

Problem Solving Class: Geometric Tests

1 Area formula

Area of Triangle $\triangle abc$: If we want to compute the area of the triangle with vertices $a = (a_x, a_y)$, $b = (b_x, b_y)$ and $c = (c_x, c_y)$, there are two ways to do it: one is to use

$$\text{SignArea}(\triangle abc) = \frac{1}{2}(a_x b_y - a_y b_x + b_x c_y - c_x b_y + c_x a_y - a_x c_y).$$

and the other is just to use the following formula

$$\text{SignArea}(\triangle abc) = \frac{1}{2} \begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix}. \quad (1)$$

Note that

- $\text{SignArea}(\triangle abc) > 0$ if a, b, c are in *counterclockwise order*,
- $\text{SignArea}(\triangle abc) < 0$ if a, b, c are in *clockwise order*,
- $\text{SignArea}(\triangle abc) = 0$ if a, b, c are *collinear*.

Area of a Convex Quadrilateral $\square abcd$:

$$\begin{aligned} \text{SignArea}(\square abcd) &= \text{SignArea}(\triangle abc) + \text{SignArea}(\triangle acd) \\ &= \text{SignArea}(\triangle dab) + \text{SignArea}(\triangle dbc) \\ &= a_x b_y - a_y b_x + b_x c_y - c_x b_y + c_x d_y - d_x c_y + d_x a_y - a_x d_y. \end{aligned} \quad (2)$$

Area of a Nonconvex Quadrilateral $\square abcd$: Same as (2). $\text{SignArea}(\triangle acd) < 0$.

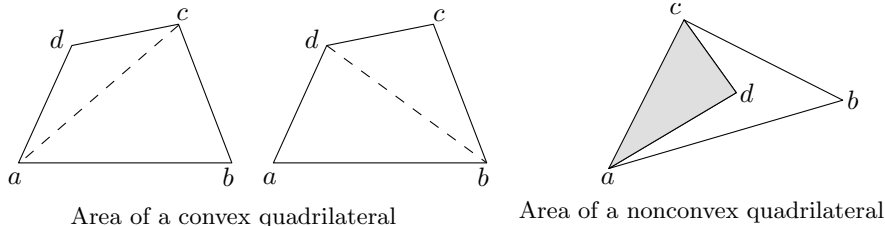
Area from an Arbitrary Center:

Theorem 1.1 If $\triangle abc$ is a triangle with vertices oriented counterclockwise, and $p \in \mathbb{R}^2$ is any point in the plane, then

$$\text{SignArea}(\triangle abc) = \text{SignArea}(\triangle pab) + \text{SignArea}(\triangle pbc) + \text{SignArea}(\triangle pca).$$

Theorem 1.2 Let a polygon (convex or nonconvex) P have vertices v_1, v_2, \dots, v_n labeled counterclockwise, and let $p \in \mathbb{R}^2$ be any point in the plane. Then

$$\text{SignArea}(P) = \text{SignArea}(\triangle pv_1 v_2) + \dots + \text{SignArea}(\triangle pv_n v_1) = \sum_{i=1}^{i=n} \text{SignArea}(\triangle pv_i v_{i+1}). \quad (3)$$



In the following figure, $\text{SignArea}(\triangle p12), \text{SignArea}(\triangle p67), \text{SignArea}(\triangle p70)$ are negative.

Volume in Three and Higher Dimensions: Formula (1) generalizes to d -dimension as the volume of d -dimensional simplex with $d!$ in the formula

$$\text{SignArea}(\text{Tetrahedron } abcd \text{ in } \mathbb{R}^3) = \frac{1}{3!} \begin{vmatrix} a_x & a_y & a_z & 1 \\ b_x & b_y & b_z & 1 \\ c_x & c_y & c_z & 1 \\ d_x & d_y & d_z & 1 \end{vmatrix}$$

and thus the volume of any d -dimensional polyhedron is similar as (3).

2 $O(n \log n)$ Convex Hull Algorithm

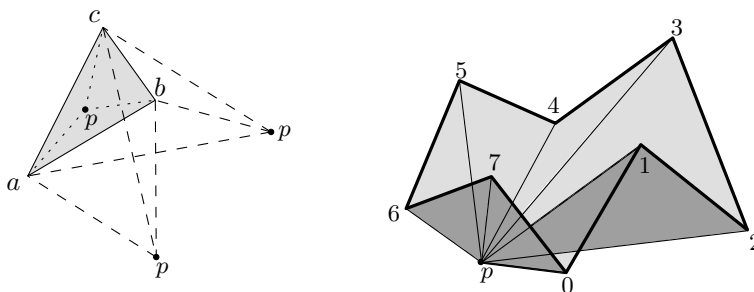
Graham's Scan Algorithm.

1. Find the rightmost lowest point; label it p_0 .
2. Sort all other points angularly about p_0 . In case of tie, delete the point closer to p_0 . If there are multiple copies, delete all but one.
3. Stack $S = (p_1, p_0) = (p_t, p_{t-1})$; t indexes top.
4. $i = 2$;
while $i < n$ do
if p_i is strictly left of $p_{t-1}p_t$
then Push(p_i, S) and set $i \rightarrow i + 1$.
else Pop(S)

3 Triangulation of a Simple Polygon

Let \mathcal{P} be a simple polygon with n vertices v_1, v_2, \dots, v_n in counterclockwise order.

- A *triangulation* of \mathcal{P} is a decomposition of the polygon into a set of triangles.
- A *diagonal* of P is a line segment between two of its vertices a and b that are clearly visible to one another. That is, \overline{ab} is inside \mathcal{P} .



Area from arbitrary center p

- A vertex v of given polygon \mathcal{P} is called an *ear* of \mathcal{P} if its two neighbors (predecessor, successor) generate a diagonal, i.e., the segment $\text{prev}(v)\text{next}(v)$ is inside \mathcal{P} . Note that if v is an ear then the vertices v , $\text{prev}(v)$ and $\text{next}(v)$ form a triangle.

3.1 Existence of Triangulation and Properties

Lemma 3.1 Every polygon must have at least one strictly convex vertex.

Proof. The lowest part of the polygon must contain a strictly convex vertex. \square

Lemma 3.2 (Meisters) Every polygon of $n \geq 4$ vertices has a diagonal.

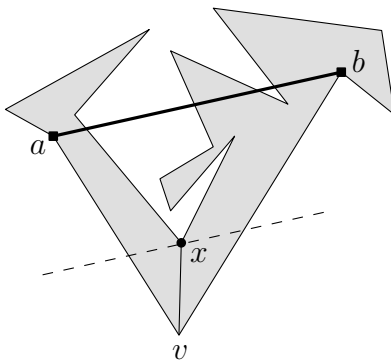
Proof. By the above lemma, there is a strictly convex vertex v . Let a and b be the vertices adjacent to v . If \overline{ab} is a diagonal, we are done. Otherwise, the triangle $\triangle vab$ contains at least one vertex of \mathcal{P} other than a, v, b . Let x be the first point in $\triangle vab$ hit by a line parallel to \overline{ab} moving from v to \overline{ab} . Then x is a vertex of \mathcal{P} and \overline{vx} must be a diagonal. \square

Theorem 3.3 (Triangulation) Every polygon \mathcal{P} can be partitioned into triangles by the addition of diagonals.

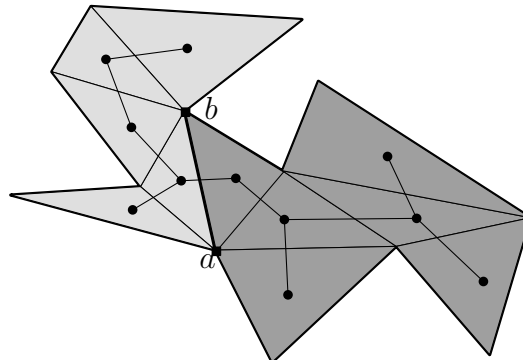
Proof. The proof is by induction. If $n = 3$, trivial. Assume that every polygon with at most k vertices can be triangulated and consider a polygon \mathcal{P} with $k + 1$ vertices. By Meisters' Lemma, there is a diagonal \overline{ab} in \mathcal{P} . By the diagonal, \mathcal{P} is partitioned into \mathcal{P}' and \mathcal{P}'' sharing ab as edges. \mathcal{P}' and \mathcal{P}'' have at most k vertices each, so we can triangulate them. This gives us a triangulation of \mathcal{P} . \square

Theorem 3.4 Given a simple polygon \mathcal{P} with $n \geq 4$ vertices,

- Every triangulation of \mathcal{P} has exactly $n - 3$ diagonals and $n - 2$ triangles.
- The dual graph of any triangulation of \mathcal{P} is a tree.



vx is a diagonal



Triangulation of \mathcal{P} and its dual

- There exist at least two nonoverlapping ears.

Proof. - The proof is by induction. For $n = 3$, trivially true. Consider a polygon \mathcal{P} with k vertices. A diagonal \overline{ab} partitions \mathcal{P} into \mathcal{P}' and \mathcal{P}'' sharing ab as edges. \mathcal{P}' has $k_1 \leq k - 1$ vertices and \mathcal{P}'' has $k_2 \leq k - 1$ vertices, where $k_1 + k_2 = k + 2$. By the assumption, the triangulation of \mathcal{P}' consists of $k_1 - 3$ diagonals and $k_1 - 2$ triangles, and the triangulation of \mathcal{P}'' consists of $k_2 - 3$ diagonals and $k_2 - 2$ triangles. So \mathcal{P} has $(k_1 - 3) + (k_2 - 3) + 1 = k - 3$ diagonals and $(k_1 - 2) + (k_2 - 2) = k - 2$ triangles.

- Dual graph of the triangulation is the graph, one node per triangle and one edge per diagonal.
 - A leaf node in a triangulation dual corresponds to an ear. A tree having at least two nodes has at least two leaves. \square

3.2 $O(n^2)$ algorithm for a triangulation of \mathcal{P}

1. For each vertex v , check if v is an ear and store the information;
 How to check if v is an ear :
 - Check if v is convex ($\text{Left}(\text{prev}(v), v, \text{next}(v)) = \text{TRUE}$) and then
 - Check if the segment $\text{prev}(v)\text{next}(v)$ does not intersect any edge of \mathcal{P} , i.e., check if $\text{Diagonal}(\text{prev}(v), \text{next}(v)) = \text{TRUE}$.
2. While there's any ear left in the polygon, find an ear and do
 - Cut the ear off and list it to a triangle in our triangulation.
 - Update the information of *earness* for $\text{prev}(v)$ and $\text{next}(v)$.

Analysis:

Step 1: $O(n^2)$

Step 2: Whenever we cut off an ear (in total $n - 2$ ears)

- Finding an ear: $O(n)$
- Updating the information for $\text{prev}(v)$ and $\text{next}(v)$: $2 O(n)$

So Step 2 has time complexity $O(n^2)$.

3.3 Implementation of Triangulation

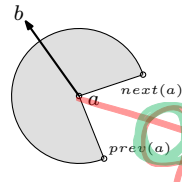
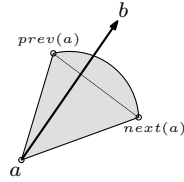
Let $\text{SignArea}(a, b, c) = \frac{1}{2}(a_x b_y - a_y b_x + b_x c_y - c_x b_y + c_x a_y - a_x c_y)$. Then

- $\text{Left}(a, b, c)$: TRUE if c is left of ab ;
 c is left of ab if $\text{SignArea}(a, b, c) > 0$, right if $\text{SignArea}(a, b, c) < 0$ and c is on ab if $\text{SignArea}(a, b, c) = 0$.
- $\text{Diagonal}(a, b)$: TRUE if ab is a diagonal of \mathcal{P} .
 - return $(\text{InCone}(a, b) \ \&\& \ \text{InCone}(b, a) \ \&\& \ \text{Diagonal}(a, b))$;

Handwritten: \overline{ab} 가 구간밖?

Handwritten: \overline{ba} 가 구간밖?
4내부?

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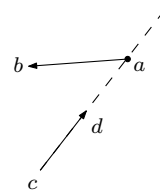
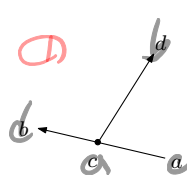
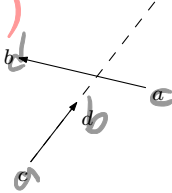
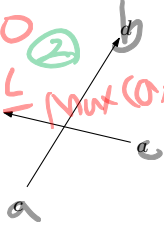


InCone(a,b) when a is convex/reflex

Inter

Inter

① $SA(a,b,c) = 0$
 $\&\& 1 \leq C_d \leq \max(\alpha, \beta)$
 $\min(\alpha, \beta)$



Proper Intersection

Improper Intersection

Collinearity is not sufficient

- InCone(a,b): TRUE if ab is inside \mathcal{P} in the vicinity of vertex a .
 - Let $\text{next}(a)$ and $\text{prev}(a)$ be the next and the preceding vertices of a .
 - if (LeftOn(a, next(a), prev(a)))
 return (Left(a, b, prev(a)) && Left(b, a, next(a)));
 - else
 return !(LeftOn(a, b, next(a)) && LeftOn(b, a, prev(a)));
- Diagonal(a,b): TRUE if ab does not intersect any edge of \mathcal{P} .
 - For all i , do the following:
 if $((v_i, v_{i+1} \neq a, b) \&\& \text{Intersect}(a, b, v_i, v_{i+1}))$ return FALSE;
 - return TRUE;
- Intersect(a,b,c,d): TRUE if ab and cd intersect.
 - if (Between(a,b,c) || Between(a,b,d) || Between(c,d,a) || Between(c,d,b)) return TRUE;
 - return Xor(Left(a,b,c), Left(a,b,d)) && Xor(Left(c,d,a), Left(c,d,b));
- Between(a,b,c): TRUE if collinear(a,b,c) is TRUE and c is between a and b . (how can check this?)