The Winding Number of a Closed Curve

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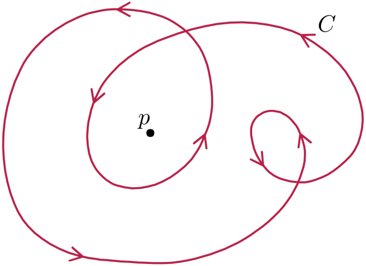
Formalisation project in LEAN4



January 21, 2025

Reminder on complex curves

Definition

A **curve** is a map

$$\gamma : [0,1] \longrightarrow \mathbb{C}$$

which is continuously differentiable, i.e., $\gamma \in \mathcal{C}^1([0,1])$. The curve is said to be **closed** if $\gamma(0) = \gamma(1)$.

In LEAN4, we have defined it as follows:

```
structure curve where
toFun : ℝ → ℂ
class_c1 : ContDiff ℝ 1 toFun
```

```
instance :
CoeFun curve fun _ ↦ ℝ → ℂ := ⟨fun f ↦ f.toFun⟩
```

```
structure closed_curve extends curve where
closed : toFun 0 = toFun 1.
```

Definition of ω, the winding number

Definition

Let γ be a closed curve and $z_0 \in \mathbb{C}$ satisfying $z_0 \notin \text{im}\gamma$. The **winding number** of γ around z_0 is given by

$$\omega(z_0, \gamma) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z_0} = \int_0^1 \frac{\gamma'(t)}{\gamma(t) - z_0} dt.$$

noncomputable def $\omega \ (z : \mathbb{C})\ (\gamma : \texttt{closed\_curve}) : \mathbb{C} :=$

$$\frac{1}{2 \cdot \texttt{Real.pi} \cdot \texttt{Complex.I}} \int \texttt{t in I,}\ \frac{\texttt{deriv}\ \gamma(t)}{\gamma(t) - z}.$$

Some results formalised

- A list of useful lemmas

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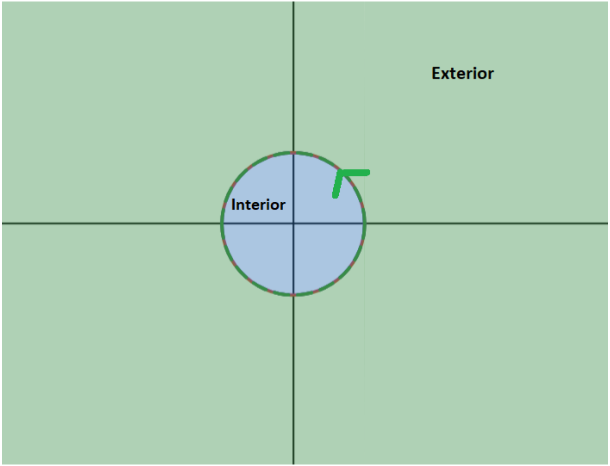
- A list of useful lemmas
- The interior and exterior of a curve



Some results formalised

- Winding numbers in the circle

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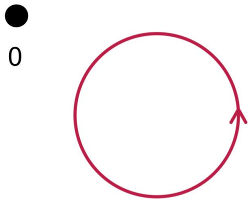
Theorem (Winding outside the circle)

Let $\gamma(t) = e^{2\pi i t}$ and $z_0 \in \mathbb{C}$ satisfying $\|z_0\| > 1$. Then,

$$\omega(z_0, \gamma) = 0.$$



Some results formalised

- Winding numbers in the circle

Theorem (Winding outside the circle)

Let $\gamma(t) = e^{2\pi it}$ *and* $z_0 \in \mathbb{C}$ *satisfying* $\|z_0\| > 1$*. Then,*

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Theorem (Winding inside the circle)

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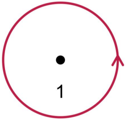

The winding number of a closed curve is an integer

Theorem

Let γ be a closed curve and $z_0 \in \mathbb{C}$ satisfying $z_0 \notin im\gamma$. Then,

$$\omega(z_0, \gamma) \in \mathbb{Z}.$$

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1. Define $g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-z_0} ds$, $\psi(t) = e^{-g(t)} \cdot (\gamma(t) - z_0)$.

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2. Prove $\psi'(t) = 0, \forall t \in [0, 1]$.
3. $(2) + \psi$ continuous $\implies \psi(0) = \psi(1)$.
4. $e^{-g(1)} = 1$.
5. $(4) \implies \exists k \in \mathbb{Z}, g(1) = 2k\pi i$, and so

$$\omega(z_0, \gamma) = \frac{g(1)}{2\pi i} = k \in \mathbb{Z}.$$

Future work

Theorem (Winding number is continuous)

The function $z \mapsto \omega(z,\gamma)$ *is continuous in* $\mathbb{C} \setminus im\gamma$.

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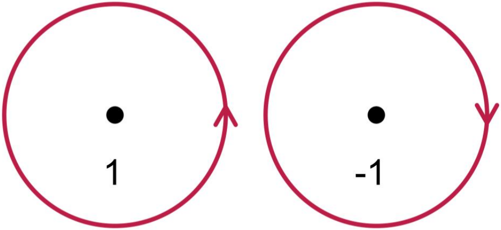
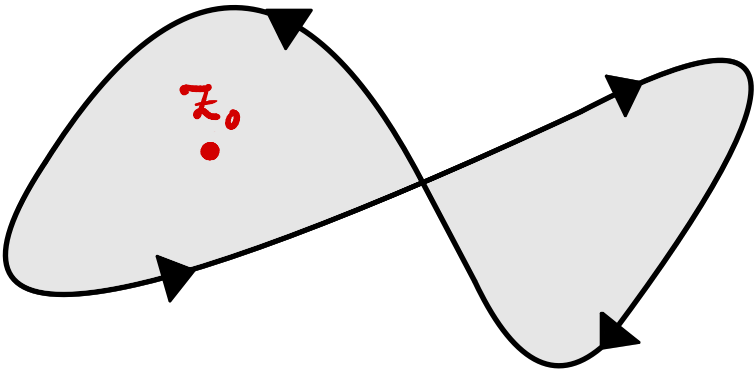
Theorem

The winding number of the **reverse curve** *is equal to the winding number of the original curve, but with opposite sign.*



Conclusion



Conclusion

Thank you for listening!