M 327J - Differential Equations with Linear Algebra

November 14, 2022

Practice Problems

1. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} - 3t^2 y = e^{t^3}\cos(t) \\ y(0) = 7. \end{cases}$$

Solution

This is an integrating factor problem. We wish to find a function Φ such that

$$(\Phi y)' = \Phi y' + \Phi' y = \Phi(y' - 3t^2 y).$$

This is only possible if

$$\Phi' = -3t^2.$$

This equation is separable

$$\log|\Phi| = \int \frac{1}{\Phi}\, d\Phi = \int -3t^2\, dx = -t^3 + C$$

so we may take

$$\Phi = e^{-t^3}$$

as our integrating factor. This turns the problem into

$$(e^{-t^3} y)' = \cos(t)$$

and integrating both sides in t gives

$$y(t) = e^{t^3}(\sin(t) + C).$$

From our initial condition we find

$$y(0) = C = 7$$

so our particular solution is

$$y(t) = e^{t^3}(\sin(t) + 7).$$

2. Find the implicit general solution to

$$\frac{dy}{dt} = te^{-y} + e^{7t-y}$$

Solution

Factoring we see

$$\frac{dy}{dt} = te^{-y} + e^{7t-y} = \left(t + e^{7t}\right) e^{-y}$$

which shows that this equation is separable. Integrating we compute the solution as

$$\int e^y \frac{dy}{dt}\, dt = \int t + e^{7t}\, dt$$

which evaluates to

$$e^y = \frac{1}{2}t^2 + \frac{1}{7}e^{7t} + C.$$

3. Find the solution to

$$\begin{cases} \dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0 \\ y(2) = 0 \\ y'(2) = 1. \end{cases}$$

Hint: Write your general solution in terms of $e^{\lambda(x-x_0)}$*; see problems 6 and 8 in section 2.2 in Braun and follow the remark.*

Solution

The equation has characteristic polynomial

$$r^2 + 2r + 1 = (r+1)^2 = 0$$

so any solution is of the form

$$y(t) = C_1 e^{-(x-2)} + C_2 (x-2) e^{-(x-2)}.$$

Using our initial value we find

$$\begin{aligned} y(2) &= C_1 + C_2 \cdot 0 && = 0 \\ y'(2) &= -C_1 + C_2 && = 1 \end{aligned}$$

which gives constants

$$C_1 = 0 \quad C_2 = 1$$

hence our particular solution is

$$y(t) = (x-2)e^{-(x-2)}.$$

4. Find the general solution to

$$t^2y'' + 2ty' - 6y = 0.$$

Hint: Since the coefficients are not constants we cannot use the typical characteristic equation derived from $y = e^{rt}$. What happens when we let $y = t^r$?

Solution

Note: This will not be on the final.

This is an Euler equation. To find its corresponding characteristic equation substitute $y(t) = t^r$ into the equation to find

$$t^2y'' + 2ty' - 6y = t^2r(r-1)t^{r-2} + 2trt^{r-1} - 6t^r = \left(r^2 + r - 6\right)t^r = 0.$$

We see that

$$r^2 + r - 6 = 0$$

when $r = -3, 2$ hence our general solution is

$$y(t) = C_1t^{-3} + C_2t^2.$$

5. Find the general solution to

$$y'' + 2y' + y = 0.$$

Solution

The equation has characteristic polynomial

$$r^2 + 2r + 1 = (r+1)^2 = 0$$

so $r = -1$ is a double root of the equation. This means e^{rt} and te^{rt} are linearly independent solutions, giving us general solution

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}.$$

6. Consider the problem

$$y'' + 2y' + y = f(t).$$

What should your judicious guess be if

a) $f(x) = t^2 + t$

b) $f(x) = \cos(t)$

c) $f(x) = e^t \sin(t)$

d) $f(x) = 3te^t$

You do not need to check your guess. This problem will not take long if you know the rules of judicious guessing.

Solution
From problem 5 we know the general solution is

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}.$$

We will use this help determine what our guess should be.

a) If $f(x) = t^2 + t$ we see that there are no factors e^{-t} within this function, so we guess

$$y(t) = At^2 + Bt + C.$$

b) If $f(x) = \cos(t)$ we again see there are no factors e^{-t} within this function, so we guess

$$y(t) = A\cos(t) + B\sin(t).$$

c) The function $f(x) = e^{-t} t \sin(t)$ does have a factor of e^{-t}, but since $\sin(t)$ is not a polynomial we still stick with the typical guess of

$$e^{-t}(A\cos(t) + B\sin(t)).$$

d) The function $f(x) = 3te^{-t}$ is a polynomial times e^{-t} which is a solution corresponding to a double root of characteristic equation. As a result, we must take our typical guess

$$(At + B)e^{-t}$$

and multiply it by t^2 to get our guess

$$y(t) = (At^3 + Bt^2)e^{-t}$$

7. Suppose L is some linear operator satisfying

$$L[1] = 2$$
$$L[6t] = 6$$
$$L[-t^2] = -2t$$

Find a function y such that

$$L[y] = 4 - 4t \quad \text{and} \quad y(0) = -2.$$

Solution

Note: This will not be on the final.

Using superposition we find

$$L[At^2 + Bt + C] = 2At + B + 2C.$$

To find y we must now search for A, B, C such that

$$2At + B + 2C = 4 - 4t$$
$$y(0) = C = -2$$

which is solved by $A = -2$, $B = 8$, $C = -2$ so our solution is

$$y(t) = -2t^2 + 8t - 2.$$

8. Use power series to solve

$$\begin{cases} \dfrac{d^2y}{dt^2} + 3t\dfrac{dy}{dx} + 3y = 0 \\ \qquad\qquad y(1) = 0 \\ \qquad\qquad y'(1) = 1. \end{cases}$$

Solution
Letting

$$y = \sum_{n=0}^{\infty} a_n (t-1)^n$$

we find that

$$\begin{aligned} \frac{d^2y}{dt^2} &= \sum_{n=0}^{\infty} n(n-1)a_n(t-1)^{n-2} \\ &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(t-1)^n \\ 3t\frac{dy}{dx} &= 3[(t-1)+1]\sum_{n=0}^{\infty} a_n n(t-1)^{n-1} \\ &= 3\sum_{n=0}^{\infty} a_n n(t-1)^n + 3\sum_{n=0}^{\infty} a_n n(t-1)^{n-1} \\ &= 3\sum_{n=0}^{\infty} \left(a_n n + a_{n+1}(n+1)\right)(t-1)^n \end{aligned}$$

so the differential equation becomes

$$(t-1)^2\frac{d^2y}{dt^2} + 3t\frac{dy}{dx} + y = \sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} + 3a_n n + 3a_{n+1}(n+1) + 3a_n\right](t-1)^n = 0$$

which is satisfied whenever we have

$$(n+1)(n+2)a_{n+2} + 3a_n n + 3a_{n+1}(n+1) + 3a_n = 0$$

for all n. Solving this equation for a_{n+2} gives us the recurrence

$$a_{n+2} = -\frac{3a_n n + 3a_{n+1}(n+1) + 3a_n}{(n+1)(n+2)} = -3\frac{a_n + a_{n+1}}{n+2}.$$

We can then use this with our initial condition (which implies $a_0 = 0$, $a_1 = 1$) to find the first few terms of our series solution:

$$y(t) = 0 + (t-1) - \frac{3}{2}(t-1)^2 + \frac{1}{2}(t-1)^3 + \frac{3}{4}(t-1)^4 + \cdots$$

9. An object of mass 8 kg is attached to a spring with spring constant 8 N/m and the object is immersed within a vat of maple syrup giving this system a damping constant of 16 N s/m.

a) What is the differential equation that describes the motion of this system?

b) If initially the mass is 2 meters away from its equilibrium position and given an initial velocity of .1 m/s in the direction of this equilibrium will it

   i. never cross the equilibrium,

   ii. overshoot the equilibrium once, or

   iii. cross the equilibrium an infinite number of times?

Hint: Solve the IVP corresponding to this situation.

Solution

a) The equation is

$$8y'' + 16y' + 8y = 0$$

where the value $y(t)$ is the number of meters the mass is from its equilibrium position at time t.

b) We find the general solution to this equation by first solving the characteristic equation

$$8r^2 + 16r + 8 = 8(r+1)^2 = 0$$

giving $r = -1$ as a double root. So our general solution is then

$$C_1 e^{-t} + C_2 t e^{-t}.$$

The situation in part (b) tells us that $y(0) = 2$ and $y'(0) = -.1$ which forces $C_1 = 2, C_2 = 1.9$ giving the particular solution

$$y(t) = e^{-t}(2 + 1.9t).$$

Solving $y(t) = 0$ we find the only solution is $t = -2/1.9$ which is negative, so we are in case (i): **the mass never crosses the equilibrium.**

10. Find the inverse of the matrix
$$\begin{pmatrix} 1 & -1 & 5 \\ 2 & -1 & 6 \\ 3 & -1 & 5 \end{pmatrix}$$

Solution
The final solution is
$$\begin{pmatrix} 1 & -1 & 5 \\ 2 & -1 & 6 \\ 3 & -1 & 5 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \\ -8 & 10 & -4 \\ -1 & 2 & 1 \end{pmatrix}$$
and you can find this by placing the augmented matrix
$$(A|I)$$
in RREF which is always
$$\left(I|A^{-1}\right)$$
when A is an invertible matrix.

11. Determine if the following three vectors are linearly dependent or independent:

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \begin{pmatrix}5\\1\\-5\end{pmatrix}, \quad \begin{pmatrix}8\\7\\4\end{pmatrix}$$

Bonus: Attempt this using a different method.

Solution
Method 1: If we take the determinant of the matrix which has these vectors as columns we find that

$$\det\begin{pmatrix}1 & 5 & 8\\2 & 1 & 7\\3 & -5 & 4\end{pmatrix} = 0$$

hence we can conclude that the vectors are linearly dependent.

Method 2: The RREF of the matrix from method 1 is exactly

$$\begin{pmatrix}1 & 0 & 3\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}$$

which has a single free variable. This indicates the vectors are linearly dependent.

Method 3 (not recommended): Observe that

$$3\begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}5\\1\\-5\end{pmatrix} = \begin{pmatrix}8\\7\\4\end{pmatrix}$$

showing the vectors are linearly dependent. This method is not recommended because if the vectors are linearly independent we won't be able to construct a sum like the one above, so on the test it could lead to a lot of wasted time.

12. Let V be the set of cubic polynomials p satisfying $p(0) = 2p'(0)$; that is,

$$V = \{p : p(t) = a_3t^3 + \cdots + a_0 \text{ and } p(0) = 2p'(0)\}.$$

a) Show that V is a vector space.

b) Find a basis of V.

c) What is the dimension of V?

Solution

a) V is a subset of the set of cubic polynomials which we know is a vector space. To show V is a subspace we must show i) $0 \in V$, ii) $v, w \in V \implies v + w \in V$, and iii) $c \in \mathbb{R}$, $v \in V \implies cv \in V$.

   i) The zero in the set of cubics is the cubic with zero coefficients: $z(t) = 0t^3 + 0t^2 + 0t + 0$. We see that $p(0) = 0, p'(0) = 0$ so clearly $p(0) = 2p'(0)$ allowing us to conclude $z(t) \in V$.

   ii) If $p_1, p_2 \in V$ we compute the derivative of $p_1 + p_2$ to directly evaluate our rule: $(p_1 + p_2)(0) = p_1(0) + p_2(0) = 2p_1'(0) + 2p_2'(0) = 2(p_1 + p_1)'(0)$. Therefore $p_1 + p_2 \in V$.

   iii) For scalar multiplication we repeat what we showed for (ii): $(cp)(0) = cp(0) = c(2p'(0)) = 2(cp)'(0)$ hence $cp \in V$.

   This suffices to show that V is a vector space.

b) Any arbitrary $p \in V$ is a cubic so it can be written as

$$p(t) = a_3t^3 + \cdots + a_0.$$

   Evaluating our rule on $p(t)$ shows us that these coefficients satisfy

$$a_0 = p(0) = 2p'(0) = 2a_1$$

   so we can rewrite

$$p(t) = a_3t^3 + a_2t^2 + (2t + 1)a_0$$

   We note that $\{t^3, t^2, 2t + 1\}$ is a set of elements of V (note - the original sum did not have this property because $t, 1 \notin V$) spanning V. We see that they are all of different orders so clearly they are linearly independent, hence

$$\{t^3, t^2, 2t + 1\}$$

   is a basis of V.

c) $\dim V = |\{t^3, t^2, 2t + 1\}| = 3$.

13†. Show that the map

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \int_{-1}^{1} ax^2 + bx + c\cos(x) + d\,dx$$

is linear.

Solution
For this problem we must check linearity over addition and scalar multiplication; that is, i) for M, N two 2×2 matrices we have $T(M+N) = T(M) + T(N)$ and ii) for M a 2×2 matrix and $c \in \mathbb{R}$ we have $T(cM) = cT(M)$.

i) We write M, N as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad N = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

We see that

$$\begin{aligned} T(M+N) &= T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) \\ &= T\begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix} \\ &= \int_{-1}^{1} (a+a')x^2 + (b+b')x + (c+c')\cos(x) + (d+d')\,dx \\ &= \int_{-1}^{1} ax^2 + bx + c\cos(x) + d\,dx + \int_{-1}^{1} a'x^2 + b'x + c'\cos(x) + d'\,dx \\ &= T(M) + T(N) \end{aligned}$$

ii) For a $\lambda \in \mathbb{R}$ we see

$$\begin{aligned} T(\lambda M) &= T\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \\ &= \int_{-1}^{1} \lambda ax^2 + \lambda bx + \lambda c\cos(x) + \lambda d\,dx \\ &= \lambda \int_{-1}^{1} ax^2 + bx + c\cos(x) + d\,dx = \lambda T(M) \end{aligned}$$

14. Which of the following are linear maps:

a) The map from $C(\mathbb{R}) \to \mathbb{R}$ defined as

$$A(f) = f(0)$$

(where $C(\mathbb{R})$ is the set of all continuous functions $\mathbb{R} \to \mathbb{R}$.)

b) The map from $M^2 \to \mathbb{R}^2$ defined as

$$B\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \sin b \\ c + d \end{pmatrix}$$

c) † The map from $C([0,1]) \to C([0,1])$ defined by

$$D(f)(x) = \int_0^1 f(y) e^{x-y}\, dx$$

d) The map from $\mathbb{R} \to \mathbb{R}$ defined by

$$E(x) = 0$$

Solution

a) **Linear** We know for any two functions $(f+g)(0) = f(0) + g(0)$ and $(cf)(0) = c(f(0))$ which suffices to show linearity.

b) **Nonlinear** This is not linear due to the $a \sin b$ term. In particular

$$B\left(c\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} c \sin c \\ 0 \end{pmatrix} \neq c\begin{pmatrix} \sin 1 \\ 0 \end{pmatrix}$$

because the function $c \sin c = 0$ for $c = \pi$ while $c \sin(1)$ is nonzero at $c = \pi$.

c) **Linear** This is a special operation between functions called the "convolution." Using linearity of multiplication and integration it is straightforward to prove this is linear.

d) **Linear** Clearly $E(x+y) = 0 = 0+0 = E(x)+E(y)$ and $E(cx) = 0 = c \cdot 0 = cE(x)$.

15. Convert the third order IVP

$$\begin{cases} y''' + 12y'' - 6y' + 7y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = 3 \end{cases}$$

into a first order systems IVP

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} \\ \mathbf{x}(0) = x_0 \end{cases}$$

Solution
To convert the differential equation above into a first order we must track the three variables y'', y', and y with our new variables x_1, x_2, x_3 (note: we do not need to track y''' because $y''' = (y'')'$, the first derivative of a function will track.) Setting $x_1 = y$, $x_2 = y'$, $x_3 = y''$ we find the first order system

$$\begin{aligned} x_1' &= y' && = x_2 \\ x_2' &= y'' && = x_3 \\ x_3' &= y''' && = -12x_3 + 6x_2 - 7x_1 \end{aligned}$$

which is equivalent to the first order system of equations

$$\dot{\mathbf{x}} = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 6 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A\mathbf{x}$$

From the initial conditions on y, y', y'' we find that

$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \\ y''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

giving the initial value for the system.

16. Find the solution sets to the following augmented matrices:

a)

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & -1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

b)

$$\left(\begin{array}{ccc|c} 3 & 0 & -6 & 18 \\ 3 & 1 & -5 & 25 \\ 3 & 0 & -6 & 16 \end{array}\right)$$

Solution

1. This matrix is already in RREF. We see there are no pivots in the x_2, x_4 columns so they are free variables, we will let $x_2 = t, x_4 = s$. The rows of this matrix are equivalent to the equations

$$\begin{aligned} x_1 + 2x_2 &= 6 \\ x_3 - x_4 &= 7 \\ 0 &= 0 \end{aligned}$$

which gives us the solution

$$\begin{pmatrix} 6 - 2t \\ t \\ 7 + s \\ s \end{pmatrix}$$

2. We find that this matrix has RREF

$$\left(\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

which has a pivot in the right most column and hence has no solutions (another way to interpret this is the last row requires $0 = 1$ which is obviously false.)

17. Let V be the vector space of solutions to the equation

$$\dot{\mathbf{x}} = A\mathbf{x}$$

and suppose it has the basis elements

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} \sin(t) - \cos(t) \\ \cos(t) \end{pmatrix}, \quad \mathbf{x}_1(t) = e^t \begin{pmatrix} \sin(t) + \cos(t) \\ -\sin(t) \end{pmatrix}.$$

We define the functions ψ_1, ψ_2 as solutions to this same equation satisfying the initial conditions

$$\psi_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_1(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Express ψ_1, ψ_2 as linear combinations of $\mathbf{x}_1, \mathbf{x}_2$.

Solution
Note: This will not be on the final.

18. Find the general solution to

$$\dot{\mathbf{x}} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

Solution

We find that $\det(A - \lambda I) = (2-\lambda)^2(1-\lambda)$ giving $\lambda = 1$ as a single root and $\lambda = 2$ as a double root.

For $\lambda = 1$ we see

$$A - I = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

giving e_3 as the corresponding eigenvector.

For $\lambda = 2$ we have

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which has the single eigenvector e_1. Since this is a double root we must find a generalized eigenvector (that is, w such that $(A-2I)w = e_1$.) To do this we put the following matrix in RREF

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which tells us that e_2 is a generalized eigenvector. The general solution is then

$$\mathbf{x}(t) = C_1 e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 e^{At} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where

$$e^{At} C_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{2t} e^{(A-2I)t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{2t} \left(I + t(A - 2I)\right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{2t} \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

19. Solve the IVP

$$\dot{\mathbf{x}} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) \quad = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

Solution
We find

$$\det(A - \lambda I) = [(2-\lambda)^2 + 1](2 - \lambda)$$

which gives us eigenvalues $\lambda = 2, 2 \pm i$. When $\lambda = 2$ we find

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

giving us eigenvector e_3.
For $\lambda = 2 - i$ we see

$$A - (2-i) = \begin{pmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & i \end{pmatrix} \sim \begin{pmatrix} 1 & -i & 0 \\ -1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which has free variable in x_2, so the general solution is

$$\begin{pmatrix} it \\ t \\ 0 \end{pmatrix}$$

so we may take $ie_1 + e_2$ as our eigenvector.
We now observe that

$$e^{(2-i)t} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = e^{2t} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} (\cos(-t) + i\sin(-t)) = e^{2t} \begin{pmatrix} i\cos(t) + \sin(t) \\ \cos(t) - i\sin(t) \\ 0 \end{pmatrix} = e^{2t} \begin{pmatrix} \sin(t) \\ \cos(t) \\ 0 \end{pmatrix} + ie^{2t} \begin{pmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{pmatrix}$$

so our general solution is

$$\mathbf{x}(t) = C_1 e^{2t} \begin{pmatrix} \sin(t) \\ \cos(t) \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

To find our particular solution we observe that

$$\mathbf{x}(0) = \mathbf{x}(t) = C_1 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

which can be solved by putting the matrix in RREF

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

giving

$$C_1 = 1 \quad C_2 = 1 \quad C_3 = 3.$$

20[†]**.** Consider the family of first order linear systems

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ \alpha & \alpha \end{pmatrix} \mathbf{x}.$$

Determine which values of α cause the solutions to be asymptotically stable, stable, and unstable.

Solution
We find that

$$\det(A - \lambda I) = -\lambda(\alpha - \lambda) - \alpha = \lambda^2 - \lambda\alpha - \alpha = 0$$

has solutions

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 + 4\alpha}}{2}$$

These can be separated into three cases depending on α:

Distinct real roots when $\alpha^2 + 4\alpha > 0$ which happens when $\alpha > 0$, $\alpha < -4$. When $\alpha > 0$ if we take the positive square root we find an eigenvalue $\lambda > 0$ so solutions are **unstable** for $\alpha > 0$. When $\alpha < -4$ we see that $\alpha^2 + 4\alpha < \alpha^2$ so both eigenvalues < 0 telling us that solutions are **asymptotically stable**.

Imaginary roots when $\alpha^2 + 4\alpha < 0$ when happens when $-4 < \alpha < 0$. In this region we find $\mathrm{Re}(\lambda) = \alpha/2 < 0$ so for these values of α solutions are **asymptotically stable**.

Repeated roots when $\alpha^2 + 4\alpha = 0$ which happens when $\alpha = 0, -4$. When $\alpha = 0$ we find both $\lambda = 0$. The matrix defining the problem has RREF with just one free variable, so these solutions are **unstable**. When $\alpha = -4$ the double root is for $\lambda = -4$ which is negative, so solutions are **asymptotically stable**.

In summary,

$$\text{Solutions to the system are } \begin{cases} \textbf{asymptotically stable} & \text{when } \alpha < 0 \\ \textbf{unstable} & \text{when } \alpha \geq 0 \end{cases}$$

21†. Find the orbits of

$$\dot{x} = 5\sin(x+y)$$
$$\dot{y} = 15\sin(x+y)$$

Hint: Recall that $\sin(\theta) = 0$ *whenever* $\theta = n\pi$ *for* n *an integer.*

Solution
We see that $\dot{x} = \dot{y} = 0$ whenever $\sin(x+y) = 0$. This happens whenever

$$x + y = n\pi$$

for an integer n. This causes us to have equilibrium points on the infinite family of lines

$$y = n\pi - x.$$

Solving the corresponding equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{15}{5} = 3 \implies y = 3x + C.$$

So, our orbits are every equilibrium point (x, y) satisfying $x+y = n\pi$ for n an integer and the line segments of the lines $y = 3x + C$ which lie between the equilibrium lines $x + y = n\pi$.

22†. Find all eigenvalues for the boundary value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dx} + \lambda y = 0 \\ y'(0) = 0 \\ y(5) = 0. \end{cases}$$

Solution

Note: This will not appear on the final

23†. Use separation of variables to solve

$$\begin{cases} u_{xx} + u = u_{tx} \\ u(0,x) = xe^x \end{cases}$$

Solution
If we assume our solution takes the form $u(t,x) = T(t)X(x)$ the PDE becomes

$$T(t)X''(x) + T(t)X(x) = T'(t)X'(x).$$

Factoring we find

$$T(X'' + X) = T'X'$$

so if we divide by TX' we separate the equation into

$$\frac{X'' + X}{X'} = \frac{T'}{T} = \lambda \in \mathbb{R}$$

where we deduce since the left and right hand sides depend on x and t, respectively, they must be constant. This gives us the system

$$\begin{aligned} X'' &= \lambda X' - X \\ T' &= \lambda T. \end{aligned}$$

Taking the second derivative of $X = xe^x$ we find

$$X'' = (xe^x + e^x)' = xe^x + 2e^x = 2(xe^x + e^x) - xe^x = 2X' - X$$

giving us $\lambda = 2$. Therefore,

$$T(t) = e^{2t}$$

so our solution is

$$u(t,x) = T(t)X(x) = xe^{x+2t}.$$

24. Suppose a 2 m long metal rod is fully insulated (including ends), has $\alpha^2 = 0.86$, and at time $t = 0$ the temperature at point x on the bar is given by

$$f(x) = 2x + 2.$$

a) Write the IVP/heat equation modeling this situation.

b) Solve the IVP from (a).

a)

$$\begin{cases} \dfrac{\partial u}{\partial t} = .86\dfrac{\partial^2 u}{\partial t^2} \\ u(0,x) = 2x + 2 \\ u_x(t,0) = u_x(0,t) = 0 \end{cases}$$

b) Since we have insulated ends we must find a cosine series of our initial temperature distribution $f(x) = 2x + 2$. We see that

$$\begin{aligned} b_0 &= \frac{2}{2}\int_0^2 2x + 2\,dx = 8 \\ b_n &= \frac{2}{2}\int_0^2 (2x+2)\cos\left(\frac{n\pi}{2}x\right)\,dx \\ &= \frac{2}{n\pi}(2x+2)\sin\left(\frac{n\pi}{2}x\right)\Big|_0^2 - \frac{4}{n\pi}\int_0^2 \sin\left(\frac{n\pi}{2}x\right) \\ &= 0 + \frac{8}{(n\pi)^2}\cos\left(\frac{n\pi}{2}x\right)\Big|_0^2 \\ &= \frac{8}{(n\pi)^2}\left((-1)^n - 1\right) \end{aligned}$$

so our final solution is

$$u(t,x) = \frac{8}{2} + \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi}{2}x\right)e^{-.86\frac{n^2\pi^2}{4}}.$$