Charges on a Sphere under an Electric Field

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Abstract

By expressing the potential of a conducting sphere of radius 1m under an electric field in terms of the positions of four charges of 1C on its surface we have written a computer program to evaluate and find the optimum configurations of the system. We found that system switches from a three-fold symmetry tetrahedron configuration to a two-fold symmetry tetrahedron configuration at an electric field of $0.54123\mathrm{NC}^{-1}$ and the latter becomes indistinguishable from a square configuration to an accuracy of 10^{-6}J at an electric field of $1.21326\mathrm{NC}^{-1}$.

1 Introduction

When point charges are confined to a surface of a conducting sphere they repel each other across the surface until they are equally spaced and in equilibrium. Introducing a uniform electric field across this system distorts the arrangement of the charges until they are once again in equilibrium, and at large enough field strengths the arrangement will change drastically. In this report we will study how four charges switch between three arrangements, each with different symmetries, when varying the electric field and find the critical values for the electric field.

It is a complicated task to produce an expression for the minimum potential of the system under a given electric field analytically, and for that reason this project takes the form a computational evaluation. In the next section we shall express the potentials in terms of the charges' latitude to the electric field and in section 3 build an algorithm to find the minimum potential for a given electric field. In section 4 we use the algorithm to find the critical values for the electric field at which the system transitions between symmetries.

2 Evaluating Configuration Potential

The configurations of the charges are variations of three different arrangements which are detailed in the project notes [1]. Both the tetrahedron with three-fold symmetry and square configuration potentials can be described in terms of two variables, the electric field strength and the charges' latitude to the electric field.

It is left to us to describe the potential of the two-fold symmetry tetrahedron, which has two pairs of charges at different latitudes, introducing a second angle to consider. There are three contributions to this potential: the repulsion between each pair of charges at the same latitude; the repulsion between charges at different latitudes and the effect of the electric field on each charge.

Finding the effect of the electric field is the most trivial step as it is the same as for the other two configurations, i.e. dependent on each particle's vertical distance from the 'north pole'.

$$2qER(1 - \cos\alpha) + 2qER(1 - \cos\beta)$$

where α is the latitude of one pair of particles and β is the latitude of the other pair. The repulsion between the particles at the same latitude is also straightforward to find as it dependent on the reciprocal of twice the distance of each particle from the central axis.

$$\frac{q^2}{2R\sin\alpha} + \frac{q^2}{2R\sin\beta}$$

To find the potential due to the repulsion between the particles at different latitudes let us consider one particle from each pair. They are separated by a longitude of 90° so their positions can be described by the two vectors

$$\vec{A} = \begin{pmatrix} R\sin\alpha \\ R\cos\alpha \\ 0 \end{pmatrix}, \vec{B} = \begin{pmatrix} 0 \\ R\cos\beta \\ R\sin\beta \end{pmatrix}$$

The effect of the repulsion between these charges will be dependent on the reciprocal of the magnitude of the difference between these two vectors which simplifies to

$$\frac{q^2}{|\vec{A} - \vec{B}|} = \frac{q^2}{R\sqrt{2 - 2\cos\alpha\cos\beta}}$$

Finally we factor in the other two particles and add our other two terms to give the total potential

$$W_{tetr2} = \frac{2\sqrt{2}q^2}{R\sqrt{1 - \cos\alpha\cos\beta}} + \frac{q^2}{2R}(\csc\alpha + \csc\beta) + 2qER(2 - \cos\alpha - \cos\beta)$$

We can perform a simple check to see that this correct by considering that when $\alpha = \beta$ this potential reduces to that of the four-fold symmetry square configuration.

There is one simplification to be made to all three expressions for potential that will make their evaluation more efficient and useful, and that is to state our electric field in terms of q/R^2. This reduces the number of steps in each calculation and removes dependences on the size of the sphere and the charges. From here on we shall be studying a sphere of radius 1m and charges of 1C, but the potentials we obtain can be scaled to spheres and charges of any size by multiplying by the factor q/R^2. Thus the three potentials we shall be evaluating are

$$W_{tetr4} = \frac{\sqrt{3}}{\sin\alpha} + \frac{3}{\sqrt{2 - 2\cos\alpha}} + 3E(1 - \cos\alpha)$$

$$W_{tetr2} = \frac{2\sqrt{2}}{\sqrt{1 - \cos\alpha\cos\beta}} + \frac{1}{2}(\csc\alpha + \csc\beta) + 2E(2 - \cos\alpha - \cos\beta)$$

$$W_{sq} = \frac{2\sqrt{2} + 1}{\sin\alpha} + 4E(1 - \cos\alpha)$$

W_{tetr4} and W_{sq} are slightly modified versions of those in the project notes [1]. Expanding the bottom of the second term in W_{tetr4} yields a 40% reduction in the processing time of that particular term and combining the first and second term in W_{sq} yields a 20% reduction. This is possible because computers are faster at basic arithmetic than evaluating trigonometric functions.

3 Finding Minimum Potential

We can find the minimum potential of a configuration for a given electric field by tabulating small intervals over a range of angles. In the case of the two-fold symmetry configuration we would be tabulating over two angles and would require a number of samples equal to the product of the number of intervals in the two angle ranges. This would require a lot of processing time but the problem can be reduced by first finding the approximate position of the minimum and then tabulating that region in detail. Figure 1 illustrates how we can 'zoom in' by taking small data sets repeatedly at increasingly smaller intervals to find the minimum to a high accuracy in just a few steps. In this arbitrary example an accuracy normally achieved by taking $17 \times 17 = 289$ samples is achieved in just $5 \times 5 \times 3 = 75$ samples. Each time we want to double the accuracy we can just take another 25 samples rather than quadrupling the number of samples. This method can also be applied to the single angle data sets for some, but not as great, a gain in efficiency.

There is a drawback to this method which is that if there are multiple troughs in the data it is possible that at a low resolution we could test near but not directly in the deepest trough, whilst testing directly in another trough. This other trough could be deep enough to appear deeper than the off-centre measurement of the deepest trough and end up stealing the samples taken for higher accuracy. This is the likely reason for the anomalies in Figure 2. To combat this problem we can control how many samples our 'zooming' algorithm uses for each accuracy level, i.e. the resolution, and increase it until all anomalies are gone.

4 Finding Critical Electric Field Strength

The minimum potentials for the three different configurations under different electric fields have been calculated using the method detailed in the previous section and plotted in Figure 3. It's hard to read off this plot the electric fields at which each configuration becomes optimum. Instead we can iteratively test whether an electric field is higher or lower than a critical value by each time setting our electric field to be the average of the new high and low values and testing which potential is greater. We can set our program to do this repeatedly until the difference between the high and low values is less than a specified accuracy.

Using this method with a specified accuracy of 10^{-5}NC^{-1} we find the first transition to be at 0.54123NC^{-1} and the second at 1.20182NC^{-1}. Both these values were found using the 'zooming' algorithm detailed in section 3 using a 40×40 resolution and a depth of 10 iterations. If we measure again at a higher resolution of 100×100 samples per level we still get the same electric field for the first transition but 1.21327NC^{-1} for the second, which is quite a large difference.

We already know that the expression for the two-fold symmetry configuration reduces to the expression for the four-fold symmetry when all the charges are at the same latitude, so their minimum potentials should converge as field strength increases rather than cross. The crossing is likely to be due to precision error and the limited accuracy in finding the optimum angles. Bearing this in mind it might make more sense to determine the electric field at which the two plots become indistinguishable. We can do this by using same iterative technique as for finding the transition but instead testing for a difference less than a specified error. Using this method and calling a difference in potential less than 10^{-6}J indistinguishable we determine that beyond an electric field of 1.21326NC^{-1} the two configurations are indistinguishable. This value is consistent for all zooming resolutions beyond 40×40 samples with a depth of 10 iterations.

This isn't an ideal representation of the critical value for the second transition. If we had more time it might be useful to study the gradients of the two and four-fold trends to see when they are no longer converging as it might indicate as to when they would have been equal if weren't for the errors in our calculation.

5 Conclusions

We have written expressions for the potential of four charges on the surface of a conducting sphere under a uniform electric field in terms of the charges' latitude to the electric field for configurations of three different symmetries. We have also built a 'zooming' algorithm to efficiently determine each expression's lowest possible potential by iteratively evaluating the potential over a range angles and then evaluating a more detailed range about the found optimum angles. Using this tool we could determine the critical electric field for the transition of the three-fold symmetry tetrahedron configuration into the two-fold symmetry tetrahedron to be 0.54123NC^{-1} to five decimal places.

The critical value for the transition from the two-fold symmetry tetrahedron configuration into the four-fold symmetry square configuration could not be reliably found so instead we determined that their minimum potentials become indistinguishable to an accuracy of 10^{-6}J beyond an electric field of 1.21326NC^{-1}.

References

[1] COMPUTER TECHNIQUES IN PHYSICS - Stability of electrostatic charges on a sphere

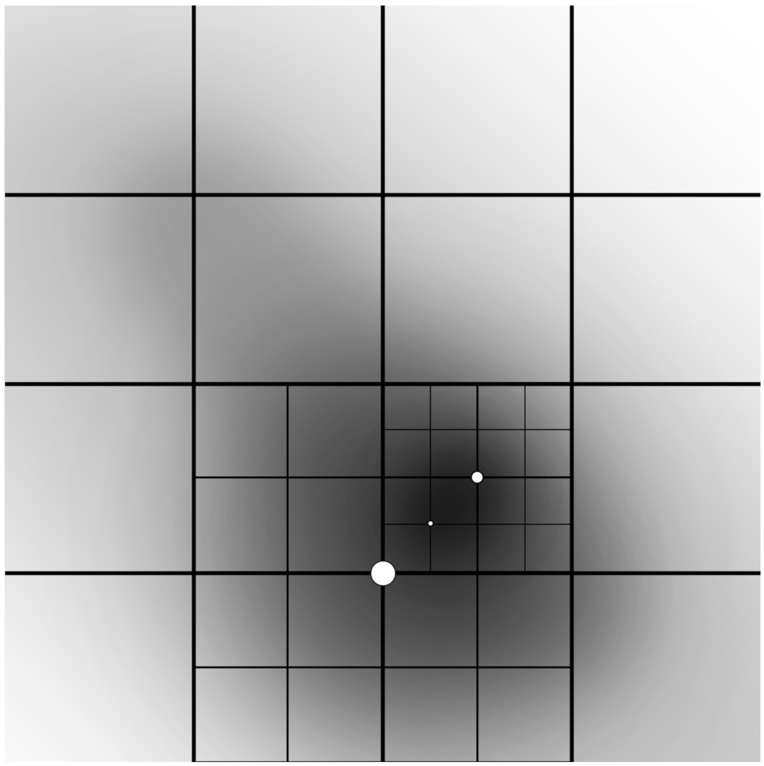

Figure 1: Finding the minimum in an arbitrary scalar field to a high accuracy without requiring a large data set. The brightness represents field magnitude and the white dots represent the lowest value found at different resolutions.

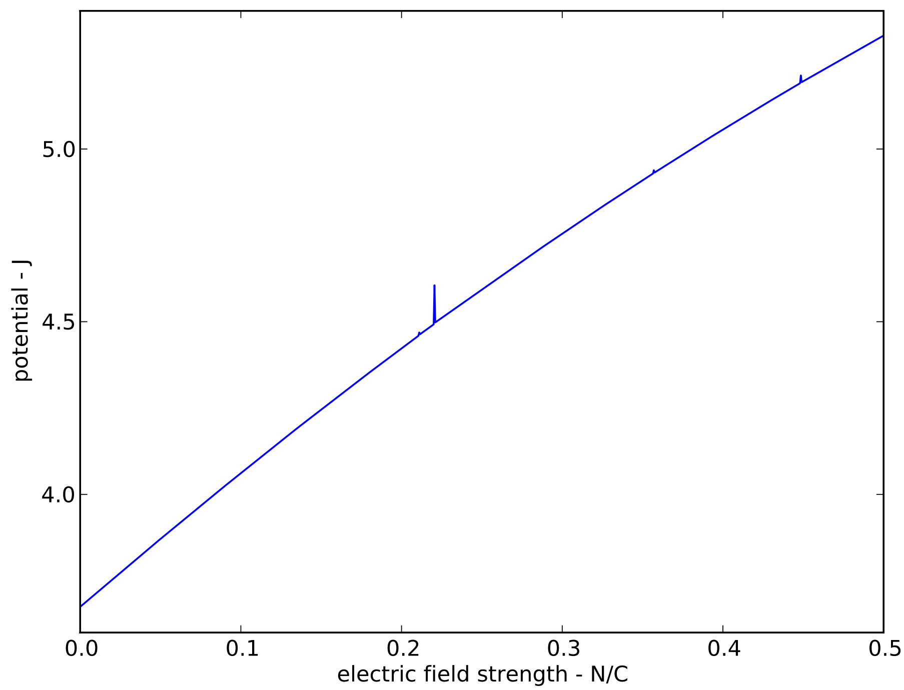

Figure 2: Anomalies found in the two-fold symmetry configuration minimum potential when using a 'zooming' technique to find the minimum.

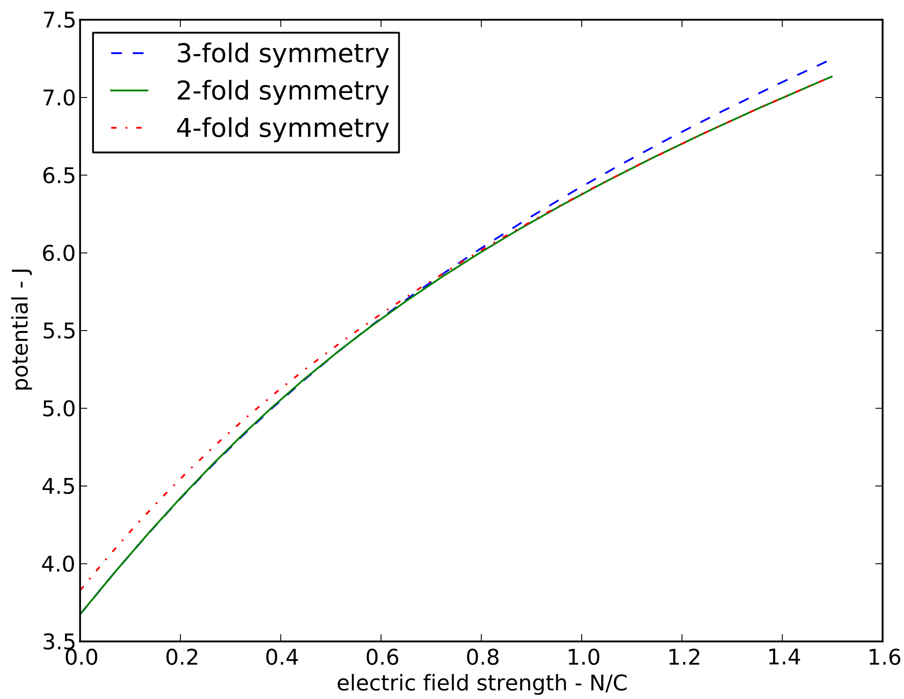

Figure 3: Minimum potential for three different configurations against electric field strength.