```
> options(digits=5)
> mu.f = 3.5/52;   ## weekly risk-free rate
> cat("weekly risk-free rate:", mu.f)
weekly risk-free rate: 0.067308
```

1. The multivariate t distribuion of dimension 4.

```
> cat("Starting:");head(y4,2);cat("Ending:");tail(y4,2);
Starting:
               CPB      CVS        K     PG
2011-01-07 -0.7800 0.802068 -0.15673 0.2639
2011-01-14  1.3539 0.085578  0.00000 1.5843
Ending:
               CPB     CVS       K      PG
2022-09-23  1.8902 -3.7028  2.7201 -1.9719
2022-09-30 -3.0926 -3.0768 -4.7381 -7.1298
> n = dim(y4)[1];N = dim(y4)[2]; c(n = n, N = N)
  n   N
613   4
> mu.f = 3.5/52;   ## weekly risk-free rate
> cat("weekly risk-free rate:", mu.f)
weekly risk-free rate: 0.067308
```

(a) The MLE of DF is 4.45. The estimates of mean and scale matrix are in the output.

```
> library(MASS);library(mnormt)
> df = seq(3, 6, .01) ## candidate value of DF
> loglik_p = c()
> for(i in 1:length(df)){
+   fit = cov.trob(y4,nu=df[i])
+   loglik_p[i] = sum(dmt(y4,mean=fit$center, S = fit$cov,df=df[i], log = T))
+ }
> nu = df[which.max(loglik_p)];
> cat("The MLE of degrees of freedom:", paste(nu) )
The MLE of degrees of freedom: 4.45
> mt = cov.trob(y4, nu = nu)
> cat("The MLE of degrees of freedom:", paste(nu) );cat("MLE of mean:");
+        mt$center;cat("MLE of scale matrix Lambda:");mt$cov
The MLE of degrees of freedom: 4.45
MLE of mean:
```

```
    CPB     CVS       K      PG
0.19055 0.31394 0.23753 0.27516
```

```
MLE of scale matrix Lambda:
       CPB    CVS      K      PG
CPB 5.3787 1.7505 2.6807 1.6335
CVS 1.7505 6.6288 1.7212 1.7108
K   2.6807 1.7212 3.6778 1.6335
PG  1.6335 1.7108 1.6335 3.2327
```

(b)
```
> mus = mt$center;  m.ex = mus - mu.f; ## excess returns
> Lambda = mt$cov; IL = solve(Lambda);  ## Lambda and its inverse
> ones = rep(1,N)
> w4.T = (IL%*%m.ex)[,1]/(t(ones)%*%IL%*%m.ex)[1,1]
> names(w4.T) = syb4; w4.T
     CPB      CVS        K       PG
-0.11581  0.27115  0.27587  0.56880
```

(c) The portfolio with 0.2 risk-free asset and 0.8 tangency portfolio has a t-distribution withe mean $\hat{\mu}_p = 0.24153$, scale $\hat{\lambda}_p = 1.31129$, degrees of freedom $\hat{\nu} = 4.45$.

```
> mu.p = 0.2*mu.f + 0.8*sum(w4.T*mus); lambda.p = 0.8*sqrt(t(w4.T)%*%Lambda%*%w4.T)
> c(mean = mu.p, lambda = lambda.p, nu = nu)
   mean  lambda      nu
0.24153 1.31129 4.45000
```

(d) Applying VaR and ES formulas of a univariate t-distribution, we get both risk measures.

```
> alpha = c(.05,.01); names(alpha) = paste0(100*alpha, "%"); S = 50000
> q = qt(alpha, df = nu); VaRp = -S/100*(mu.p + lambda.p*q);
> ESp = S/100*(-mu.p+lambda.p*dt(q,nu)/alpha*(nu+q^2)/(nu-1))
> rbind(VaR = VaRp, ES = ESp)
        5%     1%
VaR 1237.3 2204.2
ES  1870.8 3032.4
```

2.
```
> cat("Starting:");head(y8,2);cat("Ending:");tail(y8,2);N = dim(y8)[2];
Starting:
             AMZN       KO      NKE    PFE    TSLA    UNH     URI       V
2011-01-07 3.0044 -4.42994 -2.23744 4.6313  5.8701 6.2788 10.4261  3.6002
2011-01-14 1.7422  0.33322  0.64439 0.0000 -9.2305 5.8588  4.7941 -2.5542
Ending:
               AMZN      KO      NKE      PFE    TSLA     UNH     URI       V
2022-09-23 -8.22173 -1.5914  -7.0627 -4.32871 -9.6917 -1.4324 -8.2572 -4.9525
2022-09-30 -0.68789 -4.5026 -15.4632 -0.72861 -3.7298 -1.6827  2.5306 -3.4903
```

```
> n = dim(y8)[1];N = dim(y8)[2]; c(n = n, N = N);
  n   N
613   8
```

(a) VaR and ES estimates of each asset.

```
> alpha =0.05; S = 50000
> VaRs = ESs = c()
> for(i in 1:N){
+   est = fitdistr(y8[,i], "t")$est
+   mu = est["m"]; lambda = est["s"]; nu = est["df"]
+   q = qt(alpha, df = est["df"]);
+   VaRs[i] = -(mu + lambda*q)
+   ESs[i] = -mu + lambda*dt(q,nu)/alpha*(nu + q^2)/(nu-1)
+ }
> VaRs = S/100*VaRs; ESs = S/100*ESs;
> names(VaRs) = names(ESs) = syb8;
> rbind(Var = VaRs, ES = ESs)
      AMZN     KO    NKE    PFE   TSLA    UNH    URI      V
Var 3134.0 1726.0 2716.7 2207.6 5568.6 2373.8 4699.1 2181.5
ES  4470.9 2694.6 4007.2 3324.6 7883.6 3728.3 7483.1 3490.2
```

(b) Tangency Portfolio.

```
> mu = apply(y8,2,mean); m.ex = mu-mu.f;  ## excess return
> S = var(y8); IS = solve(S);  ## var and its inverse
> ones = rep(1,N)
> w8.T = (IS%*%m.ex)[,1]/(t(ones)%*%IS%*%m.ex)[1,1]
> names(w8.T) = syb8
> w8.T
     AMZN        KO       NKE       PFE      TSLA       UNH       URI         V
 0.191148 -0.686255 -0.298009  0.014457  0.293056  0.819579 -0.120018  0.786043
```

(c) The formula is quadrtic form of a vector, say $\tilde{\boldsymbol{w}}$ and the Spearman correlation matrix $\boldsymbol{R}$, where the jth element of $\tilde{\boldsymbol{w}}$ is $\tilde{w}_j = w_{T,j} \times \text{VaR}_j$. The computation is simple, we get VaR $= 3252.1$ and ES $= 4957.4$.

```
> Rho = cor(y8, method = "s")
> w = w8.T*VaRs
> VaR = sqrt(t(w)%*%Rho%*%w)
> w = w8.T*ESs
> ES = sqrt(t(w)%*%Rho%*%w)
> cat("5% VaR and ES of the tangency portfolio:"); c(VaR = VaR, ES = ES)
5% VaR and ES of the tangency portfolio:
   VaR     ES
3252.1 4957.4
```

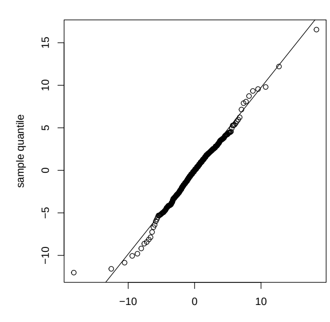
3.
```
> cat("Starting:");head(y1,2);cat("Ending:");tail(y1,2);n = dim(y1)[1];cat("Sample size: n =",n )
Starting:
                ORCL
2011-01-03  1.01719
2011-01-04 -0.44373
Ending:
                ORCL
2022-09-29 -2.68354
2022-09-30 -0.53891
Sample size: n = 2957
> alpha = c("5%" = 0.05, "1%" = 0.01); S = 50000
```

(a)
```
> q = quantile(y1,alpha)
> VaR = -S/100*q; ES = -S/100*c(mean(y1[y1 < q[1]]), mean(y1[y1<q[2]]))
> cat("Nonparametric estimates for risks:"); cbind(quantile = q, VaR = VaR, ES = ES)
Nonparametric estimates for risks:
   quantile    VaR     ES
5%  -2.4613 1230.6 1981.4
1%  -4.6604 2330.2 3467.8
```

(b)
```
> est = fitdistr(y1,"t")$est
> mu = est["m"];lambda = est["s"]; nu = est["df"]
> ps = ((1:n)-0.5)/n; qs = mu + lambda*qt(ps,nu)
> par(pty = "s")
> plot(qs,quantile(y1,ps),xlab = "t-quantile",ylab = "sample quantile")
> abline(lsfit(qs, quantile(y1,ps))$coef)

> q = qt(alpha,nu);
> VaR = -S/100*(mu + lambda*q);
> ES = S/100*(-mu+lambda*dt(q,nu)/alpha*(nu+q^2)/(nu-1))
> cat("MLE for t parameters:"); est;cat("parametric estimates for risks");
+        cbind(quantile = q, VaR = VaR, ES = ES)

MLE for t parameters:
       m        s       df
0.078239 0.985156 3.005963
parametric estimates for risks
   quantile    VaR     ES
5%  -2.3515 1119.2 1866.2
1%  -4.5335 2194.0 3401.9
```



(c)
```
> par(mfrow = c(3,3), pty = "s", mex = 0.8, mar = c(4,2,3,1))
> s = seq(0.5,0.8,0.05)
> m = round(n^s); names(m) = paste0("n^",s);m
 n^0.5 n^0.55  n^0.6 n^0.65  n^0.7 n^0.75  n^0.8
    54     81    121    180    269    401    598
> ys = log(-sort(as.numeric(y1))[1:max(m)])
> xs = log((1:max(m))/n)
> out = matrix(nrow = 3, ncol = length(m))
> rownames(out) = c("slope", "se","sig.e")
> colnames(out) = paste("m", m, sep = " = ")
> for(i in 1:length(m)){
+   x = xs[1:m[i]]; y = ys[1:m[i]]
+   lse = lm(y~x)
+   plot(x,y, main = colnames(out)[i], xlab = "", ylab = "", ylim = range(ys), xlim = range(xs),
+        cex = 0.75, pch = 20)
+   abline(coef(lse))
+   out[,i] = c(coef(lse)[2],sqrt(vcov(lse)[2,2]),sigma(lse) )
+ }
> out = rbind(out,ahat = -1/out["slope", ])
> round(out,5)
        m = 54   m = 81  m = 121  m = 180  m = 269  m = 401  m = 598
slope -0.35977 -0.37182 -0.38056 -0.39054 -0.42041 -0.45576 -0.51198
se     0.00992  0.00681  0.00458  0.00324  0.00339  0.00335  0.00408
sig.e  0.06461  0.05592  0.04697  0.04124  0.05337  0.06518  0.09750
ahat   2.77952  2.68944  2.62771  2.56054  2.37862  2.19416  1.95319
```

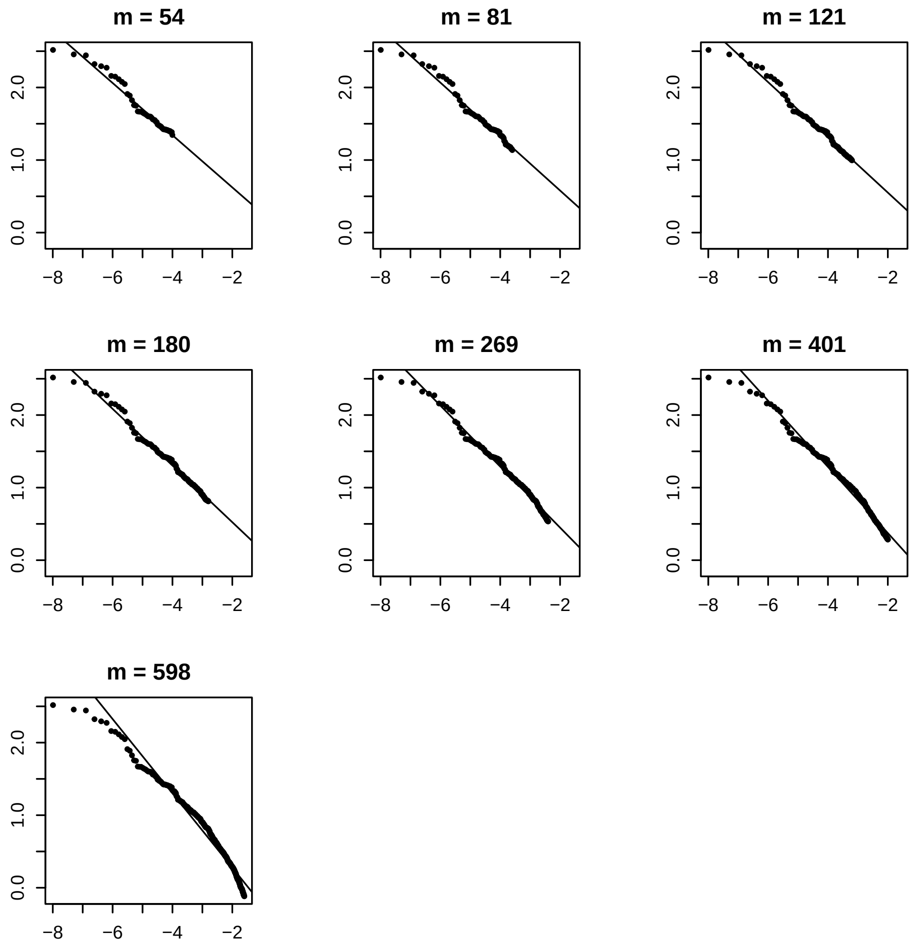
From the plots, the tail index estimates with bandwidths $m = 180$ and 269 are suitable candidate estimates. The two plots appear to be straight line and bandwidths are not too small. There is no “correct” answer among the two bandwidths. The bandwidth $m = 269$ is the larger of the two as we usually prefer larger bandwidth. The tail index estimate with $m = 269$ is 2.379. The estimate with $m = 180$ has the smallest standard error for slope and the smallest $\hat{\sigma}_\varepsilon$. We list VaR and ES for both estimates of tail index.

Bandwidth $m = 180$.

```
> a = out["ahat", m == 180]
> alpha0 = 0.1;
> VaR0 = -S/100*quantile(y1,alpha0)
> VaR = VaR0*(alpha0/alpha)^(1/a)
> cbind(VaR = VaR, ES = a/(a-1)*VaR)
      VaR     ES
5% 1064.4 1746.4
1% 1995.6 3274.3
```

Bandwidth $m = 269$

```
> a = out["ahat", m == 269]
> alpha0 = 0.1;
> VaR0 = -S/100*quantile(y1,alpha0)
> VaR = VaR0*(alpha0/alpha)^(1/a)
> cbind(VaR = VaR, ES = a/(a-1)*VaR)
      VaR     ES
5% 1086.6 1874.8
1% 2137.6 3688.2
```


m = 54
m = 81
m = 121
m = 180
m = 269
m = 401
m = 598
0.0
1.0
2.0
−8
−6
−4
−2