Homework Assignment #4
(Due May 15, 2020)

Modeling Volatility: GARCH Models

1. Objective of the assignment

The objective of this assignment is to highlight the so-called "volatility timing", i.e. the additional return an investor can expect when she is able to forecast the dynamics of expected returns and volatility correctly.

The dataset includes the total return index of the S&P500 index (S&PCOMP) for US stocks and the JP Morgan index (JPMUSU$) for US government bonds, and the one-week interest rate (FRFEDFD) for cash, from January 2001 to December 2018 at weekly frequency. The one-week interest rate is annualized and expressed in %.

Compute returns for stocks and bonds ($R_{s,t}$ and $R_{b,t}$) and compute the weekly returns for the risk-free asset ($R_{f,t}$). Be careful, do not multiply returns by 100.

2. Static asset allocation with constant expected returns and volatility

An investor would like to invest her wealth in a portfolio composed of US stocks, bonds and cash. For this purpose, she uses a mean-variance criterion, defined as

$$\max_{\{\alpha\}} \quad \mu_p - \frac{\lambda}{2}\sigma_p^2$$

where $\mu_p = \alpha'\mu + (1 - e'\alpha)R_f$ and $\sigma_p^2 = \alpha'\Sigma\alpha$ are the expected portfolio return and variance of the portfolio return. We denote $\mu = (\mu_s, \mu_b)'$ the (2,1) vector of expected returns for stocks and bonds, e the (2,1) vector of ones, Σ the (2,2) covariance matrix of returns, and λ is the degree of risk aversion. Weights $\alpha = (\alpha_s, \alpha_b)'$ are unconstrained, and the weight on the risk-free asset is given by $1 - e'\alpha$. R_f is the average risk-free rate over the sample.

2a. Give the expression for the optimal portfolio weights. (Hint: compute the first-order derivative of the mean-variance criterion with respect to α.)

2b. Assume that expected returns are given by sample means and that the covariance matrix Σ is given by the sample covariance matrix. Compute the optimal weight vector α^* for $\lambda = 2$ and 10.

3. **Estimation of a GARCH model**

3a. For both stocks and bonds, provide evidence on the non-normality (Kolmogorov-Smirnov test) and the auto-correlation (Ljung-Box test, with 4 lags) of the excess returns.

3b. Estimate an AR(1) model on stocks and bonds returns, to filter out autocorrelation. We now denote $\hat{\varepsilon}_t$ the residuals of the AR(1) model.

3c. For $\hat{\varepsilon}_t$, test the ARCH effect using the LM test of Engle (1982), using 4 lags.

3d. We consider the following model, for $i = s, b$ (for stocks and bonds):

$$\hat{\varepsilon}_{i,t} = \sigma_{i,t} z_{i,t}$$
$$\sigma_{i,t}^2 = \omega_i + a_i \hat{\varepsilon}_{i,t-1}^2 + b_i \sigma_{i,t-1}^2$$

where the usual assumptions apply. Estimate the model using the conditional ML technique (using the Sheppard's toolbox). Comment the parameter estimates (in particular, the sum $a_i + b_i$).

3e. At the end of the sample *T*, make forecasts for $\sigma_{i,T+k}^2$ for $k = 1, 2, \ldots, 52$ and $i = s, b$. Plot your results.

4. **Dynamic asset allocation with time-varying expected return and volatility**

4a. For each date $t = 2, \ldots, T$ and for the 2 assets, $i = 1, 2$, compute the expected returns $\mu_{i,t}$ using the estimated AR(1) processes and the expected variances $\sigma_{i,t}^2$ using the estimated GARCH(1,1) processes for stocks and bonds. Assume that the correlation ρ_{sb} between residuals $\hat{\varepsilon}_t$ is constant over time. Then, the expected covariance between residuals is given by $\sigma_{sb,t} = \rho_{sb} \times \sigma_{s,t} \times \sigma_{b,t}$. The covariance matrix at time *t* is

$$\Sigma_t = \begin{pmatrix} \sigma_{s,t}^2 & \sigma_{sb,t} \\ \sigma_{sb,t} & \sigma_{b,t}^2 \end{pmatrix}$$

4b. Then, using the same procedure as in 2, compute the optimal weight vector $\tilde{\alpha}_t$ for stocks and bonds for each date *t*, that maximizes the mean-variance criterion

$$\max_{\{\tilde{\alpha}_t\}} \quad \mu_{p,t} - \frac{\lambda}{2} \sigma_{p,t}^2$$

where $\mu_{p,t} = \alpha_t' \mu_t + (1 - e' \alpha_t) R_{f,t}$ and $\sigma_{p,t}^2 = \alpha_t' \Sigma_t \alpha_t$.

4c. Plot the vector of optimal weights for stocks and bonds for the two approaches: with constant expected returns and volatility on the one hand (question 2b) and for time-varying volatility on the other hand (question 4b). Comment your results for $\lambda = 2$ and 10.

4d. Compute the cumulative returns of the optimal portfolio for the two approaches (you use the optimal portfolio weights and the realized returns). For instance, for the dynamic approach, you have

$$CR_t = \prod_{j=1}^{t} (1 + R_{p,j})$$

where $R_{p,t} = \tilde{\alpha}_{s,t} R_{s,t} + \tilde{\alpha}_{b,t} R_{b,t} + (1 - \tilde{\alpha}_{s,t} - \tilde{\alpha}_{b,t}) R_{f,t}$. (If you get something explosive, take the log of the portfolio returns by $r_{p,t} = \log(1 + R_{p,t})$ and calculate the cumulative returns as the cumulative sum of $r_{p,t}$.)

Plot the two time series of cumulative returns. Which one performs better? What factor(s) could change your opinion?

4e. We introduce now the transaction cost you have to pay for the dynamic portfolio as a percentage τ of a simplified portfolio turnover proxy given by the following formula:

$$TC_t = \left[\left| \tilde{\alpha}_{s,t} - \tilde{\alpha}_{s,t-1} \right| + \left| \tilde{\alpha}_{b,t} - \tilde{\alpha}_{b,t-1} \right| \right] \tau$$

(There are no transaction costs for the risk-free asset.) At each period, the actual return of the dynamic is therefore the realized portfolio return minus the transaction cost TC_t.

At what τ are the static (question 2b) and the dynamic (question 4b) portfolio allocations equally performing?