Chapter 15, Solution 1.

(a) $$\cosh(at) = \frac{e^{at} + e^{-at}}{2}$$

$$\mathcal{L}\left[\cosh(at)\right] = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{\mathbf{s}}{\mathbf{s^2 - a^2}}$$

(b) $$\sinh(at) = \frac{e^{at} - e^{-at}}{2}$$

$$\mathcal{L}\left[\sinh(at)\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{\mathbf{a}}{\mathbf{s^2 - a^2}}$$

Chapter 15, Solution 2.

(a) $f(t) = \cos(\omega t)\cos(\theta) - \sin(\omega t)\sin(\theta)$

$F(s) = \cos(\theta)\,\mathcal{L}\left[\,\cos(\omega t)\,\right] - \sin(\theta)\,\mathcal{L}\left[\,\sin(\omega t)\,\right]$

$F(s) = \dfrac{\mathbf{s\cos(\theta) - \omega\sin(\theta)}}{\mathbf{s^2 + \omega^2}}$

(b) $f(t) = \sin(\omega t)\cos(\theta) + \cos(\omega t)\sin(\theta)$

$F(s) = \sin(\theta)\,\mathcal{L}\left[\,\cos(\omega t)\,\right] + \cos(\theta)\,\mathcal{L}\left[\,\sin(\omega t)\,\right]$

$F(s) = \dfrac{\mathbf{s\sin(\theta) - \omega\cos(\theta)}}{\mathbf{s^2 + \omega^2}}$

Chapter 15, Solution 3.

(a) $$L\left[e^{-2t}\cos(3t)\,u(t)\right] = \mathbf{\frac{s+2}{(s+2)^2+9}}$$

(b) $$L\left[e^{-2t}\sin(4t)\,u(t)\right] = \mathbf{\frac{4}{(s+2)^2+16}}$$

(c) Since $L\left[\cosh(at)\right] = \dfrac{s}{s^2 - a^2}$

$$L\left[e^{-3t}\cosh(2t)\,u(t)\right] = \mathbf{\frac{s+3}{(s+3)^2-4}}$$

(d) Since $L\left[\sinh(at)\right] = \dfrac{a}{s^2 - a^2}$

$$L\left[e^{-4t}\sinh(t)\,u(t)\right] = \mathbf{\frac{1}{(s+4)^2-1}}$$

(e) $$L\left[e^{-t}\sin(2t)\right] = \frac{2}{(s+1)^2+4}$$

If $\quad f(t) \longleftrightarrow F(s)$

$\quad\quad t\,f(t) \longleftrightarrow \dfrac{-d}{ds}F(s)$

Thus, $$L\left[t\,e^{-t}\sin(2t)\right] = \frac{-d}{ds}\left[2\left((s+1)^2+4\right)^{-1}\right]$$

$$= \frac{2}{((s+1)^2+4)^2}\cdot 2(s+1)$$

$$L\left[t\,e^{-t}\sin(2t)\right] = \mathbf{\frac{4(s+1)}{((s+1)^2+4)^2}}$$

Chapter 15, Solution 4.

Design a problem to help other students better understand how to find the Laplace transform of different time varying functions.

Although there are many ways to solve this problem, this is an example based on the same kind of problem asked in the third edition.

Problem

Find the Laplace transforms of the following:

(a) $g(t) = 6\cos(4t - 1)$
(b) $f(t) = 2tu(t) + 5e^{-3(t-2)}u(t-2)$

Solution

(a) $$\mathbf{G(s) = 6\frac{s}{s^2+4^2}e^{-s} = \underline{\frac{6se^{-s}}{s^2+16}}}$$

(b) $$\mathbf{F(s) = \underline{\frac{2}{s^2} + 5\frac{e^{-2s}}{s+3}}}$$

Chapter 15, Solution 5.

(a) $$\mathcal{L}\left[\cos(2t+30°)\right] = \frac{s\cos(30°) - 2\sin(30°)}{s^2+4}$$

$$\mathcal{L}\left[t^2\cos(2t+30°)\right] = \frac{d^2}{ds^2}\left[\frac{s\cos(30°)-1}{s^2+4}\right]$$

$$= \frac{d}{ds}\frac{d}{ds}\left[\left(\frac{\sqrt{3}}{2}s-1\right)\left(s^2+4\right)^{-1}\right]$$

$$= \frac{d}{ds}\left[\frac{\sqrt{3}}{2}\left(s^2+4\right)^{-1} - 2s\left(\frac{\sqrt{3}}{2}s-1\right)\left(s^2+4\right)^{-2}\right]$$

$$= \frac{\frac{\sqrt{3}}{2}(-2s)}{\left(s^2+4\right)^2} - \frac{2\left(\frac{\sqrt{3}}{2}s-1\right)}{\left(s^2+4\right)^2} - \frac{2s\left(\frac{\sqrt{3}}{2}\right)}{\left(s^2+4\right)^2} + \frac{(8s^2)\left(\frac{\sqrt{3}}{2}s-1\right)}{\left(s^2+4\right)^3}$$

$$= \frac{-\sqrt{3}s - \sqrt{3}s + 2 - \sqrt{3}s}{\left(s^2+4\right)^2} + \frac{(8s^2)\left(\frac{\sqrt{3}}{2}s-1\right)}{\left(s^2+4\right)^3}$$

$$= \frac{(-3\sqrt{3}s+2)(s^2+4)}{\left(s^2+4\right)^3} + \frac{4\sqrt{3}s^3 - 8s^2}{\left(s^2+4\right)^3}$$

$$\mathcal{L}\left[t^2\cos(2t+30°)\right] = \mathbf{\frac{8 - 12\sqrt{3}s - 6s^2 + \sqrt{3}s^3}{\left(s^2+4\right)^3}}$$

(b) $$\mathcal{L}\left[3t^4e^{-2t}\right] = 3\cdot\frac{4!}{(s+2)^5} = \mathbf{\frac{72}{(s+2)^5}}$$

(c) $$\mathcal{L}\left[2t\,u(t) - 4\frac{d}{dt}\delta(t)\right] = \frac{2}{s^2} - 4(s\cdot 1 - 0) = \mathbf{\frac{2}{s^2} - 4s}$$

(d) $$2e^{-(t-1)}u(t) = 2e^{-t}u(t)$$

$$\mathcal{L}\left[2e^{-(t-1)}u(t)\right] = \mathbf{\frac{2e}{s+1}}$$

(e) Using the scaling property,

$$\mathcal{L}\left[5u(t/2)\right] = 5\cdot\frac{1}{1/2}\cdot\frac{1}{s/(1/2)} = 5\cdot 2\cdot\frac{1}{2s} = \mathbf{\frac{5}{s}}$$

(f) $$\mathcal{L}\left[6e^{-t/3}u(t)\right] = \frac{6}{s+1/3} = \mathbf{\frac{18}{3s+1}}$$

(g) Let $f(t) = \delta(t)$. Then, $F(s) = 1$.

$$\mathcal{L}\left[\frac{d^n}{dt^n}\delta(t)\right] = \mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots$$

$$\mathcal{L}\left[\frac{d^n}{dt^n}\delta(t)\right] = \mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^n \cdot 1 - s^{n-1} \cdot 0 - s^{n-2} \cdot 0 - \cdots$$

$$\mathcal{L}\left[\frac{d^n}{dt^n}\delta(t)\right] = \mathbf{s^n}$$

Chapter 15, Solution 6.

$f(t) = 5t[u(t)–u(t–1)] – 5t[u(t–1)–u(t–2)] = 5[tu(t)–tu(t–1) – tu(t–1) + tu(t–2)]$
$= 5[tu(t) – 2tu(t–1) + tu(t–2)]$
$= 5[tu(t) – 2(t–1)u(t–1) – 2u(t–1) + (t–2)u(t–2) + 2u(t–2)]$ which leads to

$F(s) =5[(1/s^2) – (1/s^2)e^{–s} – (2/s)e^{–s} + (1/s^2)e^{–2s} + (2/s)e^{–2s}]$

Chapter 15, Solution 7.

(a) $F(s) = \dfrac{2}{s^2} + \dfrac{4}{s}$

(b) $G(s) = \dfrac{4}{s} + \dfrac{3}{s+2}$

(c) $\mathbf{H(s) = 6\dfrac{3}{s^2+9} + 8\dfrac{s}{s^2+9} = \dfrac{8s+18}{s^2+9}}$

(d) From Problem 15.1,

$$L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

$$X(s) = \frac{s+2}{(s+2)^2 - 4^2} = \frac{s+2}{s^2 + 4s - 12}$$

$$\mathbf{(a)\ \frac{2}{s^2} + \frac{4}{s}, (b) \frac{4}{s} + \frac{3}{s+2}, (c) \frac{8s+18}{s^2+9}, (d) \frac{s+2}{s^2+4s-12}}$$

Chapter 15, Solution 8.

(a) 2t=2(t-4) + 8

f(t) = 2tu(t-4) = 2(t-4)u(t-4) + 8u(t-4)

$$F(s) = \frac{2}{s^2}e^{-4s} + \frac{8}{s}e^{-4s} = \underline{\left(\frac{2}{s^2} + \frac{8}{s}\right)e^{-4s}}$$

(b) $F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty 5\cos t\,\delta(t-2)e^{-st}dt = 5\cos t e^{-st}\Big|_{t=2} =$ $\underline{\mathbf{5cos(2)e^{-2s}}}$

(c) $e^{-t} = e^{-(t-\tau)}e^{-\tau}$

$$f(t) = e^{-\tau}e^{-(t-\tau)}u(t-\tau)$$

$$F(s) = e^{-\tau}e^{-\tau s}\frac{1}{s+1} = \underline{\frac{e^{-\tau(s+1)}}{s+1}}$$

(d) $\sin 2t = \sin[2(t-\tau) + 2\tau] = \sin 2(t-\tau)\cos 2\tau + \cos 2(t-\tau)\sin 2\tau$

$$f(t) = \cos 2\tau \sin 2(t-\tau)u(t-\tau) + \sin 2\tau \cos 2(t-\tau)u(t-\tau)$$

$$F(s) = \underline{\cos 2\tau e^{-\tau s}\frac{2}{s^2+4} + \sin 2\tau e^{-\tau s}\frac{s}{s^2+4}}$$

Chapter 15, Solution 9.

(a) $f(t) = (t-4)u(t-2) = (t-2)u(t-2) - 2u(t-2)$

$$F(s) = \mathbf{\frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^2}}$$

(b) $g(t) = 2e^{-4t}u(t-1) = 2e^{-4}e^{-4(t-1)}u(t-1)$

$$G(s) = \mathbf{\frac{2e^{-s}}{e^4(s+4)}}$$

(c) $h(t) = 5\cos(2t-1)u(t)$

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\cos(2t-1) = \cos(2t)\cos(1) + \sin(2t)\sin(1)$$

$$h(t) = 5\cos(1)\cos(2t)u(t) + 5\sin(1)\sin(2t)u(t)$$

$$H(s) = 5\cos(1)\cdot\frac{s}{s^2+4} + 5\sin(1)\cdot\frac{2}{s^2+4}$$
$$H(s) = \mathbf{\frac{2.702s}{s^2+4} + \frac{8.415}{s^2+4}}$$

(d) $p(t) = 6u(t-2) - 6u(t-4)$

$$P(s) = \mathbf{\frac{6}{s}e^{-2s} - \frac{6}{s}e^{-4s}}$$

Chapter 15, Solution 10.

(a) By taking the derivative in the time domain,

$$g(t) = (-t e^{-t} + e^{-t})\cos(t) - t e^{-t}\sin(t)$$
$$g(t) = e^{-t}\cos(t) - t e^{-t}\cos(t) - t e^{-t}\sin(t)$$

$$G(s) = \frac{s+1}{(s+1)^2+1} + \frac{d}{ds}\left[\frac{s+1}{(s+1)^2+1}\right] + \frac{d}{ds}\left[\frac{1}{(s+1)^2+1}\right]$$

$$G(s) = \frac{s+1}{s^2+2s+2} - \frac{s^2+2s}{(s^2+2s+2)^2} - \frac{2s+2}{(s^2+2s+2)^2} =$$

$$\mathbf{\frac{s^2(s+2)}{(s^2+2s+2)^2}}$$

(b) By applying the time differentiation property,

$$G(s) = sF(s) - f(0)$$

where $f(t) = t e^{-t}\cos(t)$, $f(0) = 0$

$$G(s) = (s)\cdot\frac{-d}{ds}\left[\frac{s+1}{(s+1)^2+1}\right] = \frac{(s)(s^2+2s)}{(s^2+2s+2)^2} =$$

$$\mathbf{\frac{s^2(s+2)}{(s^2+2s+2)^2}}$$

Chapter 15, Solution 11.

(a) Since $\mathcal{L}\left[\cosh(at)\right] = \dfrac{s}{s^2 - a^2}$

$$F(s) = \frac{6(s+1)}{(s+1)^2 - 4} = \mathbf{\frac{6(s+1)}{s^2 + 2s - 3}}$$

(b) Since $\mathcal{L}\left[\sinh(at)\right] = \dfrac{a}{s^2 - a^2}$

$$\mathcal{L}\left[3e^{-2t}\sinh(4t)\right] = \frac{(3)(4)}{(s+2)^2 - 16} = \frac{12}{s^2 + 4s - 12}$$

$$F(s) = \mathcal{L}\left[t \cdot 3e^{-2t}\sinh(4t)\right] = \frac{-d}{ds}\left[12(s^2 + 4s - 12)^{-1}\right]$$

$$F(s) = (12)(2s+4)(s^2 + 4s - 12)^{-2} = \mathbf{\frac{24(s+2)}{(s^2 + 4s - 12)^2}}$$

(c) $\cosh(t) = \dfrac{1}{2} \cdot (e^t + e^{-t})$

$$f(t) = 8e^{-3t} \cdot \frac{1}{2} \cdot (e^t + e^{-t})\,u(t-2)$$

$$= 4e^{-2t}\,u(t-2) + 4e^{-4t}\,u(t-2)$$

$$= 4e^{-4}\,e^{-2(t-2)}\,u(t-2) + 4e^{-8}\,e^{-4(t-2)}\,u(t-2)$$

$$\mathcal{L}\left[4e^{-4}\,e^{-2(t-2)}\,u(t-2)\right] = 4e^{-4}\,e^{-2s} \cdot \mathcal{L}\left[e^{-2}\,u(t)\right]$$

$$\mathcal{L}\left[4e^{-4}\,e^{-2(t-2)}\,u(t-2)\right] = \frac{4e^{-(2s+4)}}{s+2}$$

Similarly, $\mathcal{L}\left[4e^{-8}\,e^{-4(t-2)}\,u(t-2)\right] = \dfrac{4e^{-(2s+8)}}{s+4}$

Therefore,

$$F(s) = \frac{4e^{-(2s+4)}}{s+2} + \frac{4e^{-(2s+8)}}{s+4} = \mathbf{\frac{e^{-(2s+6)}\left[(4e^2 + 4e^{-2})s + (16e^2 + 8e^{-2})\right]}{s^2 + 6s + 8}}$$

Chapter 15, Solution 12.

$$G(s) = \frac{s+2}{(s+2)^2 + 4^2} = \underline{\frac{s+2}{s^2 + 4s + 20}}$$

Chapter 15, Solution 13.

(a) $tf(t) \quad \longleftrightarrow \quad -\dfrac{d}{ds}F(s)$

If f(t) = cost, then $F(s) = \dfrac{s}{s^2+1}$ and $-\dfrac{d}{ds}F(s) = -\dfrac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}$

$$\underline{\mathcal{L}(t\cos t) = \frac{s^2-1}{(s^2+1)^2}}$$

(b) Let f(t) = e^{-t} sin t.

$$F(s) = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$$

$$\frac{dF}{ds} = \frac{(s^2+2s+2)(0)-(1)(2s+2)}{(s^2+2s+2)^2}$$

$$\underline{\mathcal{L}(e^{-t}t\sin t) = -\frac{dF}{ds} = \frac{2(s+1)}{(s^2+2s+2)^2}}$$

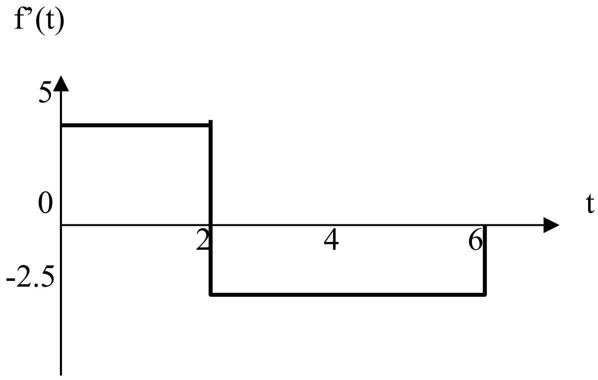
(c) $\dfrac{f(t)}{t} \quad \longleftrightarrow \quad \displaystyle\int_s^\infty F(s)ds$

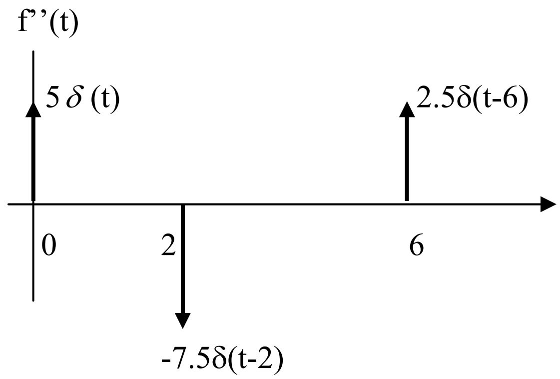
Let $f(t) = \sin\beta t$, then $F(s) = \dfrac{\beta}{s^2+\beta^2}$

$$\mathcal{L}\left[\frac{\sin\beta t}{t}\right] = \int_s^\infty \frac{\beta}{s^2+\beta^2}ds = \beta\frac{1}{\beta}\tan^{-1}\frac{s}{\beta}\Big|_s^\infty = \frac{\pi}{2}-\tan^{-1}\frac{s}{\beta} = \underline{\tan^{-1}\frac{\beta}{s}}$$

Chapter 15, Solution 14.

Taking the derivative of f(t) twice, we obtain the figures below.





$f'' = 5\delta(t) - 7.5\delta(t-2) + 2.5\delta(t-6)$

Taking the Laplace transform of each term,

$s^2F(s) = 5 - 7.5e^{-2s} + 2.5e^{-6s}$ or $\underline{\mathbf{F(s) = \frac{5}{s} - 7.5\frac{e^{-2s}}{s^2} + 2.5\frac{e^{-6s}}{s^2}}}$

Please note that we can obtain the same answer by representing the function as,

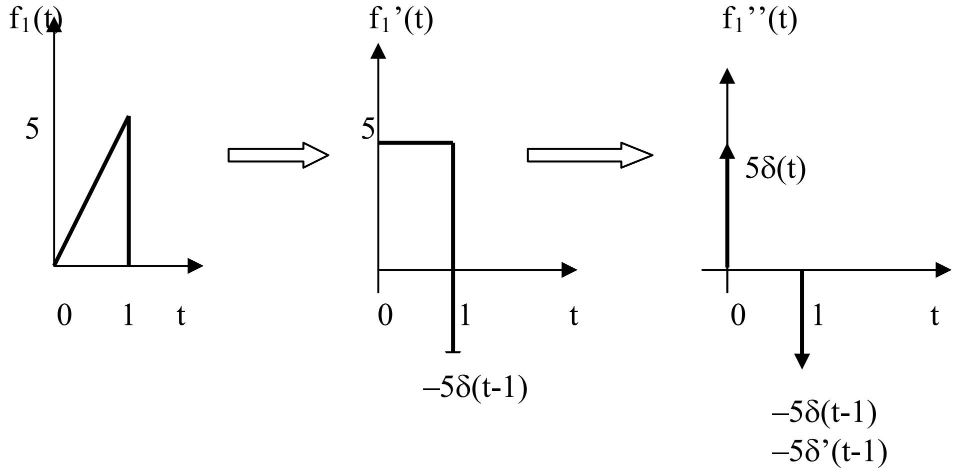
$f(t) = 5tu(t) - 7.5u(t-2) + 2.5u(t-6).$

Chapter 15, Solution 15.

This is a periodic function with T=3.

$$F(s) = \frac{F_1(s)}{1-e^{-3s}}$$

To get $F_1(s)$, we consider f(t) over one period.



$f_1'' = 5\delta(t) - 5\delta(t-1) - 5\delta'(t-1)$

Taking the Laplace transform of each term,

$s^2F_1(s) = 5 - 5e^{-s} - 5se^{-s}$ or $F_1(s) = 5(1 - e^{-s} - se^{-s})/s^2$

Hence,

$$F(s) = \underline{\mathbf{5\frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-3s})}}}$$

Alternatively, we can obtain the same answer by noting that $f_1(t) = 5tu(t) - 5tu(t-1) - 5u(t-1)$.

Chapter 15, Solution 16.

$$f(t) = 5u(t) - 3u(t-1) + 3u(t-3) - 5u(t-4)$$

$$F(s) = \mathbf{\frac{1}{s}\left[5 - 3e^{-s} + 3e^{-3s} - 5e^{-4s}\right]}$$

Chapter 15, Solution 17.

Using Fig. 15.29, design a problem to help other students to better understand the Laplace transform of a simple, non-periodic waveshape.

Although there are many ways to solve this problem, this is an example based on the same kind of problem asked in the third edition.

Problem

Find the Laplace transform of f(t) shown in Fig. 15.29.

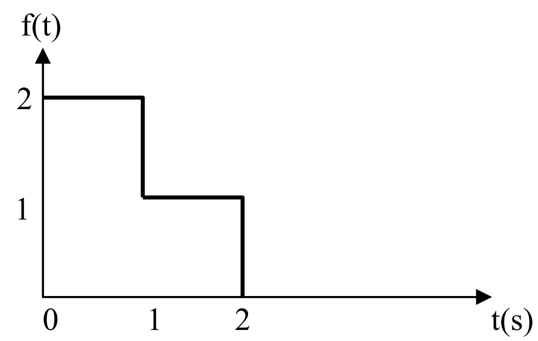
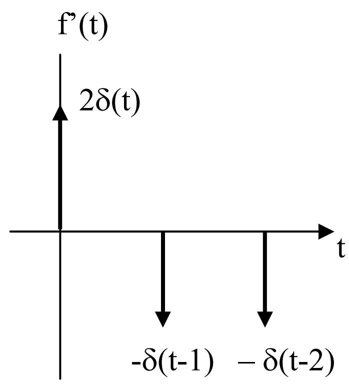


Figure 15.29 For Prob. 15.17.

Solution

Taking the derivative of f(t) gives f'(t) as shown below.



$$f'(t) = 2\delta(t) - \delta(t-1) - \delta(t-2)$$

Taking the Laplace transform of each term,

$$sF(s) = 2 - e^{-s} - e^{-2s}$$ which leads to

$$F(s) = \mathbf{[2 - e^{-s} - e^{-2s}]/s}$$

We can also obtain the same answer noting that $f(t) = 2u(t) - u(t-1) - u(t-2)$.

Chapter 15, Solution 18.

(a) $$g(t) = u(t) - u(t-1) + 2\left[u(t-1) - u(t-2)\right] + 3\left[u(t-2) - u(t-3)\right]$$
$$= u(t) + u(t-1) + u(t-2) - 3u(t-3)$$

$$G(s) = \mathbf{\frac{1}{s}(1 + e^{-s} + e^{-2s} - 3e^{-3s})}$$

(b) $$h(t) = 2t\left[u(t) - u(t-1)\right] + 2\left[u(t-1) - u(t-3)\right]$$
$$+ (8-2t)\left[u(t-3) - u(t-4)\right]$$

$$= 2t\,u(t) - 2(t-1)u(t-1) - 2u(t-1) + 2u(t-1) - 2u(t-3)$$
$$- 2(t-3)u(t-3) + 2u(t-3) + 2(t-4)u(t-4)$$

$$= 2t\,u(t) - 2(t-1)u(t-1) - 2(t-3)u(t-3) + 2(t-4)u(t-4)$$

$$H(s) = \frac{2}{s^2}(1 - e^{-s}) - \frac{2}{s^2}e^{-3s} + \frac{2}{s^2}e^{-4s} = \mathbf{\frac{2}{s^2}(1 - e^{-s} - e^{-3s} + e^{-4s})}$$

Chapter 15, Solution 19.

Since $\mathcal{L}\left[\delta(t)\right] = 1$ and $T = 2$, $F(s) = \dfrac{\mathbf{1}}{\mathbf{1 - e^{-2s}}}$

Chapter 15, Solution 20.

Using Fig. 15.32, design a problem to help other students to better understand the Laplace transform of a simple, periodic waveshape.

Although there are many ways to solve this problem, this is an example based on the same kind of problem asked in the third edition.

Problem

The periodic function shown in Fig. 15.32 is defined over its period as

$$g(t) = \begin{cases} \sin \pi t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

Find $G(s)$.

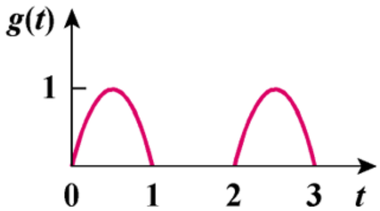


Figure 15.32

Solution

Let $\quad g_1(t) = \sin(\pi t), \quad 0 < t < 1$

$$= \sin(\pi t)\left[u(t) - u(t-1)\right] \; 0 < t < 2$$

$$= \sin(\pi t)\,u(t) - \sin(\pi t)\,u(t-1)$$

Note that $\sin(\pi(t-1)) = \sin(\pi t - \pi) = -\sin(\pi t)$.

So, $\quad g_1(t) = \sin(\pi t)\,u(t) + \sin(\pi(t-1))\,u(t-1)$

$$G_1(s) = \frac{\pi}{s^2 + \pi^2}(1 + e^{-s})$$

$$G(s) = \frac{G_1(s)}{1 - e^{-2s}} = \boldsymbol{\frac{\pi(1 + e^{-s})}{(s^2 + \pi^2)(1 - e^{-2s})}}$$

Chapter 15, Solution 21.

$$T = 2\pi$$

Let $$f_1(t) = \left(1 - \frac{t}{2\pi}\right)\left[u(t) - u(t-2\pi)\right]$$

$$f_1(t) = u(t) - \frac{t}{2\pi}u(t) + \frac{1}{2\pi}(t-2\pi)\,u(t-2\pi)$$

$$F_1(s) = \frac{1}{s} - \frac{1}{2\pi s^2} + \frac{e^{-2\pi s}}{2\pi s^2} = \frac{2\pi\, s + \left[-1 + e^{-2\pi s}\right]}{2\pi s^2}$$

$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}} = \mathbf{\frac{2\pi s - 1 + e^{-2\pi s}}{2\pi s^2\,(1 - e^{-2\pi s})}}$$

Chapter 15, Solution 22.

(a) Let $g_1(t) = 2t, \quad 0 < t < 1$

$$= 2t\left[u(t) - u(t-1)\right]$$
$$= 2t\,u(t) - 2(t-1)u(t-1) + 2u(t-1)$$

$$G_1(s) = \frac{2}{s^2} - \frac{2e^{-s}}{s^2} + \frac{2}{s}e^{-s}$$

$$G(s) = \frac{G_1(s)}{1 - e^{-sT}}, \quad T = 1$$

$$G(s) = \mathbf{\frac{2(1 - e^{-s} + s\,e^{-s})}{s^2(1 - e^{-s})}}$$

(b) Let $h = h_0 + u(t)$, where h_0 is the periodic triangular wave.

Let h_1 be h_0 within its first period, i.e.

$$h_1(t) = \begin{cases} 2t & 0 < t < 1 \\ 4 - 2t & 1 < t < 2 \end{cases}$$

$$h_1(t) = 2t\,u(t) - 2t\,u(t-1) + 4u(t-1) - 2t\,u(t-1) - 2(t-2)u(t-2)$$
$$h_1(t) = 2t\,u(t) - 4(t-1)u(t-1) - 2(t-2)u(t-2)$$

$$H_1(s) = \frac{2}{s^2} - \frac{4}{s^2}e^{-s} - \frac{2e^{-2s}}{s^2} = \frac{2}{s^2}(1 - e^{-s})^2$$

$$H_0(s) = \frac{2}{s^2}\frac{(1 - e^{-s})^2}{(1 - e^{-2s})}$$

$$H(s) = \mathbf{\frac{1}{s} + \frac{2}{s^2}\frac{(1 - e^{-s})^2}{(1 - e^{-2s})}}$$

Chapter 15, Solution 23.

(a) Let $f_1(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \end{cases}$

$$f_1(t) = \left[\, u(t) - u(t-1) \right] - \left[\, u(t-1) - u(t-2) \right]$$
$$f_1(t) = u(t) - 2\,u(t-1) + u(t-2)$$

$$F_1(s) = \frac{1}{s}(1 - 2\,e^{-s} + e^{-2s}) = \frac{1}{s}(1 - e^{-s})^2$$

$$F(s) = \frac{F_1(s)}{(1 - e^{-sT})}, \quad T = 2$$
$$F(s) = \boldsymbol{\frac{(1 - e^{-s})^2}{s\,(1 - e^{-2s})}}$$

(b) Let $h_1(t) = t^2 \left[\, u(t) - u(t-2) \right] = t^2\, u(t) - t^2\, u(t-2)$

$$h_1(t) = t^2\, u(t) - (t-2)^2\, u(t-2) - 4\,(t-2)\,u(t-2) - 4\,u(t-2)$$

$$H_1(s) = \frac{2}{s^3}(1 - e^{-2s}) - \frac{4}{s^2}\,e^{-2s} - \frac{4}{s}\,e^{-2s}$$

$$H(s) = \frac{H_1(s)}{(1 - e^{-Ts})}, \quad T = 2$$
$$H(s) = \boldsymbol{\frac{2\,(1 - e^{-2s}) - 4s\,e^{-2s}\,(s + s^2)}{s^3\,(1 - e^{-2s})}}$$

Chapter 15, Solution 24.

Design a problem to help other students to better understand how to find the initial and final values of a transfer function.

Although there are many ways to solve this problem, this is an example based on the same kind of problem asked in the third edition.

Problem

Given that

$$F(s) = \frac{s^2 + 10s + 6}{s(s+1)^2(s+2)}$$

Evaluate f(0) and f(∞) if they exist.

Solution

f(0) = $\left[\lim_{n \to \infty} \left(\frac{s(s^2 + 10s + 6)}{s(s + 1)^2 (s + 2)}\right)\right] = \mathbf{0}$

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s^2 + 10s + 6}{(s+1)^2(s+2)} = \frac{6}{(1)(2)} = \underline{3} = \mathbf{3}$$

Chapter 15, Solution 25.

(a) $f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{5s(s+1)}{(s+2)(s+3)} = \lim_{s\to\infty} \frac{5(1+1/s)}{(1+2/s)(1+3/s)} = \underline{5}$

$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{5s(s+1)}{(s+2)(s+3)} = \underline{0}$

(b) $F(s) = \frac{5(s+1)}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$

$$A = \frac{5(-1)}{1} = -5, \qquad B = \frac{5(-2)}{-1} = 10$$

$$F(s) = \frac{-5}{s+2} + \frac{10}{s+3} \qquad \longrightarrow \qquad f(t) = -5e^{-2t} + 10e^{-3t}$$

f(0) = -5 + 10 = **5**
f(∞)= -0 + 0 = **0**

Chapter 15, Solution 26.

(a) $$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{5s^3 + 3s}{s^3 + 4s^2 + 6} = \mathbf{5}$$

Two poles are not in the left-half plane.
$\mathrm{f}(\infty)$ **does not exist**

(b) $$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s^3 - 2s^2 + s}{4(s-2)(s^2 + 2s + 4)}$$

$$= \lim_{s\to\infty} \frac{1 - \frac{2}{s} + \frac{1}{s^2}}{\left(1 - \frac{2}{s}\right)\left(1 + \frac{2}{s} + \frac{4}{s^2}\right)} = \mathbf{0.25}$$

One pole is not in the left-half plane.
$\mathrm{f}(\infty)$ **does not exist**

Chapter 15, Solution 27.

(a) $f(t) = \mathbf{u(t) + 2e^{-t}u(t)}$

(b) $G(s) = \dfrac{3(s+4) - 11}{s+4} = 3 - \dfrac{11}{s+4}$

$g(t) = \mathbf{3\delta(t) - 11e^{-4t}u(t)}$

(c) $H(s) = \dfrac{4}{(s+1)(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{s+3}$

$A = 2, \qquad B = -2$

$H(s) = \dfrac{2}{s+1} - \dfrac{2}{s+3}$

$h(t) = \mathbf{[2e^{-t} - 2e^{-3t}]u(t)}$

(d) $J(s) = \dfrac{12}{(s+2)^2(s+4)} = \dfrac{A}{s+2} + \dfrac{B}{(s+2)^2} + \dfrac{C}{s+4}$

$B = \dfrac{12}{2} = 6, \quad C = \dfrac{12}{(-2)^2} = 3$

$12 = A(s+2)(s+4) + B(s+4) + C(s+2)^2$

Equating coefficients :

$s^2: \quad 0 = A + C \longrightarrow A = -C = -3$

$s^1: \quad 0 = 6A + B + 4C = 2A + B \longrightarrow B = -2A = 6$

$s^0: \quad 12 = 8A + 4B + 4C = -24 + 24 + 12 = 12$

$J(s) = \dfrac{-3}{s+2} + \dfrac{6}{(s+2)^2} + \dfrac{3}{s+4}$

$j(t) = \mathbf{[3e^{-4t} - 3e^{-2t} + 6te^{-2t}]u(t)}$

Chapter 15, Solution 28.

Design a problem to help other students to better understand how to find the inverse Laplace transform.

Although there are many ways to solve this problem, this is an example based on the same kind of problem asked in the third edition.

Problem

Find the inverse Laplace transform of the following functions:

(a) $F(s) = \dfrac{20(s+2)}{s(s^2+6s+25)}$

(b) $P(s) = \dfrac{6s^2+36s+20}{(s+1)(s+2)(s+3)}$

Solution

(a) $F(s) = \dfrac{20(s+2)}{s(s^2+6s+25)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+6s+25}$

$20(s+2) = A(s^2+6s+25s) + Bs^2 + Cs$

Equating components,

s^2 : 0 = A + B or B= - A

s: 20 = 6A + C

constant: 40 – 25 A or A = 8/5, B = -8/5, C= 20 – 6A= 52/5

$$F(s) = \frac{8}{5s} + \frac{-\frac{8}{5}s + \frac{52}{5}}{(s+3)^2 + 4^2} = \frac{8}{5s} + \frac{-\frac{8}{5}(s+3) + \frac{24}{5} + \frac{52}{5}}{(s+3)^2 + 4^2}$$

$$\underline{f(t) = \frac{8}{5}u(t) - \frac{8}{5}e^{-3t}\cos 4t + \frac{19}{5}e^{-3t}\sin 4t}$$

(b) $P(s) = \dfrac{6s^2+36s+20}{(s+1)(s+2)(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{s+3}$

$$A = \frac{6-36+20}{(-1+2)(-1+3)} = -5$$

$$B = \frac{24-72+20}{(-1)(1)} = 28$$

$$C = \frac{54-108+20}{(-2)(-1)} = -17$$

$$P(s) = \frac{-5}{s+1} + \frac{28}{s+2} - \frac{17}{s+3}$$

$$p(t) = \underline{(-5e^{-t} + 28e^{-2t} - 17e^{-3t})u(t)}$$

Chapter 15, Solution 29.

$$V(s) = \frac{2}{s} + \frac{As + B}{(s+2)^2 + 3^2}; \; 2s^2 + 8s + 26 + As^2 + Bs = 2s + 26 \to A = -2 \text{ and } B = -6$$

$$V(s) = \frac{2}{s} - \frac{2(s+2)}{(s+2)^2 + 3^2} - \frac{2}{3}\frac{3}{(s+2)^2 + 3^2}$$

$$v(t) = \mathbf{(2 - 2e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t)u(t), \; t \geq 0}$$

Chapter 15, Solution 30.

(a) $$F_1(s) = \frac{6s^2 + 8s + 3}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5}$$

$$6s^2 + 8s + 3 = A(s^2 + 2s + 5) + Bs^2 + Cs$$

We equate coefficients.

s^2 : 6 = A + B

s: 8= 2A + C

constant: 3=5A or A=3/5

B=6-A = 27/5, C=8-2A = 34/5

$$F_1(s) = \frac{3/5}{s} + \frac{27s/5 + 34/5}{s^2 + 2s + 5} = \frac{3/5}{s} + \frac{27(s+1)/5 + 7/5}{(s+1)^2 + 2^2}$$

$$\underline{f_1(t) = \left[\frac{3}{5} + \frac{27}{5}e^{-t}\cos 2t + \frac{7}{10}e^{-t}\sin 2t\right]u(t)}$$

(b) $$F_2(s) = \frac{s^2 + 5s + 6}{(s+1)^2(s+4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+4}$$

$$s^2 + 5s + 6 = A(s+1)(s+4) + B(s+4) + C(s+1)^2$$

Equating coefficients,

s^2 : 1=A+C

s: 5=5A+B+2C

constant: 6=4A+4B+C

Solving these gives

A=7/9, B= 2/3, C=2/9

$$F_2(s) = \frac{7/9}{s+1} + \frac{2/3}{(s+1)^2} + \frac{2/9}{s+4}$$

$$\underline{f_2(t) = \left[\frac{7}{9}e^{-t} + \frac{2}{3}te^{-t} + \frac{2}{9}e^{-4t}\right]u(t)}$$

(c) $$F_3(s) = \frac{10}{(s+1)(s^2 + 4s + 8)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 4s + 8}$$

$$10 = A(s^2 + 4s + 8) + B(s^2 + s) + C(s+1)$$

s^2 : 0 = A + B or B = -A

s: 0=4A+ B + C

constant: 10=8A+C

Solving these yields

A=2,   B= -2, C= -6

$$F_3(s) = \frac{2}{s+1} + \frac{-2s-6}{s^2+4s+8} = \frac{2}{s+1} - \frac{2(s+1)}{(s+1)^2+2^2} - \frac{4}{(s+1)^2+2^2}$$

$f_3(t)$ = **$(2e^{-t} - 2e^{-t}\cos(2t) - 2e^{-t}\sin(2t))u(t)$**.

Chapter 15, Solution 31.

(a) $$F(s) = \frac{10s}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$A = F(s)\,(s+1)\Big|_{s=-1} = \frac{-10}{2} = -5$$
$$B = F(s)\,(s+2)\Big|_{s=-2} = \frac{-20}{-1} = 20$$
$$C = F(s)\,(s+3)\Big|_{s=-3} = \frac{-30}{2} = -15$$

$$F(s) = \frac{-5}{s+1} + \frac{20}{s+2} - \frac{15}{s+3}$$

$$f(t) = \mathbf{(-5e^{-t} + 20e^{-2t} - 15e^{-3t})u(t)}$$

(b) $$F(s) = \frac{2s^2+4s+1}{(s+1)(s+2)^3} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

$$A = F(s)\,(s+1)\Big|_{s=-1} = -1$$
$$D = F(s)\,(s+2)^3\Big|_{s=-2} = -1$$
$$2s^2+4s+1 = A(s+2)(s^2+4s+4) + B(s+1)(s^2+4s+4)$$
$$+ C(s+1)(s+2) + D(s+1)$$

Equating coefficients :

s^3: $0 = A + B \longrightarrow B = -A = 1$

s^2: $2 = 6A + 5B + C = A + C \longrightarrow C = 2 - A = 3$

s^1: $4 = 12A + 8B + 3C + D = 4A + 3C + D$

$4 = 6 + A + D \longrightarrow D = -2 - A = -1$

s^0: $1 = 8A + 4B + 2C + D = 4A + 2C + D = -4 + 6 - 1 = 1$

$$F(s) = \frac{-1}{s+1} + \frac{1}{s+2} + \frac{3}{(s+2)^2} - \frac{1}{(s+2)^3}$$

$$f(t) = -e^{-t} + e^{-2t} + 3t\,e^{-2t} - \frac{t^2}{2}e^{-2t}$$
$$f(t) = \mathbf{\left(-e^{-t} + \left(1 + 3t - \frac{t^2}{2}\right)e^{-2t}\right)u(t)}$$

(c) $$F(s) = \frac{s+1}{(s+2)(s^2+2s+5)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+5}$$

$$A = F(s)\,(s+2)\Big|_{s=-2} = \frac{-1}{5}$$

$$s+1 = A\,(s^2+2s+5) + B\,(s^2+2s) + C\,(s+2)$$

Equating coefficients :

$$s^2: \quad 0 = A + B \longrightarrow B = -A = \frac{1}{5}$$

$$s^1: \quad 1 = 2A + 2B + C = 0 + C \longrightarrow C = 1$$

$$s^0: \quad 1 = 5A + 2C = -1 + 2 = 1$$

$$F(s) = \frac{-1/5}{s+2} + \frac{1/5 \cdot s + 1}{(s+1)^2 + 2^2} = \frac{-1/5}{s+2} + \frac{1/5\,(s+1)}{(s+1)^2+2^2} + \frac{4/5}{(s+1)^2+2^2}$$

$$f(t) = \mathbf{(-0.2\,e^{-2t} + 0.2\,e^{-t}\cos(2t) + 0.4\,e^{-t}\sin(2t))u(t)}$$

Chapter 15, Solution 32.

(a) $$F(s) = \frac{8(s+1)(s+3)}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$A = F(s)\, s\Big|_{s=0} = \frac{(8)(3)}{(2)(4)} = 3$$
$$B = F(s)\,(s+2)\Big|_{s=-2} = \frac{(8)(-1)}{(-4)} = 2$$
$$C = F(s)\,(s+4)\Big|_{s=-4} = \frac{(8)(-1)(-3)}{(-4)(-2)} = 3$$

$$F(s) = \frac{3}{s} + \frac{2}{s+2} + \frac{3}{s+4}$$

$$f(t) = \mathbf{3u(t) + 2e^{-2t} + 3e^{-4t}}$$

(b) $$F(s) = \frac{s^2 - 2s + 4}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$s^2 - 2s + 4 = A(s^2 + 4s + 4) + B(s^2 + 3s + 2) + C(s+1)$$

Equating coefficients :

s^2: $\quad 1 = A + B \longrightarrow B = 1 - A$

s^1: $\quad -2 = 4A + 3B + C = 3 + A + C$

s^0: $\quad 4 = 4A + 2B + C = -B - 2 \longrightarrow B = -6$

$A = 1 - B = 7 \qquad C = -5 - A = -12$

$$F(s) = \frac{7}{s+1} - \frac{6}{s+2} - \frac{12}{(s+2)^2}$$

$$f(t) = \mathbf{7e^{-t} - 6(1 + 2t)e^{-2t}}$$

(c) $$F(s) = \frac{s^2 + 1}{(s+3)(s^2 + 4s + 5)} = \frac{A}{s+3} + \frac{Bs + C}{s^2 + 4s + 5}$$

$$s^2 + 1 = A(s^2 + 4s + 5) + B(s^2 + 3s) + C(s+3)$$

Equating coefficients :

s^2: $\quad 1 = A + B \longrightarrow B = 1 - A$

$$s^1: \quad 0 = 4A + 3B + C = 3 + A + C \longrightarrow A + C = -3$$

$$s^0: \quad 1 = 5A + 3C = -9 + 2A \longrightarrow A = 5$$

$$B = 1 - A = -4 \qquad C = -A - 3 = -8$$

$$F(s) = \frac{5}{s+3} - \frac{4s+8}{(s+2)^2+1} = \frac{5}{s+3} - \frac{4(s+2)}{(s+2)^2+1}$$

$$f(t) = \mathbf{5e^{-3t} - 4e^{-2t}\cos(t)}$$

Chapter 15, Solution 33.

(a) $$F(s) = \frac{6(s-1)}{s^4 - 1} = \frac{6}{(s^2+1)(s+1)} = \frac{As+B}{s^2+1} + \frac{C}{s+1}$$

$$6 = A(s^2+s) + B(s+1) + C(s^2+1)$$

Equating coefficients :

s^2: $0 = A + C \longrightarrow A = -C$

s^1: $0 = A + B \longrightarrow B = -A = C$

s^0: $6 = B + C = 2B \longrightarrow B = 3$

$A = -3$, $B = 3$, $C = 3$

$$F(s) = \frac{3}{s+1} + \frac{-3s+3}{s^2+1} = \frac{3}{s+1} + \frac{-3s}{s^2+1} + \frac{3}{s^2+1}$$

$f(t) = \mathbf{(3e^{-t} + 3\sin(t) - 3\cos(t))u(t)}$

(b) $$F(s) = \frac{s e^{-\pi s}}{s^2+1}$$

$f(t) = \mathbf{\cos(t-\pi)u(t-\pi)}$

(c) $$F(s) = \frac{8}{s(s+1)^3} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$A = 8$, $D = -8$

$$8 = A(s^3 + 3s^2 + 3s + 1) + B(s^3 + 2s^2 + s) + C(s^2 + s) + Ds$$

Equating coefficients :

s^3: $0 = A + B \longrightarrow B = -A$

s^2: $0 = 3A + 2B + C = A + C \longrightarrow C = -A = B$

s^1: $0 = 3A + B + C + D = A + D \longrightarrow D = -A$

s^0: $A = 8, \quad B = -8, \quad C = -8, \quad D = -8$

$$F(s) = \frac{8}{s} - \frac{8}{s+1} - \frac{8}{(s+1)^2} - \frac{8}{(s+1)^3}$$

$f(t) = \mathbf{8\left[1 - e^{-t} - t e^{-t} - 0.5 t^2 e^{-t}\right]u(t)}$

(a) $(3e^{-t} + 3\sin(t) - 3\cos(t))u(t)$, (b) $\cos(t-\pi)u(t-\pi)$, (c) $8\left[1 - e^{-t} - t e^{-t} - 0.5 t^2 e^{-t}\right]u(t)$

Chapter 15, Solution 34.

(a) $$F(s) = 10 + \frac{s^2 + 4 - 3}{s^2 + 4} = 11 - \frac{3}{s^2 + 4}$$

$$f(t) = \mathbf{11\delta(t) - 1.5\sin(2t)}$$

(b) $$G(s) = \frac{e^{-s} + 4e^{-2s}}{(s+2)(s+4)}$$

Let $$\frac{1}{(s+2)(s+4)} = \frac{A}{s+2} + \frac{B}{s+4}$$

$$A = 1/2 \qquad B = 1/2$$

$$G(s) = \frac{e^{-s}}{2}\left(\frac{1}{s+2} + \frac{1}{s+4}\right) + 2e^{-2s}\left(\frac{1}{s+2} + \frac{1}{s+4}\right)$$

$$g(t) = \mathbf{0.5\left[e^{-2(t-1)} - e^{-4(t-1)}\right]u(t-1) + 2\left[e^{-2(t-2)} - e^{-4(t-2)}\right]u(t-2)}$$

(c) Let $$\frac{s+1}{s(s+3)(s+4)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$A = 1/12, \qquad B = 2/3, \qquad C = -3/4$$

$$H(s) = \left(\frac{1}{12}\cdot\frac{1}{s} + \frac{2/3}{s+3} - \frac{3/4}{s+4}\right)e^{-2s}$$

$$h(t) = \mathbf{\left(\frac{1}{12} + \frac{2}{3}e^{-3(t-2)} - \frac{3}{4}e^{-4(t-2)}\right)u(t-2)}$$

Chapter 15, Solution 35.

(a) Let $G(s) = \dfrac{s+3}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

$A = 2, \qquad B = -1$

$G(s) = \dfrac{2}{s+1} - \dfrac{1}{s+2} \longrightarrow g(t) = 2e^{-t} - e^{-2t}$

$F(s) = e^{-6s}\,G(s) \longrightarrow f(t) = g(t-6)\,u(t-6)$

$f(t) = \mathbf{\left[2e^{-(t-6)} - e^{-2(t-6)}\right]u(t-6)}$

(b) Let $G(s) = \dfrac{1}{(s+1)(s+4)} = \dfrac{A}{s+1} + \dfrac{B}{s+4}$

$A = 1/3, \qquad B = -1/3$

$G(s) = \dfrac{1}{3(s+1)} - \dfrac{1}{3(s+4)}$

$g(t) = \dfrac{1}{3}\left[e^{-t} - e^{-4t}\right]$

$F(s) = 4\,G(s) - e^{-2t}\,G(s)$

$f(t) = 4\,g(t)\,u(t) - g(t-2)\,u(t-2)$

$f(t) = \mathbf{\dfrac{4}{3}\left[e^{-t} - e^{-4t}\right]u(t) - \dfrac{1}{3}\left[e^{-(t-2)} - e^{-4(t-2)}\right]u(t-2)}$

(c) Let $G(s) = \dfrac{s}{(s+3)(s^2+4)} = \dfrac{A}{s+3} + \dfrac{Bs+C}{s^2+4}$

$A = -3/13$

$s = A(s^2+4) + B(s^2+3s) + C(s+3)$

Equating coefficients :

$s^2: \quad 0 = A + B \longrightarrow B = -A$

$s^1: \quad 1 = 3B + C$

$$s^0: \qquad 0 = 4A + 3C$$

$$A = -3/13, \qquad B = 3/13, \qquad C = 4/13$$

$$13G(s) = \frac{-3}{s+3} + \frac{3s+4}{s^2+4}$$

$$13g(t) = -3e^{-3t} + 3\cos(2t) + 2\sin(2t)$$

$$F(s) = e^{-s}\, G(s)$$

$$f(t) = g(t-1)\,u(t-1)$$

$$f(t) = \frac{\mathbf{1}}{\mathbf{13}}\left[\mathbf{-3e^{-3(t-1)} + 3\cos(2(t-1)) + 2\sin(2(t-1))}\right]\mathbf{u(t-1)}$$

Chapter 15, Solution 36.

(a) $$X(s) = 3\frac{1}{s^2(s+2)(s+3)} = 3\left\{\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}\right\}$$

$$\text{B} = 1/6, \qquad \text{C} = 1/4, \qquad \text{D} = \text{-}1/9$$

$$1 = \text{A}(\text{s}^3 + 5\text{s}^2 + 6\text{s}) + \text{B}(\text{s}^2 + 5\text{s} + 6) + \text{C}(\text{s}^3 + 3\text{s}^2) + \text{D}(\text{s}^3 + 2\text{s}^2)$$

Equating coefficients :

s^3: $\quad 0 = \text{A} + \text{C} + \text{D}$

s^2: $\quad 0 = 5\text{A} + \text{B} + 3\text{C} + 2\text{D} = 3\text{A} + \text{B} + \text{C}$

s^1: $\quad 0 = 6\text{A} + 5\text{B}$

s^0: $\quad 1 = 6\text{B} \longrightarrow \text{B} = 1/6$

$$\text{A} = \text{-}5/6\,\text{B} = \text{-}5/36$$

$$X(s) = 3\left(\frac{\text{-}5/36}{s} + \frac{1/6}{s^2} + \frac{1/4}{s+2} - \frac{1/9}{s+3}\right)$$

$$\text{x}(\text{t}) = \left(\frac{\mathbf{-5}}{\mathbf{12}}\mathbf{u(t)} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{t} + \frac{\mathbf{3}}{\mathbf{4}}\mathbf{e}^{\mathbf{-2t}} - \frac{\mathbf{1}}{\mathbf{3}}\mathbf{e}^{\mathbf{-3t}}\right)\mathbf{u(t)}$$

(b) $$Y(s) = 2\frac{1}{s(s+1)^2} = 2\left(\frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}\right)$$

$$\text{A} = 1, \qquad \text{C} = \text{-}1$$

$$1 = \text{A}(\text{s}^2 + 2\text{s} + 1) + \text{B}(\text{s}^2 + \text{s}) + \text{C}\,\text{s}$$

Equating coefficients :

s^2: $\quad 0 = \text{A} + \text{B} \longrightarrow \text{B} = \text{-A}$

s^1: $\quad 0 = 2\text{A} + \text{B} + \text{C} = \text{A} + \text{C} \longrightarrow \text{C} = \text{-A}$

s^0: $\quad 1 = \text{A}, \quad \text{B} = \text{-}1, \quad \text{C} = \text{-}1$

$$Y(s) = 2\left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$\text{y}(\text{t}) = \left(\mathbf{2 - 2e^{-t} - 2t\,e^{-t}}\right)\mathbf{u(t)}$$

(c) $$Z(s) = 5\left(\frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+6s+10}\right)$$

$$A = 1/10, \qquad B = -1/5$$

$$1 = A(s^3 + 7s^2 + 16s + 10) + B(s^3 + 6s^2 + 10s) + C(s^3 + s^2) + D(s^2 + s)$$

Equating coefficients :

s^3: $\quad 0 = A + B + C$

s^2: $\quad 0 = 7A + 6B + C + D = 6A + 5B + D$

s^1: $\quad 0 = 16A + 10B + D = 10A + 5B \longrightarrow B = -2A$

s^0: $\quad 1 = 10A \longrightarrow A = 1/10$

$$A = 1/10, \qquad B = -2A = -1/5, \qquad C = A = 1/10, \qquad D = 4A = \frac{4}{10}$$

$$\frac{10}{5} Z(s) = \frac{1}{s} - \frac{2}{s+1} + \frac{s+4}{s^2+6s+10}$$

$$2Z(s) = \frac{1}{s} - \frac{2}{s+1} + \frac{s+3}{(s+3)^2+1} + \frac{1}{(s+3)^2+1}$$

$$z(t) = \mathbf{0.5\left[\,1 - 2\,e^{-t} + e^{-3t}\cos(t) + e^{-3t}\sin(t)\right]\,u(t)}$$

Chapter 15, Solution 37.

(a) $H(s) = \dfrac{s+4}{s(s+2)} = \dfrac{A}{s} + \dfrac{B}{s+2}$

s+4 =A(s+2) + Bs
Equating coefficients,
s: 1 = A + B
constant: 4= 2 A $\longrightarrow$ A =2, B=1-A = -1

$$H(s) = \frac{2}{s} - \frac{1}{s+2}$$
$$h(t) = 2u(t) - e^{-2t}u(t) = \underline{(2 - e^{-2t})u(t)}$$

(b) $G(s) = \dfrac{A}{s+3} + \dfrac{Bs+C}{s^2+2s+2}$

$$s^2 + 4s + 5 = (Bs + C)(s+3) + A(s^2 + 2s + 2)$$

Equating coefficients,
s^2: 1= B + A (1)
s: 4 = 3B + C + 2A (2)
Constant: 5 =3C + 2A (3)

Solving (1) to (3) gives
$$A = \frac{2}{5}, \quad B = \frac{3}{5}, \quad C = \frac{7}{5}$$
$$G(s) = \frac{0.4}{s+3} + \frac{0.6s + 1.4}{s^2 + 2s + 2} = \frac{0.4}{s+3} + \frac{0.6(s+1) + 0.8}{(s+1)^2 + 1}$$
$$g(t) = \underline{0.4e^{-3t} + 0.6e^{-t}\cos t + 0.8e^{-t}\sin t}\ \mathrm{u(t)}$$

(c) $f(t) = \underline{e^{-2(t-4)}u(t-4)}$

(d) $D(s) = \dfrac{10s}{(s^2+1)(s^2+4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4}$

$$10s = (s^2 + 4)(As + B) + (s^2 + 1)(Cs + D)$$

Equating coefficients,
s^3 : 0 = A + C
s^2 : 0 = B + D
s: 10 = 4A + C
constant: 0 = 4B+D
Solving these leads to
A = -10/3, B = 0, C = -10/3, D = 0

$$D(s) == \frac{10s/3}{s^2+1} - \frac{10s/3}{s^2+4}$$

$$d(t) = \underline{\frac{10}{3}\cos t - \frac{10}{3}\cos 2t}\,\mathrm{u(t)}$$

Chapter 15, Solution 38.

(a) $$F(s) = \frac{s^2 + 4s}{s^2 + 10s + 26} = \frac{s^2 + 10s + 26 - 6s - 26}{s^2 + 10s + 26}$$

$$F(s) = 1 - \frac{6s + 26}{s^2 + 10s + 26}$$

$$F(s) = 1 - \frac{6(s+5)}{(s+5)^2 + 1^2} + \frac{4}{(s+5)^2 + 1^2}$$

$$f(t) = \mathbf{\delta(t) - 6e^{-t}\cos(5t) + 4e^{-t}\sin(5t)}$$

(b) $$F(s) = \frac{5s^2 + 7s + 29}{s(s^2 + 4s + 29)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 29}$$

$$5s^2 + 7s + 29 = A(s^2 + 4s + 29) + Bs^2 + Cs$$

Equating coefficients :

s^0: $\quad 29 = 29A \longrightarrow A = 1$

s^1: $\quad 7 = 4A + C \longrightarrow C = 7 - 4A = 3$

s^2: $\quad 5 = A + B \longrightarrow B = 5 - A = 4$

$A = 1, \qquad B = 4, \qquad C = 3$

$$F(s) = \frac{1}{s} + \frac{4s + 3}{s^2 + 4s + 29} = \frac{1}{s} + \frac{4(s+2)}{(s+2)^2 + 5^2} - \frac{5}{(s+2)^2 + 5^2}$$

$$f(t) = \mathbf{u(t) + 4e^{-2t}\cos(5t) - e^{-2t}\sin(5t)}$$

Chapter 15, Solution 39.

(a) $$F(s) = \frac{2s^3 + 4s^2 + 1}{(s^2 + 2s + 17)(s^2 + 4s + 20)} = \frac{As + B}{s^2 + 2s + 17} + \frac{Cs + D}{s^2 + 4s + 20}$$

$$s^3 + 4s^2 + 1 = A(s^3 + 4s^2 + 20s) + B(s^2 + 4s + 20)$$
$$+ C(s^3 + 2s^2 + 17s) + D(s^2 + 2s + 17)$$

Equating coefficients :

s^3: $\quad 2 = A + C$

s^2: $\quad 4 = 4A + B + 2C + D$

s^1: $\quad 0 = 20A + 4B + 17C + 2D$

s^0: $\quad 1 = 20B + 17D$

Solving these equations (Matlab works well with 4 unknowns),

$A = -1.6, \quad B = -17.8, \quad C = 3.6, \quad D = 21$

$$F(s) = \frac{-1.6s - 17.8}{s^2 + 2s + 17} + \frac{3.6s + 21}{s^2 + 4s + 20}$$
$$F(s) = \frac{(-1.6)(s + 1)}{(s + 1)^2 + 4^2} + \frac{(-4.05)(4)}{(s + 1)^2 + 4^2} + \frac{(3.6)(s + 2)}{(s + 2)^2 + 4^2} + \frac{(3.45)(4)}{(s + 2)^2 + 4^2}$$

$f(t) =$

$\mathbf{[-1.6e^{-t}\cos(4t) - 4.05e^{-t}\sin(4t) + 3.6e^{-2t}\cos(4t) + 3.45e^{-2t}\sin(4t)]u(t)}$

(b) $$F(s) = \frac{s^2 + 4}{(s^2 + 9)(s^2 + 6s + 3)} = \frac{As + B}{s^2 + 9} + \frac{Cs + D}{s^2 + 6s + 3}$$

$$s^2 + 4 = A(s^3 + 6s^2 + 3s) + B(s^2 + 6s + 3) + C(s^3 + 9s) + D(s^2 + 9)$$

Equating coefficients :

s^3: $\quad 0 = A + C \longrightarrow C = -A$

s^2: $\quad 1 = 6A + B + D$

s^1: $\quad 0 = 3A + 6B + 9C = 6B + 6C \longrightarrow B = -C = A$

s^0: $\quad 4 = 3B + 9D$

Solving these equations,

$A = 1/12, \quad B = 1/12, \quad C = -1/12, \quad D = 5/12$

$$12F(s) = \frac{s + 1}{s^2 + 9} + \frac{-s + 5}{s^2 + 6s + 3}$$

$$s^2+6s+3=0 \longrightarrow \frac{-6\pm\sqrt{36-12}}{2} = -0.551, -5.449$$

Let $$G(s)=\frac{-s+5}{s^2+6s+3}=\frac{E}{s+0.551}+\frac{F}{s+5.449}$$

$$E=\frac{-s+5}{s+5.449}\bigg|_{s=-0.551}=1.133$$

$$F=\frac{-s+5}{s+0.551}\bigg|_{s=-5.449}=-2.133$$

$$G(s)=\frac{1.133}{s+0.551}-\frac{2.133}{s+5.449}$$

$$12\,F(s)=\frac{s}{s^2+3^2}+\frac{1}{3}\cdot\frac{3}{s^2+3^2}+\frac{1.133}{s+0.551}-\frac{2.133}{s+5.449}$$

$f(t)=$

$$\mathbf{[\,0.08333\cos(3t)+0.02778\sin(3t)+0.0944\,e^{-0.551t}-0.1778\,e^{-5.449t}\,]u(t)}$$

Chapter 15, Solution 40.

Let $H(s) = \left[\dfrac{4s^2 + 7s + 13}{(s+2)(s^2 + 2s + 5)}\right] = \dfrac{A}{s+2} + \dfrac{Bs + C}{s^2 + 2s + 5}$

$$4s^2 + 7s + 13 = A(s^2 + 2s + 5) + B(s^2 + 2s) + C(s + 2)$$

Equating coefficients gives:

s^2 : $\quad 4 = A + B$

s : $\quad 7 = 2A + 2B + C \longrightarrow C = -1$

constant : $\quad 13 = 5A + 2C \longrightarrow 5A = 15 \text{ or } A = 3,\ B = 1$

$$H(s) = \frac{3}{s+2} + \frac{s-1}{s^2 + 2s + 5} = \frac{3}{s+2} + \frac{(s+1) - 2}{(s+1)^2 + 2^2}$$

Hence,

$$h(t) = 3e^{-2t} + e^{-t}\cos 2t - e^{-t}\sin 2t = 3e^{-2t} + e^{-t}(A\cos\alpha\cos 2t - A\sin\alpha\sin 2t)$$

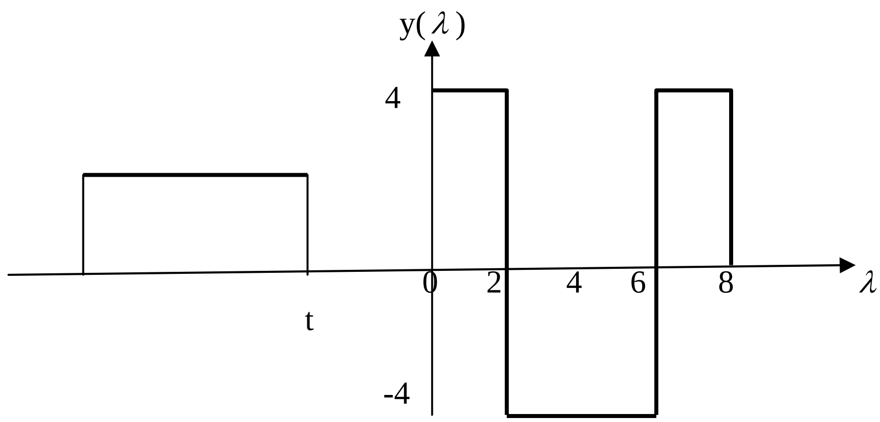
where $A\cos\alpha = 1, \quad A\sin\alpha = 1 \longrightarrow A = \sqrt{2}, \quad \alpha = 45^o$

Thus,

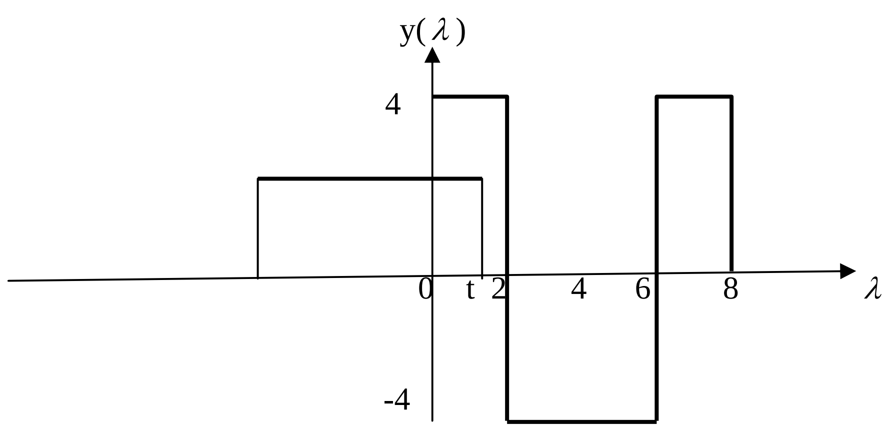
$$\mathbf{h(t) = \underline{\left[\sqrt{2}e^{-t}\cos(2t + 45^o) + 3e^{-2t}\right]u(t)}}$$

Chapter 15, Solution 41.

We fold x(t) and slide on y(t). For t<0, no overlapping as shown below. x(t) =0.

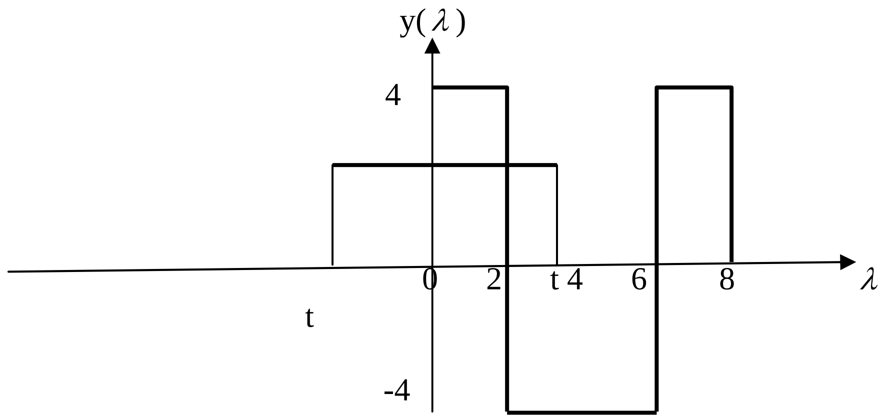


For 0 < t < 2, there is overlapping, as shown below.



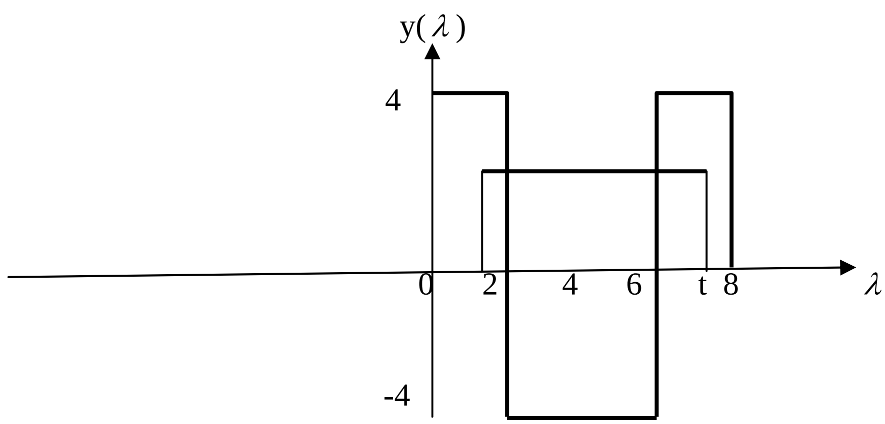
$$z(t) = \int_0^t (2)(4)dt = 8t$$

For 2 < t < 6, the two functions overlap, as shown below.



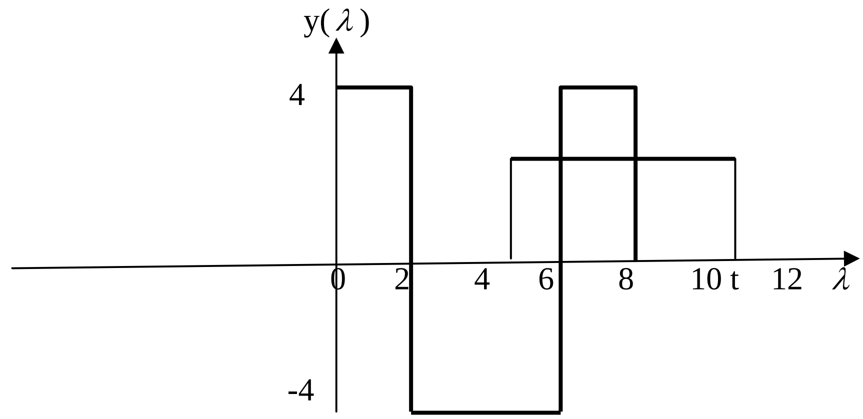
$$z(t)=\int_{0}^{2}(2)(4)d\lambda+\int_{0}^{t}(2)(-4)d\lambda=16-8t$$

For 6<t<8, they overlap as shown below.



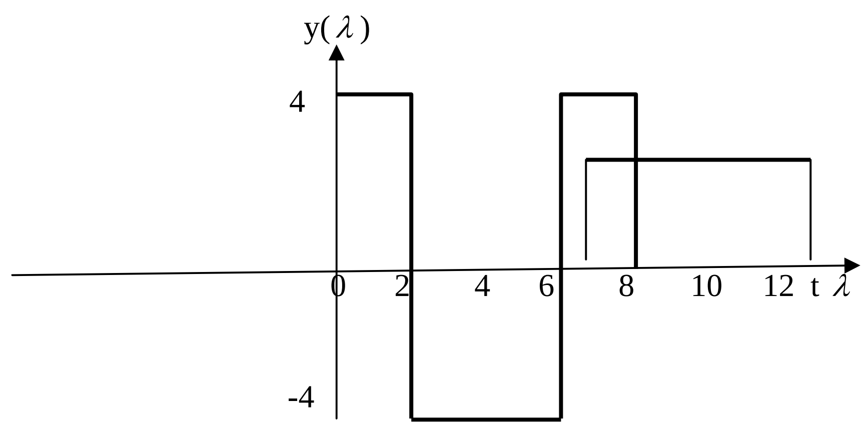
$$z(t)=\int_{t-6}^{2}(2)(4)d\lambda+\int_{2}^{6}(2)(-4)d\lambda+\int_{6}^{t}(2)(4)d\lambda=8\lambda\Big|_{t-6}^{2}-8\lambda\Big|_{2}^{6}+8\lambda\Big|_{6}^{t}=-16$$

For 8< t <12, they overlap as shown below.



$$z(t) = \int_{t-6}^{6} (2)(-4)d\lambda + \int_{6}^{8} (2)(4)d\lambda = -8\lambda \Big|_{t-6}^{6} + 8\lambda \Big|_{6}^{8} = 8t - 80$$

For 12 < t < 14, they overlap as shown below.



$$z(t) = \int_{t-6}^{8} (2)(4)d\lambda = 8\lambda \Big|_{t-6}^{8} = 112 - 8t$$

Hence,

$$z(t) = \begin{cases} \mathbf{8t,} & \mathbf{0<t<2} \\ \mathbf{16-8t,} & \mathbf{2<t<6} \\ \mathbf{-16,} & \mathbf{6<t<8} \\ \mathbf{8t-80,} & \mathbf{8<t<12} \\ \mathbf{112-8t,} & \mathbf{12<t<14} \\ \mathbf{0,} & \textbf{otherwise}. \end{cases}$$

Chapter 15, Solution 42.

Design a problem to help other students to better understand how to convolve two functions together.

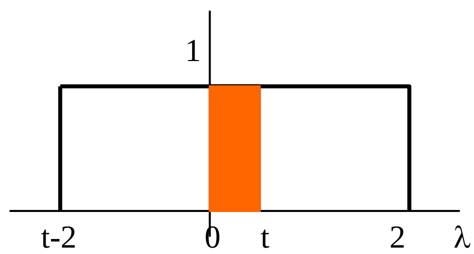
Although there are many ways to solve this problem, this is an example based on the same kind of problem asked in the third edition.

Problem

Suppose that f(t) = u(t) – u(t-2). Determine f(t)*f(t).

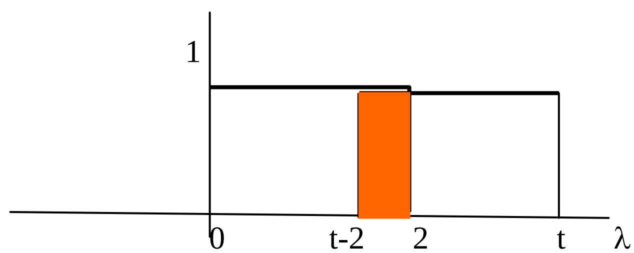
Solution

For 0<t<2, the signals overlap as shown below.



$$y(t) = f(t) * f(t) = \int_0^t (1)(1)d\lambda = t$$

For 2 < t< 4, they overlap as shown below.



$$y(t) = \int_{t-2}^{2} (1)(1)d\lambda = t\Big|_{t-2}^{2} = 4 - t$$

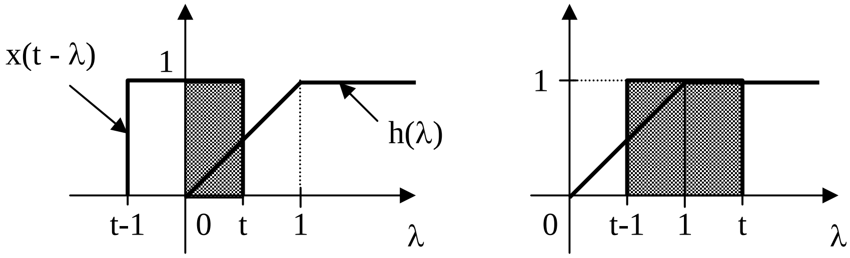
Thus,

$$y(t) = \begin{cases} t, & 0 < t < 2 \\ 4 - t, & 2 < t < 4 \\ 0, & \text{otherwise} \end{cases}$$

Chapter 15, Solution 43.

(a) For $0 < t < 1$, $x(t-\lambda)$ and $h(\lambda)$ overlap as shown in Fig. (a).

$$y(t) = x(t) * h(t) = \int_0^t (1)(\lambda)\, d\lambda = \frac{\lambda^2}{2}\Big|_0^t = \frac{t^2}{2}$$



(a) **(b)**

For $1 < t < 2$, $x(t-\lambda)$ and $h(\lambda)$ overlap as shown in Fig. (b).

$$y(t) = \int_{t-1}^1 (1)(\lambda)\, d\lambda + \int_1^t (1)(1)\, d\lambda = \frac{\lambda^2}{2}\Big|_{t-1}^1 + \lambda\Big|_1^t = \frac{-1}{2}t^2 + 2t - 1$$

For $t > 2$, there is a complete overlap so that

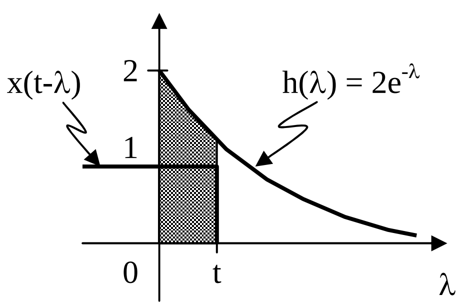
$$y(t) = \int_{t-1}^t (1)(1)\, d\lambda = \lambda\Big|_{t-1}^t = t - (t-1) = 1$$

Therefore,

$$y(t) = \begin{cases} \mathbf{t^2/2,} & \mathbf{0 < t < 1} \\ \mathbf{-(t^2/2) + 2t - 1,} & \mathbf{1 < t < 2} \\ \mathbf{1,} & \mathbf{t > 2} \\ \mathbf{0,} & \mathbf{otherwise} \end{cases}$$

(b) For $t > 0$, the two functions overlap as shown in Fig. (c).

$$y(t) = x(t) * h(t) = \int_0^t (1)\, 2e^{-\lambda}\, d\lambda = -2e^{-\lambda}\Big|_0^t$$



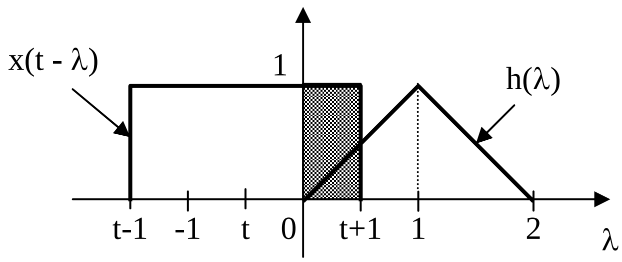
(c)

Therefore,

$$y(t) = \mathbf{2(1 - e^{-t}), \quad t > 0}$$

(c) For $-1 < t < 0$, $x(t-\lambda)$ and $h(\lambda)$ overlap as shown in Fig. (d).

$$y(t) = x(t) * h(t) = \int_0^{t+1} (1)(\lambda)\, d\lambda = \frac{\lambda^2}{2}\Big|_0^{t+1} = \frac{1}{2}(t+1)^2$$

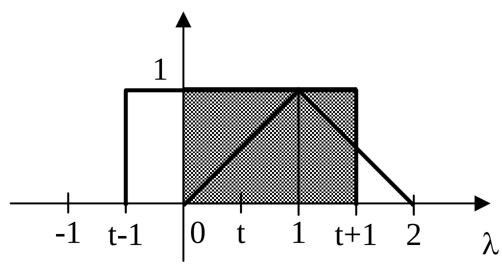


(d)

For $0 < t < 1$, $x(t-\lambda)$ and $h(\lambda)$ overlap as shown in Fig. (e).

$$y(t) = \int_0^1 (1)(\lambda)\, d\lambda + \int_1^{t+1} (1)(2-\lambda)\, d\lambda$$

$$y(t) = \frac{\lambda^2}{2}\Big|_0^1 + \left(2\lambda - \frac{\lambda^2}{2}\right)\Big|_1^{t+1} = \frac{-1}{2}t^2 + t + \frac{1}{2}$$

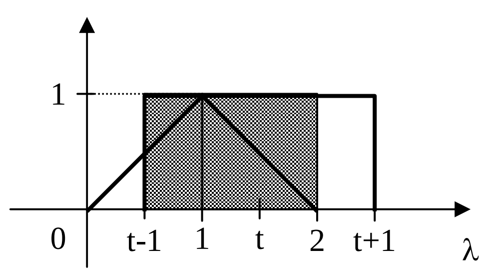


(e)

For $1 < t < 2$, $x(t-\lambda)$ and $h(\lambda)$ overlap as shown in Fig. (f).

$$y(t) = \int_{t-1}^1 (1)(\lambda)\, d\lambda + \int_1^2 (1)(2-\lambda)\, d\lambda$$

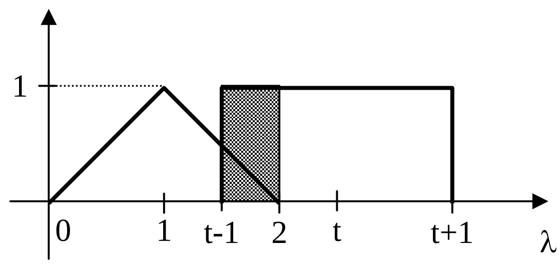
$$y(t) = \frac{\lambda^2}{2}\Big|_{t-1}^1 + \left(2\lambda - \frac{\lambda^2}{2}\right)\Big|_1^2 = \frac{-1}{2}t^2 + t + \frac{1}{2}$$



(f)

For $2 < t < 3$, $x(t-\lambda)$ and $h(\lambda)$ overlap as shown in Fig. (g).

$$y(t) = \int_{t-1}^{2} (1)(2-\lambda)\, d\lambda = \left(2\lambda - \frac{\lambda^2}{2}\right)\Big|_{t-1}^{2} = \frac{9}{2} - 3t + \frac{1}{2}t^2$$



(g)

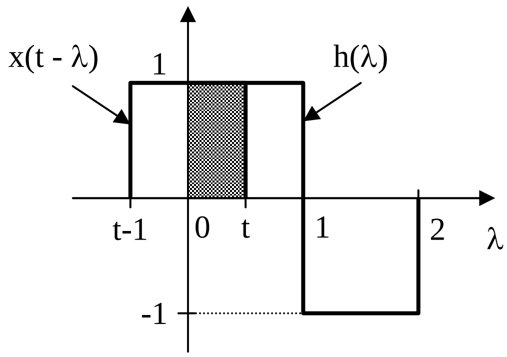
Therefore,

$$y(t) = \begin{cases} \mathbf{(t^2/2) + t + 1/2,} & \mathbf{-1 < t < 0} \\ \mathbf{-(t^2/2) + t + 1/2,} & \mathbf{0 < t < 2} \\ \mathbf{(t^2/2) - 3t + 9/2,} & \mathbf{2 < t < 3} \\ \mathbf{0,} & \mathbf{otherwise} \end{cases}$$

Chapter 15, Solution 44.

(a) For $0 < t < 1$, $x(t-\lambda)$ and $h(\lambda)$ overlap as shown in Fig. (a).

$$y(t) = x(t) * h(t) = \int_0^t (1)(1)\, d\lambda = t$$



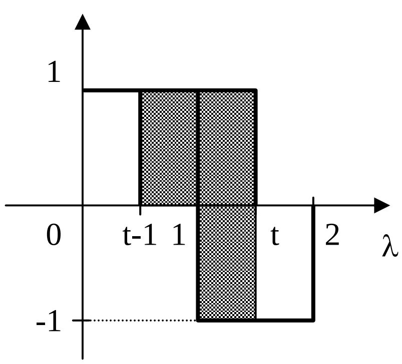
(a)

For $1 < t < 2$, $x(t-\lambda)$ and $h(\lambda)$ overlap as shown in Fig. (b).

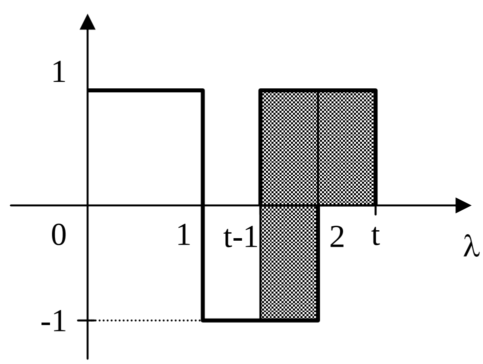
$$y(t) = \int_{t-1}^{1} (1)(1)\, d\lambda + \int_1^t (-1)(1)\, d\lambda = \lambda\Big|_{t-1}^{1} - \lambda\Big|_1^t = 3 - 2t$$

For $2 < t < 3$, $x(t-\lambda)$ and $h(\lambda)$ overlap as shown in Fig. (c).

$$y(t) = \int_{t-1}^{2} (1)(-1)\, d\lambda = -\lambda\Big|_{t-1}^{2} = t - 3$$



(b)



(c)

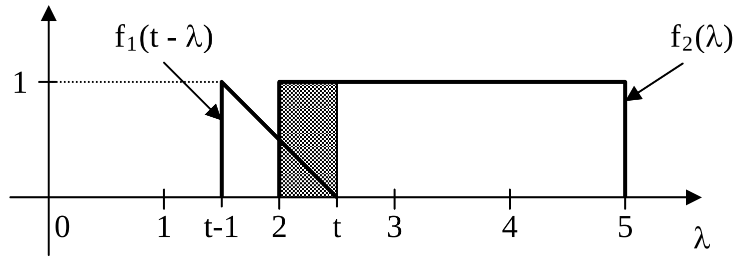
Therefore,

$$y(t) = \begin{cases} \mathbf{t}, & \mathbf{0 < t < 1} \\ \mathbf{3 - 2t}, & \mathbf{1 < t < 2} \\ \mathbf{t - 3}, & \mathbf{2 < t < 3} \\ \mathbf{0}, & \textbf{otherwise} \end{cases}$$

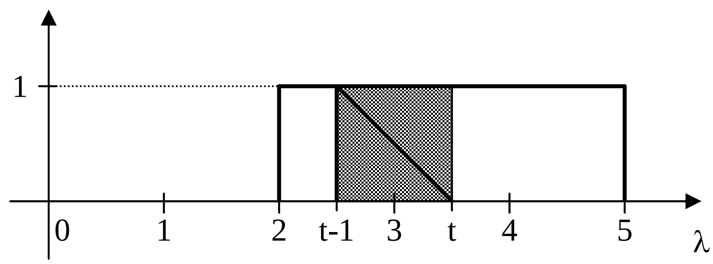
(b) For $t < 2$, there is no overlap. For $2 < t < 3$, $f_1(t-\lambda)$ and $f_2(\lambda)$ overlap, as shown in Fig. (d).

$$y(t) = f_1(t) * f_2(t) = \int_2^t (1)(t-\lambda)\, d\lambda$$

$$= \left(\lambda t - \frac{\lambda^2}{2} \right) \Big|_2^t = \frac{t^2}{2} - 2t + 2$$



(d)



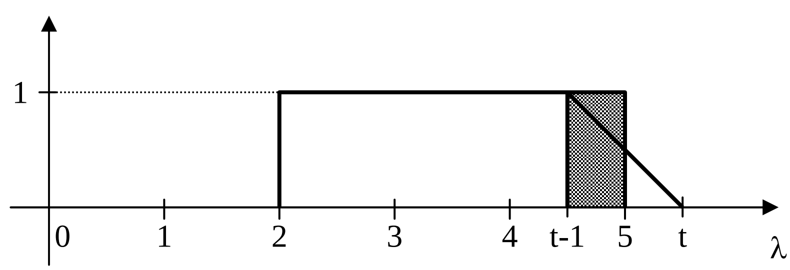
(e)

For $3 < t < 5$, $f_1(t-\lambda)$ and $f_2(\lambda)$ overlap as shown in Fig. (e).

$$y(t) = \int_{t-1}^t (1)(t-\lambda)\, d\lambda = \left(\lambda t - \frac{\lambda^2}{2} \right) \Big|_{t-1}^t = \frac{1}{2}$$

For $5 < t < 6$, the functions overlap as shown in Fig. (f).

$$y(t) = \int_{t-1}^5 (1)(t-\lambda)\, d\lambda = \left(\lambda t - \frac{\lambda^2}{2} \right) \Big|_{t-1}^5 = \frac{-1}{2} t^2 + 5t - 12$$



(f)

Therefore,

$$y(t) = \begin{cases} \mathbf{(t^2/2) - 2t + 2,} & \mathbf{2 < t < 3} \\ \mathbf{1/2,} & \mathbf{3 < t < 5} \\ \mathbf{-(t^2/2) + 5t - 12,} & \mathbf{5 < t < 6} \\ \mathbf{0,} & \mathbf{otherwise} \end{cases}$$

Chapter 15, Solution 45.

$$y(t) = h(t) * x(t) = \left[4e^{-2t}u(t)\right] * \left[\delta(t) - 2e^{-2t}u(t)\right]$$

$$= 4e^{-2t}u(t) * \delta(t) - 4e^{-2t}u(t) * 2e^{-2t}u(t) = 4e^{-2t}u(t) - 8e^{-2t}\int_0^t e^o d\lambda$$

$$= \underline{4e^{-2t}u(t) - 8te^{-2t}u(t)}$$

Chapter 15, Solution 46.

(a) $x(t)*y(t) = 2\delta(t)*4u(t) = \underline{8u(t)}$

(b) $x(t)*z(t) = 2\delta(t)*e^{-2t}u(t) = \underline{2e^{-2t}u(t)}$

(c) $y(t)*z(t) = 4u(t)*e^{-2t}u(t) = 4\int_0^t e^{-2\lambda}d\lambda = \left.\frac{4e^{-2\lambda}}{-2}\right|_0^t = \underline{2(1-e^{-2t})}$

(d) $y(t)*[y(t)+z(t)] = 4u(t)*[4u(t)+e^{-2t}u(t)] = 4\int[4u(\lambda)+e^{-2\lambda}u(\lambda)]d\lambda$

$$= 4\int_0^t [4+e^{-2\lambda}]d\lambda = 4[4t+\left.\frac{e^{-2\lambda}}{-2}]\right|_0^t = \underline{16t-2e^{-2t}+2}$$

Chapter 15, Solution 47.

(a) $H(s) = \dfrac{s}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

s=A(s+2) + B(s+1)

We equate the coefficients.

s: 1= A+B

constant: 0 =2A +B

Solving these, A = -1, B= 2.

$$H(s) = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$h(t) = \underline{(-e^{-t} + 2e^{-2t})u(t)}$$

(b) $H(s) = \dfrac{Y(s)}{X(s)} \longrightarrow Y(s) = H(s)X(s) = \dfrac{s}{(s+1)(s+2)}\dfrac{1}{s}$

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{C}{s+1} + \frac{D}{s+2}$$

C=1 and D=-1 so that

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$y(t) = \underline{(e^{-t} - e^{-2t})u(t)}$$

Chapter 15, Solution 48.

(a) Let $G(s) = \dfrac{2}{s^2 + 2s + 5} = \dfrac{2}{(s+1)^2 + 2^2}$

$$g(t) = e^{-t}\sin(2t)$$

$$F(s) = G(s)\,G(s)$$

$$f(t) = \mathsf{L}^{-1}\left[\,G(s)\,G(s)\right] = \int_0^t g(\lambda)\,g(t-\lambda)\,d\lambda$$

$$f(t) = \int_0^t e^{-\lambda}\sin(2\lambda)\,e^{-(t-\lambda)}\sin(2(t-\lambda))\,d\lambda$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\,\cos(A-B) - \cos(A+B)\right]$$

$$f(t) = \frac{1}{2}e^{-t}\int_0^t e^{-\lambda}\left[\,\cos(2t) - \cos(2(t-2\lambda))\right]d\lambda$$

$$f(t) = \frac{e^{-t}}{2}\cos(2t)\int_0^t e^{-2\lambda}\,d\lambda - \frac{e^{-t}}{2}\int_0^t e^{-2\lambda}\cos(2t-4\lambda)\,d\lambda$$

$$f(t) = \frac{e^{-t}}{2}\cos(2t)\cdot\frac{e^{-2\lambda}}{-2}\Big|_0^t - \frac{e^{-t}}{2}\int_0^t e^{-2\lambda}\left[\,\cos(2t)\cos(4\lambda) + \sin(2t)\sin(4\lambda)\right]d\lambda$$

$$f(t) = \frac{1}{4}e^{-t}\cos(2t)\,(-e^{-2t}+1) - \frac{e^{-t}}{2}\cos(2t)\int_0^t e^{-2\lambda}\cos(4\lambda)\,d\lambda$$
$$-\frac{e^{-t}}{2}\sin(2t)\int_0^t e^{-2\lambda}\sin(4\lambda)\,d\lambda$$

$$f(t) = \frac{1}{4}e^{-t}\cos(2t)\,(1-e^{-2t})$$
$$-\frac{e^{-t}}{2}\cos(2t)\left[\,\frac{e^{-2\lambda}}{4+16}\left(-2\cos(4\lambda) - 4\sin(4\lambda)\right)\right]\Big|_0^t$$
$$-\frac{e^{-t}}{2}\sin(2t)\left[\,\frac{e^{-2\lambda}}{4+16}\left(-2\sin(4\lambda) + 4\cos(4\lambda)\right)\right]\Big|_0^t$$

$$f(t)=\frac{e^{-t}}{2}\cos(2t)-\frac{e^{-3t}}{4}\cos(2t)-\frac{e^{-t}}{20}\cos(2t)+\frac{e^{-3t}}{20}\cos(2t)\cos(4t)$$
$$+\frac{e^{-3t}}{10}\cos(2t)\sin(4t)+\frac{e^{-t}}{10}\sin(2t)$$
$$+\frac{e^{-t}}{20}\sin(2t)\sin(4t)-\frac{e^{-t}}{10}\sin(2t)\cos(4t)$$

(b) Let $X(s)=\dfrac{2}{s+1}$, $\qquad Y(s)=\dfrac{s}{s+4}$

$x(t)=2\,e^{-t}\,u(t)$, $\qquad y(t)=\cos(2t)\,u(t)$

$F(s)=X(s)\,Y(s)$

$$f(t)=L^{-1}\left[\,X(s)\,Y(s)\right]=\int_0^{\infty}y(\lambda)\,x(t-\lambda)\,d\lambda$$
$$f(t)=\int_0^{t}\cos(2\lambda)\cdot 2\,e^{-(t-\lambda)}\,d\lambda$$
$$f(t)=2\,e^{-t}\cdot\frac{e^{\lambda}}{1+4}\left(\cos(2\lambda)+2\sin(2\lambda)\right)\Big|_0^t$$
$$f(t)=\frac{2}{5}e^{-t}\left[\,e^{t}\left(\cos(2t)+2\sin(2t)-1\right)\right]$$
$$f(t)=\frac{2}{5}\cos(2t)+\frac{4}{5}\sin(2t)-\frac{2}{5}e^{-t}$$

Chapter 15, Solution 49.

(a) $t*e^{\alpha t}u(t) =$

$$\int_0^t e^{a\lambda}(t-\lambda)d\lambda = t\frac{e^{a\lambda}}{a}\Bigg|_0^t - \frac{e^{a\lambda}}{a^2}(a\lambda-1)\Bigg|_0^t = \underline{\frac{t}{a}(e^{at}-1) - \frac{1}{a^2} - \frac{e^{at}}{a^2}(at-1)u(t)}$$

(b) $$\cos t * \cos tu(t) = \int_0^t \cos\lambda\cos(t-\lambda)d\lambda = \int_0^t \{\cos t\cos\lambda\cos\lambda + \sin t\sin\lambda\cos\lambda\}d\lambda$$

$$= \left[\cos t\int_0^t \frac{1}{2}[1+\cos 2\lambda]d\lambda + \sin t\int_0^t \cos\lambda d(-\cos\lambda)\right] = \left[\frac{1}{2}\cos t[\lambda + \frac{\sin 2\lambda}{2}]\Bigg|_0^t - \sin t\frac{\cos\lambda}{2}\Bigg|_0^t\right]$$

=**[0.5cos(t)(t+0.5sin(2t)) – 0.5sin(t)(cos(t) – 1)]u(t)**.

Chapter 15, Solution 50.

Take the Laplace transform of each term.

$$\left[s^2 V(s) - s\,v(0) - v'(0)\right] + 2\left[s\,V(s) - v(0)\right] + 10\,V(s) = \frac{3s}{s^2+4}$$

$$s^2 V(s) - s + 2 + 2s\,V(s) - 2 + 10\,V(s) = \frac{3s}{s^2+4}$$

$$(s^2 + 2s + 10)\,V(s) = s + \frac{3s}{s^2+4} = \frac{s^3+7s}{s^2+4}$$

$$V(s) = \frac{s^3+7s}{(s^2+4)(s^2+2s+10)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+2s+10}$$

$$s^3 + 7s = A\,(s^3 + 2s^2 + 10s) + B\,(s^2 + 2s + 10) + C\,(s^3 + 4s) + D\,(s^2 + 4)$$

Equating coefficients :

$$s^3: \quad 1 = A + C \longrightarrow C = 1 - A$$

$$s^2: \quad 0 = 2A + B + D$$

$$s^1: \quad 7 = 10A + 2B + 4C = 6A + 2B + 4$$

$$s^0: \quad 0 = 10B + 4D \longrightarrow D = \text{-}2.5B$$

Solving these equations yields

$$A = \frac{9}{26}, \qquad B = \frac{12}{26}, \qquad C = \frac{17}{26}, \qquad D = \frac{-30}{26}$$

$$V(s) = \frac{1}{26}\left[\frac{9s+12}{s^2+4} + \frac{17s-30}{s^2+2s+10}\right]$$

$$V(s) = \frac{1}{26}\left[\frac{9s}{s^2+4} + 6\cdot\frac{2}{s^2+4} + 17\cdot\frac{s+1}{(s+1)^2+3^2} - \frac{47}{(s+1)^2+3^2}\right]$$

$$v(t) = \mathbf{\frac{9}{26}\cos(2t) + \frac{6}{26}\sin(2t) + \frac{17}{26}e^{-t}\cos(3t) - \frac{47}{78}e^{-t}\sin(3t)}$$

Chapter 15, Solution 51.

Taking the Laplace transform of the differential equation yields

$$\left[s^2 V(s) - sv(0) - v'(0)\right] + 5[sV(s) - v(0)] + 6V(s) = \frac{10}{s+1}$$

or $\left(s^2 + 5s + 6\right)V(s) - 2s - 4 - 10 = \dfrac{10}{s+1} \quad \longrightarrow \quad V(s) = \dfrac{2s^2 + 16s + 24}{(s+1)(s+2)(s+3)}$

Let $V(s) = \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{s+3}$, $\quad A = 5, \quad B = 0, \quad C = -3$

Hence,

$$v(t) = \mathbf{\left(5e^{-t} - 3e^{-3t}\right) u(t)}.$$

Chapter 15, Solution 52.

Take the Laplace transform of each term.

$$\left[s^2 \, I(s) - s\,i(0) - i'(0) \right] + 3\left[s\,I(s) - i(0) \right] + 2\,I(s) + 1 = 0$$

$$(s^2 + 3s + 2)\,I(s) - s - 3 - 3 + 1 = 0$$

$$I(s) = \frac{s+5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = 4, \qquad B = -3$$

$$I(s) = \frac{4}{s+1} - \frac{3}{s+2}$$

$$i(t) = \mathbf{(4e^{-t} - 3e^{-2t})u(t)}$$

Chapter 15, Solution 53.

Transform each term.

We begin by noting that the integral term can be rewritten as,

$\int_0^t x(\lambda)e^{-(t-\lambda)}d\lambda$ which is convolution and can be written as $e^{-t}*x(t)$.

Now, transforming each term produces,

$$X(s) = \frac{s}{s^2+1} + \frac{1}{s+1}X(s) \rightarrow \left(\frac{s+1-1}{s+1}\right)X(s) = \frac{s}{s^2+1}$$

$$X(s) = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

x(t) = **cos(t) + sin(t)**.

If partial fraction expansion is used we obtain,

x(t) = **1.4142cos(t–45˚)**.

This is the same answer and can be proven by using trigonometric identities.

Chapter 15, Solution 54.

Design a problem to help other students to better understand solving second order differential equations with a time varying input.

Although there are many ways to solve this problem, this is an example based on the same kind of problem asked in the third edition.

Problem

Using Laplace transform, solve the following differential equation for t >0

$$\frac{d^2 i}{dt^2} + 4\frac{di}{dt} + 5i = 2e^{-2t}$$

subject to i(0)=0, i'(0)=2.

Solution

Taking the Laplace transform of each term gives

$$\left[s^2 I(s) - si(0) - i'(0)\right] + 4\left[sI(s) - i(0)\right] + 5I(s) = \frac{2}{s+2}$$

$$\left[s^2 I(s) - 0 - 2\right] + 4\left[sI(s) - 0\right] + 5I(s) = \frac{2}{s+2}$$

$$I(s)(s^2 + 4s + 5) = \frac{2}{s+2} + 2 = \frac{2s+6}{s+2}$$

$$I(s) = \frac{2s+6}{(s+2)(s^2+4s+5)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4s+5}$$

$$2s + 6 = A(s^2 + 4s + 5) + B(s^2 + 2s) + C(s+2)$$

We equate the coefficients.

s^2 : 0 = A+ B
s: 2= 4A + 2B + C
constant: 6 = 5A + 2C

Solving these gives
A = 2, B= -2, C = -2

$$I(s) = \frac{2}{s+2} - \frac{2s+2}{s^2+4s+5} = \frac{2}{s+2} - \frac{2(s+2)}{(s+2)^2+1} + \frac{2}{(s+2)^2+1}$$

Taking the inverse Laplace transform leads to:

$$i(t) = \left(2e^{-2t} - 2e^{-2t}\cos t + 2e^{-2t}\sin t\right)u(t) = \underline{2e^{-2t}(1 - \cos t + \sin t)u(t)}$$

Chapter 15, Solution 55.

Take the Laplace transform of each term.

$$\left[s^3\,Y(s) - s^2\,y(0) - s\,y'(0) - y''(0)\right] + 6\left[s^2\,Y(s) - s\,y(0) - y'(0)\right]$$
$$+ 8\left[s\,Y(s) - y(0)\right] = \frac{s+1}{(s+1)^2 + 2^2}$$

Setting the initial conditions to zero gives

$$(s^3 + 6s^2 + 8s)\,Y(s) = \frac{s+1}{s^2 + 2s + 5}$$

$$Y(s) = \frac{(s+1)}{s\,(s+2)(s+4)(s^2+2s+5)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4} + \frac{Ds+E}{s^2+2s+5}$$

$$A = \frac{1}{40}, \qquad B = \frac{1}{20}, \qquad C = \frac{-3}{104}, \qquad D = \frac{-3}{65}, \qquad E = \frac{-7}{65}$$

$$Y(s) = \frac{1}{40}\cdot\frac{1}{s} + \frac{1}{20}\cdot\frac{1}{s+2} - \frac{3}{104}\cdot\frac{1}{s+4} - \frac{1}{65}\cdot\frac{3s+7}{(s+1)^2+2^2}$$

$$Y(s) = \frac{1}{40}\cdot\frac{1}{s} + \frac{1}{20}\cdot\frac{1}{s+2} - \frac{3}{104}\cdot\frac{1}{s+4} - \frac{1}{65}\cdot\frac{3(s+1)}{(s+1)^2+2^2} - \frac{1}{65}\cdot\frac{4}{(s+1)^2+2^2}$$

$$y(t) = \left(\mathbf{\frac{1}{40} + \frac{1}{20}e^{-2t} - \frac{3}{104}e^{-4t} - \frac{3}{65}e^{-t}\cos(2t) - \frac{2}{65}e^{-t}\sin(2t)}\right)\mathbf{u(t)}$$

Chapter 15, Solution 56.

Taking the Laplace transform of each term we get:

$$4\left[s\,V(s) - v(0) \right] + \frac{12}{s} V(s) = 0$$

$$\left[4s + \frac{12}{s} \right] V(s) = 8$$

$$V(s) = \frac{8s}{4s^2 + 12} = \frac{2s}{s^2 + 3}$$

$$v(t) = \mathbf{2\cos\left(\sqrt{3}t\right)}$$

Chapter 15, Solution 57.

Although there is no correct way to work this problem, this is an example based on the same kind of problem asked in the third edition.

Problem

Solve the following integrodifferential equation using the Laplace transform method:

$$\frac{dy(t)}{dt} + 9\int_0^t y(t)dt = \cos 2t, \qquad y(0) = 1$$

Solution

Take the Laplace transform of each term.

$$\left[\, s\,Y(s) - y(0)\right] + \frac{9}{s}Y(s) = \frac{s}{s^2+4}$$

$$\left(\frac{s^2+9}{s}\right)Y(s) = 1 + \frac{s}{s^2+4} = \frac{s^2+s+4}{s^2+4}$$

$$Y(s) = \frac{s^3+s^2+4s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$s^3+s^2+4s = A\,(s^3+9s) + B\,(s^2+9) + C\,(s^3+4s) + D\,(s^2+4)$$

Equating coefficients :

s^0 : $\quad 0 = 9B + 4D$

s^1 : $\quad 4 = 9A + 4C$

s^2 : $\quad 1 = B + D$

s^3 : $\quad 1 = A + C$

Solving these equations gives

$$A = 0\,, \qquad B = -4/5\,, \qquad C = 1\,, \qquad D = 9/5$$

$$Y(s) = \frac{-4/5}{s^2+4} + \frac{s+9/5}{s^2+9} = \frac{-4/5}{s^2+4} + \frac{s}{s^2+9} + \frac{9/5}{s^2+9}$$

$$y(t) = [\,\mathbf{-0.4\sin(2t) + \cos(3t) + 0.6\sin(3t)\,]u(t)}$$

Chapter 15, Solution 58.

We take the Laplace transform of each term.

$$[sV(s) - v(0)] + 2V(s) + \frac{5}{s}V(s) = \frac{4}{s}$$

$$[sV(s) + 1] + 2V(s) + \frac{5}{s}V(s) = \frac{4}{s} \quad \longrightarrow \quad V(s) = \frac{4 - s}{s^2 + 2s + 5}$$

$$V(s) = \frac{-(s+1) + 5}{(s+1)^2 + 2^2} = \frac{-(s+1)}{(s+1)^2 + 2^2} + \frac{5}{2}\frac{2}{(s+1)^2 + 2^2}$$

$$v(t) = \underline{(-e^{-t}\cos 2t + 2.5e^{-t}\sin 2t)u(t)}$$

Chapter 15, Solution 59.

Take the Laplace transform of each term of the integrodifferential equation.

$$\left[s\,Y(s) - y(0) \right] + 4\,Y(s) + \frac{3}{s}Y(s) = \frac{6}{s+2}$$

$$(s^2 + 4s + 3)\,Y(s) = s\left(\frac{6}{s+2} - 1\right)$$

$$Y(s) = \frac{s\,(4-s)}{(s+2)(s^2+4s+3)} = \frac{(4-s)\,s}{(s+1)(s+2)(s+3)}$$

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$A = –2.5$, $B = 12$, $C = -10.5$

$$Y(s) = \frac{-2.5}{s+1} + \frac{12}{s+2} - \frac{10.5}{s+3}$$

$$y(t) = \mathbf{[-2.5e^{-t} + 12e^{-2t} - 10.5e^{-3t}]u(t)}$$

Chapter 15, Solution 60.

Take the Laplace transform of each term of the integrodifferential equation.

$$2\left[\,s\,X(s) - x(0)\right] + 5\,X(s) + \frac{3}{s}X(s) + \frac{4}{s} = \frac{4}{s^2+16}$$

$$(2s^2+5s+3)\,X(s) = 2s - 4 + \frac{4s}{s^2+16} = \frac{2s^3-4s^2+36s-64}{s^2+16}$$

$$X(s) = \frac{2s^3-4s^2+36s-64}{(2s^2+5s+3)(s^2+16)} = \frac{s^3-2s^2+18s-32}{(s+1)(s+1.5)(s^2+16)}$$

$$X(s) = \frac{A}{s+1} + \frac{B}{s+1.5} + \frac{Cs+D}{s^2+16}$$

$$A = (s+1)\,X(s)\big|_{s=-1} = -6.235$$

$$B = (s+1.5)\,X(s)\big|_{s=-1.5} = 7.329$$

When $s = 0$,

$$\frac{-32}{(1.5)(16)} = A + \frac{B}{1.5} + \frac{D}{16} \quad \longrightarrow \quad D = 0.2579$$

$$s^3-2s^2+18s-32 = A\,(s^3+1.5s^2+16s+24) + B\,(s^3+s^2+16s+16)$$
$$+\,C\,(s^3+2.5s^2+1.5s) + D\,(s^2+2.5s+1.5)$$

Equating coefficients of the s^3 terms,

$$1 = A + B + C \quad \longrightarrow \quad C = -0.0935$$

$$X(s) = \frac{-6.235}{s+1} + \frac{7.329}{s+1.5} + \frac{-0.0935s+0.2579}{s^2+16}$$

$$x(t) = \mathbf{-6.235\,e^{-t} + 7.329\,e^{-1.5t} - 0.0935\cos(4t) + 0.0645\sin(4t)}$$

Chapter 16, Solution 61.

Solve the following differential equations subject to the specified initial conditions

(a) $d^2v/dt^2 + 4v = 12$, $v(0) = 0$, $dv(0)/dt = 2$

(b) $d^2i/dt^2 + 5di/dt + 4i = 8$, $i(0) = -1$, $di(0)/dt = 0$

(c) $d^2v/dt^2 + 2dv/dt + v = 3$, $v(0) = 5$, $dv(0)/dt = 1$

(d) $d^2i/dt^2 + 2di/dt + 5i = 10$, $i(0) = 4$, $di(0)/dt = -2$

8.29

Solution

(a) Converting into the s-domain we get

$s^2V(s)–sv(0^–)–v'(0^–)+4V(s) = 12/s = s^2V(s)–s0–2+4V(s)$ or
$(s^2+4)V(s) = 2+12/s = 2(s+6)/s$ or $V(s) = 2(s+6)/[s(s+j2)(s–j2)]$
$= [A/s] + [B/(s+j2)] + [C/(s–j2)]$ where $A = 12/4 = 3$, $B = 2(–j2+6)/[–j2(–j4)]$
$= 2(6.325\angle–18.43^\circ)/(8\angle180^\circ) = 1.5812\angle161.57^\circ$ and C
$= s(j2+6)/[j2(j4)] = 2(6.325\angle18.43^\circ)/(8\angle180^\circ) = 1.5812\angle–161.57^\circ$

$v(t) = [3+1.5812e^{161.57^\circ}e^{–j2t}+1.\ 5812e^{–161.57^\circ}e^{j2t}]u(t)$ volts
= **$[3+3.162\cos(2t–161.12^\circ)]u(t)$ volts**.

(b) Converting into the s-domain we get

$s^2I(s)–si(0^–)–i'(0^–)+5sI(s)–5i(0^–)+4I(s) = 8/s$
$= s^2I(s)–s(–1)–0+5sI(s)–5(–1)+4I(s)$ or
$(s^2+5s+4)I(s) = (–s–5) + 8/s = –(s^2+5s–8)/s$
$I(s) = –(s^2+5s–8)/[s(s+1)(s+4)] = [A/s] + [B/(s+1)] + [C/(s+4)]$ where
$A = 8/[(1)(4)] = 2$; $B = –[(–1)^2+5(–1)–8]/[(–1)(–1+4)] = 12/(–3) = –4$;
$C = –[(–4)^2+5(–4)–8]/[(–4)(–4+1)] = 12/(12) = 1$ therefore

$i(t) =$ **$[2–4e^{–t}+e^{–4t}]u(t)$ amps**.

(c) $s^2V(s)–sv(0^–)–v'(0^–)+2sV(s)–2v(0^–)+V(s) = 3/s$
$= s^2V(s)–s5–1+2sV(s)–2x5+V(s) = (s^2+2s+1)V(s) – (5s+11)$ or
$(s^2+2s+1)V(s) = (5s+11) + 3/s = (5s^2+11s+3)/s$ or
$V(s) = (5s^2+11s+3)/[s(s+1)^2] = [A/s] + [B/(s+1)] + [C/(s+1)^2]$ where
$A = 3$; $C = [5(–1)^2+11(–1)+3]/(–1) = (–3)/(–1) = 3$; going back to the original and eliminating the denominators we get $5s^2+11s+3 = 3(s^2+2s+1)+Bs^2+Bs+3s$ or
$B = 2$, thus,
$v(t) =$ **$[3+2e^{–t}+3te^{–t}]u(t)$ volts**.

(d) $s^2I(s)–si(0^-)–i'(0^-)+2sI(s)–2i(0^-)+5I(s) = 10/s$

$= s^2I(s)–s(4)–(–2)+2sI(s)–2(4)+5I(s)$ or
$(s^2+2s+5)I(s) – (4s–2+8) = 10/s$ or $(s^2+2s+5)I(s) = (4s^2+6s+10)/s$ or

$I(s) = (4s^2+6s+10)/[s(s+1+j2)(s+1–j2)] = [A/s] + [B/(s+1+j2)] + [C/(s+1–j2)]$
where $A = 10/5 = 2$; $B = [4(–1–j2)^2+6(–1–j2)+10]/[(–1–j2)(–j4)]$
$= [4(1+j4–4)–6–j12+10]/[–8+j4] = [–12+j16–6–j12+10]/(8.944\angle 153.43^\circ)$
$= [–8+j4]/(8.944\angle 153.43^\circ) = 1$; $C = [4(–1+j2)^2+6(–1+j2)+10]/[(–1+j2)(j4)]$
$= [4(1–j4–4)–6+j12+10]/[–8–j4] = [–12–j16–6+j12+10]/(8.944\angle –153.43^\circ)$
$= [–8–j4]/(8.944\angle 153.43^\circ) = 1$ thus

$i(t) = [2+e^{-t}e^{-j2t}+ e^{-t}e^{j2t}]u(t)$ amps = $\mathbf{[2+2e^{-t}\cos(2t)]u(t)}$ **amps**.

(a) $\mathbf{[3+3.162\cos(2t–161.12^\circ)]u(t)}$ **volts**, (b) $\mathbf{[2–4e^{-t}+e^{-4t}]u(t)}$ **amps**,
(c) $\mathbf{[3+2e^{-t}+3te^{-t}]u(t)}$ **volts**, (d) $\mathbf{[2+2e^{-t}\cos(2t)]u(t)}$ **amps**