M 327J - Differential Equations with Linear Algebra

September 12, 2022

Quiz 2

1. [**4 points**] Find the particular solution to the initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + y = e^{t-2} \\ y(2) = 1 \\ y'(2) = 0 \end{cases}$$

Solution

The corresponding homogeneous problem has characteristic equation

$$r^2 - 2r + 1 = (r-1)^2 = 0$$

which has the single root $r = 1$. A general solution of the homogeneous equation can be written in the form

$$C_1 e^{t-2} + C_2(t-2)e^{t-2}.$$

To create our judicious guess we first note that $e^{t-2}, (t-2)e^{t-2}$ are both solutions to the homogenous problem. So if we guess $p(t)e^t$ for a polynomial p then p must be at least degree two. Guessing $\psi(t) = A(t-2)^2 e^t$ we find

$$\begin{aligned} \psi''(t) - 2\psi'(t) + \psi(t) &= [2Ae^{t-2} + 4A(t-2)e^{t-2} + A(t-2)^2 e^{t-2}] \\ &\quad - 2[2A(t-2)e^{t-2} + A(t-2)^2 e^{t-2}] = 2Ae^{t-2} \end{aligned}$$

which shows selecting $A = \frac{1}{2}$ gives us a particular solution to the nonhomogeneous equation. Hence we can write any general solution in the form

$$y(t) = C_1 e^{t-2} + C_2(t-2)e^{t-2} + \frac{1}{2}(t-2)^2 e^{t-2}.$$

To find our particular solution we compute

$$\begin{aligned} y(t) &= C_1 + C_2(t-2)e^{t-2} + \frac{1}{2}(t-2)^2 e^{t-2} \\ y'(t) &= C_1 e^{t-2} + C_2 e^{t-2} + C_2(t-2)e^{t-2} + (t-2)e^{t-2} + \frac{1}{2}(t-2)^2 e^{t-2} \end{aligned}$$

which when evaluated at $t = 2$ gives the system of equations

$$\begin{aligned} 1 &= C_1 \\ 0 &= C_1 + C_2 \end{aligned}$$

which has solution $(C_1, C_2) = (1, -1)$. Our particular solution is then

$$\begin{aligned} y(t) &= e^{t-2} - (t-2)e^{t-2} + \frac{1}{2}(t-2)^2 e^{t-2} \\ &= \frac{t^2}{2}e^{t-2} - 3te^{t-2} + 5e^{t-2}. \end{aligned}$$

2. [3 points] Find the general solution to

$$\frac{d^2y}{dt^2} + 4y = 0.$$

Solution
The equation has characteristic equation

$$r^2 + 4 = 0$$

which has roots $r = \pm 2i$. Hence the equation has general solution

$$y(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

3. [3 points] The functions

$$\begin{aligned}\psi_1(t) &= \sin(2t) + te^t \\ \psi_2(t) &= \cos(2t) + 8\sin(2t) + te^t \\ \psi_3(t) &= te^t - \cos(2t)\end{aligned}$$

are all solutions to some second order nonhomogeneous linear equation. Find the general solution to this equation.

Solution
We know that the difference of any two solutions will solve the corresponding homogeneous equation, so we have

$$\begin{aligned}\psi_2(t) - \psi_1(t) &= \cos(2t) + 7\sin(2t) \\ \psi_3(t) - \psi_1(t) &= -\cos(2t) - \sin(2t)\end{aligned}$$

as two solutions to the homogenous equation. This gives general solution to the homogenous equation

$$C_1(\cos(2t) + 7\sin(2t)) + C_2(\cos(2t) + \sin(2t))$$

which can be rewritten into the form

$$C_1' \cos(2t) + C_2' \sin(2t)$$

after collecting terms. Adding any of the three nonhomogeneous solutions to this general solution will give us the general solution to the nonhomogeneous equation. Selecting ψ_1 we find the general solution

$$y(t) = C_1' \cos(2t) + C_2' \sin(2t) + \sin(2t) + te^t.$$

Note: In general we need to show $\cos(2t), \sin(2t)$ *are linearly independent to show that they form a fundamental set of solutions for the homogeneous equation. I decided to ignore this step when grading.*