# 5 | Copulas

For a given a random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_d)'$, its joint distribution function $F_{\boldsymbol{Y}}(y_1, \ldots, y_d)$ fully describes its components' marginal behaviour and dependence structure. Copulas are functions that separate the dependence structure and the marginal distributions.

## Definition and Families

The word "copula" was first used in a mathematical or statistical sense by Sklar (1959). But the functions themselves can be traced back to Hoeffding (1940). Copulas differ not so much in the degree of association they provide, but rather in which part of the distributions the association is strongest.

### CDF and Inverse CDF of a Univariate Random Variable

If $Y$ has a CDF $F$, The inverse of $F$ or the quantile function associated with $F$ is $F^{-1} : (0,1) \to \mathbb{R}$, $F^{-1}(p) = \inf\{y : F(y) \geq p\}$. If $F$ is continuous and strictly increasing then $F(F^{-1}(p)) = p$ and $F^{-1}(F(y)) = y$, $\forall\ y, p$.

**Quantile transformation** Let $F$ be a CDF function on $\mathbb{R}$ and $U \sim$ Uniform$(0,1)$ then the random variable $F^{-1}(U)$ has a CDF $F$. It is easier to see with a continuous and strictly increasing $F$,

$$P(F^{-1}(U) \leq y) = P\{U \leq F(y)\} = F(y).$$

The result holds for any CDF $F$ but slightly more involved to prove in the general case.

**Probability transformation** If $Y$ has continuous CDF $F$, then the random variable $F(Y) \sim$ Uniform(0, 1). This is again simpler to see with a continuous and strictly increasing $F$. For $p \in [0,1]$,

$$P\{F(Y) \le p\} = P\{Y \le F^{-1}(p)\} = F(F^{-1}(p)) = p\,.$$

**Definition**

For clarity of presentation we start with a definition of the bivariate copula functions.

**Two-dimensional Copulas** A 2-dimensional copula is the CDF of a bivariate random variable whose univariate marginal distributions are Uniform(0, 1). Let $\boldsymbol{Y} = (Y_1\, Y_2)^T$ that has a CDF $F_{\boldsymbol{Y}}$ with continuous marginal univariate CDFs, $F_{Y_1}$ and $F_{Y_2}$ . The CDF of joint distribution of $\left\{F_{Y_1}(Y_1), F_{Y_2}(Y_2)\right\}^T$ is a copula and denoted by $C_Y$. This CDF is called the *copula* of $\boldsymbol{Y}$ .

The joint distribution of $C_Y : [\,0,1\,]^2 \to [\,0,1\,]$ is thus the joint distribution of two uniform random variates,

$$\begin{aligned} C_Y(u_1, u_2) &= P\left\{F_{Y_1}(Y_1) \le u_1, F_{Y_2}(Y_2) \le u_2\right\} \\ &= P\left\{Y_1 \le F_{Y_1}^{-1}(u_1), Y_2 \le F_{Y_2}^{-1}(u_2)\right\} \\ &= F_Y\left\{F_{Y_1}^{-1}(u_1), F_{Y_2}^{-1}(u_2)\right\}. \end{aligned} \tag{5.1}$$

We have, for simplicity, assumed the marginal CDF's $F_{Y_1}$ and $F_{Y_2}$ are





strictly increasing. Let $u_i = F_{Y_i}(y_i)$ , $j = 1, 2$ , we get

$$F_Y(y_1, y_2) = C_Y\left\{F_{Y_1}(y_1), F_{Y_2}(y_2)\right\}. \tag{5.2}$$

Equation (5.2) is part of the famous Sklar's theorem. It states that a multivariate CDF $F_Y$ can be decomposed into the copula $C_Y$ , which contains all information about the dependencies among $(Y_1, Y_2)^T$ , and the univariate marginal CDFs $F_{Y_1}$ and $F_{Y_1}$ , which contain all information about the univariate marginal distributions.

**Densities of $F_Y$ and $C_Y$** The density of $\boldsymbol{Y}$ is obtained by differentiating $F_Y$, we find from (5.2) that,

$$f_Y(y_1, y_2) = c_Y\left\{F_{Y_1}(y_1), F_{Y_2}(y_2)\right\} f_{Y_1}(y_1) f_{Y_2}(y_2), \tag{5.3}$$

where $c_Y(\cdot)$ is the density of $C_Y$,

$$c_Y(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C_Y(u_1, u_2)\,.$$

**$d$-Dimensional Copulas** The same definition can be extended to a $d$-dimensional copula, $d \ge 2$ of a $d$-dimensional $\boldsymbol{Y} = (Y_1, \ldots, Y_d)^T$. Let $F_{\boldsymbol{Y}}$ be the CDF of $\boldsymbol{Y}$ and $F_{Y_i}$, $i = 1, \ldots, d$ are the marginal CDFs of $Y_i$'s . The copula of $\boldsymbol{Y}$ is

$$C_Y(u_1, \ldots, u_d) = F_Y\left\{F_{Y_1}^{-1}(u_1), \ldots, F_{Y_d}^{-1}(u_d)\right\}.$$

By the Sklar's Theorem,

$$F_Y(y_1, \ldots, y_d) = C_Y\left\{F_{Y_1}(y_1), \ldots, F_{Y_d}(y_d)\right\}.$$

and the density of $F_{\boldsymbol{Y}}$ is the product of $c_{\boldsymbol{Y}}$ and the marginal PDF $f_{Y_i}$'s ,

$$f_Y(y_1,\ldots,y_d)=c_Y\{F_{Y_1}(y_1),\ldots,F_{Y_d}(y_d)\}f_{Y_1}(y_1)\cdots f_{Y_d}(y_d).$$

**Invariant property of of copulas** One of the most important properties of the copula is that a copula is invariant under increasing and continuous transformations of the variates. Suppose $g_j$, $j=1,\ldots,d$ are strictly increasing functions and $X_j=g_j(Y_j)$. Then $\boldsymbol{X}=(X_1,\ldots,X_d)$ and $\boldsymbol{Y}$ have the same copulas. That is,

$$C_X(u_1,\ldots,u_d)=C_Y\{u_1,\ldots,u_d\}\ .$$

### Special Copulas

There are three copulas of special interest because they represent independence and two extremes of dependence.

**The $d$-dimensional independent copula $C_0$** This is the CDF of $d$ mutually independent Uniform$(0,1)$ random variables. It is

$$C_0(u_1,\ldots,u_d)=u_1\cdots u_d\,,$$

and the associate density is $c_0(u_1,\ldots,u_d)=1$ for $(u_1,\ldots,u_d)^T\in[0,1]^d$ and 0 otherwise. The 3rd column of Figure 5.1 is an example of bivariate $C_0(u_1,u_2)$.

**The $d$-dimensional co-monotonicity copula $C_+$** This copula characterizes perfect positive dependence. Let $U$ be Uniform(0,1). Then





$C_+$ is the CDF that contains $d$ copies of $U$. That is

$$C_+(u_1,\ldots,u_d)=P(U\le u_1,\ldots,U\le u_d)=P\{U\le\min_i u_i\}=\min_i u_i\,.$$

The co-monotonicity copula is also an upper bound for all copula functions: $C(u_1,\ldots,u_d)\le C_+(u_1,\ldots,u_d)$, $\forall\,(u_1,\ldots,u_d)^T\in[0,1]^d$.

**The 2-dimensional counter-monotonicity copula $C_-$** The copula is defined as the CDF of $(U,1-U)^T$, which has perfect negative dependence,

$$\begin{aligned}C_-(u_1,u_2)&=P(U\le u_1,1-U\le u_2)=P(1-u_2\le U\le u_1)\\&=\max\{u_1+u_2-1,0\}\ .\end{aligned}$$

All two-dimensional copula functions are bounded below by $C_-$. It is not possible to have a counter-monotonic copula with $d>2$. However, a lower bound for all copula functions is: $\max(u_1+\cdots+u_d+1-d,0)\le C(u_1,\ldots,u_d)$, $\forall\,(u_1,\ldots,u_d)^T\in[0,1]^d$. This lower bound is obtainable only point-wise, but it is not itself a copula function for $d>2$.

### Copulas of Parametric Families

To use copulas to model multivariate dependencies, we next consider parametric families of copulas.

### Gaussian and $t$-copulas

Multivariate normal and multivariate $t$-distributions offer a convenient way to generate families of copulas. Let $\boldsymbol{Y}=(Y_1,\ldots,Y_d)^T$ have a multivariate normal distribution. Then $C_Y$ depends only on the $d \times d$ correlation matrix of $\boldsymbol{Y}$, which will be denoted by $\boldsymbol{\Omega}$. Therefore, there is one-to-one correspondence between correlation matrix and Gaussian copulas and will be denoted $C_{\text{Gauss}}(u_1, \ldots, u_d \,|\, \boldsymbol{\Omega})$.

If a random vector $\boldsymbol{Y}$ has a Gaussian copula, then $\boldsymbol{Y}$ is said to have a *meta-Gaussian distribution*, which is NOT necessary a multivariate Gaussian distribution. A $d$-dimensional copula with an identity correlation matrix is the $d$-dimensional independence copula. A Gaussian copula converges to the co-monotonicity copula $C_+$ if all correlations in $\boldsymbol{\Omega}$ converges to $1$. In the bivariate case, as the correlation converges to $-1$, the copula converges to the counter-monotonicity copula $C_-$.

Similarly, let $C_t(u_1, \ldots, u_d \,|\, \boldsymbol{\Omega}, \nu)$ denote the copula of a random vector that has a multivariate $t$-distribution with tail index $\nu$ and correlation matrix $\boldsymbol{\Omega}$.

The tail index $\nu$ determines the amount of tail dependence of random vectors that have a $t$-copula. Such a vector is said to have a meta $t$-distribution if it has a $t$-copula.

## Archimedean copulas

The Archimedean copulas are the most popular class of copulas because, unlike the Gaussian or $t$ copula, they admit an explicit formula. The formulas depend only on one parameter governing the strength of dependence. The have the form

$$C(u_1, \ldots, u_d) = \varphi^{-1}\left\{\varphi(u_1) + \cdots + \varphi(u_d)\right\}, \qquad (5.4)$$





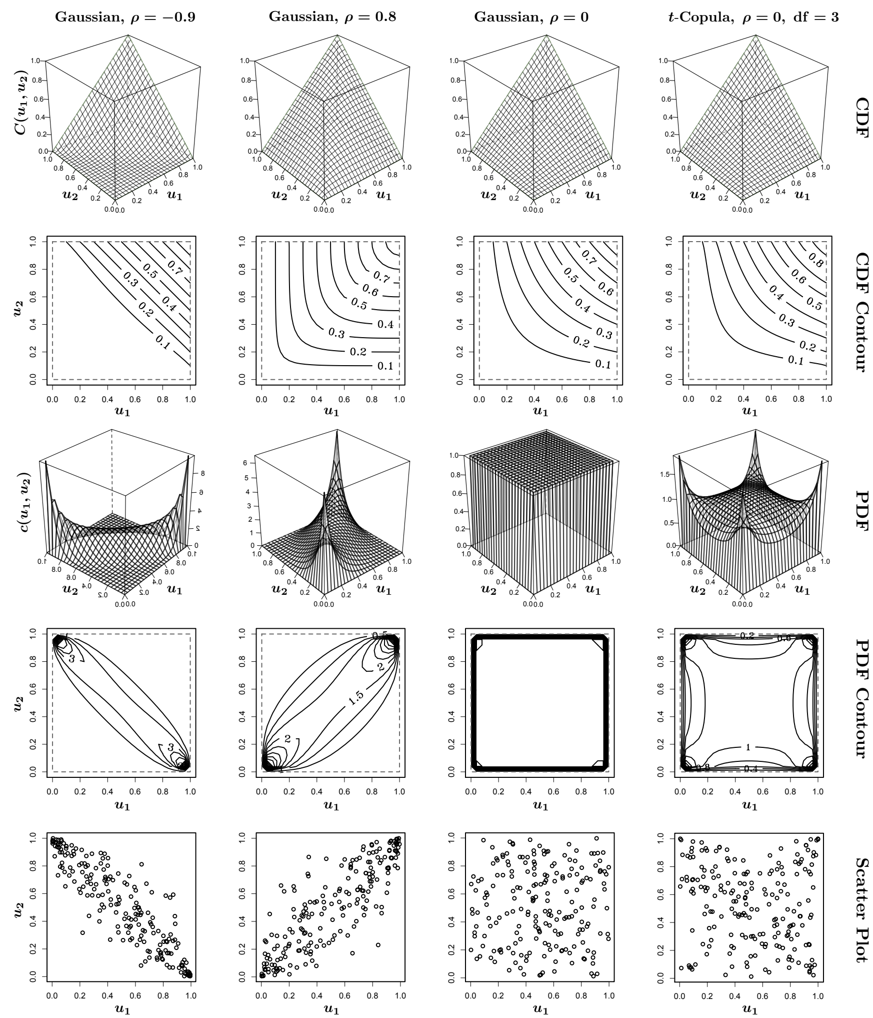


**Figure 5.1:** The CDF, PDF and their contours as well as scatter plots of simulated bivariate Gaussian copulas with $\rho = -0.9, 0, 0.8$. The 4th column are plots of the uncorrelated $t$-copula with $\mathrm{df} = 3$.

where $\varphi$ is the generator function satisfies the following conditions,

1. $\varphi : [0,1] \to [0,\infty]$ is continuous, strictly decreasing and convex;
2. $\varphi(0) = \infty$; and
3. $\varphi(1) = 0$.

There are four Archimedian copulas in common use: the Clayton, Frank, Gumbel and Joy. The top panel shows examples of these copulas with their probability density contours. The bottom panel are the density contours of bivariate normals with each of these copulas.

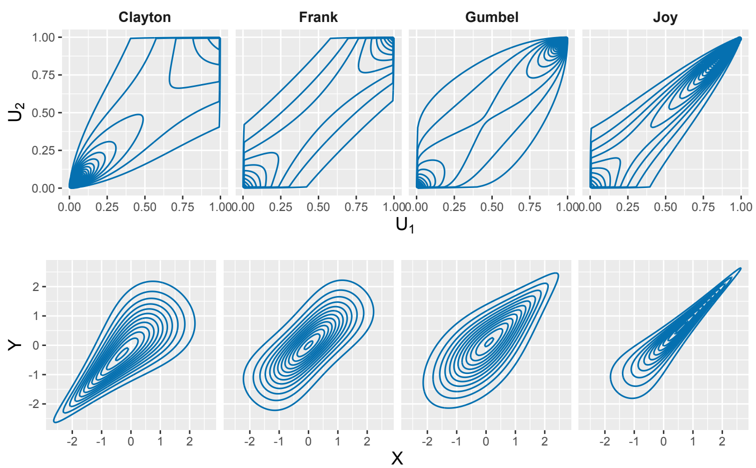


**Figure 5.2:** The PDF contours of Clayton, Frank, Joe and Gumbel copulas.

## Rank Correlation

For each variable, the ranks of variable are determined by ordering the observations from smallest to largest and giving the smallest





rank 1, the next-smallest rank 2, and so forth. Mathematically,

$$\text{rank}(Y_i) = \sum_{j=1}^{n} I\{Y_j \le Y_i\}$$

A rank statistic is a statistic that depends on the data only through the ranks. A key property of ranks is that they are unchanged by strictly monotonic transformations of the variables. In particular, the ranks are unchanged by transforming each variable by its CDF, so the distribution of any rank statistic depends only on the copula of the observations, not on the univariate marginal distributions.

Rank correlations are rank statistics that measure statistical association between pairs of variables. The two most widely used rank correlation coefficients are Kendall's tau and Spearman's rho.

### Kendall's tau

Suppose that $\boldsymbol{Y}^*$ is an independent copy of the bivariate $\boldsymbol{Y} = (Y_1, Y_2)^T$, they are called a concordant pair if $(Y_1 - Y_1^*)(Y_2 - Y_2^*) > 0$ and a discordant pair if $(Y_1 - Y_1^*)(Y_2 - Y_2^*) < 0$. Kendall's tau is defined as

$$\begin{aligned}\rho_\tau(Y_1, Y_2) &= P\big\{(Y_1 - Y_1^*)(Y_2 - Y_2^*) > 0\big\} - P\big\{(Y_1 - Y_1^*)(Y_2 - Y_2^*) < 0\big\} \\ &= E\Big[\text{sign}\big\{(Y_1 - Y_1^*)(Y_2 - Y_2^*)\big\}\Big].\end{aligned}$$

Kendall's tau $\rho_\tau$ is symmetric in its arguments and takes values in $[-1, 1]$. If $g$ and $h$ are increasing functions, then

$$\rho_\tau\big\{g(Y_1), h(Y_2)\big\} = \rho_\tau(Y_1, Y_2). \tag{5.5}$$

If $g$ and $h$ are CDF of $Y_1$ and $Y_2$ then the LHS of (5.5) is Kendall's tau for a pair of random variables according to the copula of $\boldsymbol{Y}$. Kendall's tau depends only on he copula of a bivariate random vector. For a random vector $\boldsymbol{Y}$ of dimension $d$, Kendall's tau correlation matrix $\boldsymbol{\Omega}_\tau$ is the $d \times d$ matrix whose $(j,k)$th entry is $\rho_\tau(Y_j, Y_k)$.

The sample Kendall's tau based on a bivariate sample $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_n$ is

$$\hat{\rho}_\tau(\boldsymbol{Y}_{1:n}) = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \operatorname{sign}\left\{(Y_{i,1} - Y_{j,1})(Y_{i,2} - Y_{j,2})\right\}.$$

This is the average signs over all distinct pairs of observations.

### Spearment's rho

Spearmen's rho for a bivariate $\boldsymbol{Y}$ is defined to be the correlation coefficient of the variates transformed by their univariate marginal CDFs,

$$\rho_s(Y_1, Y_2) = \operatorname{Corr}\left\{F_{Y_1}(Y_1), F_{Y_2}(Y_2)\right\}.$$

The joint CDF of $\left\{F_{Y_1}(Y_1), F_{Y_2}(Y_2)\right\}$ is the copula of $\boldsymbol{Y}$, thus like Kendall's tau, depends only on the copula function. For a random vector $\boldsymbol{Y}$ of dimension $d$, Spearment's rho correlation matrix $\boldsymbol{\Omega}_S$ is the $d \times d$ matrix whose $(j,k)$th entry is $\rho_S(Y_j, Y_k)$.

The sample version of Spearman's rho based on a sample $\boldsymbol{Y}_1 \dots Y_n$ is

$$\hat{\rho}_S(\boldsymbol{Y}_{1:n}) = \frac{12}{n(n^2-1)} \sum_{i=1}^{n} \left\{\operatorname{rank}(Y_{i,1}) - \frac{n+1}{2}\right\}\left\{\operatorname{rank}(Y_{i,2}) - \frac{n+1}{2}\right\}.$$

The rank function takes integers 1 to $n$, the mean of a set of ranks is





$(n+1)/2$. It can be shown that $\hat{\rho}_S$ is the sample correlation between the ranks of $\{Y_{i,1}\}$ and $\{Y_{i,2}\}$.

The sample versions of both Kenall's tau and Spearment's rho can be computed with R's `cor()` function by setting the `method` argument.

```
args(cor)

## function (x, y = NULL, use = "everything", method = c("pearson",
##     "kendall", "spearman"))
```

### Tail Dependence

One interesting application of copulas is measure tail dependence between variables, that is, the probability of simultaneous extremes. For a bivariate $\boldsymbol{Y}$ with copula $C_Y$, the coefficient of lower tail dependence is defined as

$$\begin{aligned}
\lambda_\ell &= \lim_{q \downarrow 0} P\left\{Y_2 \le F_{Y_2}^{-1}(q) | Y_1 \le F_{Y_1}^{-1}(q)\right\} \\
&= \lim_{q \downarrow 0} P\left\{F_{Y_2}(Y_2) \le q | F_{Y_1}(Y_1) \le q\right\} \\
&= \lim_{q \downarrow 0} \frac{P\left\{F_{Y_1}(Y_1) \le q, F_{Y_2}(Y_2) \le q\right\}}{P\left\{F_{Y_1}(Y_1) \le q\right\}} = \lim_{q \downarrow 0} \frac{C_Y(q,q)}{q}.
\end{aligned}$$

A bivariate Gaussian copula $C_{\text{Gauss}}$ doest not have tail dependence unless $\rho = 1$, that is, $\lambda_\ell = 0, \rho \ne 1$. For a bivariate $t$-copula $C_t$ with tail index $\nu$ and correlation $\rho$,

$$\lambda_\ell = 2 F_{t,\nu+1}\left\{-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right\}, \tag{5.6}$$

where $F_{t,\nu+1}$ is the CDF of the $t$-distribution with tail index $\nu+1$.

As $\nu \to \infty$, $\lambda_\ell \to 2F_{t,\nu+1}(-\infty) = 0$. This makes sense because the $t$-copula coverages to a Gaussian copula as $\nu \to \infty$. Also $\lambda_\ell \to 0$ as $\rho \to -1$, this is because $\lambda_\ell$ measures only positive tail dependence as we can see from the definition, it gives 0 for a perfect negative dependence.

The coefficient of upper tail dependence $\lambda_u$ is defined as

$$\lambda_u = \lim_{q\uparrow 1} P\left\{Y_2 \geq F_{Y_2}^{-1}(q) | Y_1 \geq F_{Y_1}^{-1}(q)\right\} = \lim_{q\uparrow 1} \frac{1-2q+C_Y(q,q)}{1-q}.$$

For Gaussian and $t$-copula, $\lambda_u = \lambda_\ell$.

## Calibrating Copulas

Suppose that we have a $d$-dimensional sample $\boldsymbol{Y}_1, \ldots, Y_n$ and we wish to estimate the copula of $\boldsymbol{Y}$ and perhaps its univariate marginal distributions as well.

### The MLE and Pseudo-MLE

Suppose we have $F_{Y_1}(\cdot|\boldsymbol{\theta}_1), \ldots, F_{Y_d}(\cdot|\boldsymbol{\theta}_d)$ for the marginal CDFs as well as a $c_Y(\cdot|\boldsymbol{\theta}_C)$ for the copula density. Let. $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \ldots, \boldsymbol{\theta}_d^T)^T$ the log-likelihood is

$$\log\{L(\boldsymbol{\theta})\} = \sum_{i=1}^{n} \Bigg( \log\Big[c_Y\left\{F_{Y_1}(Y_{i,1}|\boldsymbol{\theta}_1), \ldots, F_{Y_d}(Y_{i,d}|\boldsymbol{\theta}_d)\right\}\Big] + \log\left\{f_{Y_1}(Y_{i,1}|\boldsymbol{\theta}_1)\right\} + \cdots + \log\left\{f_{Y_d}(Y_{i,d}|\boldsymbol{\theta}_d)\right\} \Bigg). \tag{5.7}$$





When the number of parameters is large, the MLE is numerically challenging. The pseudo-maximum likelihoods are easier to compute. The pseudo ML estimates can also serve as the starting values for the full ML estimation.

The Pseudo-Maximum Likelihood estimation is a two-step procedure. In the first step, each of the d univariate marginal distribution functions is estimated, one at a time. Let $\hat{F}_{Y_j}$ be the estimate of the $j$th univariate marginal CDF, $j = 1, \ldots, d$. In the second step,

$$\sum_{i=1}^{n} \log\Big[c_Y\left\{\hat{F}_{Y_1}(Y_{i,1}), \ldots, \hat{F}_{Y_d}(Y_{i,d}) \Big| \boldsymbol{\theta}_C\right\}\Big].$$

is maximized over $\boldsymbol{\theta}_C$.

There are two approaches to the first step, parametric and nonparametric. In the parametric approach, parametric models. In the parametric approach, the MLE of each $F_{Y_j}(\cdot|\boldsymbol{\theta}_j)$ is estimated by the MLE using $Y_{1,j}, \ldots, Y_{n,j}$. In the nonparametric approach, $\hat{F}_{Y_j}$ is estimated by the empirical CDF of $Y_{1,j}, \ldots, Y_{n,j}$ with a modification of the denominator,

$$\hat{F}_{\boldsymbol{Y}_j}(y) = \frac{1}{n+1} \sum_{i=1}^{n} I\left\{Y_{i,j} \leq y\right\}. \tag{5.8}$$

With this modification, the maximum value of $\hat{F}_{Y_j}$ is $n/(n+1) < 1$ rather than 1. Avoiding the value of 1 is essential because very often $c_y(u_1, \ldots, u_d) = \infty$ if some of $u_1 \ldots, u_d$ are equal to 1.

When both steps are parametric, the estimation method is called parametric pseudo-maximum likelihood. The combination of a nonparametric first step and a parametric second step is called semi-parametric pseudo-maximum likelihood.

The values $\hat{F}_{Y_j}(Y_{i,j})$, $i = 1, \dots, n$, $j = 1, \dots, d$, will be called the uniform-transformed variables, since they should be distributed as Uniform(0,1) random variables. The multivariate empirical CDF of the uniform transformed variables is called the empirical copula and is a nonparametric estimate of the copula function. It can be used for checking goodness of fit of parametric copula models.

### Calibrating Meta-Gaussian and Meta-$t$ Distributions

*Gaussian Copulas*. Rank correlation can be useful for estimating the parameters of a copula. Suppose $\boldsymbol{Y}$ has a meta-Gaussian distribution. Then its copula is $C_{\text{Gauss}}(\cdot|\boldsymbol{\Omega})$ for some correlation matrix $\boldsymbol{\Omega}$. The marginal distribution can be estimated by the MLE. The following result shows that $\boldsymbol{\Omega}$ can be estimated by the sample Spearman's correlation matrix.

**Result 5.1.** *Suppose that $\boldsymbol{Y} = (Y_1, \dots, Y_d)$ have a meta-Gaussian distribution with continuous univariate marginal distributions and copula $C_{\text{Gauss}}(\cdot|\boldsymbol{\Omega})$. Let $\omega_{ij}$ be the $(i,j)$th entry of $\boldsymbol{\Omega}$, then*

$$\rho_\tau(Y_i, Y_j) = \frac{2}{\pi}\arcsin(\omega_{ij}), \quad \textit{and} \tag{5.9}$$

$$\rho_S(Y_i, Y_j) = \frac{6}{\pi}\arcsin(\omega_{ij}/2) \approx \omega_{ij}\,. \tag{5.10}$$

*Suppose, instead, that $\boldsymbol{Y}$ has a meta-$t$-distribution with continuous univariate marginal distribution and copula $C_t(\cdot|\boldsymbol{\Omega}, \nu)$. Then* (5.9) *still holds but* (5.10) *does not hold.*





The approximation in (5.10) is due to $6/\pi \arcsin(x) \approx x$ for $|x| \le 1$ with equality when $x = -1, 0, 1$ and their maximum difference over the range $[-1, 1]$ is 0.018. However the relative error can be large as much as 0.047 and is largest near $x = 0$.

By (5.10), the sample Spearman's rank correlation matrix can be used as an estimate of the correlation matrix $\boldsymbol{\Omega}$ or used as a starting value for the MLE or pseudo-MLE.

*t-Copulas*. If $\boldsymbol{Y}$ has a $t$-coupla, result 5.1 suggests that we can use the sample Kendall's tau correlation matrix $\hat{\boldsymbol{\Omega}}_\tau$. However, the resulting matrix is no longer positive definite. The tail index can be obtained by plugging $\tilde{\boldsymbol{\Omega}}$ into $\sum_i \log\left[c_Y\left\{\hat{F}(Y_{i,1}), \dots, \hat{F}(Y_{i,d}) \,|\, \nu\right\}\right]$ and maximizing over $\nu$.

### Implementing with `R`'s `Copula` Package

The copula related computation, including simulation, estimation and plotting, is implemented with the `R` package `copula`, which is of S4 method. The MLE of full model is possible to obtained by using `R`'s `mvdc` functions. The likelihood function can be calculated with `dmvdc()`, this requires to specify the configuration of the model with `mvdc()`.

```
args(mvdc)

## function (copula, margins, paramMargins, marginsIdentical = FALSE,
##     check = TRUE, fixupNames = TRUE)

args(dmvdc)

## function (mvdc, x, log = FALSE)
```

We will show the parametric (semi-parametric) pseudo-MLE and the full MLE in the following bivariate example.

**Eg 5.1.** In this example, we will analyze weekly log returns of Apple and Qualcomm from August 1, 2010 to August 31, 2024, $n = 735$.

```
head(yt,2)

##                  AAPL       QCOM
## 2010-08-06 -0.6743217 -0.2841869
## 2010-08-13 -4.3172875 -1.8277002

tail(yt,2)

##                 AAPL     QCOM
## 2024-08-23 0.3488680 0.763718
## 2024-08-30 0.9477097 1.032125

n = dim(yt)[1]; cat("Sample size:", n)

## Sample size: 735

c(Person = cor(yt, method = "p")[1,2], Kendall = cor(yt, method = "k")[1,2],
  Spearman = cor(yt, method = "s")[1,2])

##    Person   Kendall  Spearman
## 0.5171591 0.3703831 0.5274642
```

From the three correlations, the two return series are dependent. We can also see it from the scatter plot of Figure 5.3, which is with the kernel density estimates imposed. The QQ plot on the left plots the sample quantiles of Qualcomm returns against that of Apple's weekly returns. The plot indicates the tails of one sample has heavier tails than the other.

We first fit a set of candidate univariate models to both series and select a suitable model for each series. The $t$ distribution is selected





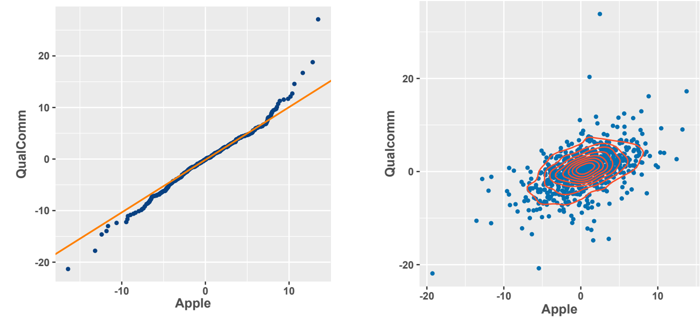


**Figure 5.3:** Left: The sample quantiles of weekly returns of Apple and Qualcomm. Right: the scatter plot of the two return series with the joint KDE superimposed.

for both series, the details are omitted. The MLEs are calculated with MASS's `fitdistr()` with fGarch's `dstd()` as the density function.

```
syb ## ticker symbols

## [1] "AAPL" "QCOM"

library(MASS); library(fGarch)
lower =  c(-100,0.001,0.1); est = c()
for(i in 1:d){
        start = list(mean = mean(yt[,i]), sd = sd(yt[,i]), nu = 4)
        est = rbind(est,fitdistr(yt[,i],dstd, start,lower = lower)$est)
}
rownames(est) = syb; est

##           mean       sd       nu
## AAPL 0.5106911 3.814130 6.821894
## QCOM 0.3992866 4.547079 3.647967
```

The estimates of degrees of freedom are not close, we will consider the class of copula models.

**The Pseudo ML estimation** This is a 2-step procedure, the first step can be either parametric or non-parametric.

*Step 1.* The parametric step 1 is taking the probability transformation to get the uniform variates with the model selected for the marginal distribution, $\hat{\boldsymbol{U}} = F_{\hat{\boldsymbol{\theta}}}(\boldsymbol{Y})$.

```
ut = c()
for(i in 1:d) ut = cbind(ut, pstd(yt[,i], mean = est[i,1], sd = est[i,2],
                                  nu = est[i,3]));
colnames(ut) = syb
```

Alternatively, the uniform bivariates can be obtained non-parametrically by the empirical CDF of (5.8) or uniform transformation.

```
ut.0 = apply(yt, 2, function(x) rank(x)/(n + 1))
```

Figure 5.4 5.5 gives the histograms of both $\hat{\boldsymbol{U}}$ series , the scatter plot of them and the contours of their PDF with both approached. The upper 4 plots are based on the estimated $t$-CDF of each series. Both histograms show the probability transform $\hat{U}$'s are approximately Uniform distributed. The lower plots are $\hat{\boldsymbol{U}}$ based on the nonparametric empirical distributions. By definition (5.8), the histograms are exactly uniform distributions.

*Step 2.* Fitting parametric copulas using $\hat{\boldsymbol{U}}$'s from step 1 is done by `fitCopula()` function. Six copulas are considered, the Gaussian, $t$, Gumbel, Frank, Joe and Clayton copulas. When estimating $t$- and Gaussian copulas, the initial value can be set with the Kendall's tau or Spearman's rho estimates for $\boldsymbol{\Omega}$.





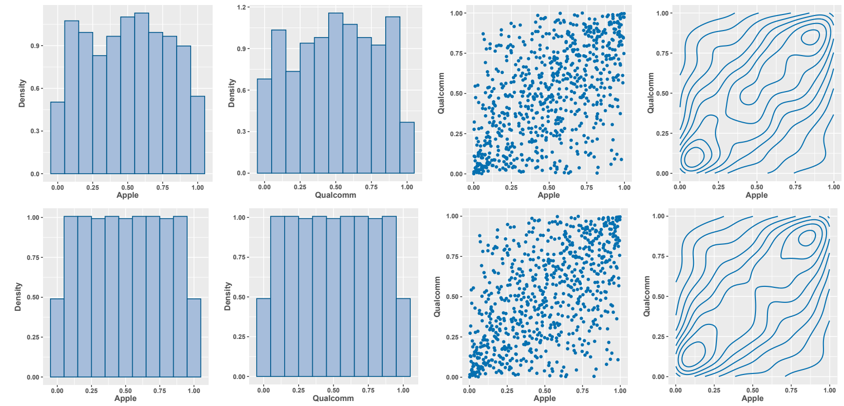


**Figure 5.4:** Histogram, scatterplot and probability density contours of probability transformation of weekly log returns. The upper panel is based on the estimated $t$-CDFs, the lower panel is based on nonparametric empirical CDF's.

```
library(copula)
omega = cor(ut, method = "s")[1,2]
tCop = fitCopula(copula=tCopula(dim = 2), data=ut, method="ml",
start=c(omega, 4))
tCop
```

```
## Call: fitCopula(tCopula(dim = 2), data = ut, ...
##    = pairlist(method = "ml", start = c(omega,
##     4)))
## Fit based on "maximum likelihood" and 735 2-dimensional observations.
## Copula: tCopula
##  rho.1     df
## 0.5424 9.3493
## The maximized loglikelihood is 129
## Optimization converged
```

The return object of `fitCopula` is an S4 object (instead of S3). We can see the element names of an S4 object with `R` command `slotNames()` and the symbol "@", instead of "$", to retrieve the values we desire.

```
slotNames(tCop)

## [1] "copula"        "estimate"      "var.est"       "loglik"
## [5] "nsample"       "method"        "call"          "fitting.stats"

aic = -2*tCop@loglik + 2*length(tCop@estimate);
bic = -2*tCop@loglik + log(n)*length(tCop@estimate);
cat("aic:", aic, "\t\t\tbic", bic)

## aic: -254.0812  bic -244.8815
```

The AICs and BICs of fitting the 6 copulas are summarized. The output below lists those of using the parametric $\hat{U}$'s.

```
## AIC and BIC of copula fits using parametric ut estimates:
##            max_loglik p       AIC       BIC
## t-Copula   129.04062 2 -254.0812 -244.8815
## Gaussian   124.89668 1 -247.7934 -243.1935
## Gumbel     115.22406 1 -228.4481 -223.8482
## Frank      121.72368 1 -241.4474 -236.8475
## Joe         83.16288 1 -164.3258 -159.7259
## Clayton     83.80908 1 -165.6182 -161.0183
```

The $t$-Copula has the lowest AIC and BIC values. Figure 5.5 shows contours of the CDFs of the six estimated parametric copulas superimposed with the empirical copula. These plots are in agreement with the AIC and BIC values, the $t$-Copula fits the best.





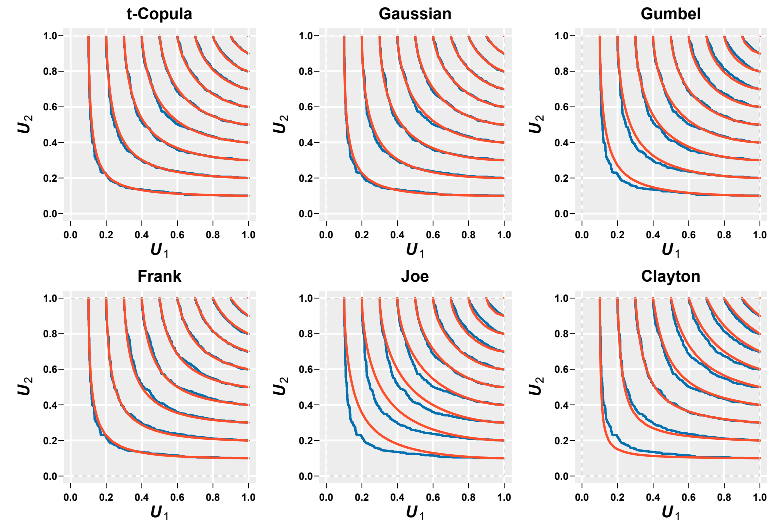


**Figure 5.5:** Apple and Qualcomm weekly returns. CDF contours of the 6 fitted copulas using parametric $\hat{U}$'s in red curves, the blue curves are the empirical copula.

The AICs and BICs of copula fits using the nonparametric $\hat{U}$'s have also been computed, they are close to those in the above output, The same contours based on the non-parametric $\hat{U}$'s are also plotted and they are almost identical to those in Figure 5.5. Thus the semi-parametric pseudo estimation related results are not presented here.

Both probability and cumulate density, of the two return series using Skylar's formulas (5.2) and (5.3). For example the pseudo-MLE joint density is

$$\hat{f}_{pML}(y_1, y_2) = c_Y\{\tilde{F}_1(y_1), \tilde{F}_2(y_2)|\hat{\rho}, \hat{\nu}\}\tilde{f}_1(y_1)\tilde{f}_2(y_2), \tag{5.11}$$

where $\tilde{f}_i(\dot{)}$ and $\tilde{F}_i(\cdot)$, $i = 1, 2$ are MLE of univariate $t$ distribution. The parametric pseudo-ML estimates and semi-parametric pseudo-

ML estimates are expected to be very similar.

**The Full ML estimation** Fitting a full model with the log-likelihood in (5.7) is often computationally difficult, but can be eased with the use of pseudo ML estimations. The AIC and BIC of semi-parametric pseudo copula fits can be used to select a small number of candidate models. Furthermore, the pseudo MLE can be the starting value for evaluating the log-likelihood.

With the pseudo MLE as starting values, we fit all six models using `fitMvdc()` from `copula` package. This requires to specify the model to be estimated first. The code is lengthy that we do not present it here. Interested readers can refer to the `R` script fo R lab 8.11.2 from the book website.

The AIC and BIC of the six full models are shown here.

```
## AIC and BIC of MLE of full models:
##             max_loglik p      AIC      BIC
## t-Copula    -3960.912 8 7937.824 7974.623
## Gaussian    -3964.970 7 7943.940 7976.139
## Gumbel      -3972.936 7 7959.872 7992.071
## Frank       -3967.566 7 7949.133 7981.332
## Joe         -4001.263 7 8016.527 8048.726
## Clayton     -3988.985 7 7991.971 8024.170
```

The meta-$t$ distribution has the lowest AIC and BIC values. Figure 5.6 shows the parametric contours of the six full parametric models comparing with the kernel density contour. The plots confirm that the meta-$t$ is the best fit among the six copula constructed bivariate models. These plots are zoomed in for clarity, see Figure 5.3.





The joint density estimates is again by the Skylar's formula ,(5.3),

$$\hat{f}_{ML}(y_1, y_2) = c_Y\left\{F_1(y_1 \mid \hat{\mu}_1, \hat{\sigma}_1, \hat{\nu}_1), F_2(y_2 \mid \hat{\mu}_2, \hat{\sigma}_2, \hat{\nu}_2) \mid \hat{\rho}, \hat{\nu}_C\right\} \times f_1(y_1 \mid \hat{\mu}_1, \hat{\sigma}_1, \hat{\nu}_1) f_2(y_2 \mid \hat{\mu}_2, \hat{\sigma}_2, \hat{\nu}_2). \quad (5.12)$$

Both (5.11) and (5.12) can be obtained using `copula`'s `dmvdc()`.

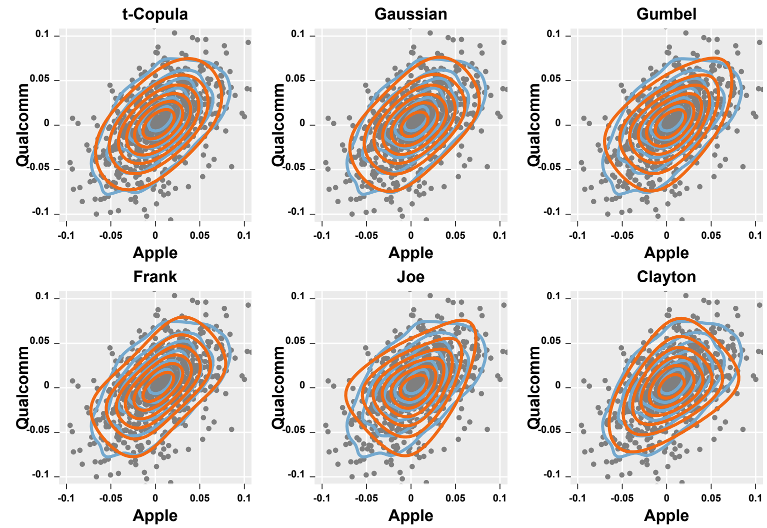


**Figure 5.6:** Apple and Qualcomm weekly returns. Scatter plots, PDF contours (red) of full parametric models and kernel density contours (blue).

The tail dependence index $\lambda_\ell$ of $t$-Copula (5.6) and Kendall's $\tau$ $t$-Copula (5.9) can be estimated with the $t$-Copula estimates.

```
round(EST,4)

##              rho   nu_C mu_aapl sig_aapl nu_aapl mu_qcom sig_qcom nu_qcom
## pseudo-ML 0.5424 9.3448  0.5107   3.8141  6.8219  0.3993   4.5471  3.6480
## ML        0.5446 9.3353  0.3717   3.8554  6.5709  0.1000   4.5565  3.6249
```

```
nu.hat = EST[,"nu_C"]; rho.hat = EST[,"rho"]
lambda.hat = 2*pt(-sqrt((nu.hat + 1)*(1 - rho.hat)/(1 + rho.hat)),
                 df = nu.hat + 1)
cat("tail dependence:"); round(lambda.hat,4);
cat("\t\tKendall's tau:");round(2/pi*asin(rho.hat),4)

## tail dependence:
## pseudo-ML        ML
##    0.1093    0.1105
##  Kendall's tau:
## pseudo-ML        ML
##    0.3650    0.3666
```





Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.