M 327J - Differential Equations with Linear Algebra

September 19, 2022

Quiz 3

1. [5 points] Find the particular solution to the initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} - 2t^2y = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

For your answer write the first four terms of the solution's series.

Solution
Letting

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

we find that

$$y''(t) = \sum_{n=0}^{\infty} a_n n(n-1)t^{n-2}.$$

In our equation the second term becomes

$$-2t^2y(t) = -2\sum_{n=0}^{\infty} a_n t^{n+2}$$

and, after reindexing, we have

$$-2t^2y(t) = -2\sum_{k=2}^{\infty} a_{k-2}t^k \qquad (k = n+2)$$

$$y''(t) = \sum_{n=0}^{\infty} a_{k+2}(k+2)(k+1)t^k \qquad (k = n-2)$$

This puts our equation into the form

$$\begin{aligned} y''(t) - 2t^2y &= \sum_{k=2}^{\infty} t^k \left[a_{k+2}(k+2)(k+1) - 2a_{k-2}\right] \\ &+ 6a_3t + 2a_2 = 0 \end{aligned}$$

revealing $a_2, a_3 = 0$. We also note that our recurrence relation is

$$a_{n+4} = \frac{2a_n}{(n+4)(n+3)}.$$

Our initial conditions give us that

$$\begin{aligned} a_0 &= y(0) = 1 \\ a_2 &= y'(0) = 2 \end{aligned}$$

which gives us the first two terms

$$y(t) = 1 + 2t + \cdots$$

Since $a_2, a_3 = 0$ we compute a_4 and a_5, which by our recurrence is

$$\begin{aligned} a_4 &= \frac{2a_0}{4 \cdot 3} = \frac{1}{6} \\ a_5 &= \frac{2a_1}{5 \cdot 4} = \frac{1}{5} \end{aligned}$$

giving us the next two terms,

$$y(t) = 1 + 2t + \frac{1}{6}t^4 + \frac{1}{5}t^5 + \cdots$$

2. [5 points] Suppose a weight of mass 5 kg is attached to a spring with spring constant $k = 5$ N/m and damping constant $c = 10$ Ns/m. At $t = 0$ the mass is 0.1 m below its equilibrium position with velocity 0.2 m/s toward the equilibrium position. How many times will the mass return to its equilibrium position?
Hint: either never, once, or infinitely many times.

Solution
This situation is modeled by the IVP

$$\begin{cases} 5\dfrac{d^2y}{dt^2} + 10\dfrac{dy}{dt} + 5y = 0 \\ y(0) = 0.1 \\ y'(0) = -0.2 \end{cases}$$

Note: the signs on $y(0), y'(0)$ are dependent on which direction you describe as positive. As long as $y(0)$ and $y'(0)$ point in opposite directions and have the correct magnitude you will have an appropriate model. Taking the characteristic equation we find

$$5r^2 + 10r + 5 = 5(r+1)^2$$

giving us a double root at $r = -1$, so our solution is of the form

$$y(t) = e^{-t}\left(C_1 + C_2 t\right).$$

Evaluating $y(0), y'(0)$ we find

$$\begin{aligned} y(0) &= 0.1 = C_1 \\ y'(0) &= -0.2 = -.1 + C_2 \\ C_2 &= -0.1 \end{aligned}$$

hence

$$y(t) = e^{-t}\left(\frac{1}{10} - \frac{1}{10}t\right)$$

and solving $y(t) = 0$ we see the mass returns to its equilibrium position once at time

$$t = 1.$$