M 327J - Differential Equations with Linear Algebra

November 14, 2022

Quiz 7

1. [5 points] Find the orbits of the system

$$\begin{aligned}\dot{x} &= y(x^2 - y^2)\\ \dot{y} &= x(y^2 - x^2)\end{aligned}$$

Solution
To find the equilibria we must find all solutions to

$$\begin{aligned}y(x^2 - y^2) &= 0\\ x(y^2 - x^2) &= 0\end{aligned}$$

The first equation is zero if either $y = 0$ or $x^2 = y^2$. In the case that $y = 0$ the second equation becomes

$$x^3 = 0$$

giving $(0, 0)$ as our first equilibrium. If $x^2 = y^2$ we see the second equation is immediately equal to 0, so all points $(x, \pm x)$ are also equilibria of this system.

To find the non-stationary orbits we solve the differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x(y^2 - x^2)}{y(x^2 - y^2)} = -\frac{x}{y}.$$

This is a separable equation; multiplying by y and integrating both sides in x we find

$$\frac{1}{2}y^2 = \int y\frac{dy}{dx}\,dx = -\int x\,dx = -\frac{1}{2}x^2 + C$$

hence the solution lies on the curves described by

$$x^2 + y^2 = 2C$$

which are circles.

We notice that for any $C > 0$ these circles intersect the lines $y = x, y = -x$ at four points, so each of these circles contain four orbits.

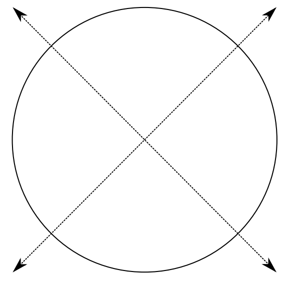

Figure 1: The diagonal lines $y = \pm x$ of equilibrium points divide the circle into four orbits, each of which is a quarter arc.

2. [**5 points**] Solve the problem

$$\begin{cases} \frac{\partial}{\partial t}u = 4\frac{\partial^2}{\partial x^2}u \\ u(x,0) = 2\sin\left(\frac{5\pi x}{2}\right) \\ u(0,t) = u(2,t) = 0 \end{cases}$$

Solution
Letting

$$u(x,t) = X(x)T(t)$$

the differential equation becomes

$$X(x)T'(t) = 4X''(x)T(t).$$

Dividing this by XT we find

$$\frac{T'}{T} = 4\frac{X''}{X} = \lambda$$

for some constant λ. This means that T, X must satisfy the equations

$$T' = \lambda T$$
$$X'' = \frac{\lambda}{4}X$$

Setting X equal to our initial condition

$$X(x) = u(0,x) = 2\sin\left(\frac{5\pi x}{2}\right)$$

we find that

$$X'' = -2\left(\frac{5\pi}{2}\right)^2 \sin\left(\frac{5\pi x}{2}\right) = -\frac{25\pi^2}{4}X$$

verifying our second equation when

$$\lambda = -25\pi^2.$$

Solving the first equation with this λ we find

$$T(t) = e^{-25\pi^2 t}$$

is the unique solution with $T(0) = 1$. Hence our final solution is

$$u(x,t) = X(x)T(t) = 2e^{-25\pi^2 t}\sin\left(\frac{5\pi x}{2}\right).$$