M 327J - Differential Equations with Linear Algebra

October 17, 2022

Maybe

1. [7 points] Let V be the vector space containing all solutions to the differential equation

$$y'' + 2y' + 2y = 0.$$

(a) Find a basis of V.

(b) Define $D : V \to V$ as the differentiation operator, i.e. for $f \in V$ we define

$$(Df)(t) = f'(t).$$

Since D is a linear transformation we know we can write it as a matrix. Find this matrix in terms of the basis found in part (a).

Solution

(a) From our knowledge of linear second order equations we know any $y \in V$ can be written as

$$y(t) = C_1 e^{-t}\sin(t) + C_2 e^{-t}\cos(t).$$

This shows that the functions

$$f_1(t) = e^{-t}\sin(t), \quad f_2(t) = e^{-t}\cos(t)$$

both span V and we can conclude they are linearly independent, as $f_1(0) = 0 \neq 1 = f_2(0)$. So we can select our basis to be the functions $\{f_1, f_2\}$.

(b) For $f \in V$ we can write $f = af_1 + bf_2$ for constants a, b. Applying D to this expression we find

$$Df = D(af_1 + bf_2) = aD(f_1) + bD(f_2) = \begin{pmatrix} \vdots & \vdots \\ D(f_1) & D(f_2) \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

so to find our matrix we just need to compute the coordinates of $D(f_1)$ and $D(f_2)$ in our basis. These values are

$$D(f_1)(t) = -e^{-t}\sin(t) + e^{-t}\cos(t) \text{ has coordinates vector } \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$D(f_2)(t) = -e^{-t}\sin(t) - e^{-t}\cos(t) \text{ has coordinates vector } \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

so we find the matrix describing D is

$$\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}.$$

2. [**3 points**] Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Solution

$$A^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$