-like curve in Figure 6.1 is the values of $(\sigma_{R_p}, E(R_p))$ when $0 \le w \le 1$. The leftmost point on this

curve achieves the minimum variance portfolio. The points that have an expected return at least as large as the minimum variance portfolio are called the *efficient frontier* as shown in red. Portfolios on the efficient frontier are called efficient portfolios or mean-variance efficient portfolios. The points labeled R1 and R2 correspond to $w = 1$ and $w = 0$, respectively. The other features of this figure will be explained later.

Combining Two Risky Assets with a Risk Free Asset

Our goal is to find optimal portfolios combining many risky assets with a risk-free asset. First consider the simplest case when there are only two risky assets.

The point F in Figure 6.1 gives $(\sigma_{R_p}, E(R_p))$ for the risk-free asset, $\sigma_{R_p} = 0$ at F. The possible $(\sigma_{R_p}, E(R_p))$ for a portfolio consisting of the fixed w portfolio of two risky assets and the risk-free asset is a line connecting the point F with a point on the efficient frontier, for example the dashed green line.

Tangency portfolio The solid line connecting F with the point labeled T lies above the dashed green line connecting F and the typical portfolio. For any value of σ_{R_p} the solid line gives a higher expected return than the dashed green line. The slope of each line is called its Sharpe's ratio, named after William Sharpe. If $E(R_p)$ and σ_{R_p} are the expected return and standard deviation of the return on a portfolio and μ_f is the risk-free rate, then

$$\text{Sharpe's ratio} = \frac{E(R_p) - \mu_f}{\sigma_{R_p}}\,.$$

Sharpe's ratio can be thought of as a "reward-to-risk" ratio. It is the ratio of the reward quantified by the excess expected return (of the risk-free rate) to the risk as measured by the standard deviation.

A line with a larger slope gives a higher expected return for a given level of risk. The point T on the efficient frontier is the portfolio with the highest Sharpe's ratio. It is the optimal portfolio for the purpose of mixing with the risk-free asset. This portfolio is called *the tangency portfolio* since its line is tangent to the efficient frontier.

Result 6.1. *The optimal or efficient portfolios mix the tangency portfolio with the risk-free asset. Each efficient portfolio has two properties:*

1. *it has a higher expected return than any other portfolio with the same or smaller risk; and*
2. *it has a smaller risk than any other portfolio with the same or higher expected return.*

Thus we can only improve (reduce) the risk of an efficient portfolio by accepting a worse (smaller) expected return, and we can only improve (increase) the expected return of an efficient portfolio by accepting worse (higher) risk.

Note that all efficient portfolios use the same mix of the two risky assets (same w), namely, the tangency portfolio (T). Only the proportion allocated to the tangency portfolio and the proportion allocated to the risk-free asset vary.

Weight of tangency portfolio Define the excess expected returns,

$$V_1 = \mu_1 - \mu_f \qquad \text{and} \qquad V_2 = \mu_2 - \mu_f\,,$$

Then weight of the tangency portfolio of 2 risky assets are

$$w_T = \frac{V_1\sigma_2^2 - V_2\rho_{12}\sigma_1\sigma_2}{V_1\sigma_2^2 + V_2\sigma_1^2 - (V_1+V_2)\rho_{12}\sigma_1\sigma_2} \tag{6.1}$$

for the first asset and weight $1-w_T$ for the second.

Let R_T, $E(R_t)$ and σ_T be the return, expected return and standard deviation of the return on the tangency portfolio. Then

$$E(R_T) = w_T\mu_1 + (1-w_T)\mu_2$$
$$\sigma_T^2 = w_T^2\sigma_1^2 + (1-w_T)^2\sigma_2 + 2w_T(1-w_T)\rho_{12}\sigma_1\sigma_2$$

Eg 6.2. Suppose that the risk-free rate $\mu_f = 0.06$. Then in Eg. 6.1, $V_1 = \mu_1 - \mu_f = 0.14 - 0.06 = 0.08$ and $V_2 = 0.08 - 0.06 = 0.02$. Plugging these values into the formula (6.1), we get

$$w_T = 0.693\,, \qquad E(R_T) = 0.122 \qquad \text{and} \qquad \sigma_T = 0.146\,.$$

This is the point T in Figure 6.1 .

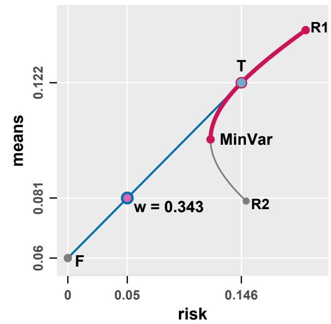
Combining the Tangency Portfolio with the Risk-Free Asset

Let R_p be the return on the portfolio that allocates a fraction w of the investment to the tangency portfolio and $1-w$ to the risk-free asset. Then $R_p = wR_T + (1-w)\mu_f = \mu_f + w(R_T - R_f)$, so that

$$E(R_p) = \mu_f + w\{E(R_T - \mu_f)\} \quad \text{and} \quad \sigma_{R_p} = w\sigma_{R_T}\,. \tag{6.2}$$

Eg 6.3. Continuation of Eg. 6.1 and Eg. 6.2. The optimal investment with $\sigma_{R_p} = 0.05$.

The maximum expected return with $\sigma_{R_p} = 0.05$ mixes the tangency portfolio and the risk-free asset such that $\sigma_{R_p} = 0.05$. Since $\sigma_T = 0.146$, we have that $w = 0.05/0.146 = 0.343$ and $1 - w = 0.657$.



Thus if one chooses a risk value $\sigma_{R_p} = 0.05$. The portfolio which maximizes the expected return under this condition consists of 65.7% of the risk-free asset, and 34.3% of the tangency portfolio. This implies that (0.343)(69.3%) = 23.7% should be in the first risky asset and (0.343)(30.7%) = 10.5% in the second risky asset due to the tangent weight $w_T = 0.693$ calculated in Eg. 6.2. They don't sum up to 100% because of rounding error.

Eg 6.4. Continuation of Eg. 6.1 and Eg. 6.2. Given the desired expect return 10%, find the best portfolio (lowest risk) of (1) only risky assets and (2) the risky assets and the risk-free asset.

The best portfolio of only risky assets uses w_1 solving $0.1 = w_1(0.14) + (1 - w_1)(0.08)$, which implies that $w_1 = 1/3$. This is the only portfolio of risky assets with $E(R_p) = 0.1$, so it is the best. Then

$$\sigma_{R_p}^2 = \frac{(0.2)^2}{9} + \frac{4(0.15)^2}{9} = 0.014444, \quad \sigma_{R_p} = 0.120.$$

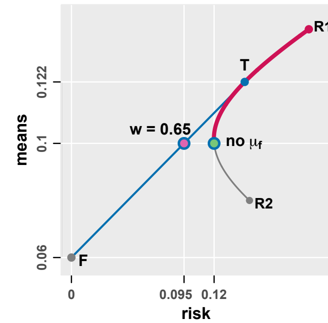
The best portfolio of the two risky assets and the risk-free asset can

be found as follows. First,

$$E(R_p) = \mu_f + w\{E(R_T) - \mu_f\}$$
$$\Rightarrow 0.1 = 0.06 + w\{0.122 - 0.06\}$$

We get $w = 0.65$, $1 - w = 0.35$ and

$$\sigma_p = w\sigma_T = 0.095.$$



So combining the risk-free asset with the two risky assets reduces σ_{R_p} from 0.120 to 0.095 while maintaining $E(R_p) = 0.1$. The reduction in risk is $(0.120 - 0.095)/0.095 = 26.69\%$.

Effects of correlation Positive correlation between the two risky assets ρ_{12} increases risk. With positive correlation, the two assets tend to move together which increases the volatility of the portfolio. Conversely, negative correlation is beneficial since it decreases risk. If the assets are negatively correlated, a negative return of one tends to occur with a positive return of the other so the volatility of the portfolio decreases.

Figure 6.2 shows the efficient frontier and tangency portfolio when $\mu_1 = 0.14$, $\mu_2 = 0.08$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$ and $\mu_f = 0.06$. The value of ρ_{12} is varied from 0.5 to -0.7. Notice that Sharpe's ratio of the tangency portfolio returns increases as ρ_{12} decreases. This means that when ρ_{12} is small, then efficient portfolios have less risk for a given expected return compared to when ρ_{12} is large.

Selling short A negative weight on an asset means that this asset is sold short. Short selling is a way to profit if a stock price goes

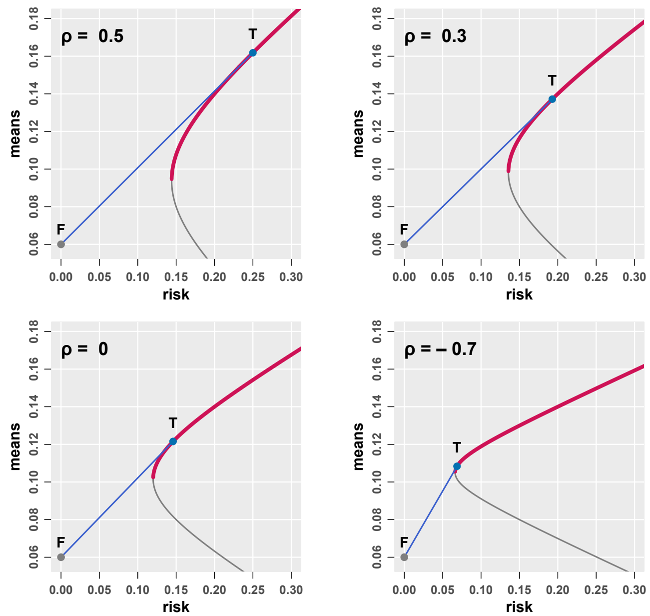


Figure 6.2: Efficient frontier (red) and tangency portfolio (T) when $\mu_1 = 0.14$, $\mu_2 = 0.08$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$ and $\mu_f = 0.06$. The value of ρ_{12} is varied from 0.5 to -0.7.

down. To sell a stock short, one sells the stock without owning it. The stock must be borrowed from a broker or another customer of the broker. At a later point in time, one buys the stock and gives it back to the lender. This closes the short position.

Consider two risky assets A and B both with current price \$25 share. A trader believes that asset A will rise in price, and asset B will decline in price. Therefore he borrows 50 shares of asset B and sells them with $\$1,250 = 50 \times \$25$, together with his own \$2,500 he purchases 150 shares of asset A with an amount of \$3,750. If R_A and R_B are the returns on assets A and B, then the return on his

portfolio would be

$$\frac{3}{2}R_A + \left(-\frac{1}{2}\right)R_B\,.$$

The portfolio weights are $3/2$ and $-1/2$.

If one sells a stock short, one is said to have a short position in that stock, and owning the stock is called a long position.

Risk-Efficient Portfolios with N Risky Assets

Assume that we have N risky assets, the goal is to choose an optimal set of weights in which the portfolio achieves a target expected return with minimal risk. Let the return on the ith risky asset be R_i and $E(R_i) = \mu_i$. Define

$$\boldsymbol{R} = (R_1, \ldots, R_N)^T, \quad \boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)^T \quad \text{and} \quad \text{var}(\boldsymbol{R}) = \boldsymbol{\Sigma}$$

and denote the set of weights to be determined by

$$\boldsymbol{w} = (w_1, \ldots, w_N)^T. \tag{6.3}$$

The optimal portfolio $\boldsymbol{w}$ can be obtained by solving a quadratic programming problem which is to minimize a quadratic objective function with constraints. The constraints depend on the strategy being entailed, the resulting portfolio $\boldsymbol{w}^T\boldsymbol{R}$ has a return μ_p and risk σ_p,

$$\mu_{R_p} = \boldsymbol{w}^T\boldsymbol{\mu} \qquad \text{and} \qquad \sigma_p = \sqrt{\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}}\,.$$

We will phrase the problem in terms of the portfolio variance and minimize the quadratic $\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}$.

Portfolio with short selling

The first type of our optimization problem is to do with portfolio allowing short sells and is as follows,

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \frac{1}{2}\boldsymbol{w}^T\Sigma\boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{A}^T\boldsymbol{w} = \boldsymbol{b}\,. \end{aligned} \tag{6.4}$$

This is a classic problem of constrained optimization and can be solved by the Lagrangian method, the explicit solutions of this type of problems are derived in the Appendix.

Eg 6.5. The data set is weekly returns (in %) of Adobe Inc. (ADBE), Advanced Micro Devices (AMD), General Mills (GIS), Netflix (NFLX), Qualcomm (QCOM) and XPO Logistics, Inc. (XPO) from July 1, 2012 to Sep 28, 2024, $n = 639$.

```
head(yt,2) # y is the matrix of weekly returns in %

##               ADBE    AMD     GIS    NFLX     QCOM     XPO
## 2012-07-06 -1.5499  0.000 0.90709 17.4609 -0.61284  -2.8341
## 2012-07-13 -2.3219 -15.996 0.71980  3.6097 -0.59843 -10.9788

tail(yt,2)

##               ADBE   AMD     GIS    NFLX    QCOM     XPO
## 2024-09-20 -2.7514 2.3618  1.07949 0.56792 0.70697  2.70287
## 2024-09-27 -1.3144 5.2463 -0.17463 0.89748 0.71377 -0.86596

n = dim(yt)[1]; N = dim(yt)[2]
cat("Sample size n =", n, "\t\t\t Number of assets N = ", N)

## Sample size n = 639   Number of assets N =  6
```

The sample mean and variance covariance matrix.


```
y.mu = apply(yt,2,mean);y.mu ## means of yt

##    ADBE     AMD     GIS    NFLX    QCOM     XPO
## 0.43564 0.52469 0.10342 0.66928 0.17488 0.46220

y.S = var(yt); y.S ## variance matrix of yt

##          ADBE     AMD      GIS    NFLX     QCOM      XPO
## ADBE 16.12347 11.5592  0.75896  9.6461  7.66093  8.73008
## AMD  11.55920 59.3892  1.76037 13.8253 13.55856 16.29992
## GIS   0.75896  1.7604  6.79618  1.5347  0.99634 -0.18254
## NFLX  9.64610 13.8253  1.53469 44.7787  6.82999  8.71693
## QCOM  7.66093 13.5586  0.99634  6.8300 20.78019 10.67188
## XPO   8.73008 16.2999 -0.18254  8.7169 10.67188 44.92548
```

Minimum variance portfolio For the portfolio that allows short selling, the minimum variance portfolio is (6.27),

$$\boldsymbol{w}_{\text{min.v}} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}}\,.$$

```
## Minimum variance portfolio
ones = rep(1,N);  ## a vector of 1's
IS = solve(y.S) ; ## inverse of variance matrix
a = as.numeric((t(ones)%*%IS%*%ones))
w.min = 1/a*(IS%*%ones);  ## the weights
mu.min = as.numeric(t(w.min)%*%y.mu);  ## return of w
s.min = sqrt(as.numeric(t(w.min)%*%y.S%*%w.min)); ## std.dev of w
cat("The minimum variance portfolio:"); w.min[,1];

## The minimum variance portfolio:
##      ADBE       AMD       GIS      NFLX      QCOM       XPO
##  0.183408 -0.019188  0.656843  0.023100  0.107090  0.048747

cat("Portfolio return is:", mu.min, "\t with risk", s.min)

## Portfolio return is: 0.19448   with risk 2.1686
```

The mean and variance are simply $\dfrac{\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}}$ and $\dfrac{1}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}}$.

Tangency portfolio Let $\mu.f$ be the weekly return of the risk free asset and $\boldsymbol{\mu}_{ex} = \boldsymbol{\mu} - \mu_f \mathbf{1}$, the tangency portfolio is given in (6.30) of Appendix,

$$\boldsymbol{w}_T = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}$$

The annul risk free return is 4.26%, we convert it to weekly.

```
## Tangency portfolio
mu.f = 4.26/52 ## weekly return of risk free asset
m.ex = y.mu -mu.f ## excess return
aT = as.numeric((t(ones)%*%IS%*%m.ex))
w.T = 1/aT*(IS%*%m.ex)
mu.T = as.numeric(t(w.T)%*%y.mu);  ## return of w.T
s.T = sqrt(as.numeric(t(w.T)%*%y.S%*%w.T)); ## std.dev of w.T
cat("Tangency portfolio:"); w.T[,1];

## The minimum variance portfolio:
##       ADBE        AMD        GIS       NFLX       QCOM        XPO
##  0.6680665  0.1226005 -0.0025751  0.3846198 -0.3638459  0.1911342

cat("Portfolio return is:", mu.T, "\t with risk", s.T)

## Portfolio return is: 0.63723   with risk 4.8167
```

The mean and variance of the tangency portfolio are also given in Appendix without the need of calculation,

$$\mu_T = \frac{\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}, \qquad \sigma_T^2 = \frac{\boldsymbol{\mu}_{ex}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}{\left(\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}\right)^2}.$$

Efficient frontier The set of efficient frontier also solvable by the Lagrangian method. For a given target portfolio risky asset return $\mu_p = m$, the explicit solutions of portfolio and its variance are derived in the Appendix. If we would like to compute the variances of

a set of target returns, it can easily be done. Let

$$\mathtt{Amat} = [\boldsymbol{\mu}, \mathbf{1}], \quad \mathbf{H} = \mathtt{Amat}^T\boldsymbol{\Sigma}^{-1}\mathtt{Amat} = \begin{pmatrix} C & B \\ B & A \end{pmatrix}, \quad \Delta = \det(\boldsymbol{H}).$$

The smallest variance for a return m is

$$\sigma_{opt}^2 = \frac{Am^2 - 2Bm + C}{\Delta}.$$

We can create a sequence of m's and compute their risk σ_{opt}.

```
## Efficient frontier
m.R = seq(-.15, 1, 0.001)
Amat = cbind(y.mu, ones)
H = t(Amat)%*%IS%*%Amat
A = H[2,2]; B = H[1,2]; C = H[1,1]; Delta = det(H)
sd.R = sqrt((A*m.R^2 - 2*B*m.R + C)/Delta)
```

The curve of portfolio returns and their risks can be plotted with the set of (`m.R`, `sd.R`).

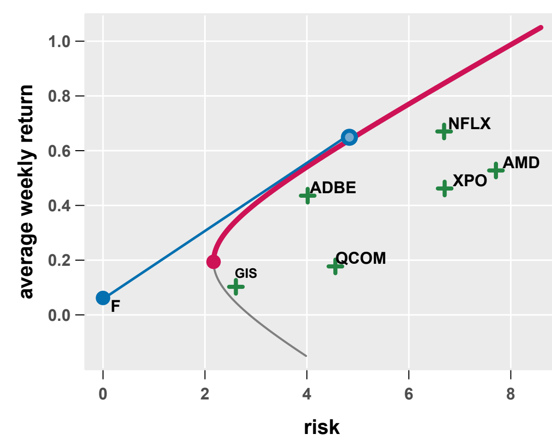


Figure 6.3: The efficient frontier (red line), tangency portfolio (T) and 6 stocks.

Finding portfolio with `solve.QP()` function. The `solve.QP()` function from R's `quadprog` package can be used for more general quadratic programming problems.

```
library(quadprog)
args(solve.QP)
## function (Dmat, dvec, Amat, bvec, meq = 0, factorized = FALSE)
```

The `solve.QP()` function solves the quadratic programming problem, by its arguments, `Dmat`, `dvec`, `Amat` and `bvec`, as follows,

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \frac{1}{2}\boldsymbol{x}^T \texttt{Dmat}\, \boldsymbol{x} - \texttt{dvec}^T \boldsymbol{x} \tag{6.5}$$

$$\text{subject to} \quad \texttt{Amat}^T \boldsymbol{x} \geq \texttt{bvec} \tag{6.6}$$

where (6.5) is the objective function, (6.6) is a set of constraints and $\boldsymbol{x}$ is the solution. We always set

$$\texttt{Dmat} = \underset{N\times N}{\boldsymbol{\Sigma}} \quad \text{and} \quad \texttt{dvec} = \underset{N\times 1}{\boldsymbol{0}}. \tag{6.7}$$

The constraints (6.6) are set with `Amat` and `bvec`, they can be equality (=) constraints or greater-or-equal (≥) constraints. The equality ones should always be specified first with the number of them indicated in the `meg` argument.

Though the explicit solutions are available when short selling is allowed, we can also solve with `solve.QP()`. We only need to specify appropriate `Amat` and `bvec` for the constraints in (6.6).

Minimum variance portfolio. For the minimum variance, the constraint is $\texttt{Amat}^T = 1$. Thus, $\texttt{Amat} = \boldsymbol{1}$ and `bvec = 1` and `meq = 1`. Note, `Amat` has to be a matrix.


```
## solve.QP()--mininum variance portfolio
zeros = rep(0,N); ones = rep(1,N)
Amat = as.matrix(ones)
bvec = 1
out = solve.QP(Dmat = y.S, dvec = zeros, Amat = Amat, bvec = 1, meq = 1)
names(out)

## [1] "solution"            "value"              "unconstrained.solution"
## [4] "iterations"          "Lagrangian"         "iact"

w.min = out$solution; names(w.min) = syb
cat("Portfolio:"); w.min

## Portfolio:
##      ADBE       AMD       GIS      NFLX      QCOM       XPO
##  0.183408 -0.019188  0.656843  0.023100  0.107090  0.048747

c(return = sum(w.min*y.mu), risk =sqrt(2*out$value))

##  return    risk
## 0.19448 2.16857
```

The portfolio $\boldsymbol{w}$ is given in `$solution`, and the resulting minimized objective function, $\frac{1}{2}\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}$ is given in `$value`.

Portfolio with a prespecified return. If There is a pre-specified risky asset return $\mu_{R_p} = m$, then the constraints are $\boldsymbol{\mu}^T\boldsymbol{w} = m$ and $\boldsymbol{1}^T\boldsymbol{w} = 1$ There are 2 equality (=) constraints, thus we set `meg = 2` and write the 2 constraints as

$$\texttt{Amat} = \underset{N\times 2}{\begin{pmatrix} \boldsymbol{\mu} & \boldsymbol{1} \end{pmatrix}}, \quad \texttt{bvec} = \underset{2\times 1}{\begin{pmatrix} m \\ 1 \end{pmatrix}}. \tag{6.8}$$

For example, our target risky asset return $m = 0.45\%$, we can compute as follows.

```r
## solve.QP() -- risky assets with a target return 0.45
Amat = cbind(y.mu, ones)
bvec = c(0.45,1)
out = solve.QP(Dmat = y.S, dvec =zeros, Amat = Amat, bvec = bvec, meq = 2)
w = out$solution; names(w) = syb
cat("Portfolio:");w

## Portfolio:
##      ADBE       AMD       GIS      NFLX      QCOM       XPO
##  0.463114  0.062641  0.276280  0.231740 -0.164696  0.130921

cat("Risk with .45% return is", sqrt(2*out$value))

## Risk with .45% return is 3.296
```

The lowest possible risk portfolio with return 0.45% is,

$$\boldsymbol{w} = (0.4631, 0.0626, 0.2763, 0.2317, -0.1647, 0.1309)^T.$$

The value of the lowest risk of this portfolio is $\hat{\sigma}_p = \sqrt{\boldsymbol{w}^T \hat{\boldsymbol{\Sigma}} \boldsymbol{w}} = 3.296$. A reminder that our objective function of the quadratic programming problem is $\frac{1}{2}\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$.

Efficient frontier plot To obtain a set of returns and their risks, we evaluate a sequence of μ_{R_p}'s with `solve.QP()` in a loop.

```r
## solve.QP() -- Efficient frontier
m.R = seq(-0.15, 1, 0.001);
Amat = cbind(y.mu, ones)
sd.R = c();
for(i in 1:length(m.R)){
        bvec = c(m.R[i],1)
        out = solve.QP(Dmat = y.S, dvec = zeros, Amat=Amat, bvec = bvec,
                       meq = 2);
        sd.R[i] = sqrt(2*out$value)
}
```


Tangency portfolio First we obtain $\boldsymbol{w}_*$ a portfolio on the tangent line with a target return m, it can be any value within the range of $\boldsymbol{\mu}$. The tangency portfolio is proportional to $\boldsymbol{w}_*$ and can be obtained by rescaling $\boldsymbol{w}_*$. The only constraint is the sum of excess return equals to m, be cause the weights of $\boldsymbol{w}_*$ do not need to sum up to 1 except the tangency portfolio.

```r
## solve.QP() -- Tangency portfolio
Amat = as.matrix(y.mu-mu.f) ## excess returns
bvec = 0.3    ##  a return value
out = solve.QP(Dmat = y.S, dvec = zeros, Amat = Amat, bvec = bvec, meq = 1)
w.star = out$solution;  names(w.star) = syb
w.T = w.star/sum(w.star); ## re-scalse s.t. sum of w is 1
cat("Portfolio:");w.T

## Portfolio:
##      ADBE        AMD        GIS       NFLX       QCOM        XPO
##  0.6680665  0.1226005 -0.0025751  0.3846198 -0.3638459  0.1911342

mu.T = sum(w.T*y.mu); s.T = sqrt(2*out$value)/sum(out$solution)
c(return = mu.T, risk = s.T)

##  return    risk
## 0.63723 4.81666
```

Combining risk-free and tangency portfolio. The portfolio with lowest risk for a target return $\mu_p < \mu_T$ and the portfolio with the optimal return for a given risk $\sigma_p < \sigma_T$ can be obtained by the estimated analogues of (6.2), namely,

$$\hat{\mu}_p = \mu_f + w(\hat{\mu}_T - \mu_f) \quad \text{and} \quad \hat{\sigma}_p = w\hat{\sigma}_T. \tag{6.9}$$

For example, if the target weekly return is $\mu_p = 0.45\%$, plug it into the first equation of (6.9) to find the total weight for the risky assets. The lowest risk can then be computed and it is 3.1927 %. The portfolio consists of 33.72% risk free asset and 66.28% tangency portfolio.

```
## Target portfolio return 0.45%
m.p = 0.45  ## target portfolio return
w.r = (m.p - mu.f)/(mu.T - mu.f) ## risky asset weight
w.p = c(risk.free = 1-w.r, w.r*w.T) ## Portfolio weights
cat("The lowest risk of portfolio with 0.45% return is:", w.r*s.T)
## The lowest risk of portfolio with 0.45% return is: 3.1927
cat("Portfolio weights:"); round(w.p,4)

## Portfolio weights:
## risk.free      ADBE       AMD       GIS      NFLX      QCOM       XPO
##    0.3372    0.4428    0.0813   -0.0017    0.2549   -0.2412    0.1267
```

If our risk tolerance is 2.5%, then apply the 2nd equation of (6.9) to get the weight for risky assets. The optimal return is 0.37% when the risk is 2.5%. The portfolio consists of 48.1% risk free asset and 51.9% risky assets.

```
## Portfolio risk allowance 2.5%
risk = 2.5  ## target portfolio 2.5%
w.r = risk/s.T   ##  risky asset weight
w.p = c(risk.free = 1-w.r, w.r*w.T) ## portfolio weights
mu.p = (1-w.r)*mu.f + w.r*mu.T  ## portfolio return
cat("The maximum return of portfolio with risk 2.5% is:", mu.p);

## The maximum return of portfolio with risk 2.5% is: 0.37014

cat("Portfolio weights:"); round(w.p,4)

## Portfolio weights:
## risk.free      ADBE       AMD       GIS      NFLX      QCOM       XPO
##    0.4810    0.3467    0.0636   -0.0013    0.1996   -0.1888    0.0992
```

We stress that equation (6.9) applies to a portfolio with return $\mu_p < mu_T$. If one's target return exceeds the return of tangency portfolio,

then the portfolio is an efficient frontier portfolio without combining with risk-free asset.

Portfolio without short selling

When no short selling is allowed, there are two major differences in configuring the quadratic programming problem. First of all, N additional constraints on weights w_i are required for non-negativity, that is, $w_i \geq 0, i = 1, \ldots, N$. Fo the case that a pre-specified risky asset return $\mu_{R_p} = m$ is given, then together with the two constraints in (6.7), there are $N+2$ constraints with

$$\texttt{Amtx} = \underset{N\times(N+2)}{\begin{pmatrix} \boldsymbol{\mu} & \mathbf{1} & \mathbf{I}_N \end{pmatrix}}, \qquad \text{and} \qquad \texttt{bvec} = \underset{(N+2)\times 1}{\begin{pmatrix} m \\ 1 \\ \mathbf{0} \end{pmatrix}} \tag{6.10}$$

The first two constraints in "$\texttt{Amat}^T \boldsymbol{w} \geq \texttt{bvec}$" are equality constraints, `meq = 2` should be specified as before.

Secondly, the candidate values for m must be in the range of $\mu_i, i = 1, \ldots, N$. This is clearly due to the nonnegative weights. Specifying a m outside of the range of $\boldsymbol{\mu}$ will have an inconsistent $\texttt{Amat}^T \boldsymbol{w} \geq \texttt{bvec}$, there exists no solution of $\boldsymbol{w}$ satisfying such constraints.

Minimum variance portfolio. The constraints would be the set of (6.10) without the first constraint.

```
## minimum variance portfolio
Amat = cbind(ones, diag(N))
bvec = c(1,zeros)
out = solve.QP(Dmat = y.S, dvec = zeros, Amat = Amat, bvec = bvec, meq = 1)
```

```
w.min = out$solution; w.min = w.min*(abs(w.min) > 10e-7);names(w.min) = syb;
mu.min = sum(w.min*y.mu); sd.min = sqrt(2*out$val)
w.min

##     ADBE      AMD      GIS     NFLX     QCOM      XPO
## 0.177275 0.000000 0.656511 0.020574 0.100156 0.045484

c(return = mu.min, risk = sd.min)

##  return    risk
## 0.19743 2.17241
```

The risk of minimum variance portfolio 2.1724% is bounded below by that of the short selling allowed case (2.16857%) due to more constraints imposed.

Portfolio with a prespecified return. If m is our intended risky asset return, the constraint equation is (6.10), say $m = 0.4$.

```
m = 0.4
Amat = cbind(y.mu, ones, diag(N))
bvec = c(m,1,zeros)
out = solve.QP(Dmat = y.S, dvec = zeros, Amat = Amat, bvec = bvec, meq = 2)
w = out$solution; w = w*(abs(w) > 10e-7); names(w) = syb;
cat("Portfolio weights for a return 0.4%:"); w;

## Portfolio weights for a return 0.4%:
##       ADBE        AMD        GIS       NFLX       QCOM        XPO
## 0.38365350 0.03717384 0.27072947 0.20667763 0.00000000 0.10176555

cat("Risk is", sqrt(2*out$val));

## Risk is 2.988273
```

Efficient frontier plot. To obtain a set of (risk, return) for plotting, we calculate a sequence of candidate μ_{R_p} between $\min_i \mu_i$ and $\max_i \mu_i$, the $\pm.0005$ term in the calculation below is to ensure this restriction.

The set of (risk, return) efficient set.

```
## (risk, return) set, no short sales
m.R = seq(round(min(y.mu)+.005,3), round(max(y.mu)-.005,3), 0.001);
Amat = cbind(y.mu, ones, diag(N)) ## for positive w
sd.R = c();
for(i in 1:length(m.R)){
        bvec = c(m.R[i],1, zeros) ## for nonnegative w
        out= solve.QP(y.S, dvec = zeros, Amat=Amat, bvec = bvec, meq = 2);
        sd.R[i] = sqrt(2*out$value)
}
```

Tangency portfolio. As before, we find a portfolio $\boldsymbol{w}_*$ first, then rescale it such that the sum of the weights is 1. The constraints consist of the non-negative weight constraints and the linear sum of excess return $\boldsymbol{w}_*^T(\boldsymbol{\mu} - \mu_f \mathbf{1})$ equal to a pre-specified value m.

```
## Tangency Portfolio
Amat = cbind(y.mu-mu.f,diag(N)) ## excess returns
bvec = c(0.4,zeros)   ##  0.3 a return value
out = solve.QP(Dmat = y.S, dvec = zeros, Amat = Amat, bvec = bvec, meq = 1)
w.star = out$solution;  names(w.star) = syb
w.T = w.star/sum(w.star); ## re-scalse s.t. sum of w is 1
w.T  = w.T*(abs(w.T) > 10e-7)
cat("Portfolio:");w.T

## Portfolio:
##     ADBE      AMD      GIS     NFLX     QCOM      XPO
## 0.483988 0.065254 0.000000 0.325551 0.000000 0.125207

mu.T = sum(w.T*y.mu); s.T = sqrt(2*out$value)/sum(out$solution)
c(return = mu.T, risk = s.T)

##  return    risk
## 0.52084 3.98398
```

Figure 6.4 plot all available portfolios, the range of portfolio return is between the return of General Mills (.1034%) and that of Neflix (.6693%). It also includes the minimum variance and tangency portfolios.

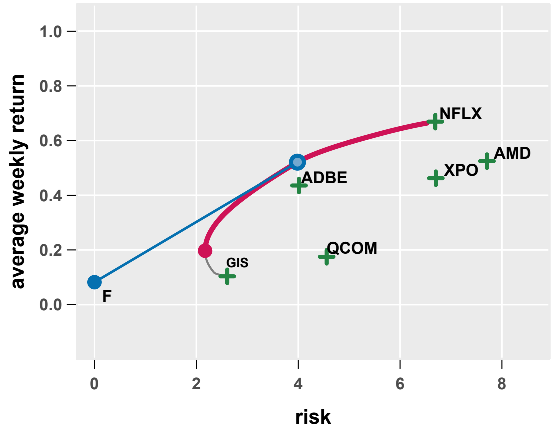


Figure 6.4: The efficient frontier (red line), tangency portfolio (T) and 6 stocks without short position.

Combining risk-free and tangency portfolio. We compute the same target return as before, $\mu_p = 0.45\%$. The first equation of (6.9) gives the weight for the risky asset, yields the risk of 3.341% which is higher than the short selling allowed case (3.1927%) as expected. The portfolio consists of 16.14% risk free asset, this is much lower than the previous case (33.72%).

```
## Target return 0.45% no short selling
w.r = (0.45-mu.f)/(mu.T-mu.f)
cat("The lowest risk of portfolio with 0.45% return is:", w.r*s.T,
    "\n The weights are:\n");c(risk_free = (1-w.r), w.r*w.T)

## The lowest risk of portfolio with 0.45% return is: 3.341
##  The weights are:
## risk_free      ADBE       AMD       GIS      NFLX      QCOM       XPO
##  0.161392  0.405876  0.054722  0.000000  0.273009  0.000000  0.104999
```


For the risk tolerance of 2.5%, the maximum portfolio return is 0.357%, lower than that of the short selling allowed case (0.370%).

```
risk = 2.5
w.r = risk/s.T # risk allowed: sd.p = 2.5%
mu.p = (1 - w.r)*mu.f + w.r*mu.T
cat("The maximum return of portfolio with risk 2.5% is:", mu.p)

## The maximum return of portfolio with risk 2.5% is: 0.35735

round(c(risk_free = (1-w.r)*mu.f, w.r*w.T),4)

## risk_free      ADBE       AMD       GIS      NFLX      QCOM       XPO
##    0.0305    0.3037    0.0409    0.0000    0.2043    0.0000    0.0786
```

Portfolios with Box Constraints

Examining the previous two examples with and without short sales, both tangency portfolios have most weight on Adobe and Netflix among the 6 stock returns. If a pool of risky assets considered has one or few stocks with much higher returns, the tangency portfolio would end up concentrating on these stocks. One way to avoid a low degree diversified portfolio is to set bounds for w_i, for example $-0.2 \le w_i \le 0.4, i = 1, \ldots, N$ so that no w_i will exceed 0.4 or go below -0.4. In general, a set of box constraints can be imposed to improve diversification. The `solve.QP()` function takes only equality "=" and greater of equal "≥". To impose a set of box constraints in to `solve.QP()`, we write $-b_2 \le w_i \le b_1$ as follows, for $b_1 > 0$ and $b_2 \ge 0$,

$$-w_i \ge -b_1 \quad \text{and} \quad w_i \ge -b_2, \qquad i = 1, \ldots, N.$$

For a pre-specified risky asset return $\mu_{R_p} = m$, the quadratic programing problem has a total of $2N+2$ constraints with

$$\underset{N\times(2N+2)}{\texttt{Amat}} = \begin{pmatrix} \boldsymbol{\mu} & \mathbf{1} & -\mathbf{I}_N & \mathbf{I}_N \end{pmatrix} \quad \text{and} \quad \texttt{bvect} = \underset{(2N+2)\times 1}{\begin{pmatrix} m \\ 1 \\ -b_1\mathbf{1} \\ -b_2\mathbf{1} \end{pmatrix}} \tag{6.11}$$

As before, `meg = 2` indicates the first two constraints are equality constraints as (6.8).

Finding the range of μ_{R_p} The question remains, what is the range of μ_p in (6.11) that would guarantee a $\boldsymbol{w}$ such that $\texttt{Amat}^T\boldsymbol{w} \geq \texttt{bvec}$ being feasible? This is a linear programing problem,

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minmize/maximize}} \quad & \boldsymbol{\mu}^T\boldsymbol{w} & (6.12) \\ \text{subject to} \quad & -b_2 \leq w_i \leq b_1, \quad i=1,\ldots,N & (6.13) \\ & \mathbf{1}^T\boldsymbol{w} = 1 & \end{aligned}$$

The range of $\mu_{R_p} = \boldsymbol{\mu}^T\boldsymbol{w}$, can be obtained by solving the linear programing problem (6.12) and (6.13) twice, once with minimization and once with maximization. This step needs to be done first before solving the quadratic programming problem of (6.11). There are several R packages for linear programming, all of them are set up simpler for $b_2 = 0$ which is the case of no short sale, but more involved for $b_2 > 0$ which is the case of short selling being allowed. We will discuss both cases separately. The R function will be used is `solveLP()` from R's `linprog` package.


```
library(linprog)
args(solveLP)
## function (cvec, bvec, Amat, maximum = FALSE, const.dir = rep("<=",
##     length(bvec)), maxiter = 1000, zero = 1e-09, tol = 1e-06,
##     dualtol = tol, lpSolve = FALSE, solve.dual = FALSE, verbose = 0)
```

The `solveLP()` is for solving the linear programming problem expressed by its arguments as, for $\boldsymbol{x} = (x_1,\ldots,x_N)^T$ with $x_i \geq 0$,

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minmize/maximize}} \quad & \texttt{cvec}^T\boldsymbol{x} & (6.14) \\ \text{subject to} \quad & \texttt{Amat}\,\boldsymbol{x} \overset{\leq}{\underset{\geq}{=}} \texttt{bvec} & (6.15) \end{aligned}$$

Set `maximum = T` for maximizing and `maximum = F` for minimizing. Notice that all elements in $\boldsymbol{x}$ are nonnegative, we cannot simply let $\boldsymbol{w}$ be $\boldsymbol{x}$ unless there is no short sales. Despite the same notations are being used for constraints (6.15) as those of `solve.QP()` in (6.6), we should mind the 2 differences, (1) `Amat` in (6.15) does not take transpose; (2) the inequality can be set for any direction individually with the `const.dir` argument, while (6.6) for `solve.QP()` can only take $=$ or $\geq$.

Case 1. No short sale with upper constraints. The objective function is $\boldsymbol{\mu}^T\boldsymbol{w}$. The box constraints with $b_2 = 0$ and $b_1 > 0$ are $0 \leq w_i \leq b_1$, $i = 1,\ldots,N$. The lower bound is giving by default, only the upper bound is required. Togeter with the sum of weights being 1, there are a total of $N+1$ constraints with

$$\texttt{cvec} = \boldsymbol{\mu}, \quad \texttt{Amat} = \underset{(N+1)\times N}{\begin{pmatrix} \mathbf{I}_N \\ \mathbf{1}^T \end{pmatrix}} \qquad \text{bvec} = \underset{(N+1)\times 1}{\begin{pmatrix} b_1\mathbf{1} \\ 1 \end{pmatrix}}. \tag{6.16}$$

Set `const.dir = c(rep("<=", N),1)` to indicate the first N constraints are with "$\leq$" and the last one is an equality constraint. When there are both equality and inequality constraints such as ours, we also need to set `lpSolve = T` to call `lpSolve` package.

Suppose the upper bound $b_1 = 0.35$. Below is the calculation of the feasible range of μ_{R_p} is $(0.2280981, 0.5565497)$.

```
# Finding the range of m.R ###
library(linprog)
b1 = 0.35
cvec = c(y.mu)
Amat.lp = rbind(diag(N),  ones)
bvec.lp = c(rep(b1,N), 1)
inequ = c(rep("<=", N), "=")
min.lp = solveLP(cvec = cvec, bvec = bvec.lp, Amat = Amat.lp,
     lpSolve=T, const.dir = inequ, maximum = F)
max.lp = solveLP(cvec = cvec, bvec = bvec.lp, Amat = Amat.lp,
     lpSolve=T, const.dir = inequ, maximum = T)
mu.lim = c(lower = min.lp$opt, upper = max.lp$opt);
cat("The range of mu.p is:\t");mu.lim;

## The range of mu.p is:
##     lower     upper
## 0.2280981 0.5565497
```

With the range of μ_{R_p}, we can calculate the mean variance efficient set using `solve.QP()` with constraints in (6.11).

```
## Efficient set (sd.R, m.R) computation
m.R = seq(round(mu.lim[1] + 0.0005,3), round(mu.lim[2] - 0.0005,3),0.001)
sd.R = c();
wmat = matrix(nrow = length(m.R), ncol = N); dimnames(wmat)[[2]] = syb;
Amat = cbind(y.mu, rep(1,N), -diag(N) ,diag(N));
for(i in 1:length(m.R)){
        bvec = c(m.R[i],1, rep(-b1, N), zeros)
        out = solve.QP(y.S, dvec = zeros, Amat=Amat, bvec = bvec, meq = 2);
        sd.R[i] = sqrt(2*out$value)
```



```
        wmat[i,] = out$solution
}
wmat = wmat*(abs(wmat) > 10e-7)
```

In the above calculation, we also store each portfolio weight for each iteration. We can find the minimum variance and tangency portfolios by finding the minimum of `sd.R` and the highest Sharpe ratio among all the portfolios computed.

```
## Short sales allowed with upper weight bound 0.35
## Minimum variance portfolio ##
i.min = which.min(sd.R);  ## index with smallest sd
w.min = wmat[i.min,];mu.min = m.R[i.min]; sd.min = sd.R[i.min];
cat("Minimum variance portfolio:"); w.min

## Minimum variance portfolio:
##       ADBE        AMD        GIS       NFLX       QCOM        XPO
## 0.32543849 0.00000000 0.35000000 0.05686910 0.20799364 0.05969877

c(return = mu.min, risk = sd.min)

##   return     risk
## 0.280000 2.516881

## Tangency Portfolio ##
i.T = which.max((m.R - mu.f)/sd.R);   ## index with highest Sharpe ratio
w.T = wmat[i.T,]; mu.T = m.R[i.T]; sd.T = sd.R[i.T]
cat("Tangency portfolio:"); w.T

## Tangency portfolio:
##       ADBE        AMD        GIS       NFLX       QCOM        XPO
## 0.35000000 0.08263675 0.07386075 0.34495059 0.00000000 0.14855191

c(return = mu.T, risk = sd.T)

##   return     risk
## 0.503000 3.854836
```

Note that the minimum variance portfolio can be computed using `solve.QP()` with (6.11) without the first constraint directly as before. However, we cannot compute the tangency portfolio directly when here is weight bound other than 0, because the bound will no longer hold after re-scalling.

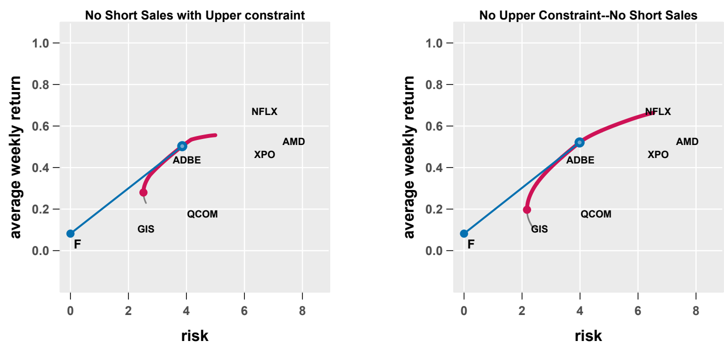


Figure 6.5: The efficient frontier (red line), tangency portfolio (T) of 6 stocks without short sales. The left plot is with additional upper weight constraints, $0 \le w_i \le 0.35$, comparing the right plot without upper weight constraint, same as Figure 6.4.

Figure 6.5 shows efficient frontier, minimum variance and tangency portfolios of no short selling allowed with upper portfolio weight bound. We also include the plot without upper weight constraint for comparison.

Case 2: Short selling with box constraints . In this case both $b_1 > 0$ and $b_2 > 0$ so that $-b_2 < w_i < b_1$. We cannot apply `solveLP()` directly as before because the solutions of (6.14) and (6.15) are only for $w_i \ge 0$ if we let $\boldsymbol{x}$ be $\boldsymbol{w}$.

In this case, we would instead let $\boldsymbol{w} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. The objective function (6.12) is now $(\boldsymbol{\mu}^T, -\boldsymbol{\mu}^T)^T \boldsymbol{x}$, where $\boldsymbol{x} = (\boldsymbol{x}_1^T, -\boldsymbol{x}_2^T)^T$.

If $-b_2 \le x_1 - x_2 < b_1$, $b_1, b_2 > 0$ with $x_1, x_2 \ge 0$ then $x_1 \le b_1$ and

$x_2 \le b_2$. Since $w_i = x_{1i} - x_{2i}$, the constraints $-b_2 \le< w_i \le b_1, i = 1, \ldots, N$ in (6.13) can be written in terms of x_{1i} and x_{2i} which are all nonnegative. We can use `solveLP()` with arguments,

$$\texttt{cvec} = \underset{2N\times 1}{\begin{pmatrix} \boldsymbol{\mu} \\ -\boldsymbol{\mu} \end{pmatrix}}, \quad \texttt{Amat} = \underset{(2N+1)\times 2N}{\begin{pmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \\ \mathbf{1}^T & -\mathbf{1}^T \end{pmatrix}}, \quad \texttt{bvec} = \underset{(2N+1)\times 1}{\begin{pmatrix} b_1 \mathbf{1} \\ b_2 \mathbf{1} \\ 1 \end{pmatrix}}. \tag{6.17}$$

Set `const.dir = c(rep("<=", 2*N), "=")` to indicate the first $2N$ inequality constraints and the last equality constraint. As before, set `lpSolve = T` for a mixed set of equality and inequality constraints.

For example, the weights are constrained in the range of $-0.1 \le w_i \le 0.5$. We set $b_1 = 0.5$ and $b_2 = 0.1$.

```
## Feasible range of m.R with b1 = 0.5 and b2 = 0.1
library(linprog)
b1 = 0.5; b2 = 0.1
cvec = c(y.mu, -y.mu)
Amat.lp = rbind(diag(2*N), c(ones, -ones))
bvec.lp = c(rep(b1,N), rep(b2,N),1)
inequ = c(rep("<=", 2*N), "=")
min.lp = solveLP(cvec = cvec, bvec = bvec.lp, Amat = Amat.lp,
     lpSolve=T, const.dir = inequ, maximum = F)
max.lp = solveLP(cvec = cvec, bvec = bvec.lp, Amat = Amat.lp,
     lpSolve=T, const.dir = inequ, maximum = T)
mu.lim = c(lower = min.lp$opt, upper = max.lp$opt); mu.lim;

##     lower     upper
## 0.1042263 0.6642508
```

The feasible range of μ_p is (0.1042263, 0.6642508). We can calculate efficient frontier using `solve.PQ()` with objective function (6.7) and constraints in (6.11).

```
## Efficient set (sd.R, m.R) computation
m.R = seq(round(mu.lim[1] + .0005, 3), round(mu.lim[2]-.0005, 3), 0.001)
sd.R = c();
wmat = matrix(nrow = length(m.R), ncol = N); colnames(wmat) = syb;
Amat = cbind(y.mu, ones, -diag(N), diag(N));
for(i in 1:length(m.R)){
        bvec = c(m.R[i],1, rep(-b1, N), rep(-b2,N))
        out = solve.QP(y.S, dvec = zeros, Amat=Amat, bvec = bvec, meq = 2);
        sd.R[i] = sqrt(2*out$value)
        wmat[i,] = out$solution
}
```

We can find the minimum variance and tangency portfolios by finding the minimum of `sd.R` and the highest Sharpe ratio among all the portfolios computed.

```
## Box constraints -0.1 < wi < 0.5
## minimum variance portfolio ##
i.min = which.min(sd.R);  ## index with smallest sd
w.min = wmat[i.min,];mu.min = m.R[i.min]; sd.min = sd.R[i.min];
cat("Minimum variance portfolio:"); w.min

## Minimum variance portfolio:
##        ADBE         AMD         GIS        NFLX        QCOM         XPO
##  0.25899546 -0.01818124  0.50000000  0.04168284  0.16158927  0.05591367

c(return = mu.min, risk = sd.min)

##   return     risk
## 0.237000 2.263979

## Tangency Portfolio ##
i.T = which.max((m.R - mu.f)/sd.R);   ## index with highest Sharpe ratio
w.T = wmat[i.T,]; mu.T = m.R[i.T]; sd.T = sd.R[i.T]
cat("Tangency portfolio:"); w.T

## Tangency portfolio:
##        ADBE         AMD         GIS        NFLX        QCOM         XPO
##  0.50000000  0.08020797  0.04665042  0.32873000 -0.10000000  0.14441162

c(return = mu.T, risk = sd.T)

##   return     risk
## 0.534000 4.005936
```

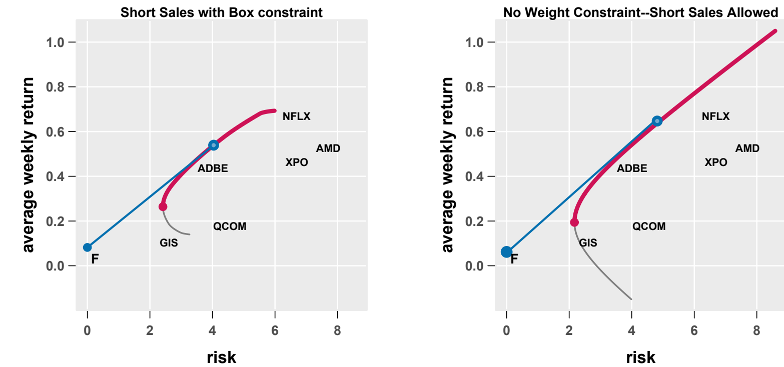



Figure 6.6: The efficient frontier (red line), tangency portfolio (T) of 6 stocks allowing short sales. The left plot is with constrained weights, $-0.1 \le w_i \le 0.5$, comparing without weight constraint at the right, same as Figure 6.3.

The left plot of Figure 6.6 shows the efficient frontier computed above with box constraints, $-0.1 < w_i < 0.5$. We also include the plot without constraint on the profolio weights at the right plot for comparison. The feasible interval of μ_{R_p} is much shorter due to the constraints imposed.

Utility

Mean-variance theory deals with the combination of assets into a portfolio. It has diversication and maximization of expected utility as foundational principles. The utility of an amount X of money is said to be $U(X)$ where the *utility function* generally has the following properties,

1. $U(0) = 0$;
2. $U(X)$ is strictly increasing or $U'(X) > 0$.
3. $U(X)$ is concave or $U''(X) < 0$.

Assumption 1 is not necessary but is reasonable and states that the utility of 0 dollars is 0. Assumption 2 states that more money is better than less. Assumption 3 implies that the more money we have the less we value an extra dollar and is called *risk aversion*. A utility function measures investor's relative preference for different levels of total wealth.

Let X_0 be an investor's initial wealth, then the wealth after one holding period is $X = X_0(1+R)$, where R is the return from the investmenst. It is assumed that a rational person will make investment decisions to maximize

$$E\{U(X)\} = E\big[U\{X_0(1+R)\}\big], \tag{6.18}$$

Each individual is assumed to have his or her own utility function and two different rational people may make different decisions because they have different utility functions.

In general, mean-variance portfolio selection and the expected utility approach are not equivalent. But if returns are normally distributed, we have the following result.

Result 6.2. *If returns on all portfolios are normally distributed and if U satisfies Assumptions 3, then the portfolio that maximizes expected utility is on the efficient frontier*.

Under the assumption of normality on portfolios, if one chose a portfolio to maximize expected utility, then a mean-variance efficient portfolio would be selected. Exactly which portfolio on the efficient frontier one chooses would depend on one's utility function.

The normality assumption in Result 6.2 can be weakened to the more realistic assumption that the vector of returns on the assets is a multivariate scale mixture. e.g., a multivariate t distribution.

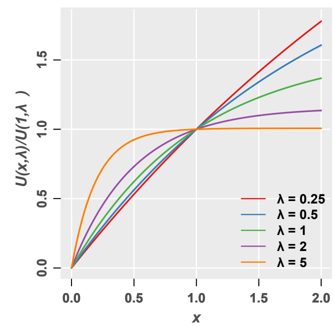
Utility function is unique up to affine transformations. That is, if $U(\cdot)$ is utility function an investor, then so does $U^{\dagger}(\cdot) = b_0 + b_1 U(\cdot)$, where $b_1 > 0$, both utility functions will reach the same decision.

A common class of utility functions is

$$U(x,\lambda) = 1 - \exp(-\lambda x), \qquad \lambda > 0 \tag{6.19}$$

The quantity λ determines the amount of risk aversion. It is easy to see that Assumptions 1, 2, and 3 are met. As $x \to \infty$, $U(x,\lambda) = 1$.

Multiplying a utility function by a positive constant willl not affect which decision maximizes utility and can standardize utility functions to make them more comparable. In the plot shown here, $\tilde{U}(x;\lambda) = U(x;\lambda)/U(0,\lambda)$ is plotted for $\lambda = 0.25, 0.5, 1, 2$ and 5 for increasing degrees of risk aversion. These utility functions are standardized so that the utility corresponding to a return of 0 is always 1.



When the utility function is given by (6.19), then (6.18) becomes

$$E\{U(X)\} = 1 - E\big[\exp\{-\lambda X_0(1+R)\}\big]. \tag{6.20}$$

If R is normally distributed, the expectation of the RHS is the expectation of a lognormal random variable. Thus (6.20) is

$$E(U(X)) = 1 - \exp\left\{ -\lambda X_0(1+E[R]) + (\lambda X_0)^2 \frac{\text{var}[R]}{2} \right\}$$

For given values of λ and X_0, the expected utility is maximized by maximizing

$$E(R) - (\lambda X_0)\frac{\text{var}(R)}{2}$$

Therefore, using the previous notions, one selects the allocation vector $\boldsymbol{w}$ of the portfolio to maximize

$$\boldsymbol{w}^T\boldsymbol{\mu} - (\lambda X_0)\frac{\boldsymbol{w}^T\Sigma\boldsymbol{w}}{2}, \qquad \text{subject to } \mathbf{1}^T\boldsymbol{w} = 1. \tag{6.21}$$

Maximizing (7.16) subject to linear constraints is a quadratic programming problem. As $\lambda \to 0$, the expected return and standard deviation of the return converge to ∞. Conversely, as $\lambda \to \infty$, the solution converges to the minimum variance portfolio. Therefore, as λ is varied from ∞ to 0, one finds all of the portfolios on the efficient frontier from left to right.

In other words, the utility optimization with the exponential utility when the return follows a multivariate normal distribution is the same as the Markowitz mean variance portfolio optimization.

Appendix

Some definitions

Margin account. A type of brokerage account which allows investors to purchase stocks or other financial products with borrowed funds. The margin account and the securities held within it serve as a collateral for the loan. The initial margin is the ratio of initial cash on the margin account to the purchase value of the securities. The maintenance margin is the minimum required ratio of cash on the margin account and current market value of securities. Once below the maintenance level, the investor receives a margin call asking her/him to deposit additional cash or liquidate some securities.

Short selling. A trader must have a margin account to engage short selling. With short-selling, one borrows a security usually from a broker-dealer hoping to sell high and later buy back low, thus profiting from a price fall. This can be viewed as borrowing money with the rate of interest being the return on the security in question, where the short-sellers wish the return to be as low as possible. Short-sellers can cover the short position at any time by purchasing the security back and return them to the broker or lender. Trader must account for any interest the broker charges or commissions on trades.

Lagrangian

In mathematical optimization, Lagrangian is the objective function plus the constraint multiplied by the Lagrange multiplier, $\boldsymbol{\ell}$. Consider the optimization problem

$$\begin{array}{ll} \text{maximize/minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{g}(\boldsymbol{x}) = \mathbf{0}\,, \end{array}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{g}(\boldsymbol{x})$ is a $m \times 1$ vector function in $\boldsymbol{g}(\boldsymbol{x})$. The Lagrangian function of the above problem is

$$L(\boldsymbol{x}, \boldsymbol{\ell}) = f(\boldsymbol{x}) + \boldsymbol{\ell}' \boldsymbol{g}(\boldsymbol{x})$$

which is a function of $n+m$ variables. The optimization problem can be solved by setting each of the $n+m$ derivatives to zero, that is,

$$\frac{\partial L}{\partial \boldsymbol{x}} = \mathbf{0}\,, \qquad \frac{\partial L}{\partial \boldsymbol{\ell}} = \mathbf{0}\,.$$

Mean-Variance Efficient Set

The portfolio selection problem is to find the portfolio $\boldsymbol{w}$ that minimizes the variance under a set of constraints, that is,

$$\begin{array}{ll} \min\limits_{\boldsymbol{w}} & \frac{1}{2} \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} \\ \text{subject to} & \boldsymbol{A}^T \boldsymbol{w} = \boldsymbol{b}\,. \end{array} \tag{6.22}$$

The corresponding Lagrangian is

$$L(\boldsymbol{w}, \boldsymbol{\ell}) = f(\boldsymbol{w}) + \boldsymbol{\ell}' \boldsymbol{g}(\boldsymbol{w}) = \frac{1}{2} \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} + \boldsymbol{\ell}^T (\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b})\,.$$

The equations to be solved are

$$\frac{\partial L}{\partial \boldsymbol{w}} = \boldsymbol{\Sigma} \boldsymbol{w} + \mathbf{A} \boldsymbol{\ell} = \mathbf{0}\,, \qquad \frac{\partial L}{\partial \boldsymbol{\ell}} = \mathbf{A}^T \boldsymbol{w} - \boldsymbol{b} = \mathbf{0} \tag{6.23}$$

or

$$\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{\ell} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{b} \end{pmatrix} \Longrightarrow \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{\ell} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ \boldsymbol{b} \end{pmatrix}.$$

IF the first matrix is full ranked. The formula of the inverse matrix for partitioned matrices gives the solution,

$$\boldsymbol{w} = \boldsymbol{\Sigma}^{-1} \boldsymbol{A} (\boldsymbol{A}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{A})^{-1} \boldsymbol{b}\,, \tag{6.24}$$

$$\boldsymbol{\ell} = -(\boldsymbol{A}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{A})^{-1} \boldsymbol{b}\,. \tag{6.25}$$

The minimized variance is

$$\boldsymbol{w}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{w} = \boldsymbol{b}^T (\boldsymbol{A}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{A})^{-1} \boldsymbol{b} = -\boldsymbol{b}^T \boldsymbol{\ell}\,. \tag{6.26}$$

Minimum variance portfolio. The constraint equation (6.22) has $\boldsymbol{A} = \mathbf{1}$, an $N \times 1$ vector and $b = 1$, a scaler. The minimum variance portfolio by (6.24) is

$$\boldsymbol{w}_{\text{min.v}} = \frac{\boldsymbol{\Sigma}^{-1} \mathbf{1}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}}\,. \tag{6.27}$$

and the variance $\sigma^2_{\text{min.v}} = 1/(\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1})$.

Portfolio with a target return. Let m denote the target return, then the constraint (6.22) has 2 linear constraints with $N \times 2$ matrix $\boldsymbol{A} = [\boldsymbol{\mu}, \mathbf{1}]$ and 2×1 vector $\boldsymbol{b} = (m, 1)^T$.

The solution (6.25) with some algebraic steps, gives

$$\ell_1 = \ell_1(m) = \frac{B - Am}{\Delta}\,, \qquad \ell_2 = \ell_2(m) = \frac{Bm - C}{\Delta}\,. \qquad (6.28)$$

where

$$A = \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\,, \qquad B = \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}, \qquad C = \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}, \qquad \Delta = AC - B^2\,.$$

The first order conditions with respect to $\boldsymbol{w}$ in (6.23) gives

$$\boldsymbol{w}_{opt}(m) = -\ell_1 \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \ell_2 \boldsymbol{\Sigma}^{-1}\mathbf{1} = -\ell_1(m)\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \ell_2(m)\boldsymbol{\Sigma}^{-1}\mathbf{1} \quad (6.29)$$

by the solution of $\boldsymbol{\ell}$ in (6.28). This is the optimal weight that yield the lowest risk for a given target return m. The variance of the portfolio $\boldsymbol{w}_{opt}(m)$ has mean m and

$$\sigma^2_{opt}(m) = \frac{Am^2 - 2Bm + C}{\Delta}$$

The efficient frontier is the set of the mean variance efficient set represented as a hyperbola in mean and standard deviation space

$$\mathscr{E} = \left\{ (\sigma_{\text{opt}}(m), m), m \geq 0 \right\}.$$

Portfolio with risk free asset. Suppose there is a risk free security with return μ_f and weight $w_0 = 1 - \mathbf{1}^T \boldsymbol{w}$. The expected return is $\mu_f + \boldsymbol{w}^T(\boldsymbol{\mu} - \mu_f \mathbf{1})$. If m is the target return, the only constraint is $\boldsymbol{w}^T(\boldsymbol{\mu} - \mu_f \mathbf{1}) = m - \mu_f$, because the weights in $\boldsymbol{w}$ do not need to sum up to 1. Define the excess return $\boldsymbol{\mu}_{ex} = \boldsymbol{\mu} - \mu_f \mathbf{1}$, then

$$\boldsymbol{A} = \boldsymbol{\mu} - \mu_f \mathbf{1} = \boldsymbol{\mu}_{ex} \qquad b = m - \mu_f\,.$$

By (6.24), the solution is

$$\boldsymbol{w}_* = \frac{(m - \mu_f)\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}{\boldsymbol{\mu}_{ex}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}\,.$$

The variance is

$$\sigma^2_*(m) = \frac{(m - \mu_f)^2}{\boldsymbol{\mu}_{ex}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}$$

The portfolio mean and standard deviation is as follows

$$\mathscr{E}_* = \left\{ (\sigma_*(m), m), 0 < m < \mu_T \right\}.$$

Tangency portfolio. The upper bound of m in $\mathscr{E}_*$ is the expected return of the tangency portfolio $\boldsymbol{w}_T$, which is the particular portfolio $\boldsymbol{w}_*$ with risky assets only. Thus $\boldsymbol{w}_T$ is proportional to $\boldsymbol{w}_*$ being rescaled so that it sums up to 1. With a few algebraic steps

$$\boldsymbol{w}_T = \frac{\boldsymbol{w}_*}{\mathbf{1}^T \boldsymbol{w}_*} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}} \qquad (6.30)$$

which has

$$\mu_T = \frac{\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}\,, \qquad \sigma_T^2 = \frac{\boldsymbol{\mu}_{ex}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}}{\left(\mathbf{1}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{ex}\right)^2}\,.$$