# STAT 631 Homework 5

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```r
load("HW05.Rdata")
```

1)

a)

In the case with short selling there are explicit solutions for the minimum variance portfolio. These are:

$$w_{\text{min.v}} = \frac{\Sigma^{-1}1}{1^T\Sigma^{-1}1}, \text{Mean} = \frac{\mu^T\Sigma^{-1}1}{1^T\Sigma^{-1}1}, \text{Variance} = \frac{1}{1^T\Sigma^{-1}1}$$

First we find the mean vector $\mu$ and covariance matrix $\Sigma$.

```r
y.mu <- apply(y,MARGIN = 2, FUN = mean)
y.mu
```

```
     AMZN        KO       NKE       PFE      TSLA       UNH       URI         V
0.4206254 0.1692590 0.2216723 0.1501095 0.6957931 0.4128102 0.5011812 0.3970618
```

```r
y.S <- var(y)
y.S
```

```
          AMZN       KO       NKE      PFE      TSLA       UNH       URI
AMZN 17.892637 2.431855  5.576128 2.564539 11.972386  2.977977  8.972939
KO    2.431855 6.238607  3.820411 2.972533  4.151542  3.972146  5.440775
NKE   5.576128 3.820411 14.638695 2.812862  9.371497  4.268385 10.709588
PFE   2.564539 2.972533  2.812862 8.787569  3.860339  4.452199  5.824990
TSLA 11.972386 4.151542  9.371497 3.860339 57.338308  6.231686 16.666210
UNH   2.977977 3.972146  4.268385 4.452199  6.231686 12.173168  7.763183
```

```
URI   8.972939 5.440775 10.709588 5.824990 16.666210  7.763183 39.914885
V     5.837551 3.829269  5.747779 3.145227  7.124705  4.292491  8.992063
            V
AMZN 5.837551
KO   3.829269
NKE  5.747779
PFE  3.145227
TSLA 7.124705
UNH  4.292491
URI  8.992063
V    9.207900
```

Now we can compute the weights.

```
one_vector <- rep(1, 8)
y.S_inv <- solve(y.S)
w_min.v <- y.S_inv%*%one_vector/as.numeric((t(one_vector)%*%y.S_inv%*%one_vector))
colnames(w_min.v) = "Weights"
w_min.v
```

```
         Weights
AMZN  0.11244506
KO    0.42307306
NKE   0.08199088
PFE   0.26278105
TSLA -0.01238153
UNH   0.07393439
URI  -0.07294053
V     0.13109762
```

The expected return of this portfolio is:

```
expected_return <- as.numeric(t(y.mu)%*%y.S_inv%*%one_vector)/
  as.numeric(t(one_vector)%*%y.S_inv%*%one_vector)
expected_return
```

```
[1] 0.2139305
```

The risk of this portfolio is:

```
risk <- sqrt(1/as.numeric(t(one_vector)%*%y.S_inv%*%one_vector))
risk
```

```
[1] 2.086775
```

b)

The tangency portfolio also has an explicit solution with short selling allowed. With $\mu_{ex} = \mu - \mu_f 1$:

$$w_T = \frac{\Sigma^{-1}\mu_{ex}}{1^T\Sigma^{-1}\mu_{ex}}$$

The annual risk-free rate is 4.37%. The weekly risk-free rate is 4.37%/52. The allocation weights of the tangency portfolio are below:

```
weekly_risk_free <- .0437/52
mu_excess = y.mu - weekly_risk_free
w_tangency = y.S_inv %*% mu_excess/
  as.numeric(one_vector^T %*% y.S_inv %*% mu_excess)
colnames(w_tangency) = "Weights"
w_tangency
```

```
          Weights
AMZN  0.200277036
KO   -0.151625522
NKE  -0.184406419
PFE  -0.151336852
TSLA  0.118670461
UNH   0.520386933
URI   0.008547379
V     0.639486985
```

The mean and variance of the tangency portfolio have the explicit solutions:

$$\text{Mean} = \frac{\mu^T\Sigma^{-1}\mu_{ex}}{1^T\Sigma^{-1}\mu_{ex}}, \text{Variance} = \frac{\mu_{ex}^T\Sigma^{-1}\mu_{ex}}{(1^T\Sigma^{-1}\mu_{ex})^2}$$

So the expected return of the tangency portfolio is:

```
expected_return_tangency <- as.numeric((t(y.mu) %*% y.S_inv %*% mu_excess)/
  (t(one_vector)%*% y.S_inv %*% mu_excess))
expected_return_tangency
```

```
[1] 0.5505735
```

And the risk of the tangency portfolio is:

```
risk_tangency <- as.numeric(sqrt((t(mu_excess)%*%y.S_inv%*%mu_excess)/
  (t(one_vector)%*%y.S_inv%*%mu_excess)^2))
risk_tangency
```

```
[1] 3.351737
```

c)

To plot the efficient frontier requires us to compute the risk for various expected returns. The explicit solution is below:

$$\text{Amat} = [\mu, 1], \text{H} = \text{Amat}^T \Sigma^{-1} \text{Amat} = \begin{bmatrix} C & B \\ B & A \end{bmatrix}, \Delta = \det(\text{H})$$

$$\text{Risk}_{opt} = \sqrt{\frac{Am^2 - 2Bm + C}{\Delta}}$$

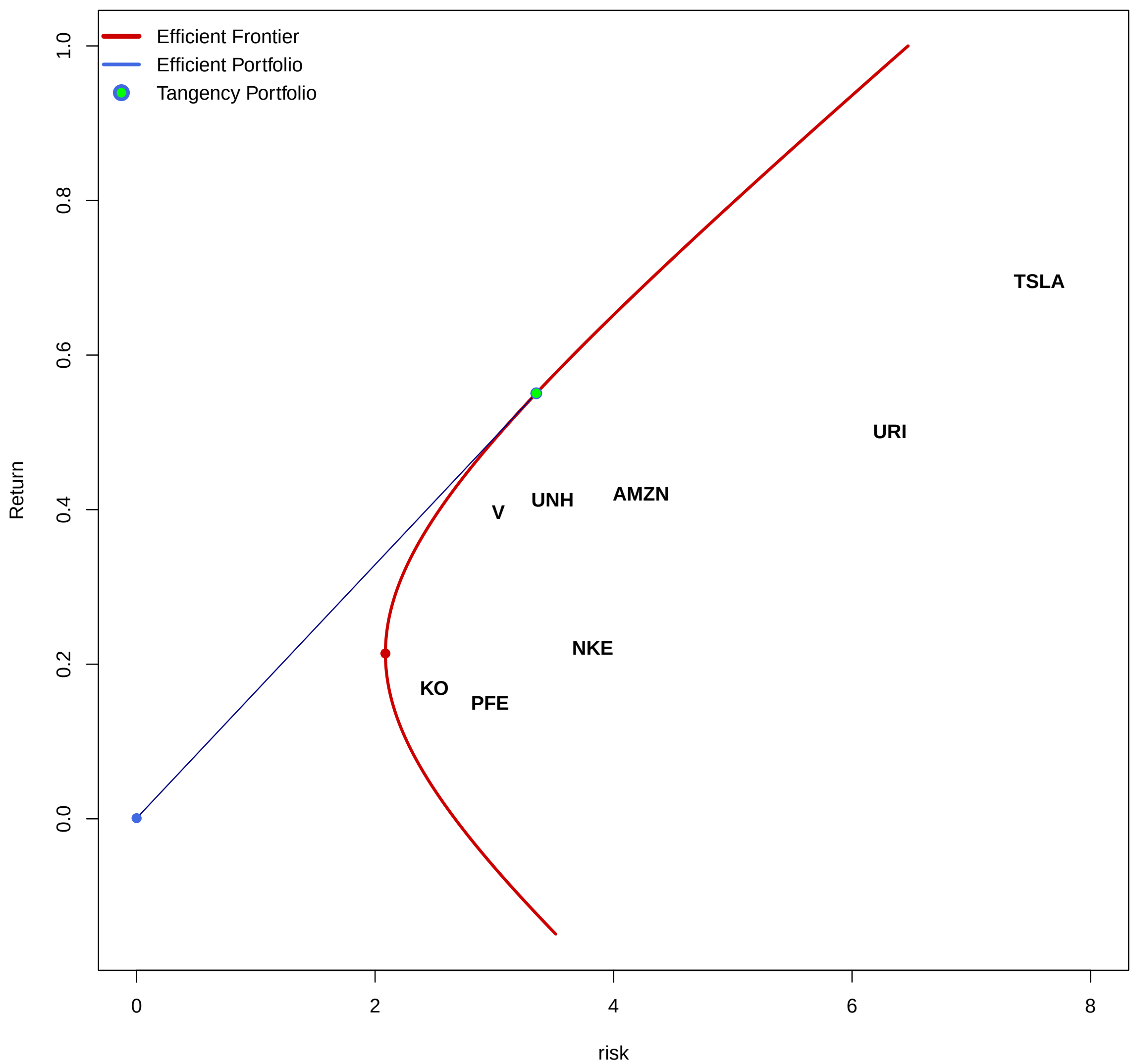
```
m.R = seq(-.15,1,0.001)
Amat = cbind(y.mu, one_vector)
H = t(Amat)%*%y.S_inv%*%Amat
A = H[2,2];B = H[1,2];C = H[1,1]; Delta = det(H)
sd.R = sqrt((A*m.R^2 - 2*B*m.R + C)/Delta)
```

Our plot is below:

```
mu.min = -.15
y.sd = sqrt(diag(y.S))
plot(sd.R,m.R, type = "l", xlim= c(0, 8), xlab = "risk", ylab = "Return")
lines(sd.R[m.R > mu.min],m.R[m.R > mu.min], lwd = 2.5, col = "red3")
lines(c(0,risk_tangency), c(weekly_risk_free, expected_return_tangency), col = "navy")
points(0, weekly_risk_free, pch = 19, col = "royalblue")
points(risk_tangency,expected_return_tangency, pch = 21, col = "royalblue", bg = "green", ce
```

```
points(risk, expected_return, pch = 19, col = "red3")
for(i in 1:8){
  text(y.sd[i], y.mu[i], syb[i], font = 2)
}

legend("topleft",c("Efficient Frontier", "Efficient Portfolio", "Tangency Portfolio"), lty =
                   y.intersp = 1.2, bty = "n", xjust = 5)
```




d)

To find the efficient portfolio with risk of 2.5% we use the fact that:

$$\sigma_p = w_r \sigma_r$$

where $\sigma_p$ is the risk of the portfolio, $w_r$ is the weight on the tangency portfolio and $\sigma_r$ is the risk of the tangency portfolio.

So then:

$$w_r = \frac{\sigma_p}{\sigma_r}$$

Where $\sigma_p$ is the allowed risk of 2.5% and $\sigma_r$ is the risk of the tangency portfolio is 3.351737%. So the weight is:

```
w_r = as.numeric(2.5/risk_tangency)
w_r
```

```
[1] 0.7458818
```

With the weight we can find the portfolio needed:

```
port_d <- rbind(1 - w_r, w_r*w_tangency)
rownames(port_d) <- c("Risk Free",syb)
port_d
```

```
               Weights
Risk Free  0.254118178
AMZN       0.149383001
KO        -0.113094721
NKE       -0.137545396
PFE       -0.112879407
TSLA       0.088514139
UNH        0.388147154
URI        0.006375335
V          0.476981717
```

Now with the weight we use the fact that the return of the portfolio is:

$$\mu_p = w_r \mu_r + (1 - w_r)\mu_{\text{rf}}$$

where $w_r$ is the weight on the tangency portfolio, $\mu_r$ is the return of the tangency portfolio and $\mu_{\text{rf}}$ is the risk free return. So the return of the efficient portfolio with allowed risk 2.5% is:

```
mu_p = w_r*expected_return_tangency + (1 - w_r)*weekly_risk_free
mu_p
```

```
[1] 0.4108763
```

e)

To find the efficient portfolio of a target return 0.55% we reverse the steps of part d. First we find the weight that corresponds to this return through:

$$\mu_p = w_r \mu_r + (1 - w_r)\mu_{\text{rf}}$$

$$w_r = \frac{\mu_p - \mu_{\text{rf}}}{\mu_r - \mu_{\text{rf}}}$$

So the weight on the tangency portfolio is:

```
w_r_e <- (.55 - weekly_risk_free)/(expected_return_tangency - weekly_risk_free)
w_r_e
```

```
[1] 0.9989568
```

With the weight we can find the portfolio:

```
port_e <- rbind(1 - w_r_e, w_r_e*w_tangency)
rownames(port_e) <- c("Risk Free",syb)
port_e
```

```
               Weights
Risk Free  0.001043194
AMZN       0.200068108
KO        -0.151467348
NKE       -0.184214047
PFE       -0.151178979
TSLA       0.118546664
UNH        0.519844068
URI        0.008538463
V          0.638819876
```

So the risk of this portfolio can be computed through:

$$\sigma_p = w_r \sigma_r$$

```
risk_e <- w_r_e*risk_tangency
risk_e
```

```
[1] 3.348241
```

f)

We go through the same procedure as part e to start:

$$w_r = \frac{\mu_p - \mu_{\text{rf}}}{\mu_r - \mu_{\text{rf}}}$$

```
w_r_f = (.85 - weekly_risk_free)/(expected_return_tangency - weekly_risk_free)
w_r_f
```

```
[1] 1.544676
```

We find that in order to get this target return of 0.85% we would need a weight over 1 in the tangency portfolio. This is not feasible as we would need to take out a loan to make up the difference. Therefore we look to the efficient frontier to find a portfolio of risky assets with the desired return. We can find the portfolio by using the two constraints:

$$\text{Amat} = \begin{bmatrix} \mu & 1 \end{bmatrix}, \text{bvec} = \begin{bmatrix} 0.85\% \\ 1 \end{bmatrix}$$

```
library(quadprog)
Amat = cbind(y.mu, one_vector)
bvec = c(.85,1)
zeros = rep(0,8)
out = solve.QP(Dmat = y.S, dvec = zeros, Amat = Amat, bvec = bvec, meq = 2)
w_f = out$solution; names(w_f) = syb
cat("Portfolio:"); w_f
```

```
Portfolio:
```

```
      AMZN          KO         NKE         PFE        TSLA         UNH
0.27839904 -0.66279023 -0.42135306 -0.51967333  0.23523444  0.91748341
       URI           V
0.08102667  1.09167306
```

We can find the risk from what we computed in part c to plot the efficient frontier. The details of the computation are in part c. The risk is:

```r
sd.R[which(m.R == .85)]
```

```
[1] 5.377229
```

2)

a)

In the case without short selling there is no explicit solution for the minimum variance portfolio. We set two constraints:

$$\text{Amat} = \begin{bmatrix} 1 & I_n \end{bmatrix}, \text{bvec} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

```r
Amat = cbind(one_vector, diag(8))
bvec = c(1, zeros)
out = solve.QP(Dmat = y.S, dvec = zeros, Amat = Amat, bvec = bvec, meq = 1)
w.min = out$solution; w.min = w.min*(abs(w.min) > 10e-7);names(w.min) = syb;
mu.min = sum(w.min*y.mu); sd.min = sqrt(2*out$val)
w.min
```

```
      AMZN         KO        NKE        PFE       TSLA        UNH        URI
0.09798502 0.43290178 0.05251957 0.25712960 0.00000000 0.05812796 0.00000000
         V
0.10133607
```

```r
c(return = mu.min, risk = sd.min)
```

```
   return      risk
0.2289597 2.1267106
```

b)

In the case without short selling there is also no explicit solution for the tangency portfolio. We find a portfolio $w_\star$ and then re-scale so the sum of weights is equal to 1. So we can choose the below constraints to find $w_\star$:

$$\text{Amat} = \begin{bmatrix} \mu - \mu_f 1 & I_n \end{bmatrix}, \text{bvec} = \begin{bmatrix} m \\ 0 \end{bmatrix}$$

```
amat = cbind(y.mu - weekly_risk_free, diag(8))
bvec = c(0.4, zeros)
out = solve.QP(Dmat = y.S, dvec = zeros, Amat = amat, bvec = bvec, meq = 1)
w.star = out$solution; names(w.star) = syb
w.T = w.star/sum(w.star)
w.T = w.T * (abs(w.T) > 10e-7)
cat("Portfolio: ");w.T
```

```
Portfolio:

      AMZN         KO        NKE        PFE       TSLA        UNH        URI 
0.15136990 0.00000000 0.00000000 0.00000000 0.08657754 0.35365583 0.00000000 
         V 
0.40839673 
```

```
mu.T = sum(w.T*y.mu); s.T = sqrt(2*out$value)/sum(out$solution)
c(return = mu.T, risk = s.T)
```

```
   return      risk 
0.4320615 2.7162153 
```

c)

First we need to create the risk-return sets. We do this by finding the portfolios with pre-specified returns ranging from the lowest per stock return (PFE with return 0.15%) and the highest per stock return (TSLA with return 0.7%) and then computing the risk of each portfolio.

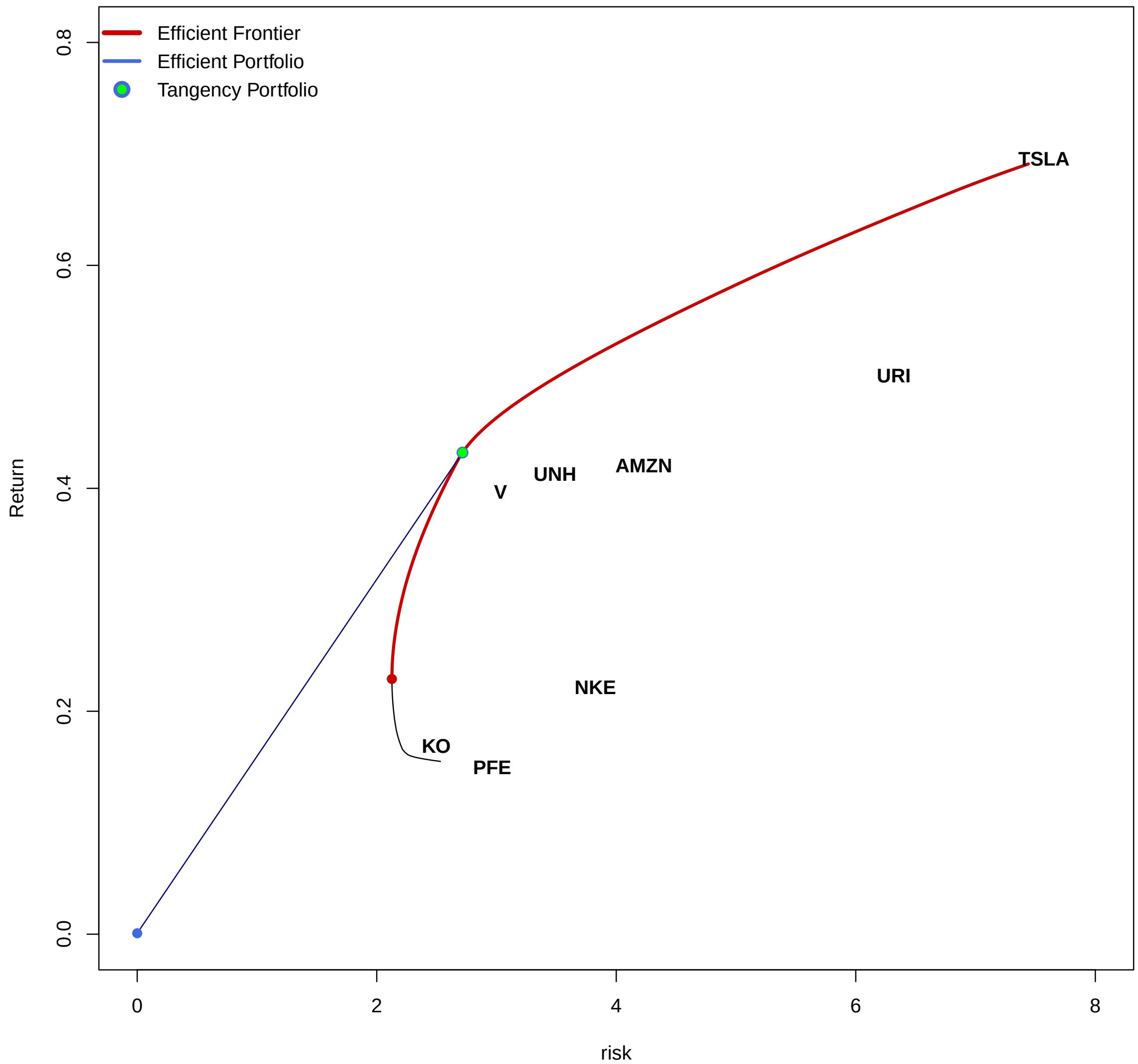
```
m.R = seq(round(min(y.mu)+.005,3), round(max(y.mu)-.005,3), 0.001);
Amat = cbind(y.mu, one_vector, diag(8)) ## for positive w
sd.R = c();
for(i in 1:length(m.R)){
```

```
  bvec = c(m.R[i],1, zeros) ## for nonnegative w
  out= solve.QP(y.S, dvec = zeros, Amat=Amat, bvec = bvec, meq = 2);
  sd.R[i] = sqrt(2*out$value)
 }
```

Now we can create our plot:

```
plot(sd.R,m.R, type = "l", xlim= c(0, 8), ylim = c(0, .8), xlab = "risk", ylab = "Return")
lines(sd.R[m.R > mu.min],m.R[m.R > mu.min], lwd = 2.5, col = "red3")
lines(c(0,s.T), c(weekly_risk_free, mu.T), col = "navy")
points(0, weekly_risk_free, pch = 19, col = "royalblue")
points(s.T,mu.T, pch = 21, col = "royalblue", bg = "green", cex = 1.2)
points(sd.min, mu.min, pch = 19, col = "red3")
for(i in 1:8){
  text(y.sd[i], y.mu[i], syb[i], font = 2)
}

legend("topleft",c("Efficient Frontier", "Efficient Portfolio", "Tangency Portfolio"), lty =
                   y.intersp = 1.2, bty = "n", xjust = 5)
```




d)

To find the portfolio with allowed risk 2.5% we follow a procedure similar to the case with short selling:

$$\sigma_p = w_r \sigma_r$$

$$w_r = \sigma_p / \sigma_r$$

So we have the weights below:

```
w_r_d <- 2.5/s.T
c(Risk_Free = 1 - w_r_d, Tangency_Portfolio = w_r_d)
```

```
         Risk_Free Tangency_Portfolio 
        0.07960167         0.92039833 
```

```
weights_d <- c(1 - w_r_d, w_r_d*w.T)
names(weights_d) = c("Risk Free", syb)
weights_d
```

```
 Risk Free       AMZN         KO        NKE        PFE       TSLA        UNH 
0.07960167 0.13932061 0.00000000 0.00000000 0.00000000 0.07968582 0.32550423 
       URI          V 
0.00000000 0.37588767 
```

And the portfolio's return and risk:

```
c(return_d = as.numeric(t(weights_d)%*% c(weekly_risk_free, y.mu)),
  risk_d = 2.5)
```

```
 return_d    risk_d 
0.3977356 2.5000000 
```

e)

Since the return of 0.55% is greater than the 0.43%, the return of the tangency portfolio, we need to find a portfolio off of the efficient frontier. The weight, return and risk are below:

```
m.e = 0.55
Amat = cbind(y.mu, one_vector, diag(8))
bvec = c(m.e,1, zeros)
out = solve.QP(y.S, dvec = zeros, Amat=Amat, bvec = bvec, meq = 2);
sd.e = sqrt(2*out$value)
portfolio_weights = out$solution;names(portfolio_weights) = syb;
portfolio_weights = portfolio_weights*(abs(portfolio_weights) > 10e-7)
cat("Portfolio Weights:");portfolio_weights
```

```
Portfolio Weights:
```

```
      AMZN          KO         NKE         PFE        TSLA         UNH         URI 
0.03685150 0.00000000 0.00000000 0.00000000 0.45942698 0.30673284 0.09598675 
         V 
0.10100193 
```

```
cat("Return: 0.55%","Risk: ",sd.e )
```

```
Return: 0.55% Risk:  4.370181
```

f)

The portfolio with the highest return that we can construct is one holding solely the highest return stock, in this case holding TSLA with a return of .7%. Since the target return of 0.85% is larger than the return of a portfolio holding only TSLA it can not be constructed without short selling.