M 327J - Differential Equations with Linear Algebra

October 3, 2022

Quiz 4

1. [5 points] Use Gaussian elimination to solve the following system of equations.

$$\begin{aligned} -x_1 + 4x_2 - 8x_3 &= 11 \\ 2x_2 - 4x_3 &= 8 \\ x_1 - x_2 + 2x_3 &= 1 \end{aligned}$$

Express your answer as a set.

Solution
At the end of Gaussian elimination we have the augmented matrix

$$\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which says we have the single free variable x_3 (due to the lack of a pivot in the third column) and it gives the values of x_1, x_2 depending only on the free variable x_3. So we can write any solution as

$$x_1 = 5,\ x_2 = 4 + 2x_3,\ x_3 \in \mathbb{R}.$$

From here we see the set of all solutions is

$$\left\{ \begin{pmatrix} 5 \\ 4 + 2x_3 \\ x_3 \end{pmatrix} \middle| \ x_3 \in \mathbb{R} \right\}.$$

2. [5 points] Which of the following is **not** a vector space? Why?

a) The set of all differentiable functions satisfying $f'(4) = -4f(2)$.

b) The set of all differentiable functions satisfying $f(0) = 1$.

c) The set of all differentiable functions satisfying

$$\lim_{t \to \infty} f'(t) = 0.$$

You **do not** have to show the other two sets are vector spaces.

Solution
The set in (b) is not a vector space. Let this set be W. We see that this set is a subset of the set of functions $\mathcal{F}$ so by theorem 5 in the notes this set is a vector space if and only if it satisfies

(i) The zero of $\mathcal{F}$ is in W ,

(ii) Closure under addition (i.e. if $f, g \in W$ then $f + g \in W$),

(iii) Closure under scalar multiplication (i.e. if c is a real number and $f \in W$ then $cf \in W$.)

In this example W violates all three of these properties, the quickest to test is (i). The zero of the space of functions $\mathcal{F}$ is the zero function, $f(t) = 0$. Evaluating this function at zero we find $f(0) = 0$ so clearly $f(0) \neq 1$ and therefore $f \notin W$. From the theorem it follows that W is not a vector space.