Winding Number of a Curve - LEAN4 Project

Jorge Carrasco Coquillat Juan Antonio Montalbán Vidal

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Abstract

In this document we aim to share our experience with this project, including the first problems we encountered, such as choosing the project topic, main results we have formalized alongside with the difficulties we have had to overcome. We finish with some conclusions and a short reflection. We will mainly follow [1].

1 Choosing our project topic

There were various options that seemed good for us, from Carmichael numbers to orientability of manifolds. Lastly, we decided to work in a complex analysis topic: the winding number.

The winding number of a curve can be defined in several ways, although we only focused on the topological and analytic ones. Our main goal was going to be to prove the equivalence between these definitions, albeit we quickly noticed it was going to be a very stretch goal due to its complexity and the time we had.

Since this project focuses on the definition and properties of the winding number, we will now give the definition of *curve* that we have used - there is no a general consensus on how to define them, especially regarding the definition interval.

For us, a curve γ will be a $\mathcal{C}^\infty(\mathrm{I}, \mathbb{C})$ function, where I stands for the unit interval.

- **Topological definition:** it uses the path-lifting property [ref]. Given a curve $\gamma : [0,1] \to \mathbb{C}$ and $z_0 \in C$ satisfying $z \notin \mathrm{im}\gamma$, the winding number of γ around z_0 is defined as
$$\omega(\gamma, z_0) = s(1) - s(0),$$
where (ρ, s) is the path written in polar coordinates, this is, the lifted path through the covering map
$$p : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$$
$$(\rho_0, s_0) \mapsto z_0 + \rho_0 e^{i2\pi s_0}.$$
- **Analytic definition:** The analytic definition is more straightforward. Consider the curve γ and the point z_0 as previously specified. The winding of γ around z_0 is defined as
$$\omega(\gamma, z_0) = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t) - z_0}\mathrm{dt}.$$

After spending half a week discussing wether we should try proving the equivalence of definitions we noticed that there was a big problem regarding this issue: there are a lot of results missing in Mathlib - especially from algebraic topology - which be would need just to get to define the winding number.

2 Main formalized results

The very first thing we had to tackle, far from a lemma or theorem, was giving the right - this is, the most accurate one - definition of a curve. First we opted for defining it for an arbitrary interval (a, b) but we rapidly noticed this was unnecesary and the vast majority of the literature simply used the unit interval, so we decided to change to that.

But that was not the unique thing we had to change about the structure definition of a (closed) curve. At the beginning, we opted for putting the continuity and differentiability conditions separately but, after hearing the given advices, we noticed the best option was to put them together using the `ContDiffOn` condition from Mathlib library.

Remark 1: we have also defined the concept of *n-piecewise curve*, i.e., a curve that is the result of the concatenation of n simple curves. This is a more general concept than a curve itself and, even though we do not formalize theorems using them, we have thought it would be of interest to get them defined.

Remark 2: the sign of the winding number depends on the orientation we choose for going along the curve or, in other words, if the curve winds *clockwise* or *counterclockwise* around the point. For the sake of simplicity, we have chosen the counterclockwise orientation as the default one. In the case a curve is given in the opposite orientation, we can always perform a change of basis to get back to desired scenario.

The main results we have formalized are the following:

Theorem 1. *Given a curve γ and a point z_0 satisfying (P), then*

$$\omega(\gamma, z_0) \in \mathbb{Z}.$$

In other words, the winding number of a curve around a point is always an integer.

Theorem 2. *Given a curve γ with winding number $n \in \mathbb{Z}$, the winding number of the reverse curve, say $\tilde{\gamma}$, is precisely $-n$.*

Theorem 3. *The winding number $\omega(\gamma, z)$, seen as a function of z, is continuous in $\mathbb{C} \setminus im\gamma$.*

3 Main difficulties

- Coming up with the most suitable definition for curves.
- As a complementary definition we thought about adding the concatenation of curves, i.e., piecewise curves, as we explained in the presentation. However, as Floris noticed, the definition was not accurate, since we had to add conditions of differentiability in the "glueing" points of the curves. For this reason, we have decided to remove this concept from our work. Nonetheless, it would be a great (maybe future work, outside the lecture) to formalize this concept properly and prove the same theorems for them, then providing a more general treatment of the winding number for piecewise curves.
- Proving continuity of the index function. Mathlib misses results missing continuity of parametric integrals (which is a fundamental concept in mathematics, namely in analysis and PDEs) so we had to do it the classical way - using the $\varepsilon - \delta$ definition of continuity. It is usually tedious to work with this concept on paper, thus doing it in LEAN4 has been, sometimes, a big trouble.

3.1 Difficulties and, in some cases, solutions

DISCUSS!!!

4 Conclusion and possible future work

As we discussed in the presentation, one of the main conclusions that arise from this project is that Mathlib desperately needs contributions - there exist several gaps regarding important results and concepts. Besides this, refining the treatment and work with integrals would be appreciated. For instance, a recurrent problem we have faced is the following: let I_1, I_2 be two definite integrals. LEAN4 doesn't interpret these expressions as the same:

$$I_1 - I_2 = (I_1) - I_2.$$

Sometimes, the RHS of the equation was directly obtained from obtaining a lemma that doesn't involve any parentheses.

References

[1] Cobos Díaz, Fernando, *Notas de clase de Análisis Complejo*, 2022.

[2] Another Author, *Another Book*, Another Publisher, Year.