Introduction to Analysis of Algorithms October 10, 2020
Boston University CS 330
Professor Adam Smith, Dora Erdos Fall 2020

Homework 5

Due Wednesday, October 7 at 11:59 PM

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Homework Guidelines

Collaboration policy Collaboration on homework problems, with the exception of programming assignments and reading quizzes, is permitted, but not encouraged. If you choose to collaborate on some problems, you are allowed to discuss each problem with at most 5 other students currently enrolled in the class. Before working with others on a problem, you should think about it yourself for at least 45 minutes. Finding answers to problems on the Web or from other outside sources (these include anyone not enrolled in the class) is strictly forbidden.

You must write up each problem solution by yourself without assistance, even if you collaborate with others to solve the problem. You must also identify your collaborators. If you did not work with anyone, you should write "Collaborators: none." It is a violation of this policy to submit a problem solution that you cannot orally explain to an instructor or TA.

Solution guidelines For problems that require you to provide an algorithm, you must give the following:

1. a precise description of the algorithm in English and, if helpful, pseudocode,
2. a proof of correctness,
3. an analysis of running time and space.

You may use algorithms from class as subroutines. You may also use any facts that we proved in class.

You should be as clear and concise as possible in your write-up of solutions.

A simple, direct analysis is worth more points than a convoluted one, both because it is simpler and less prone to error and because it is easier to read and understand. Points might be subtracted for illegible handwriting and for solutions that are too long. Incorrect solutions will get from 0 to 30% of the grade, depending on how far they are from a working solution. Correct solutions with possibly minor flaws will get 70 to 100%, depending on the flaws and clarity of the write up.

1. (**Photography shop**)

   Suppose you work at a photography shop, and at the start of your workday you are given a list of jobs $i = 1, 2, \ldots, n$, each with a processing time p_i and a "touchiness" factor $t_i > 0$ (corresponding to how mad the customer will get if job i is delayed). Your pay for finishing job i at time f_i is $(P - f_i)/t_i$, where P is the total processing time of all jobs ($P = p_1 + p_2 + \ldots p_n$). Assume you have enough time to finish all n jobs, but you get to choose the order in which you do them. You can only do one job at a time.

   The algorithmic problem is to compute an order that maximizes your total payment.

   For example, suppose you have three jobs as follows:

i	p_i	t_i
1	4.5	10
2	1.25	7
3	9	3

   Then, the order $(1, 2, 3)$ would have finishing times $f_1 = 4.5$, $f_2 = 5.75$, $f_3 = 14.75$ and a total payoff of $10.25/10 + 9/7 + 0/3 = 2.3$ On the other hand, the ordering $(3, 1, 2)$ has finishing times $f_3 = 9, f_1 = 13.5, f_2 = 14.75$ with total payoff $5.75/3 + 1.25/10 + 0/7 = 2.04$

   An idea is to find some quantity by which the tasks can be sorted, so that the optimal order is the order sorted by this quantity.

   (a) Consider each of the following orders in which to schedule jobs. For **three out of the four** of them, give a counterexample showing that it does not produce the optimal task schedule:

      i. Decreasing order of processing time p_i
      ii. Decreasing order of touchiness t_i
      iii. Increasing order of *product* of touchiness to processing time $t_i p_i$
      iv. Increasing order of *sum* of touchiness and processing time $t_i + p_i$

   (b) Give a polynomial-time greedy algorithm for this problem. [An exchange algorithm works well for the proof of correctness.]

   Hint: You may want to analyze the situation of two tasks that are scheduled next to each other: say $(2, 3)$. Under what condition on their parameters p_2, t_2, p_3, t_3 will be worth changing their order to $(3, 2)$? What is the sorting quantity suggested by this condition? (Compute the difference in payment resulting from the swap.)

- Part A
  - Two of the proposed schedules can be disproved using one example. The following example will disprove i and ii. Take the following case in which job i comes before job j in a two job set. $p_i = 100$, $t_i = 100$, $p_j = 1$ and $t_j = 1$ satisfy the following conditions:
    - i Decreasing order of processing time as $p_i > p_j$
    - ii Decreasing order of touchiness as $t_i > t_j$

    The dollar amount for ordering $\{i, j\}$ in a schedule with no idle time is represented below:
    $$A_{\{i,j\}} = \frac{(P - f_i)}{t_i} + \frac{(P - f_j)}{t_j}.$$
    and,
    $$f_i = p_i, f_j = p_i + p_j, P = p_i + p_j.$$
    Which gives:
    $$A_{\{i,j\}} = \frac{(101 - 100)}{100} + \frac{(101 - 101)}{1} = 0.01.$$
    If we calculate the reverse order $\{j, i\}$ we get:
    $$A_{\{j,j\}} = \frac{(101 - 1)}{1} + \frac{(101 - 101)}{100} = 100.$$
    Clearly, in this example, it is more advantageous to reverse the order to $\{j, i\}$ disproving i and ii.
  - The last rule, $t_i + p_i < t_j + p_j$ can be disproved with the following example in which $p_i = 5$, $t_i = 5$, $p_j = 10$, and $t_j = 2$. This set of values complies with rule iv since,
    $$5 + 5 = 10 < 10 + 2 = 12.$$
    Following the same procedure as before, the money with each ordering is:
    $$A_{\{i,j\}} = \frac{(15 - 5)}{5} + \frac{(15 - 15)}{2} = 2.$$
    $$A_{\{j,i\}} = \frac{(15 - 10)}{2} + \frac{(15 - 15)}{5} = 2.5.$$
    The reverse ordering gives a better payout so iv is disproved.

- Part B
  - A precise definition of the algorithm in English and, if helpful, pseudocode.
    * The algorithm takes in an array p with processing time p_i for jobs $i = 1, \dots, n$ and an array t with touchiness scores for jobs of the same index. $O(1)$ time.
    * We initialize an array O that contains the index values $i = 1, \dots, n$ to keep track of where each job is swapped to. $O(1)$ time.
    * Initialize an empty array *prod* that is used to hold the products of jobs $i = 1, \dots, n$. $O(1)$ time.
    * We iterate through both p and t at once and store their product in the corresponding index of *prod*. $O(n)$ time.
    * Use merge sort to iterate through *prod* and swap any two values in which the previous index is greater than the index adjacent to it. Any time an index is switched in *prod*, switch the indexes in O. $O(n \log n)$ time.
    * Return O which contains the ordering of the optimal set of jobs. $O(1)$
  - A proof of correctness.

    Take a set of jobs $S = \{0, \dots, n\}$ where job i has a process time p_i and touchiness factor t_i. There are two jobs a and b that are adjacent to each other with $b > a$. Since swapping two jobs has no impact on the rest of the jobs' start and finish times, we can analyze what happens to the payout if these jobs are switched.

    We define the following variables: s is the start time of the first job in the pair - this does not change regardless of order. f is the finish time of the pair of jobs - this also does not change and is equal to $p_a + p_b$.

    If we assume that amount $A_{\{a,b\}}$ is greater than $A_{\{b,a\}}$, we see that it implies the following:

    $$A_{\{a,b\}} = \frac{(P - s - p_a)}{t_a} + \frac{(P - s - p_a - p_b)}{t_b} > \frac{(P - s - p_b)}{t_b} + \frac{(P - s - p_a - p_b)}{t_b} = A_{\{b,a\}}.$$

    Which simplifies to:

    $$p_a t_a < p_b t_b.$$

    Which turns out to be the rule! This shows that for any pair of adjacent jobs, choosing the smaller product $p_i t_i$ will increase the payout.

    * It is clear that an optimal schedule will have no idle time as delaying a job needlessly will only decrease its payout (f_i increases).
    * All schedules with no inversions as defined by rule iii given in the problem statement have the same pay out. Any jobs that are in a different order in two job orderings must have the same payout per the algebraic simplification above. (If we assume equal instead of greater than we see that $p_a t_a = p_b t_b$)
    * There exists a job ordering with no inversions and no idle time that is optimal. If there are two jobs i and j where $p_i t_i > p_j t_j$ we know we can swap them and

increase the payout. The job set will have 1 less inversion. Any job set will have a maximum $\binom{n}{2}$ inversions.

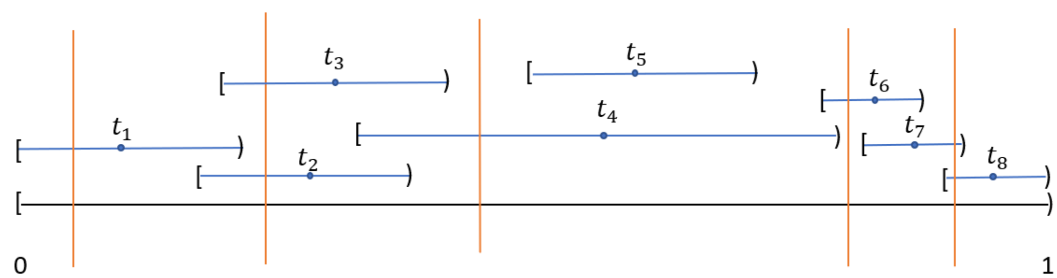
  * Since lowering the number of inversions only increases the payout and our algorithm gives the least number of inversions (none). Our algorithm is optimal.

- An analysis of running time and space.
  * Time Complexity
    The algorithm has time complexity of the steps associated above. The most time-intensive step is the sorting of the arrays done in $O(n \log n)$ time.
  * Space Complexity
    The most space consuming data structure used is the size of the input which is simply $O(n)$

2. (**Cell towers**) Suppose you are back on the long country road from Lab 3, which we can think of as a line segment with western and eastern endpoints. Let's identify points on the road with the interval $[0, 1)$. Now, suppose that a telecommunications company has invested in cell coverage for this road: they've placed n towers, each of which can cover a different radius in specific locations along this road such that every segment of the road is covered by *at least* one tower. We'll say $t_i \in [0, 1)$ is the location of the i-th tower along the road, and it has radius r_i. For technical reasons, tower i's interval of coverage will be open on the right side, giving $[t_i - r_i, t_i + r_i)$.

   As a surveyor for this telecommunications company, your job is to find the *maximum size set of points along the road* $[0, 1)$ *such that each tower serves at most one point in this set*; we'll call a set with this property *sparse*. The input for this problem is the list of pairs (t_i, r_i), for $i = 1, ..., n$.

   Below is an example of a set of towers and orange lines indicating a sparse set. (Do not hand in) Is it the largest possible sparse set?



(a) Your friend suggests the following greedy approach: sort the intervals according to their centers t_i. Moving from left to right, add t_i (the position of tower i) to your set as long it it doesn't conflict with the previous ones (meaning no interval covers both t_i and a previously selected center). Give an example showing why this will not produce the largest sparse set. [NB: this algorithm chooses tower locations, but a sparse set doesn't have to be limited to tower locationsany subset of $[0, 1)$ is ok.]

- The image below displays the algorithm described in a in red on the provided example. An example of a larger sparse set is given in green as a counter example.

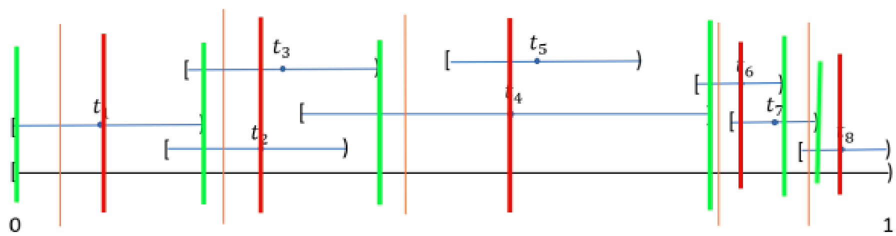


Figure 1: 2.a Counterexample in Red

Clearly, the green set shows a larger sparse set, so the approach suggested in red is not optimal.

Before we design an algorithm for finding sparse sets, let's consider a related problem: as before, you are again given the fixed locations and radii of n towers, but now suppose that the telecommunications company has decided that, in the interest of profits, they only want to keep enough cell towers running so that every point on the road $[0, 1)$ is covered by at least one tower. We will call such a set of towers *sufficient*. It's now your job to tell them the *minimum* number of towers they need to power so that the whole $[0, 1)$ interval is covered (i.e. find a sufficient set of minimum size).

(Do not hand in) How small a set of sufficient intervals can you find for the example pictured above?

(b) Prove that if there exists a sparse set $\mathcal{P}$ of locations with $|\mathcal{P}| = k$, we need at least k towers to cover $[0, 1)$ (that is, every sufficient set $\mathcal{T}$ of towers satisfies $|\mathcal{T}| \geq k$).

- We know that set $\mathcal{T}$ is a sufficient set of towers and $\mathcal{P}$ is a sparse set
- If we assume $|\mathcal{T}| < k$: we know that in a sparse set each tower can only serve one point in $\mathcal{P}$ so we can assign one tower to each location in $\mathcal{P}$. We run into an issue because there are not enough towers to cover each point without a tower serving two points.
- Therefore, by contradiction, every sufficient set of $\mathcal{T}$ of towers must satisfy $|\mathcal{T}| \geq k$).

(c) Give a polynomial-time algorithm that takes as input a list of pairs (t_i, r_i) and produces both a sparse set of positions and a sufficient set of intervals, such that the two sets have the same size. (Remember to follow the usual guidelines for algorithms.)

- The algorithm is described below:

Algorithm 1: test(A)

Input: List T of pairs of (t_i, r_i) for each tower i

L; initialize list to hold left bound of tower i ;
R; initialize list to hold right bound of tower i ;
I; initialize list to hold set of sufficient tower intervals ;
P ; initialize list to hold set of sparse points ;
$CurrentMax = 0$; initialize variable to track the current known right most interval that encompasses previous point ;
$CurrentTower$; initialize variable to track the current known best tower choice ;
$IntervalMin = 0$; Initialize minimum of interval;
$IntervalMax = 1$; Initialize max of interval;
for $i = 0$ *to* $T.length\,(\ ) - 1$ **do**
  $L[i] = T[i][0] - t[i][1]$;
  $R[i] = T[i][0] + t[i][1]$;
while $CurrentMax < IntervalMax$ **do**
  for $i = 0$ *to* $T.length$ **do**
    if $LastPoint \geq L[i]$ **then**
      if $R[i] > CurrentMax$ **then**
        $CurrentMax = R[i]$;
        $CurrentTower = i$;
  $LastPoint = CurrentMax\ I.append(CurrentTower)$;
  if $CurrentMax \neq IntervalMax$ **then**
    $P.append(CurrentMax)$;

Output: I, P

The idea of this algorithm is to start by putting a point in the sparse set at the start of the interval since we know there is a sufficient set of towers. We then chose the tower that both covers this point and reaches the furthest towards the end of the interval. We then place the next point directly at the end of that tower and repeat until we have reached the end of the interval.

This algorithm, in effect, greedily chooses the tower that reaches furthest Towards the end that also includes the previous chosen point.

- Proof of correctness

We will prove this algorithm by contradiction. The algorithm claims to find a sparse set of points and a sufficient set of intervals. To test this, let us assume we find a set of points $\{a, b\}$ that are covered by the same tower.

If a is placed into a sparse set the algorithm will then check for the tower that reaches the farthest to the right in which a falls under it. Another point is only added AFTER this interval it decides on. This makes it impossible for a and b to be placed in the same sparse set.

Similarly we can assume that we find a sparse set in which the number of towers is not equal to the number of points in the set. For this to happen a point must be added to the set that already has a corresponding tower that covers it. This is impossible because the algorithm only adds points directly AFTER every added tower.

- Time Complexity
  This algorithm must iterate through all n towers in the inner for loop. The question is - how many times does that inner for loop block get run? Since towers are only added in increasing reach to the right of the interval and there are only n towers, the outer while loop will run a maximum of n times. This makes the time complexity $O\left(n * n\right) = O\left(n^2\right)$.

(d) Argue that the algorithm from (c) finds both a sparse set of maximum possible size, and a sufficient set of the smallest possible size.

We will prove d using the greedy stays ahead approach. For finding the sufficient set of towers, the greedy rule we use is essentially - chose the next tower that has an overlapping interval (so there are no gaps) and that reaches as far right as possible. We want to prove that in an optimal algorithm the tower i's corresponding tower in our algorithm's covers at least as far to the right of the interval.

For towers $0, \ldots, n$ in our solution, $t_i + r_i$ is at least as large or larger. This is obviously true for the base case where the interval starts at zero, since the algorithm chooses the interval with the furthest range to the right. At this point we know our algorithm reaches at least as far as the optimal solution. We assume this is true up to tower $n-1$. At step n the greedy algorithm chooses the step furthest to right once again. By induction, the algorithm must find the smallest sufficient set as it will reach the end of the interval in the shortest number of steps.

Similarly we can then prove maximum sparse set size. The points in the sparse set are added as soon as the previous interval that covered the previous point ends. Whenever it is possible to place another point into he