STAT 631 Homework 2

Jack Cunningham (jgavc@tamu.edu)

9/9/24

1)

We have function $f^\star(y|\xi)$ below:

$$f^\star(y|\xi) = \begin{cases} f(y\xi) & y < 0 \\ f(y/\xi) & y \geq 0 \end{cases}$$

We are tasked with integrating $f^\star(y|\xi)$:

$$\int_{-\infty}^{\infty} f^\star(y|\xi) = \int_{-\infty}^{0} f(y\xi)dy + \int_{0}^{\infty} f(y/\xi)dy$$

Let $u = y\xi$ so $du = \xi dy$. Let $w = y/\xi$ so $dw = \frac{1}{\xi}dy$. Then we have:

$$\xi \int_{-\infty}^{0} f(u)du + \xi^{-1} \int_{0}^{\infty} f(w)dw$$

$$\xi(F(0) - F(-\infty)) + \xi^{-1}(F(\infty) - F(0))$$

Since f is a symmetric pdf about 0 $F(0) = P(y < 0) = 1/2$. Also $F(-\infty) = 0$ and $F(\infty) = 1$ from basic rules about CDFs. Then we have our desired result:

$$\xi(1/2 - 0) + \xi^{-1}(1 - 1/2) = \frac{1}{2}(\xi + \xi^{-1})$$

2)

a)

The formula of Kurtosis for a random variable with a finite fourth moment is its fourth standardized moment.

$$Kur = \frac{E[(x-\mu)^4]}{\sigma^4}$$

We know that $Kur = 3$ for a normal distribution. This implies that $E[(x-\mu)^4] = 3\sigma^4$ for the normal distribution.

We have the below discrete density mixture:

$$f(x) = .95f_1(x) + .05f_2(x)$$

Where $f_1(x)$ is the density function of $N(0,1)$ and $f_2(x) = N(0,10)$.

A helpful fact for this problem is that $\mu^{(k)} = \sum_{i=1}^{n} p_i E_{f_i}[x_i^k]$.Essentially when we are computing a moment we can compute each distribution's moment, multiply them by their weight, and sum them together.

This leads to $f(x)$ having $\sigma^2 = p_1\sigma_1^2 + p_2\sigma_2^2 = .9(1) + .1(10) = 1.9$. And $E[(x-\mu)^4] = p_1E_{f_1}[(x-\mu_1)^4] + p_2Ef_2[(x-\mu_2)^4]$. So using what we know about the fourth central moment of the normal distribution we have:

$$E[(x-\mu)^4] = p_1(3\sigma_1^4) + p_2(3\sigma_2^4)$$

Then we can compute kurtosis with the below:

$$Kur = \frac{3(p_1\sigma_1^4 + p_2\sigma_2^4)}{(p_1\sigma_1^2 + p_2\sigma_2^2)^2}$$

```r
p_1 <- .9
sigma_1 <- 1
p_2 <- .1
sigma_2 <- sqrt(10)

Kur <- 3*(p_1*sigma_1^4+p_2*sigma_2^4)/(p_1*sigma_1^2+p_2*sigma_2^2)^2
Kur
```

```
[1] 9.058172
```

The kurtosis for this discrete mixture distribution is 9.058.

b)

Building off what I worked on in the previous question we have the below after substituting $\sigma_1 = 1, \sigma_2 = \sigma, p_1 = p, p_2 = (1 - p_1)$.

$$Kur(p, \sigma) = \frac{3(p + (1-p)\sigma^4)}{(p + (1-p)\sigma^2)^2}$$

3)

```
library(quantmod)
```

```
getSymbols("^GSPC",from = "2005-01-01", to = "2024-08-01")
```

```
[1] "GSPC"
```

```
x = weeklyReturn(Ad(GSPC),type = "log")*100
n = dim(x)[1]
```

a)

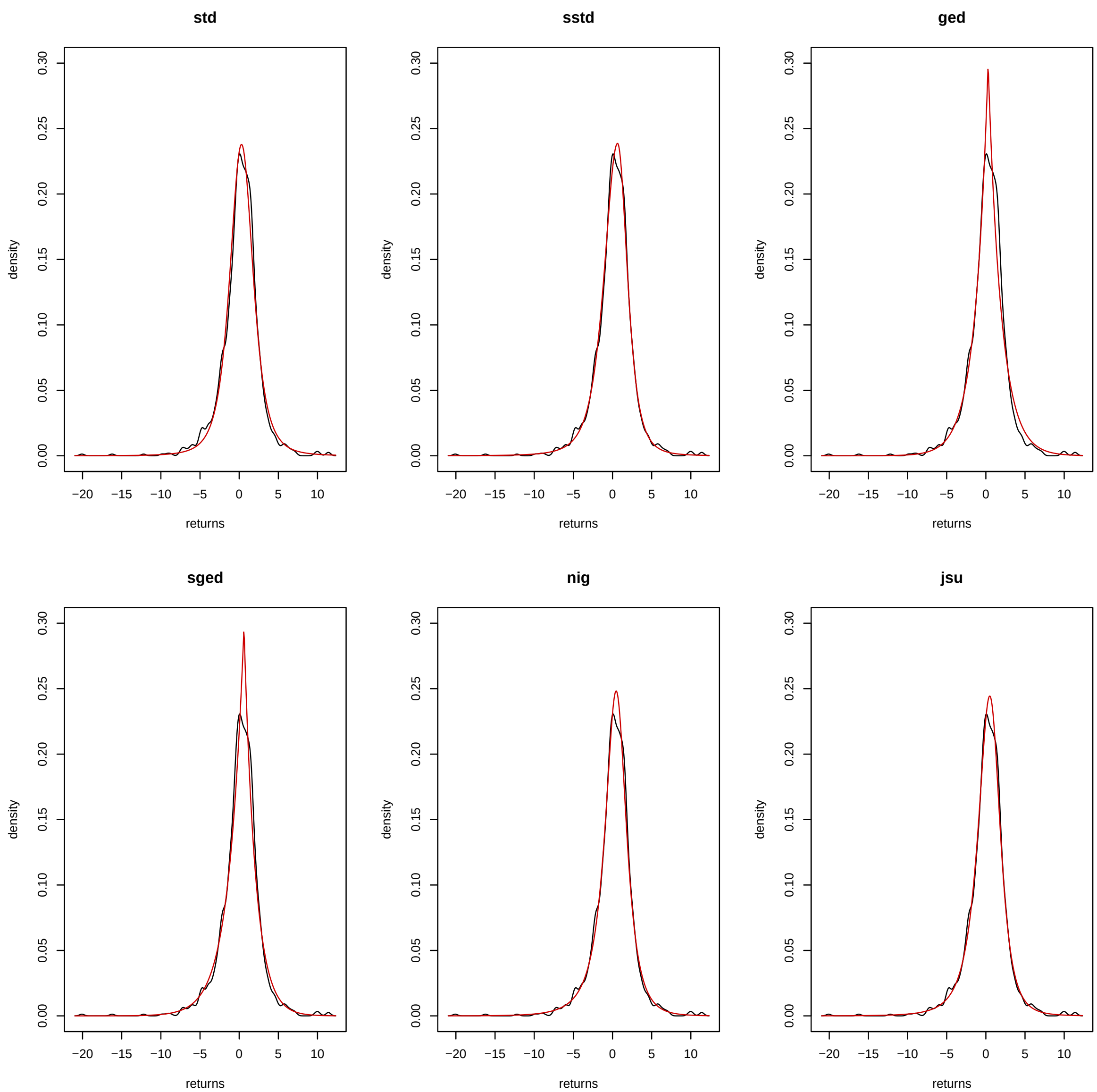
```
library(rugarch)
```

```
dists = c("std", "sstd", "ged", "sged", "nig", "jsu")
fits = vector("list", 6)
for (i in 1:6) fits[[i]]  = fitdist(dists[i], x)
```

b)

```
den = density(x, adjust = 0.75)
x0 = den$x; y0 = den$y
par(mfrow = c(2,3))

for(i in 1:length(dists)) {
  plot(x0, y0, type = "l", main = dists[i], ylim = c(0,0.3), xlab = "returns",
  ylab= "density")
  est = fits[[i]]$pars
  yi = ddist(dists[i], x0, mu = est["mu"], sigma = est["sigma"], skew =
  est["skew"], shape = est["shape"])
lines(x0, yi, col = "red3")
}
```


std
sstd
ged
sged
nig
jsu
density
returns
0.00
0.05
0.10
0.15
0.20
0.25
0.30
−20
−15
−10
−5
0
5
10