M 327J - Differential Equations with Linear Algebra

October 31, 2022

Quiz 6

1. [5 points] Suppose the matrix A has eigenvalues $1, -1$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

respectively. Use this information to solve the initial value problem

$$\begin{cases} \frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \\ \vec{x}(0) = \begin{pmatrix} 0 \\ 15 \end{pmatrix} \end{cases}$$

Solution

The general solution to this system is

$$x(t) = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Finding C_1, C_2 such that

$$x(0) = \begin{pmatrix} 0 \\ 15 \end{pmatrix}$$

can be done by solving the linear system

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 15 \end{pmatrix}.$$

After putting the associated augmented matrix into RREF we find that

$$C_1 = 6$$
$$C_2 = 3$$

giving us the particular solution

$$x(t) = 6e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

2. [5 points] Find the general solution to the problem

$$\frac{d}{dt}\vec{x}(t) = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \vec{x}(t)$$

Solution
The characteristic equation of our matrix is

$$\det \begin{pmatrix} -\lambda & -1 \\ 4 & -\lambda \end{pmatrix} = \lambda^2 + 4$$

which has roots $\lambda = \pm 2i$. Selecting $\lambda = +2i$ we wish to find the associated eigenvalues by placing $A - 2iI$ in RREF:

$$A - 2iI = \begin{pmatrix} -2i & -1 \\ 4 & -2i \end{pmatrix} \overset{\langle 1 \rangle \leftarrow \frac{1}{-2i} \langle 1 \rangle}{\sim} \begin{pmatrix} 1 & -\frac{1}{2}i \\ 4 & -2i \end{pmatrix}$$
$$\overset{\langle 2 \rangle \leftarrow \langle 2 \rangle - 4 \langle 1 \rangle}{\sim} \begin{pmatrix} 1 & -\frac{1}{2}i \\ 0 & 0 \end{pmatrix}$$

This reveals that

$$\begin{pmatrix} i \\ 2 \end{pmatrix}$$

is the associated eigenvector hence we have the complex solution

$$e^{2it} \begin{pmatrix} i \\ 2 \end{pmatrix} = (\cos(2t) + i \sin(2t)) \begin{pmatrix} i \\ 2 \end{pmatrix} = \begin{pmatrix} -\sin(2t) \\ 2\cos(2t) \end{pmatrix} + i \begin{pmatrix} \cos(2t) \\ 2\sin(2t) \end{pmatrix}.$$

Taking the real and imaginary parts of this solution we find the general solution

$$C_1 \begin{pmatrix} -\sin(2t) \\ 2\cos(2t) \end{pmatrix} + C_2 \begin{pmatrix} \cos(2t) \\ 2\sin(2t) \end{pmatrix}.$$