MULTILINEAR EXTENSIONS AND THE BANZHAF VALUE*

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ABSTRACT

It is shown that the Banzhaf value can be obtained by differentiating the multilinear extension of a game at the midpoint of the unit cube. This gives us a composition theorem for the value of compound games. As an example, the values of the electoral college and presidential election "games" are approximated by the method of extensions.

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In [5] a multilinear extension was defined for n-person games, as follows: if v is the characteristic function of a game, with player set $N = \{1, 2, \ldots, n\}$, then the multilinear extension is the function of n variables

$$f(x_1, \ldots, x_n) = \sum_{S \subset N} \left\{ \prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \right\} v(S) \tag{1}$$

for all $0 \leq x_i \leq 1$, $i = 1, \ldots, n$. It was shown in [5] how this extension is related to the Shapley value, which is obtained as the integral

$$z_i = \int_0^1 f_i(t, t, \ldots, t)dt \tag{2}$$

of the partial derivatives of f along the diagonal $x_1 = x_2 = \ldots = x_n$ of the unit cube. This method was used in [6] to approximate the value of a presidential election game.

A different value was defined, at least for simple games by Banzhaf in [2]. Given a simple game with player set N, Banzhaf considers all 2^n possible divisions of the players into two complementary sets, S and $N - S$ (the "yeas" and "nays" in a given vote). For any such division, a player i is said to be *marginal* if, by changing from one set to the other (from S to $N - S$ or vice-versa) he can change S from a winning to a losing coalition or vice-versa. Let, now, η_i denote the number of such partitions $(S, N - S)$ for which i is marginal. Then

$$\beta_i = \frac{\eta_i}{\sum_{j=1}^{n} \eta_j} \tag{3}$$

is the *Banzhaf index* or *voting value* for player i in this game.

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A generalization of this value to nonsimple games is quite straightforward. In fact, we see that i is marginal in either of two cases:

(a) $i \in S$, S is winning, $S-\{i\}$ is losing
(b) $i \notin S'$, S' is losing, $S' \cup \{i\}$ is winning.

It is easily seen (by setting $S'=S-\{i\}$ above) that these two cases reduce to the same one. Thus it would be possible to consider only partitions $(S, N-S)$, such that $i \notin S$; then case (a) above could never hold, and we would look only for case (b). This would leave out exactly one half of the marginal cases for player i, but since the value is normalized by (3) there would be no change in the value β_i. Modified in this manner, we see that we have

$$\eta_i = \sum_{\substack{S \\ i \notin S}} [v(S \cup \{i\}) - v(S)], \tag{4}$$

where it is of course understood that $v(S)=0$ for losing sets, $v(S)=1$ for winning sets. Since (4) defines the vector η entirely in terms of the characteristic function v, we have a direct generalization to all games in characteristic function form.

As mentioned above, it is only the ratios of the components η_i which matter; thus, we can multiply them by an arbitrary constant, to obtain a new vector ψ:

$$\psi_i = \frac{1}{2^{n-1}} \eta_i$$

or, equivalently,

$$\psi_i = \frac{1}{2^{n-1}} \sum_{\substack{S \\ i \notin S}} [v(S \cup \{i\}) - v(S)]. \tag{5}$$

The advantage of this representation is that there are precisely 2^{n-1} possible coalitions S. Thus the coefficient 2^{1-n} may be thought of as a probability, and we see that ψ_i is the mathematical expectation of the marginal value $v(S \cup \{i\}) - v(S)$, assuming that each of the 2^{n-1} coalitions $S(i \notin S)$ has the same probability (this is in keeping with the usual probabilistic description given for the Shapley value, which of course gives different probabilities to coalitions of different size.)

Let us now differentiate the multilinear extension (1). Letting f_i be the derivative with respect to the ith variable, we have

$$f_i(x_1, \ldots, x_n) = \sum_{\substack{S \subset N \\ i \notin S}} \left\{ \prod_{j \in S} x_j \prod_{\substack{j \notin S \\ j \neq i}} (1-x_j) \right\} [v(S \cup \{i\}) - v(S)].$$

Setting $x_1 = x_2 = \ldots = x_n = 1/2$, we now have

$$f_i(1/2, 1/2, \ldots, 1/2) = \sum_{\substack{S \subset N \\ i \notin S}} (1/2)^{n-1} [v(S \cup \{i\}) - v(S)], \tag{6}$$

which is precisely ψ_i. Thus we see that, just as the Shapley value is obtained by integrating the gradient of f along the main diagonal of the cube, so the Banzhaf value is obtained by evaluating the gradient at the midpoint of the cube.

One interesting consequence of this result is that, just as the Shapley value can be approximated by the method of multilinear extensions, so also can the Banzhaf value. In fact, it is much easier to approximate the Banzhaf value, as it requires evaluation of the partial derivatives at a single point, whereas the Shapley value requires evaluation along the entire diagonal, plus an integration.

Example: The Electoral College

As an example, consider the "electoral college game," a 51-player weighted majority game in which each player (state) has w_i (between three and 45) votes and 270 votes are needed to win. As pointed out in [5] and [6], the partial derivative $f_i(x_i, \ldots, x_n)$ can be approximated as the probability

$$f_i(x_1, \ldots, x_n) \cong \text{Prob}\{269.5 - w_i \leq Y_i \leq 269.5\}, \tag{7}$$

where Y_i is a normal random variable with mean and variance

$$\mu_i = \sum_{j \neq i} x_j w_j \tag{8}$$

$$\sigma_i^2 = \sum_{j \neq i} x_j(1 - x_j) w_j^2, \tag{9}$$

respectively. At the point $(1/2, 1/2, \ldots, 1/2)$, this is

$$\mu_i = 1/2(538 - w_i) \tag{10}$$

$$\sigma_i^2 = 1/4(9{,}942 - w_i^2) \tag{11}$$

since $\Sigma w_i = 538$ and $\Sigma w_i^2 = 9{,}942$ for the 1970 apportionment. Thus we have the approximation

$$\psi_i \cong \Phi\left(\frac{269.5 - \mu_i}{\sigma_i}\right) - \Phi\left(\frac{269.5 - \mu_i - w_i}{\sigma_i}\right), \tag{12}$$

where Φ is the standard cumulative normal distribution function. Substitution of (10) and (11) here will give us

$$\psi_i = \Phi\left(\frac{w_i + 1}{\sqrt{9{,}942 - w_i^2}}\right) + \Phi\left(\frac{w_i - 1}{\sqrt{9{,}942 - w_i^2}}\right) - 1 \tag{13}$$

as the desired approximation.

TABLE I

Electoral votes (w_i)	Banzhaf value (ψ_i)	Banzhaf ratio (β_i)	Shapley value ($\mathscr{I}_i$)
45	0.38694	0.08828	0.08831
41	0.34806	0.07941	0.07973
27	0.22151	0.05054	0.05096
26	0.21291	0.04857	0.04898
25	0.20435	0.04662	0.04700
21	0.17056	0.03891	0.03917
17	0.13736	0.03134	0.03147
14	0.11276	0.02573	0.02577
13	0.10462	0.02387	0.02388
12	0.09648	0.02201	0.02200
11	0.08838	0.02016	0.02013
10	0.08028	0.01832	0.01827
9	0.07221	0.01647	0.01641
8	0.06414	0.01463	0.01456
7	0.05610	0.01280	0.01272
6	0.04806	0.01096	0.01088
5	0.04004	0.009135	0.009053
4	0.03202	0.007305	0.007230
3	0.02402	0.005480	0.005412

In Table I, the value ψ_i is approximated by this method. Also given are the corresponding $\beta_i = \psi_i/\Sigma\psi_j$, as well as the Shapley values, for comparison.

Behavior Under Composition

In [8], Shapley defines a composition operation for simple games. This composition is generalized in [4] to nonsimple games. Essentially, if v is a simple m-person game, while $u_1, u_2, \ldots, u_m$ are simple games for $n_1, n_2, \ldots, n_m$ persons, respectively, then

$$v[u_1, u_2, \ldots, u_n] = v^* \tag{14}$$

is a simple game for $m^* = n_1 + \ldots + n_m$ persons. The persons are labeled (i, j), with $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$, and a set is winning in the game v^* if it contains a subset of the form

$$\bigcup_{i \in T} \{i\} x S_i,$$

where T is winning in v, and each S_i is winning in the corresponding u_i. A rather complicated formula, related to (1), is given in [4] to generalize this to nonsimple games.

It is shown in [5] that, if v^* is the composition (14), then the multilinear extension of v^* is the composite function

$$h = f^0(g_1, g_2, \ldots, g_m), \tag{15}$$

where $f, g_1, g_2, \ldots, g_m$ are respectively the multilinear extensions of $v, u_1, u_2, \ldots, u_m$, respectively.

Here, h is a function of m^* variables, y_{ij}, with $i=1, \ldots, m$ and $j=1, \ldots, n_i$.

Let h_{ij} be the partial derivative of h with respect to y_{ij}. We have

$$h_{ij}=\frac{\partial h}{\partial y_{ij}}=\frac{\partial f}{\partial g_i}\frac{\partial g_i}{\partial y_{ij}}$$

or

$$h_{ij}=f_i g_{ij} \tag{6}$$

e f_i is the ith partial of f, while g_{ij} is the jth partial of g_i.

For the Banzhaf value, we must evaluate this at the point $x_{11}=\ldots=x_{mn_m}=1/2$. We have then

$$h_{ij}(1/2, \ldots, 1/2)=f_i(x_1, \ldots, x_m)g_{ij}(1/2, 1/2, \ldots, 1/2) \tag{7}$$

ere the arguments x_i are given by

$$x_i=g_i(1/2, 1/2, \ldots, 1/2).$$

In general, of course, $g_i(1/2, \ldots, 1/2)$ is different from 1/2, and so the Banzhaf value does not, general, compose. It is interesting to see, however, that the value which is allotted to the members of the ith subgame is distributed among them according to the derivatives of g_i, at the point $(1/2, \ldots, 1/2)$. This is precisely the value for the component game u_i. Thus we have

THEOREM 1: In a compound game $v[u_1, \ldots, u_m]$, the value to members of a subgame u_i is proportional to their value in the subgame itself.

In some cases, we will have, for all i

$$x_i=g_i(1/2, 1/2, \ldots, 1/2)=1/2. \tag{19}$$

In this case, of course, the value composes.

In this respect, the following lemma is of interest:

LEMMA: Let u be an n-person game such that $u(N)=1$, and let g be the multilinear extension of u. If u is constant sum, then

$$g(1/2, 1/2, \ldots, 1/2)=1/2.$$

Conversely, if u is super-additive, and $g(1/2, 1/2, \ldots, 1/2)=1/2$, then u is constant-sum.

PROOF: By definition (1),

$$g(1/2, 1/2, \ldots, 1/2)=(1/2)^n \sum_{S\subset N} u(S). \tag{20}$$

This sum can be split into two sums, one containing all S such that $1\in S$, and the other containing the remaining coalitions, which are of course their complements.

$$g(1/2, 1/2, \ldots, 1/2) = (1/2)^n \sum_{\substack{S \\ 1 \in S}} [u(S) + u(N-S)].$$

Now, if u is constant-sum, we have

$$u(S) + u(N-S) = 1,$$

for all S, and so

$$g(1/2, 1/2, \ldots, 1/2) = (1/2)^n \sum_{\substack{S \\ 1 \in S}} 1.$$

There are 2^{n-1} terms in the sum, and so $g(1/2, 1/2, \ldots, 1/2) = 1/2$.

Conversely, suppose u is super-additive. Then

$$u(S) + u(1-S) \leq 1$$

for each S, and so

$$(1/2)^n \sum_{\substack{S \\ 1 \in S}} [u(S) + u(N-S)] \leq 1/2$$

with equality holding only if $u(S) + u(N-S) = 1$ for all S, i.e., if S is constant-sum.

In case of composition, the assumption $u_i(N_i) = 1$ is normally made. Thus, for constant-sum games u_i, we find that (17) takes the form

$$h_{ij}(1/2, 1/2, \ldots, 1/2) = f_i(1/2, \ldots, 1/2) g_{ij}(1/2, 1/2, \ldots, 1/2). \tag{21}$$

Thus we obtain

THEOREM 2: Let v^* be the composition $v[u_1, \ldots, u_m]$, where each u_i is a constant-sum game. Let ψ_{ij}^* be the Banzhaf value to player (i, j) in v^*; let ψ_i be the value to player i in game v, and let $\mathscr{I}_{ij}$ be the value to player j in the subgame u_i. Then

$$\psi_{ij}^* = \psi_i \mathscr{I}_{ij}. \tag{22}$$

It should be noted that it is the value ψ (given by (5) or (6)) and not the ratio β (given by (3)) which composes.

Example: The Presidential Election Game

The presidential election "game" can best be defined as the composition

$$v^* = v[u_1, u_2, \ldots, u_{5_1}]$$

where v is the electoral college game, described above, and each u_i is a simple majority game for n_i players, n_i being the number of voters in the ith state $(i = 1, \ldots, 51)$.

Strictly speaking, u_i is a constant-sum game only if n_i is odd; with an even number of voters, it is possible to obtain two complementary sets, S and $N-S$, both of which lose. This can only happen, however, if both S and $N-S$ have exactly $n_i/2$ players, and of course, the probability of this, when n_i is large, becomes negligible. Thus each u_i is, for all practical purposes, a constant-sum game. In fact, we note that, for any t,

$$g_i(t, t, \ldots, t) = \text{Prob}\left(z_i > \frac{n_i}{2}\right) \tag{23}$$

where z_i is a binomial random variable with parameters n_i and t. This can be approximated (with negligible error) by a normal variable with mean $n_i t$ and variance $n_i t(1-t)$. Thus

$$g_i(t, \ldots, t) = \Phi\left(\frac{n_i(t-1/2)}{\sqrt{n_i t(1-t)}}\right)$$

or

$$g_i(t, \ldots, t) = \Phi\left(\sqrt{n_i}\frac{t-1/2}{\sqrt{t(1-t)}}\right). \tag{24}$$

At $t=1/2$, this gives us

$$g_i(1/2, 1/2, \ldots, 1/2) = \Phi(0) = 1/2, \tag{25}$$

which is the desired result. Thus Theorem 2 is applicable here.

To obtain the partial derivatives g_{ij}, we note that g_i is symmetric in all its arguments. Thus, for a point on the main diagonal, we have

$$g_{ij}(t, t, \ldots, t) = \frac{1}{n_i}\frac{dg_i(t, t, \ldots, t)}{dt}. \tag{26}$$

Differentiating (24), we have

$$\frac{dg_i(t, \ldots, t)}{dt} = \frac{\sqrt{n_i}}{4(t-t^2)^{3/2}} \cdot \frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-n_i(t-1/2)^2}{2t(1-t)}\right\} \tag{27}$$

and, at $t=1/2$, this is

$$\frac{dg_i}{dt} = \sqrt{\frac{2n_i}{\pi}}. \tag{28}$$

Thus

$$g_{ij}(1/2, \ldots, 1/2) = \sqrt{\frac{2}{\pi n_i}} \tag{29}$$

for every $i = 1, \ldots, m$, and $j = 1, \ldots, n_i$.

Application of Theorem 2 now gives us

$$\psi^*_{ij} = \sqrt{\frac{2}{\pi n_i}}\, \psi_i\,, \tag{30}$$

and thus we see that, in the compound (election) game, the value to each voter in the ith state is proportional to the value of the ith state in the quotient (electoral college) game, divided by n_i. Generally, of course, n_i is not known, but if it may be assumed that n_i is proportional to the population p_i of the ith state, then

$$\psi^*_{ij} = \frac{k}{\sqrt{p_i}}\, \psi_i\,, \tag{31}$$

where k is a factor of proportionality. As far as the Banzhaf ratio β_{ij} is concerned, of course, this factor k is unimportant.

In Table II, the values ψ^*_{ij} are computed, setting $k = 1$. (This corresponds, ideally, to setting $n_i = 2p_i/\pi$, so that approximately 63.6 percent of the population votes). Also given are the "state values"

$$\hat{\psi}_i = p_i\, \psi_{ij} = \sqrt{p_i \psi_j}$$

which give the total power of all voters within the state, together. Also given are the ratios,

$$\psi^*_{ij}/\psi_{D.C.,j}$$

which show how the strength of a voter in a given state compares with that of a voter in the least advantaged "state," the District of Columbia, and the ratio $\hat{\beta}_i$, which shows the fraction of the total power accruing to voters in the ith state.

COMMENTS

It is of interest to compare the Banzhaf value for this game with the Shapley value, as obtained in [6].

First of all, we notice that, for the 51-player state game, the Banzhaf value gives slightly less power to the large states than does the Shapley value, though still slightly more than they would receive if the value were directly proportional to the electoral votes.

Next, we note that, in the composed game, the Banzhaf value increases this bias by a factor which is exactly proportional to the square root of the population (number of voters). In the Shapley value, this new factor is not quite equal to $\sqrt{m_i}$. Thus the two values are very nearly equal for the large game. It is not, however, a good idea to take the Shapley value for the state game, and then combine it with the Banzhaf procedure ($\sqrt{m_i}$) for composition, as this will take the "worst" of both values (from the point of view of equity), using each where it is most favorable to the large states.

Finally, there may be some question as to the probable error caused by the approximations (12). This is, of course, difficult to evaluate; we point out, however, that in [6] a similar approximation

TABLE II

State	Electoral votes	Population	Value per vote ψ^*_{ij} ($\times 10^{-5}$)	$\frac{\psi^*_{ij}}{\psi_{D.C.,j}}$	Total to state $\sqrt{m_i}\,\psi_i$	Portion to state β_i
Alabama	9	3,444,165	3.8909	1.409	134.01	0.01192
Alaska	3	302,173	4.3696	1.582	34.01	0.00117
Arizona	6	1,772,482	3.6099	1.307	63.98	0.00569
Arkansas	6	1,923,295	3.4655	1.255	66.65	0.00593
California	45	19,953,134	8.6624	3.137	1,728.42	0.15371
Colorado	7	2,207,259	3.7760	1.367	83.35	0.00741
Connecticut	9	3,032,217	4.1468	1.502	125.74	0.01118
Delaware	3	548,104	3.2445	1.175	17.78	0.00158
District of Columbia	3	765,510	2.7616	1.000	20.89	0.00186
Florida	17	6,789,443	5.2716	1.909	357.91	0.03183
Georgia	12	4,589,575	4.5035	1.631	206.69	0.01838
Hawaii	4	769,913	3.6524	1.323	28.07	0.00250
Idaho	4	713,008	3.7932	1.374	27.03	0.00240
Illinois	26	11,113,976	6.3865	2.313	709.79	0.06312
Indiana	13	5,193,669	4.5907	1.662	238.42	0.02120
Iowa	8	2,825,041	3.8161	1.382	107.81	0.00959
Kansas	7	2,249,071	3.7408	1.355	84.13	0.00748
Kentucky	9	3,219,311	4.0245	1.457	129.56	0.01152
Louisiana	10	3,643,180	4.2060	1.523	153.23	0.01363
Maine	4	993,663	3.2148	1.164	31.89	0.00284
Maryland	10	3,922,399	4.0535	1.468	159.99	0.01414
Massachusetts	14	5,689,170	4.7275	1.712	268.95	0.02392
Michigan	21	8,875,083	5.7252	2.073	508.12	0.04519
Minnesota	10	3,805,069	4.1155	1.490	156.60	0.01393
Mississippi	7	2,216,912	3.7678	1.364	85.53	0.00743
Missouri	12	4,677,399	4.4610	1.615	208.66	0.01856
Montana	4	694,409	3.8407	1.391	26.70	0.00237
Nebraska	5	1,483,791	3.2871	1.190	48.77	0.00434
Nevada	3	488,738	3.4359	1.244	16.79	0.00149
New Hampshire	4	737,681	3.7281	1.350	27.50	0.00245
New Jersey	17	7,168,164	5.1305	1.858	367.76	0.03271
New Mexico	4	1,016,000	3.1767	1.150	32.28	0.00287
New York	41	18,190,740	8.1607	2.955	1,484.50	0.13202
North Carolina	13	5,082,059	4.6408	1.680	235.85	0.02097
North Dakota	3	617,761	3.0561	1.107	18.88	0.00168
Ohio	25	10,652,017	6.2612	2.267	666.95	0.05931
Oklahoma	7	2,559,253	3.5068	1.270	89.75	0.00798
Oregon	6	2,091,385	3.3233	1.203	69.50	0.00618
Pennsylvania	27	11,793,909	6.4501	2.336	760.72	0.06765
Rhode Island	4	949,723	3.2857	1.190	31.20	0.00277
South Carolina	8	2,590,516	3.9851	1.443	103.23	0.00918
South Dakota	4	666,257	3.9250	1.421	26.12	0.00232
Tennessee	10	3,924,164	4.0526	1.467	159.03	0.01414
Texas	26	11,196,730	6.3628	2.304	712.43	0.06336
Utah	4	1,059,273	3.1111	1.127	32.96	0.00293
Vermont	3	444,732	2.6035	1.305	16.01	0.00142
Virginia	12	4,648,494	4.4749	1.620	208.01	0.01850
Washington	9	3,409,169	3.9109	1.416	133.33	0.01186
West Virginia	6	1,744,237	3.6390	1.318	63.47	0.00564
Wisconsin	11	4,417,933	4.2048	1.523	185.76	0.01652
Wyoming	3	332,416	4.1661	1.509	13.85	0.00123

to the Shapley value gave errors which were uniformly less than 0.4 percent. Thus we have reason to believe that the error here should be no greater than one half of one percent in any case.

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