M 327J - Differential Equations with Linear Algebra

November 14, 2022

Practice Problems

1. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} - 3t^2 y = e^{t^3}\cos(t) \\ \qquad y(0) = 7. \end{cases}$$

2. Find the implicit general solution to

$$\frac{dy}{dt} = te^{-y} + e^{7t-y}$$

3. Find the solution to

$$\begin{cases} \dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0 \\ \qquad y(2) = 0 \\ \qquad y'(2) = 1. \end{cases}$$

Hint: Write your general solution in terms of $e^{\lambda(x-x_0)}$*; see problems 6 and 8 in section 2.2 in Braun and follow the remark.*

4. Find the general solution to

$$t^2 y'' + 2ty' - 6y = 0.$$

Hint: Since the coefficients are not constants we cannot use the typical characteristic equation derived from $y = e^{rt}$*. What happens when we let* $y = t^r$*?*

5. Find the general solution to

$$y'' + 2y' + y = 0.$$

6. Consider the problem

$$y'' + 2y' + y = f(t).$$

What should your judicious guess be if

a) $f(x) = t^2 + t$

b) $f(x) = \cos(t)$

c) $f(x) = e^{-t}\sin(t)$

d) $f(x) = 3te^{-t}$

You do not need to check your guess. This problem will not take long if you know the rules of judicious guessing.

7. Suppose L is some linear operator satisfying

$$\begin{aligned} L[1] &= 2 \\ L[6t] &= 6 \\ L[-t^2] &= -2t \end{aligned}$$

Find a function y such that

$$L[y] = 4 - 4t \quad \text{and} \quad y(0) = -2.$$

8. Use power series to solve

$$\begin{cases} \dfrac{d^2y}{dt^2} + 3t\dfrac{dy}{dx} + 3y = 0 \\ \qquad\qquad y(1) = 0 \\ \qquad\qquad y'(1) = 1. \end{cases}$$

9. An object of mass 8 kg is attached to a spring with spring constant 8 N/m and the object is immersed within a vat of maple syrup giving this system a damping constant of 16 N s/m.

a) What is the differential equation that describes the motion of this system?

b) If initially the mass is 2 meters away from its equilibrium position and given an initial velocity of .1 m/s in the direction of this equilibrium will it

   i. never cross the equilibrium,

   ii. overshoot the equilibrium once, or

   iii. cross the equilibrium an infinite number of times?

   Hint: Solve the IVP corresponding to this situation.

10. Find the inverse of the matrix

$$\begin{pmatrix} 1 & -1 & 5 \\ 2 & -1 & 6 \\ 3 & -1 & 5 \end{pmatrix}$$

11. Determine if the following three vectors are linearly dependent or independent:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 1 \\ -5 \end{pmatrix}, \quad \begin{pmatrix} 8 \\ 7 \\ 4 \end{pmatrix}$$

Bonus: Attempt this using a different method.

12. Let V be the set of cubic polynomials p satisfying $p(0) = 2p'(0)$; that is,

$$V = \{p : p(t) = a_3 t^3 + \cdots + a_0 \text{ and } p(0) = 2p'(0)\}.$$

a) Show that V is a vector space.

b) Find a basis of V.

c) What is the dimension of V?

13†. Show that the map

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \int_{-1}^{1} ax^2 + bx + c\cos(x) + d\,dx$$

is linear.

14. Which of the following are linear maps:

a) The map from $C(\mathbb{R}) \to \mathbb{R}$ defined as

$$A(f) = f(0)$$

(where $C(\mathbb{R})$ is the set of all continuous functions $\mathbb{R} \to \mathbb{R}$.)

b) The map from $M^2 \to \mathbb{R}^2$ defined as

$$B\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\sin b \\ c + d \end{pmatrix}$$

c) † The map from $C([0,1]) \to C([0,1])$ defined by

$$D(f)(x) = \int_0^1 f(y)e^{x-y}\,dx$$

d) The map from $\mathbb{R} \to \mathbb{R}$ defined by

$$E(x) = 0$$

15. Convert the third order IVP

$$\begin{cases} y''' + 12y'' - 6y' + 7y = 0 \\ \hfill y(0) = 0 \\ \hfill y'(0) = 1 \\ \hfill y''(0) = 3 \end{cases}$$

into a first order systems IVP

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} \\ \mathbf{x}(0) = x_0 \end{cases}$$

16. Find the solution sets to the following augmented matrices:

a)

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & -1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

b)

$$\left(\begin{array}{ccc|c} 3 & 0 & -6 & 18 \\ 3 & 1 & -5 & 25 \\ 3 & 0 & -6 & 16 \end{array}\right)$$

17. Let V be the vector space of solutions to the equation

$$\dot{\mathbf{x}} = A\mathbf{x}$$

and suppose it has the basis elements

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} \sin(t) - \cos(t) \\ \cos(t) \end{pmatrix}, \quad \mathbf{x}_1(t) = e^t \begin{pmatrix} \sin(t) + \cos(t) \\ -\sin(t) \end{pmatrix}.$$

We define the functions ψ_1, ψ_2 as solutions to this same equation satisfying the initial conditions

$$\psi_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_1(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Express ψ_1, ψ_2 as linear combinations of $\mathbf{x}_1, \mathbf{x}_2$.

18. Find the general solution to

$$\dot{\mathbf{x}} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

19. Solve the IVP

$$\dot{\mathbf{x}} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \quad = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

20. Consider the family of first order linear systems

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ \alpha & \alpha \end{pmatrix} \mathbf{x}.$$

Determine which values of α cause the solutions to be asymptotically stable, stable, and unstable.

21†. Find the orbits of

$$\begin{aligned} \dot{x} &= 5\sin(x+y) \\ \dot{y} &= 15\sin(x+y) \end{aligned}$$

Hint: Recall that $\sin(\theta) = 0$ *whenever* $\theta = n\pi$ *for* n *an integer.*

22†. Find all eigenvalues for the boundary value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dx} + \lambda y = 0 \\ y'(0) = 0 \\ y(5) = 0. \end{cases}$$

23†. Use separation of variables to solve

$$\begin{cases} u_{xx} + u = u_{tx} \\ u(0,x) = xe^{x} \end{cases}$$

24. Suppose a 2 m long metal rod is fully insulated (including ends), has $\alpha^2 = 0.86$, and at time $t = 0$ the temperature at point x on the bar is given by

$$f(x) = 2x + 2.$$

a) Write the IVP/heat equation modeling this situation.

b) Solve the IVP from (a).