

Digital Design

Chapter 2: Combinational Logic Design

Slides to accompany the textbook *Digital Design, with RTL Design, VHDL, and Verilog*, 2nd Edition,
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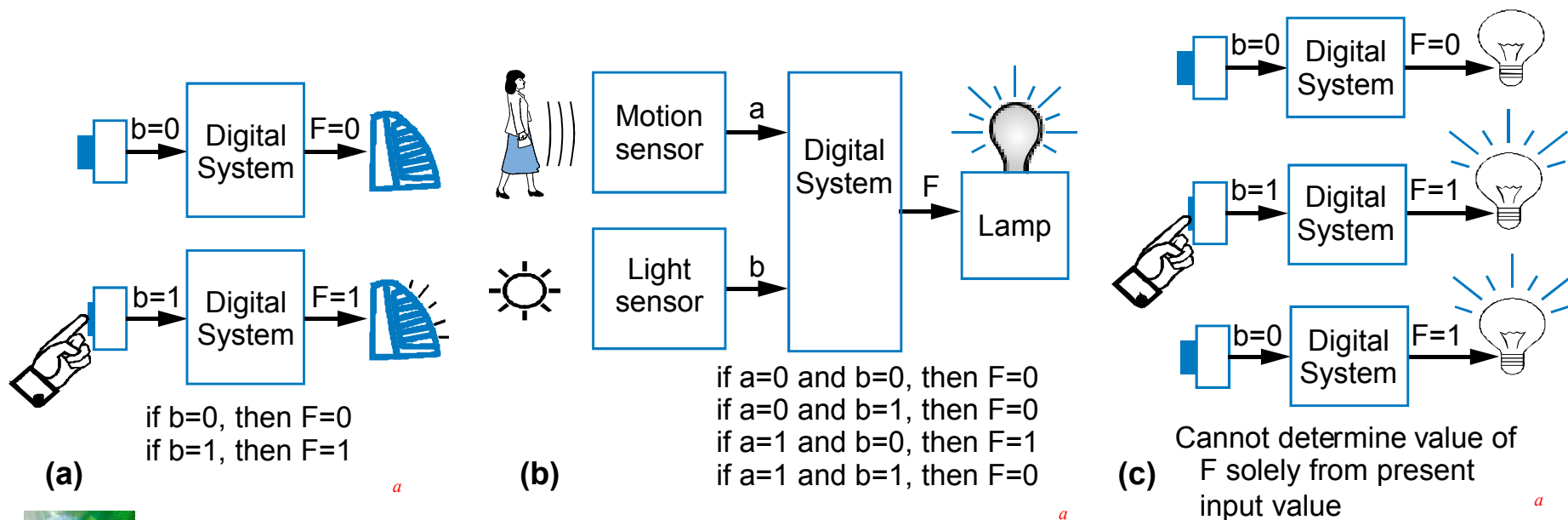
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Introduction

- Let's learn to design digital circuits, starting with a simple form of circuit:

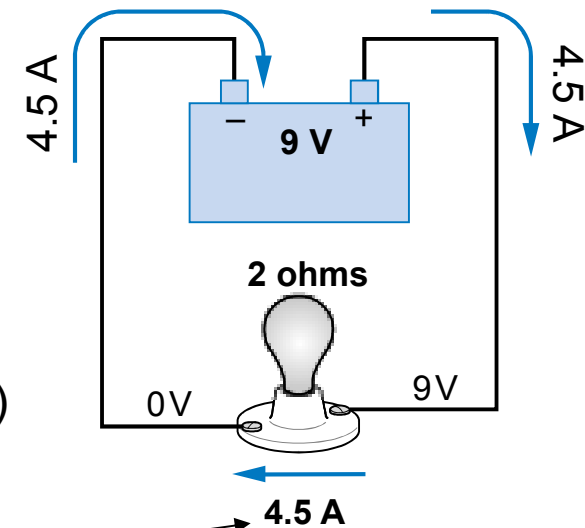
– **Combinational circuit**

- Outputs depend solely on the present combination of the circuit inputs' values
- Vs. sequential circuit: Has "memory" that impacts outputs too



Switches

- Electronic switches are the basis of binary digital circuits
 - Electrical terminology
 - Voltage**: Difference in electric potential between two points (volts, V)
 - Analogous to water pressure
 - Resistance**: Tendency of wire to resist current flow (ohms, Ω)
 - Analogous to water pipe diameter
 - Current**: Flow of charged particles (amps, A)
 - Analogous to water flow
 - $V = I * R$ (Ohm's Law)
 - $9\text{ V} = I * 2\text{ ohms}$
 - $I = 4.5\text{ A}$

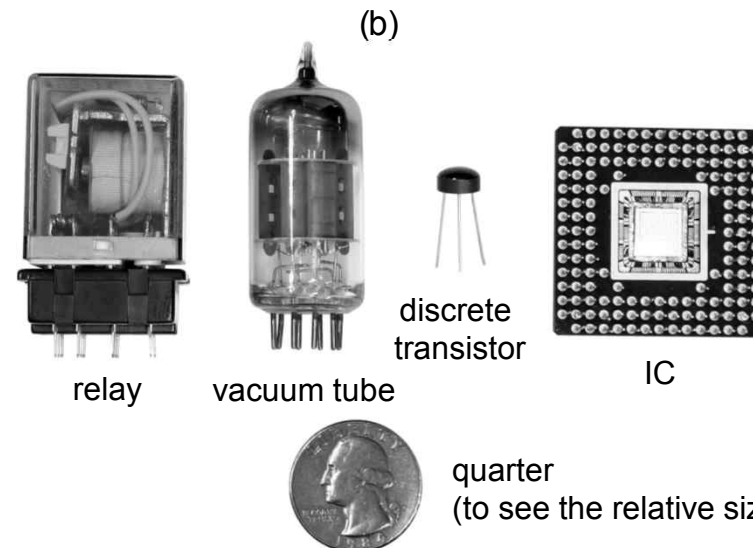
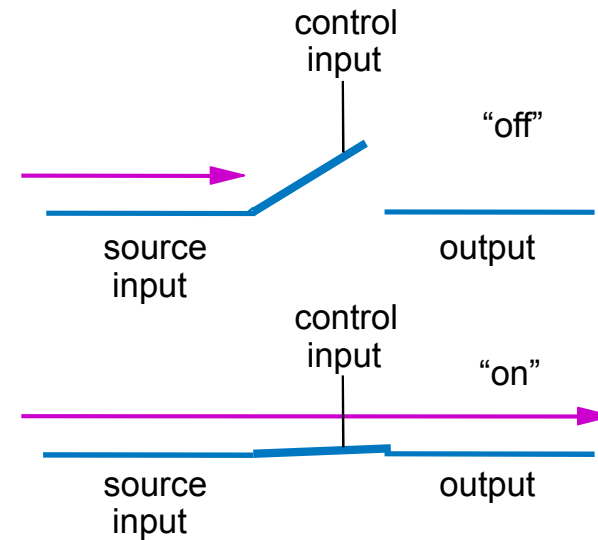


If a 9V potential difference is applied across a 2 ohm resistor, then 4.5 A of current will flow.



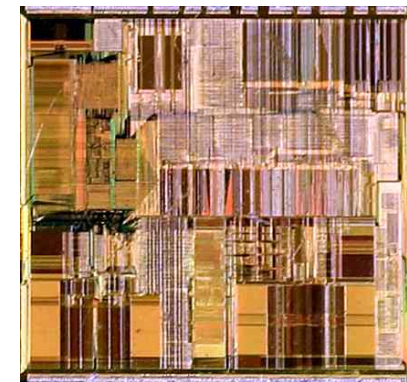
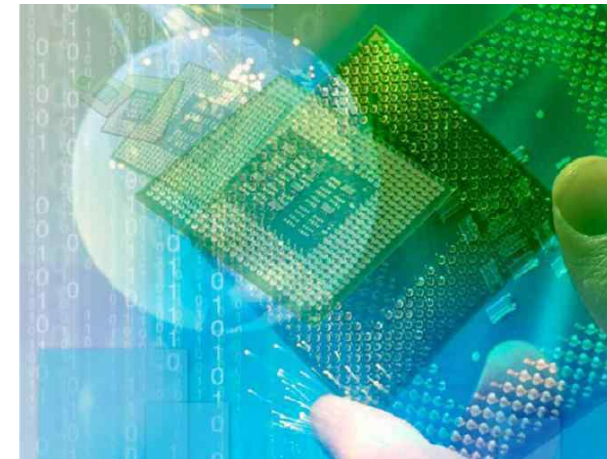
Switches

- A switch has three parts
 - Source input, and output
 - Current tries to flow from source input to output
 - Control input
 - Voltage that controls whether that current can flow
- The amazing shrinking switch
 - 1930s: Relays
 - 1940s: Vacuum tubes
 - 1950s: Discrete transistor
 - 1960s: Integrated circuits (ICs)
 - Initially just a few transistors on IC
 - Then tens, hundreds, thousands...



Moore's Law

- IC capacity doubling about every 18 months for several decades
 - Known as “Moore’s Law” after Gordon Moore, co-founder of Intel
 - Predicted in 1965 predicted that components per IC would double roughly every year or so
 - Book cover depicts related phenomena
 - For a particular number of transistors, the IC area shrinks by half every 18 months
 - Consider how much shrinking occurs in just 10 years (try drawing it)
 - Enables incredibly powerful computation in incredibly tiny devices
 - Today’s ICs hold *billions* of transistors
 - The first Pentium processor (early 1990s) needed only 3 million

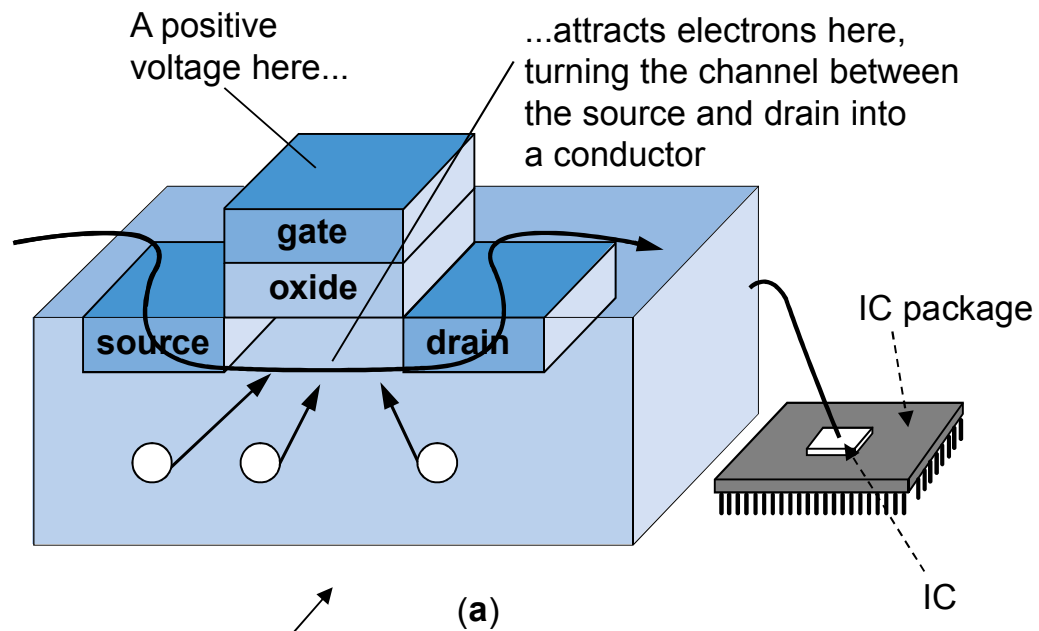


*An Intel Pentium processor IC
having millions of transistors*

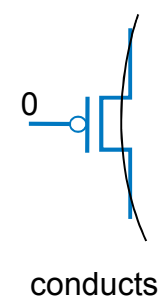
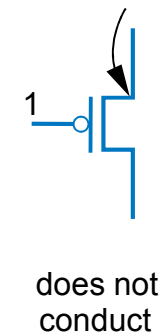
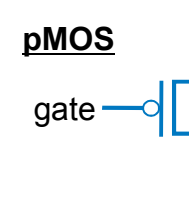
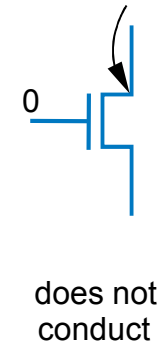
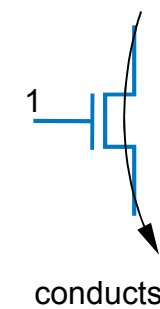
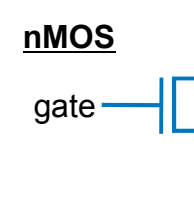


The CMOS Transistor

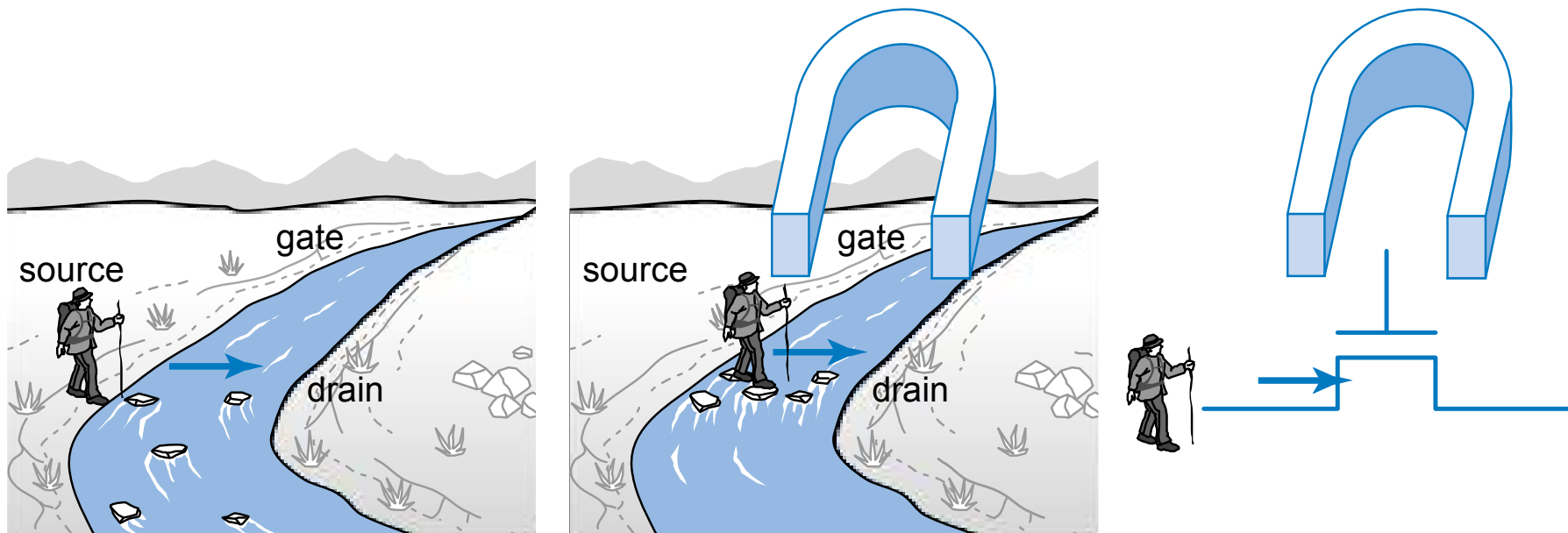
- CMOS transistor
 - Basic switch in modern ICs



Silicon -- not quite a conductor or insulator:
Semiconductor



CMOS Transistor Analogy

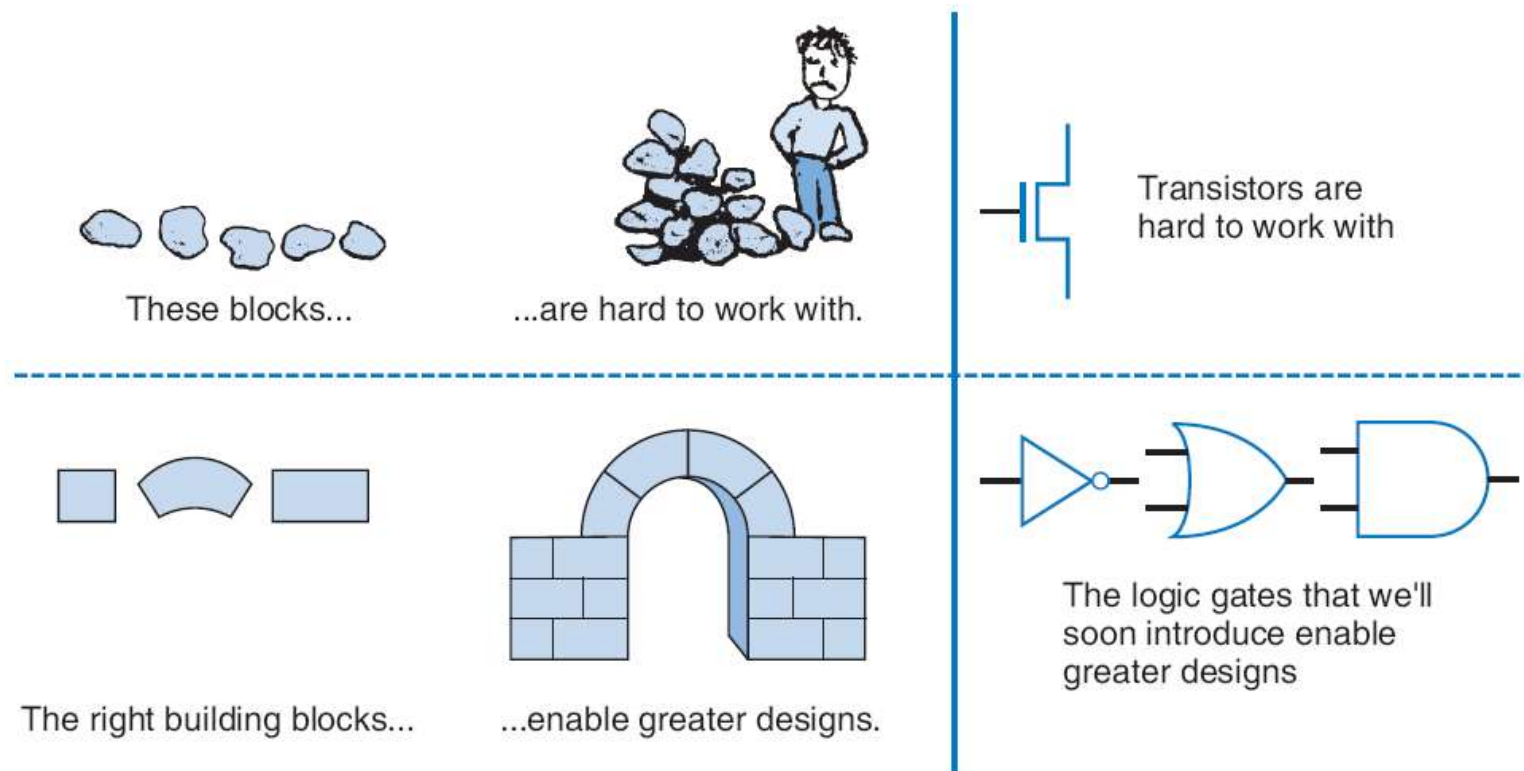


Boolean Logic Gates

2.4

Building Blocks for Digital Circuits

(Because Switches are Hard to Work With)



- “Logic gates” are better digital circuit building blocks than switches (transistors)
 - Why?...



Boolean Algebra and its Relation to Digital Circuits

- To understand the benefits of “logic gates” vs. switches, we should first understand Boolean algebra
- “Traditional” algebra
 - Variables represent real numbers (x , y)
 - Operators operate on variables, return real numbers ($2.5x + y - 3$)
- **Boolean Algebra**
 - Variables represent 0 or 1 only
 - Operators return 0 or 1 only
 - Basic operators
 - AND: $a \text{ AND } b$ returns 1 only when both $a=1$ and $b=1$
 - OR: $a \text{ OR } b$ returns 1 if either (or both) $a=1$ or $b=1$
 - NOT: $\text{NOT } a$ returns the opposite of a (1 if $a=0$, 0 if $a=1$)

a	b	AND
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR
0	0	0
0	1	1
1	0	1
1	1	1

a	NOT
0	1
1	0



Boolean Algebra and its Relation to Digital Circuits

- Developed mid-1800's by George Boole to formalize human thought
 - Ex: "I'll go to lunch if Mary goes OR John goes, AND Sally does not go."
 - Let F represent my going to lunch (1 means I go, 0 I don't go)
 - Likewise, m for Mary going, j for John, and s for Sally
 - Then **$F = (m \text{ OR } j) \text{ AND NOT}(s)$**
 - Nice features
 - Formally evaluate
 - $m=1, j=0, s=1 \rightarrow F = (1 \text{ OR } 0) \text{ AND NOT}(1) = 1 \text{ AND } 0 = \underline{0}$
 - Formally transform
 - $F = (m \text{ and NOT}(s)) \text{ OR } (j \text{ and NOT}(s))$
 - » Looks different, but same function
 - » We'll show transformation techniques soon
 - Formally prove
 - Prove that if Sally goes to lunch ($s=1$), then I don't go ($F=0$)
 - $F = (m \text{ OR } j) \text{ AND NOT}(1) = (m \text{ OR } j) \text{ AND } 0 = 0$

a	b	AND
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR
0	0	0
0	1	1
1	0	1
1	1	1

a	NOT
0	1
1	0



Evaluating Boolean Equations

- Evaluate the Boolean equation **$F = (a \text{ AND } b) \text{ OR } (c \text{ AND } d)$** for the given values of variables a, b, c, and d:
 - Q1: $a=1, b=1, c=1, d=0$.
 - Answer: $F = (1 \text{ AND } 1) \text{ OR } (1 \text{ AND } 0) = 1 \text{ OR } 0 = 1$.
 - Q2: $a=0, b=1, c=0, d=1$.
 - Answer: $F = (0 \text{ AND } 1) \text{ OR } (0 \text{ AND } 1) = 0 \text{ OR } 0 = 0$.
 - Q3: $a=1, b=1, c=1, d=1$.
 - Answer: $F = (1 \text{ AND } 1) \text{ OR } (1 \text{ AND } 1) = 1 \text{ OR } 1 = 1$.

a	b	AND
0	0	0
0	1	0
1	0	0
1	1	1

a	b	OR
0	0	0
0	1	1
1	0	1
1	1	1

a	NOT
0	1
1	0



Converting to Boolean Equations

- a • Convert the following English statements to a Boolean equation
 - Q1. a is 1 and b is 1.
 - Answer: $F = a \text{ AND } b$
 - Q2. either of a or b is 1.
 - Answer: $F = a \text{ OR } b$
 - Q3. a is 1 and b is 0.
 - Answer: $F = a \text{ AND NOT}(b)$
 - Q4. a is not 0.
 - Answer:
 - (a) Option 1: $F = \text{NOT}(\text{NOT}(a))$
 - (b) Option 2: $F = a$



Converting to Boolean Equations

- Q1. A fire sprinkler system should spray water if high heat is sensed and the system is set to enabled.
 - Answer: Let Boolean variable h represent “high heat is sensed,” e represent “enabled,” and F represent “spraying water.” Then an equation is: $F = h \text{ AND } e$.
- Q2. A car alarm should sound if the alarm is enabled, and either the car is shaken or the door is opened.
 - Answer: Let a represent “alarm is enabled,” s represent “car is shaken,” d represent “door is opened,” and F represent “alarm sounds.” Then an equation is: $F = a \text{ AND } (s \text{ OR } d)$.
 - (a) Alternatively, assuming that our door sensor d represents “door is closed” instead of open (meaning $d=1$ when the door is closed, 0 when open), we obtain the following equation: $F = a \text{ AND } (s \text{ OR NOT}(d))$.



Relating Boolean Algebra to Digital Design

Boolean algebra
(mid-1800s)

Boole's intent: formalize human thought

Switches
(1930s)

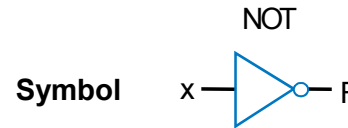
For telephone switching and other electronic uses

Shannon (1938)

Showed application of Boolean algebra to design of switch-based circuits

Digital design

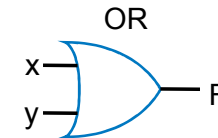
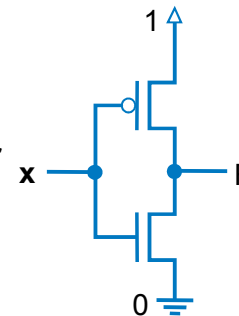
- Implement Boolean operators using transistors
 - Call those implementations **logic gates**.
 - Lets us build circuits by doing math** - powerful concept



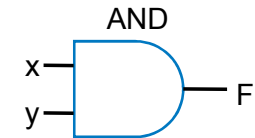
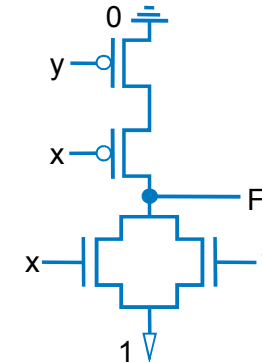
Truth table

x	F
0	1
1	0

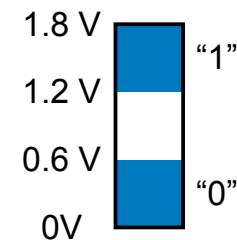
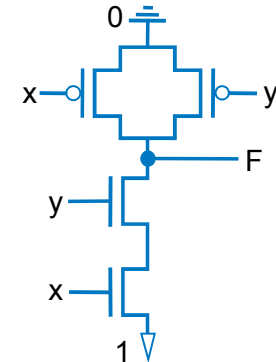
Transistor circuit



x	y	F
0	0	0
0	1	1
1	0	1
1	1	1



x	y	F
0	0	0
0	1	0
1	0	0
1	1	1

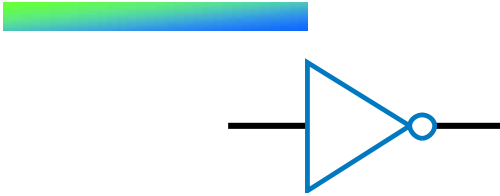


1 and 0 each actually corresponds to a voltage range

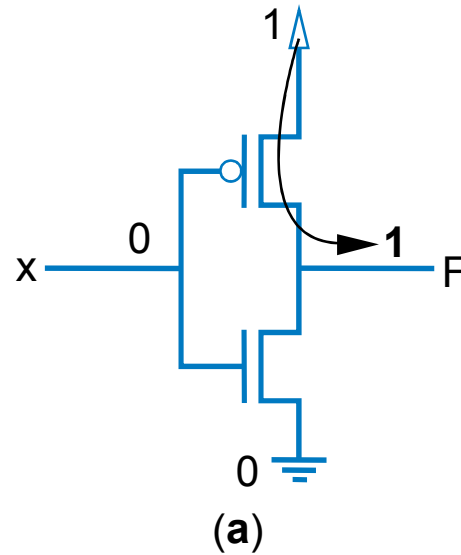
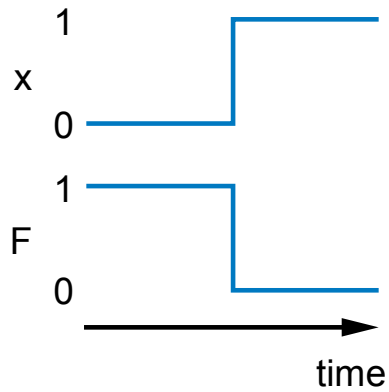
*Next slides show how these circuits work.
Note: The above OR/AND implementations are inefficient; we'll show why, and show better ones, later.*



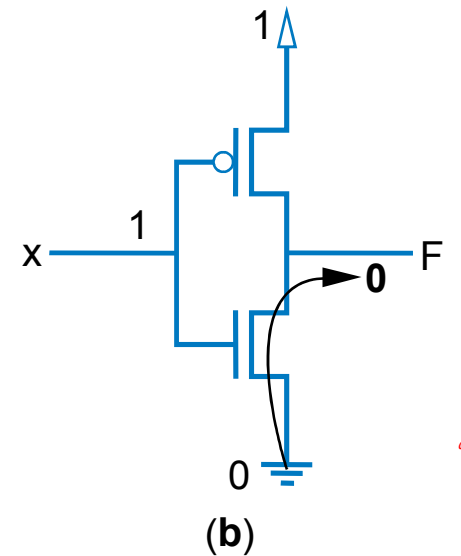
NOT gate



x	F
0	1
1	0



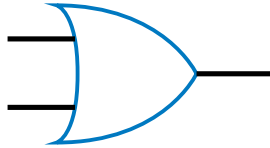
When the input is 0



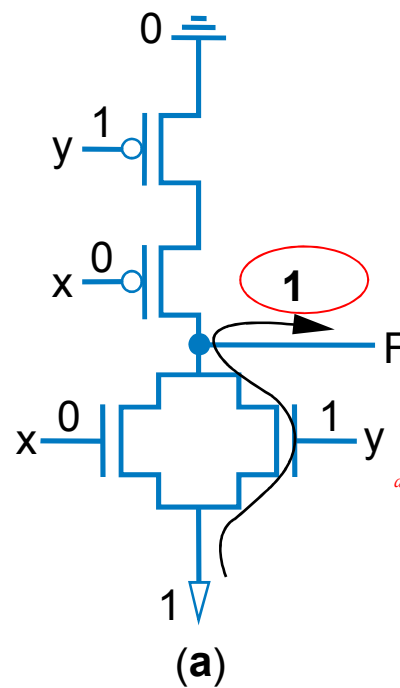
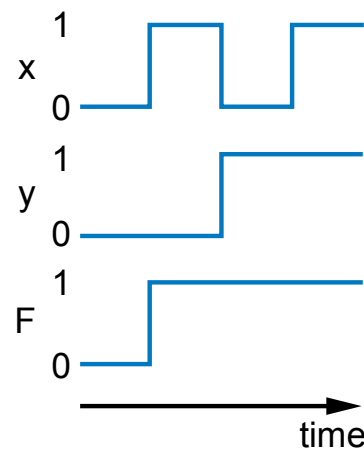
When the input is 1



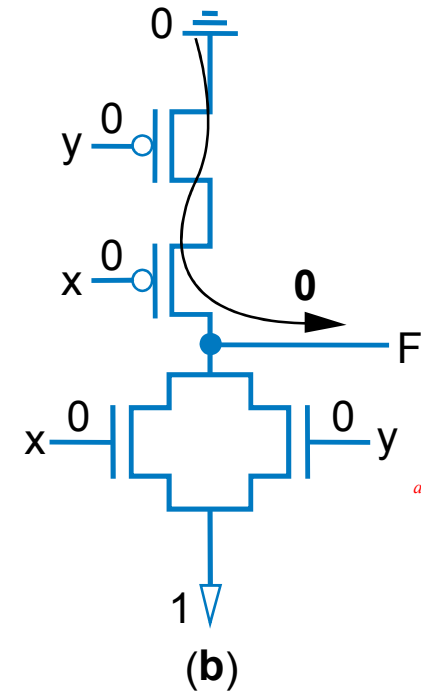
OR gate



x	y	F
0	0	0
0	1	1
1	0	1
1	1	1



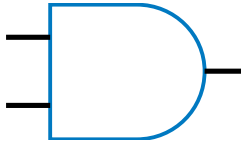
When an input is 1



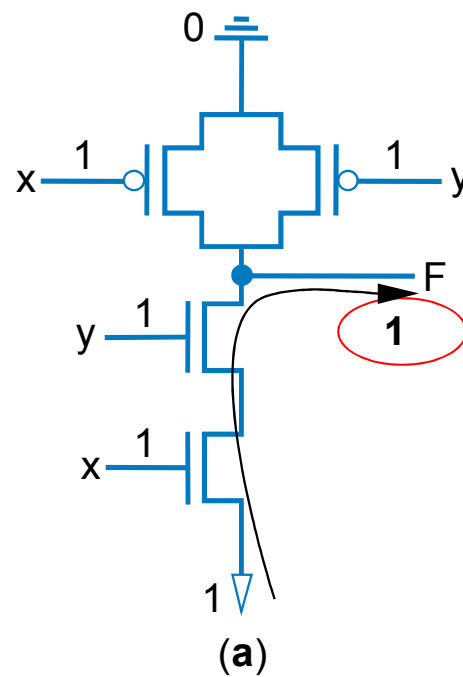
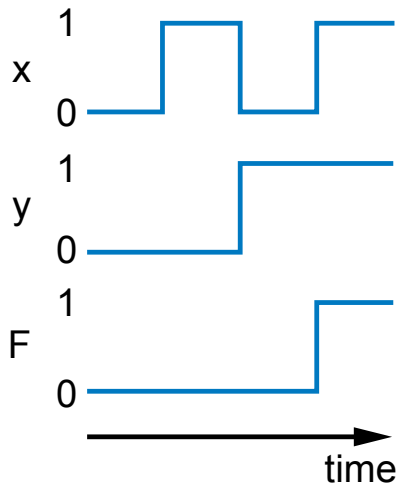
When both inputs are 0



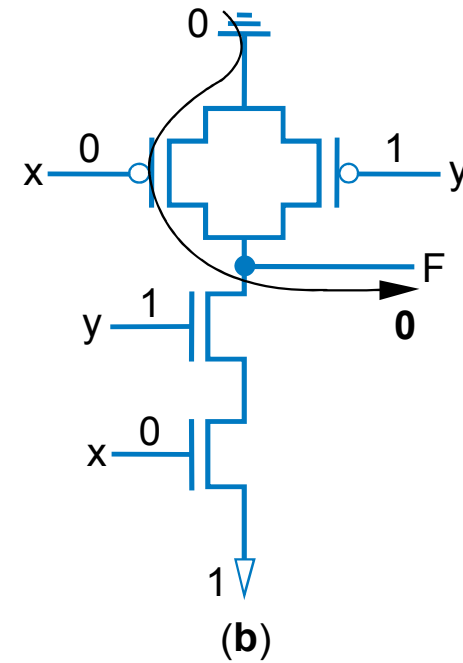
AND gate



x	y	F
0	0	0
0	1	0
1	0	0
1	1	1



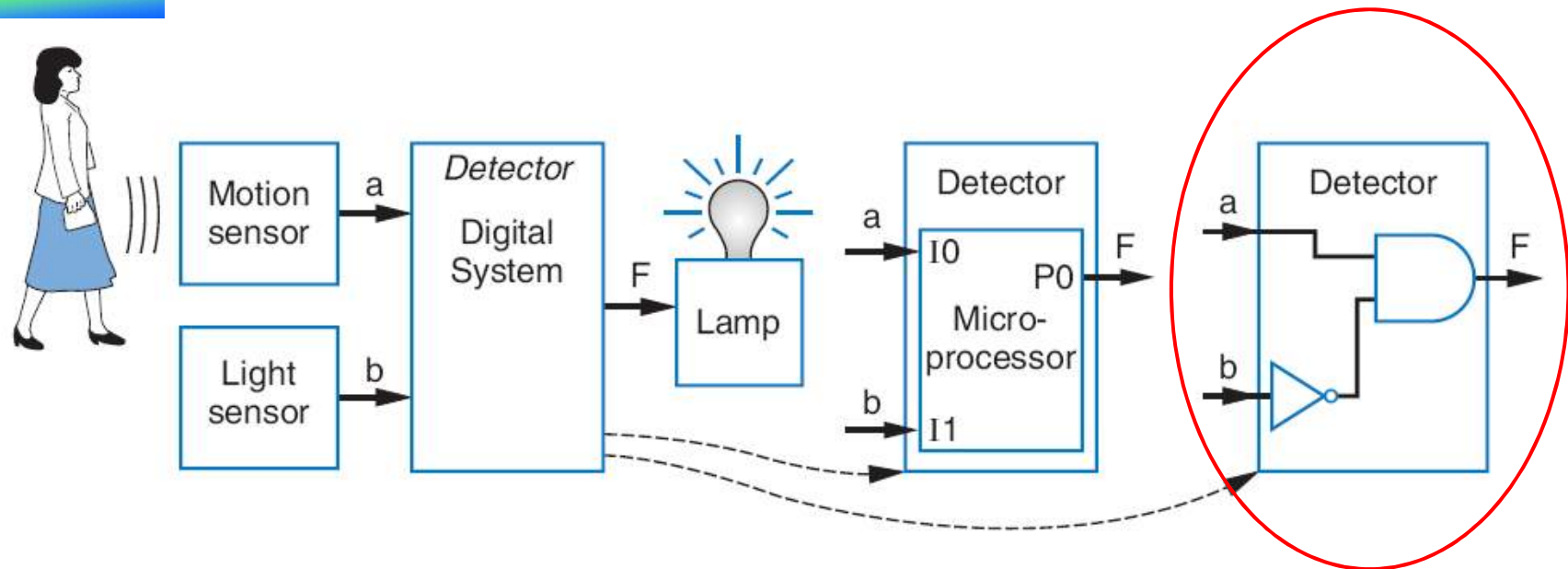
When both inputs are 1



When an input is 0



Building Circuits Using Gates



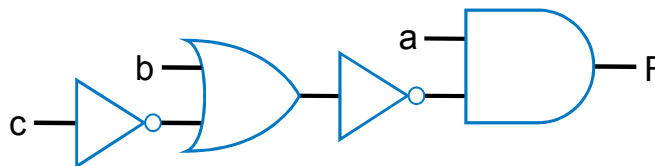
- Recall Chapter 1 motion-in-dark example
 - Turn on lamp ($F=1$) when motion sensed ($a=1$) and no light ($b=0$)
 - $F = a \text{ AND NOT}(b)$
 - Build using logic gates, AND and NOT, as shown
 - We just built our first digital circuit!



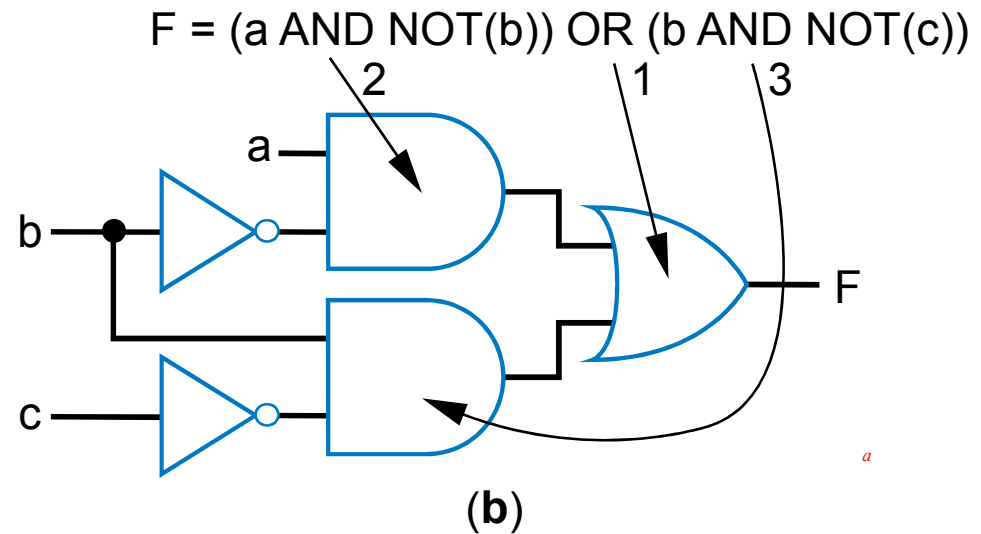
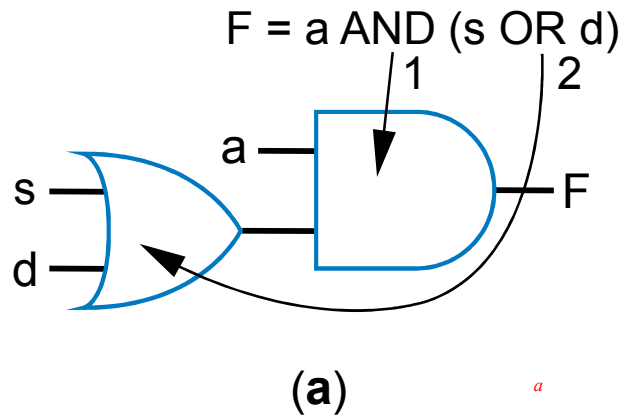
Example: Converting a Boolean Equation to a Circuit of Logic Gates

Start from the output, work back towards the inputs

- Q: Convert the following equation to logic gates:
 $F = a \text{ AND NOT}(b \text{ OR NOT}(c))$



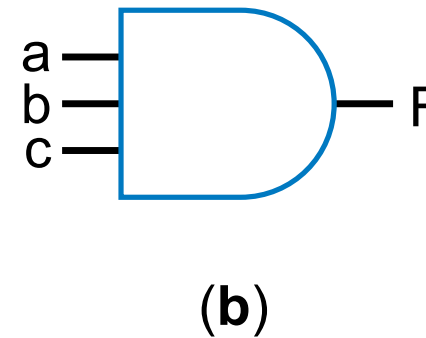
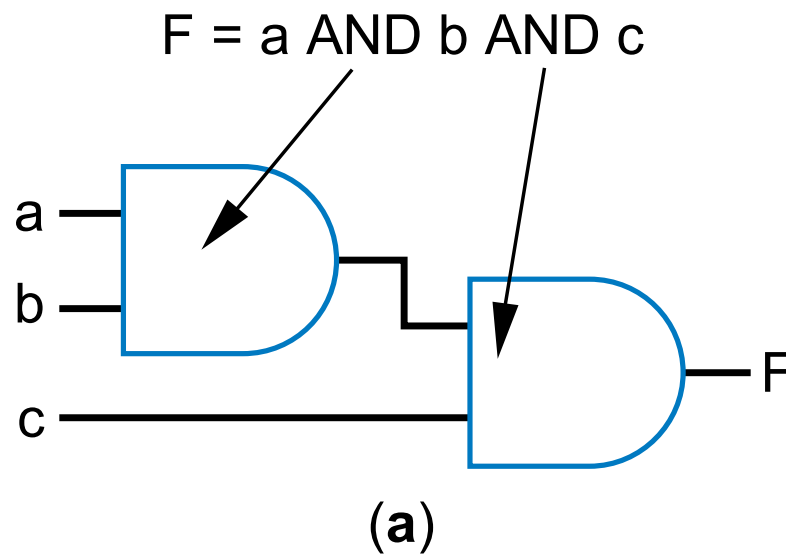
More examples



Start from the output, work back towards the inputs



Using gates with more than 2 inputs



Can think of as $AND(a,b,c)$



Example: Seat Belt Warning Light System



- Design circuit for warning light

- Sensors

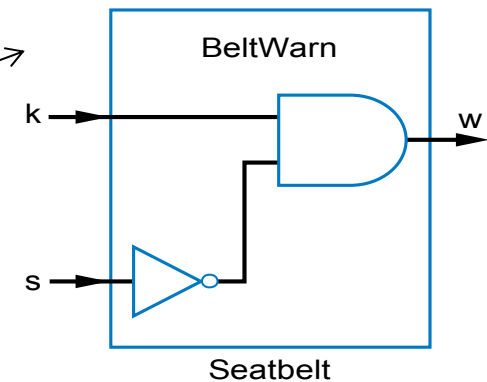
- $s=1$: seat belt fastened
- $k=1$: key inserted

- Capture Boolean equation

- seat belt not fastened, and key inserted

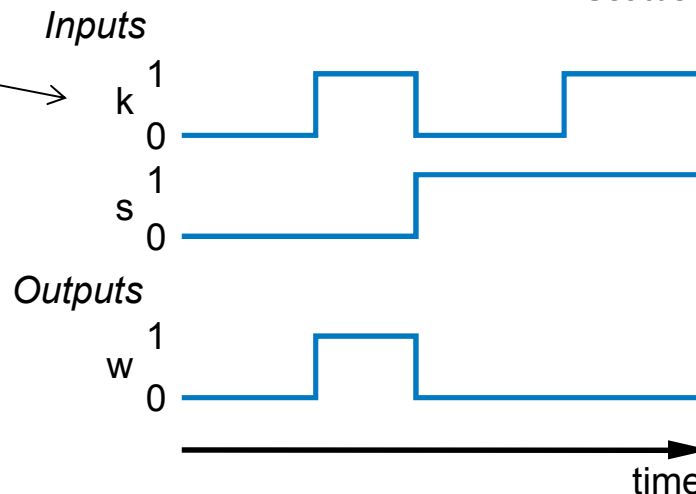
- Convert equation to circuit

$$w = \text{NOT}(s) \text{ AND } k$$



- *Timing diagram* illustrates circuit behavior

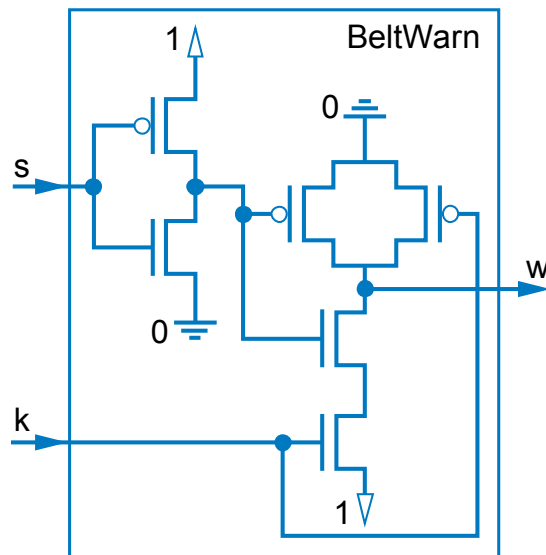
- We set inputs to any values
- Output set according to circuit



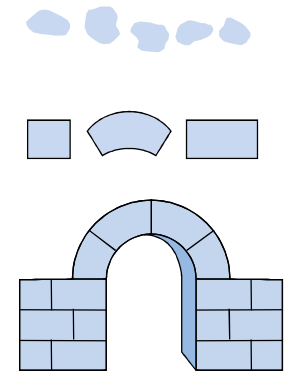
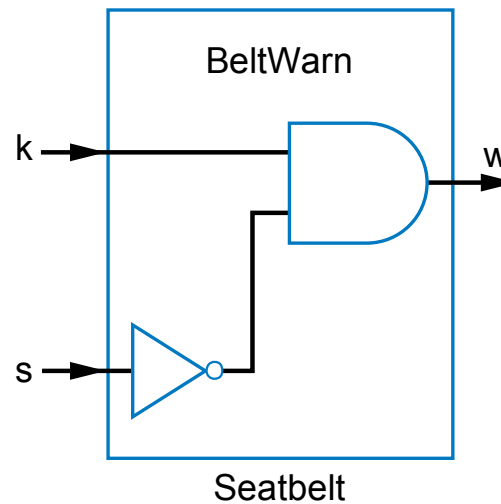
Gates vs. switches

Notice

- Boolean algebra enables easy capture as equation and conversion to circuit
 - How design with switches?
 - Of course, logic gates are built from switches, but we think at level of logic gates, not switches

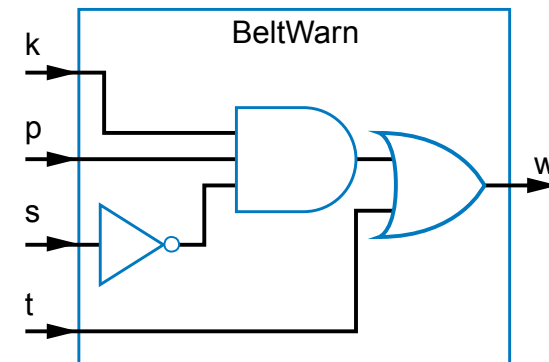
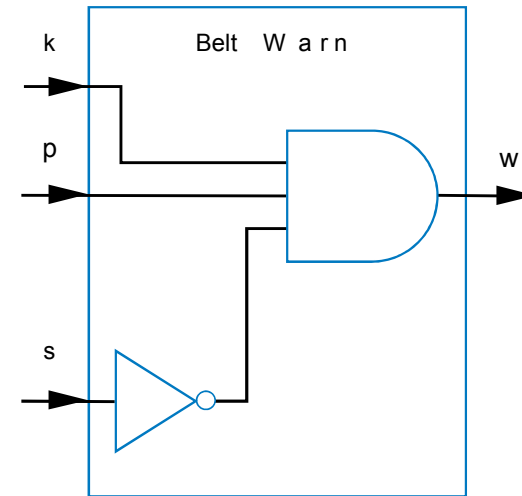


$$w = \text{NOT}(s) \text{ AND } k$$

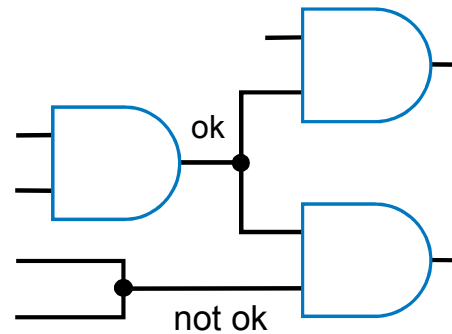
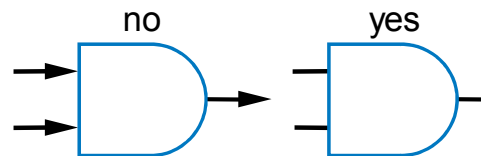
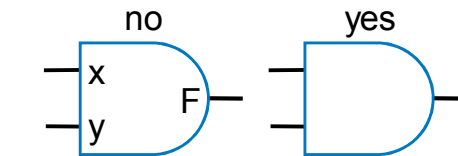


More examples: Seat belt warning light extensions

- Only illuminate warning light if person is in the seat ($p=1$), and seat belt not fastened and key inserted
- $w = p \text{ AND NOT}(s) \text{ AND } k$
- Given $t=1$ for 5 seconds after key inserted. Turn on warning light when $t=1$ (to check that warning lights are working)
- $w = (p \text{ AND NOT}(s) \text{ AND } k) \text{ OR } t$



Some Gate-Based Circuit Drawing Conventions



Boolean Algebra

- By defining logic gates based on Boolean algebra, we can *use algebraic methods to manipulate circuits*
- Notation: Writing a AND b, a OR b, NOT(a) is cumbersome
 - Use symbols: $a * b$ (or just ab), $a + b$, and a'
 - Original: $w = (p \text{ AND NOT}(s) \text{ AND } k) \text{ OR } t$
 - New: $w = ps'k + t$
 - Spoken as “w equals p and s prime and k, or t”
 - Or just “w equals p s prime k, or t”
 - s' known as “complement of s”
 - While symbols come from regular algebra, **don't** say “times” or “plus”
 - “product” and “sum” are OK and commonly used

Boolean algebra precedence, highest precedence first.

Symbol	Name	Description
()	Parentheses	Evaluate expressions nested in parentheses first
'	NOT	Evaluate from left to right
*	AND	Evaluate from left to right
+	OR	Evaluate from left to right



Boolean Algebra Operator Precedence

- Evaluate the following Boolean equations, assuming $a=1$, $b=1$, $c=0$, $d=1$.
 - Q1. $F = a * b + c$.
 - Answer: $*$ has precedence over $+$, so we evaluate the equation as $F = (1 * 1) + 0 = (1) + 0 = 1 + 0 = 1$.
 - Q2. $F = ab + c$.
 - Answer: the problem is identical to the previous problem, using the shorthand notation for $*$.
 - Q3. $F = ab'$.
 - Answer: we first evaluate b' because NOT has precedence over AND, resulting in $F = 1 * (1') = 1 * (0) = 1 * 0 = 0$.
 - Q4. $F = (ac)'$.
 - Answer: we first evaluate what is inside the parentheses, then we NOT the result, yielding $(1*0)' = (0)' = 0' = 1$.
 - Q5. $F = (a + b') * c + d'$.
 - Answer: Inside left parentheses: $(1 + (1')) = (1 + (0)) = (1 + 0) = 1$. Next, $*$ has precedence over $+$, yielding $(1 * 0) + 1' = (0) + 1'$. The NOT has precedence over the OR, giving $(0) + (1') = (0) + (0) = 0 + 0 = 0$.

Boolean algebra precedence, highest precedence first.

Symbol	Name	Description
()	Parentheses	Evaluate expressions nested in parentheses first
'	NOT	Evaluate from left to right
*	AND	Evaluate from left to right
+	OR	Evaluate from left to right



Boolean Algebra Terminology

- Example equation: $F(a,b,c) = a'bc + abc' + ab + c$
- **Variable**
 - Represents a value (0 or 1)
 - Three variables: a, b, and c
- **Literal**
 - Appearance of a variable, in true or complemented form
 - Nine literals: a', b, c, a, b, c', a, b, and c
- **Product term**
 - Product of literals
 - Four product terms: a'bc, abc', ab, c
- **Sum-of-products**
 - Equation written as OR of product terms only
 - Above equation is in sum-of-products form. “ $F = (a+b)c + d$ ” is not.



Boolean Algebra Properties

- Commutative
 - $a + b = b + a$
 - $a * b = b * a$
- Distributive
 - $a * (b + c) = a * b + a * c$
 - Can write as: $a(b+c) = ab + ac$
 - $a + (b * c) = (a + b) * (a + c)$
 - (This second one is tricky!)
 - Can write as: $a+(bc) = (ab)(ac)$
- Associative
 - $(a + b) + c = a + (b + c)$
 - $(a * b) * c = a * (b * c)$
- Identity
 - $0 + a = a + 0 = a$
 - $1 * a = a * 1 = a$
- Complement
 - $a + a' = 1$
 - $a * a' = 0$
- To prove, just evaluate all possibilities

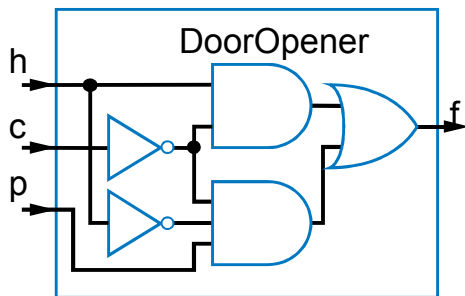
Example uses of the properties

- Show abc' equivalent to $c'ba$.
 - Use commutative property:
 - $a*b*c' = a*c'*b = c'*a*b = c'*b*a$
- Show $abc + abc' = ab$.
 - Use first distributive property
 - $abc + abc' = ab(c+c')$
 - Complement property
 - Replace $c+c'$ by 1: $ab(c+c') = ab(1)$.
 - Identity property
 - $ab(1) = ab*1 = ab$.
- Show $x + x'z$ equivalent to $x + z$.
 - Second distributive property
 - Replace $x+x'z$ by $(x+x')*(x+z)$.
 - Complement property
 - Replace $(x+x')$ by 1,
 - Identity property
 - replace $1*(x+z)$ by $x+z$.



Example that Applies Boolean Algebra Properties

- Want automatic door opener circuit (e.g., for grocery store)
 - Output: $f=1$ opens door
 - Inputs:
 - $p=1$: person detected
 - $h=1$: switch forcing hold open
 - $c=1$: key forcing closed
 - Want open door when
 - $h=1$ and $c=0$, or
 - $h=0$ and $p=1$ and $c=0$
 - Equation: $f = hc' + h'pc'$



- Can the circuit be simplified?

$$f = hc' + h'pc'$$

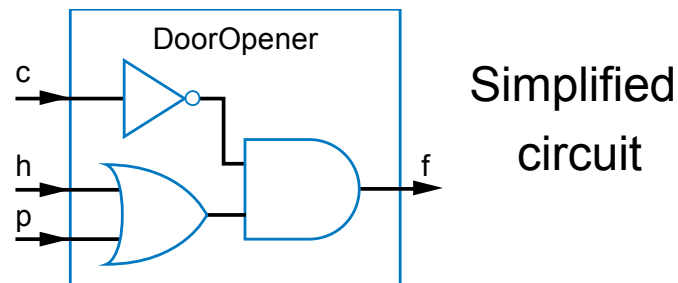
$$f = c'h + c'h'p \quad (\text{by the commutative property})$$

$$f = c'(h + h'p) \quad (\text{by the first distrib. property})$$

$$f = c'((h+h')*(h+p)) \quad (\text{2nd distrib. prop.; tricky one})$$

$$f = c'((1)*(h + p)) \quad (\text{by the complement property})$$

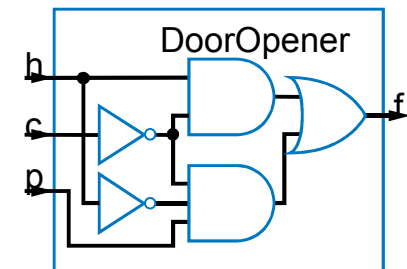
$$f = c'(h+p) \quad (\text{by the identity property})$$



Example that Applies Boolean Algebra Properties



- Found inexpensive chip that computes:
 - $f = c'h p + c'h p' + c'h' p$
 - Can we use it for the door opener?
 - Is it the same as $f = h c' + h' p c'$?
- Apply Boolean algebra:
 - Commutative
 - $a + b = b + a$
 - $a * b = b * a$
 - Distributive
 - $a * (b + c) = a * b + a * c$
 - $a + (b * c) = (a + b) * (a + c)$
 - Associative
 - $(a + b) + c = a + (b + c)$
 - $(a * b) * c = a * (b * c)$
 - Identity
 - $0 + a = a + 0 = a$
 - $1 * a = a * 1 = a$
 - Complement
 - $a + a' = 1$
 - $a * a' = 0$



$$f = c'h p + c'h p' + c'h' p$$

$$f = c'h(p + p') + c'h' p \text{ (by the distributive property)}$$

$$f = c'h(1) + c'h' p \text{ (by the complement property)}$$

$$f = c'h + c'h' p \text{ (by the identity property)}$$

$$f = h c' + h' p c' \text{ (by the commutative property)}$$

Same! Yes, we can use it.



Boolean Algebra: Additional Properties

- Null elements
 - $a + 1 = 1$
 - $a * 0 = 0$
- Idempotent Law
 - $a + a = a$
 - $a * a = a$
- Involution Law
 - $(a')' = a$
- DeMorgan's Law
 - $(a + b)' = a'b'$
 - $(ab)' = a' + b'$
 - *Very useful!*
- To prove, just evaluate all possibilities



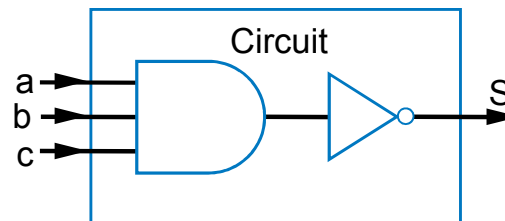
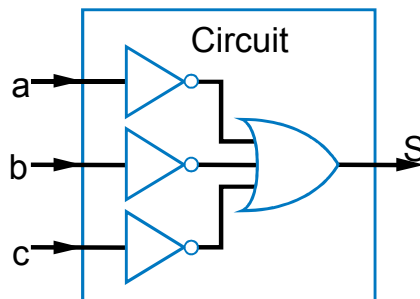
Example Applying DeMorgan's Law

$$(a + b)' = a'b'$$
$$(ab)' = a' + b'$$

Aircraft lavatory sign example



- Behavior
 - Three lavatories, each with sensor (a, b, c), equals 1 if door locked
 - Light “Available” sign (S) if any lavatory available
- Equation and circuit
 - $S = a' + b' + c'$
- Transform
 - $(abc)' = a' + b' + c'$ (by DeMorgan's Law)
 - $S = (abc)'$
- New circuit



- Alternative: Instead of lighting “Available,” light “Occupied”
- Opposite of “Available” function
 - $S = a' + b' + c'$
- So $S' = (a' + b' + c')'$
 - $S' = (a')' * (b')' * (c')'$ (by DeMorgan's Law)
 - $S' = a * b * c$ (by Involution Law)
- Makes intuitive sense
 - Occupied if all doors are locked



Example Applying Properties

- Commutative
 - $a + b = b + a$
 - $a * b = b * a$
- Distributive
 - $a * (b + c) = a * b + a * c$
 - $a + (b * c) = (a + b) * (a + c)$
- Associative
 - $(a + b) + c = a + (b + c)$
 - $(a * b) * c = a * (b * c)$
- Identity
 - $0 + a = a + 0 = a$
 - $1 * a = a * 1 = a$
- Complement
 - $a + a' = 1$
 - $a * a' = 0$
- Null elements
 - $a + 1 = 1$
 - $a * 0 = 0$
- Idempotent Law
 - $a + a = a$
 - $a * a = a$
- Involution Law
 - $(a')' = a$
- DeMorgan's Law
 - $(a + b)' = a'b'$
 - $(ab)' = a' + b'$
- For door opener $f = c'(h+p)$, *prove* door stays closed ($f=0$) when $c=1$
 - $f = c'(h+p)$
 - *Let $c = 1$* (door forced closed)
 - $f = 1'(h+p)$
 - $f = 0(h+p)$
 - $f = 0h + 0p$ (by the distributive property)
 - $f = 0 + 0$ (by the null elements property)
 - $f = 0$

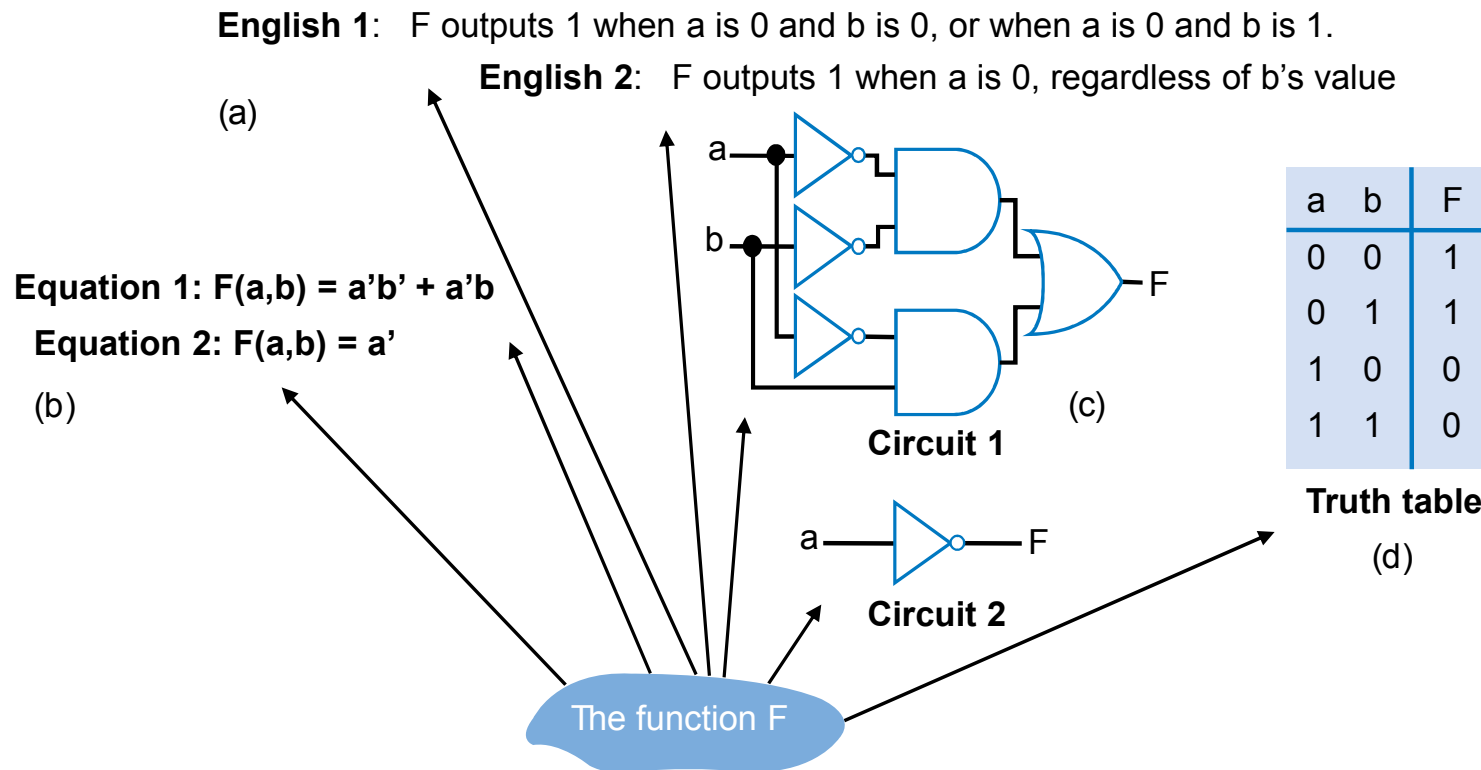


Complement of a Function

- Commonly want to find complement (inverse) of function F
 - 0 when F is 1; 1 when F is 0
- Use DeMorgan's Law repeatedly
 - Note: DeMorgan's Law defined for more than two variables, e.g.:
 - $(a + b + c)' = (abc)'$
 - $(abc)' = (a' + b' + c')$
- Complement of $f = w'xy + wx'y'z'$
 - $f' = (w'xy + wx'y'z')'$
 - $f' = (w'xy)'(wx'y'z')'$ (by DeMorgan's Law)
 - $f' = (w+x'+y')(w'+x+y+z)$ (by DeMorgan's Law)
- Can then expand into sum-of-products form



Representations of Boolean Functions



- A function can be represented in different ways
 - Above shows seven representations of the same functions $F(a,b)$, using four different methods: English, Equation, Circuit, and Truth Table

Truth Table Representation of Boolean Functions

- Define value of F for each possible combination of input values
 - 2-input function: 4 rows
 - 3-input function: 8 rows
 - 4-input function: 16 rows
- Q: Use truth table to define function $F(a,b,c)$ that is 1 when abc is 5 or greater in binary

a	b	F
0	0	
0	1	
1	0	
1	1	

(a)

a	b	c	F
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

(b)

a	b	c	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

a

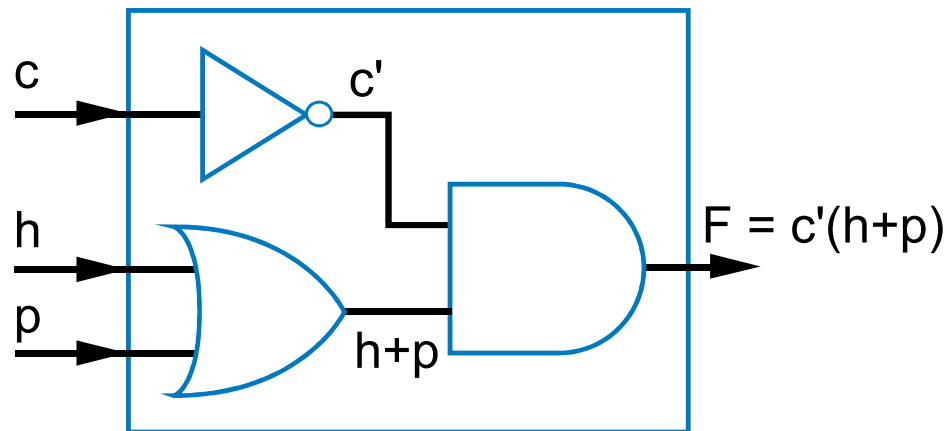
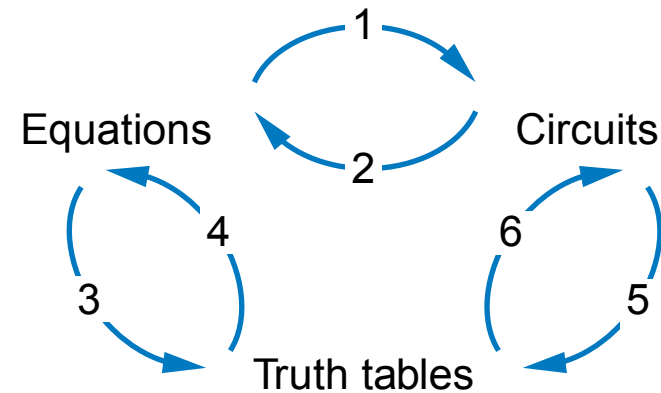
a	b	c	d	F
0	0	0	0	
0	0	0	1	
0	0	1	0	
0	0	1	1	
0	1	0	0	
0	1	0	1	
0	1	1	0	
0	1	1	1	
1	0	0	0	
1	0	0	1	
1	0	1	0	
1	0	1	1	
1	1	0	0	
1	1	0	1	
1	1	1	0	
1	1	1	1	

(c)

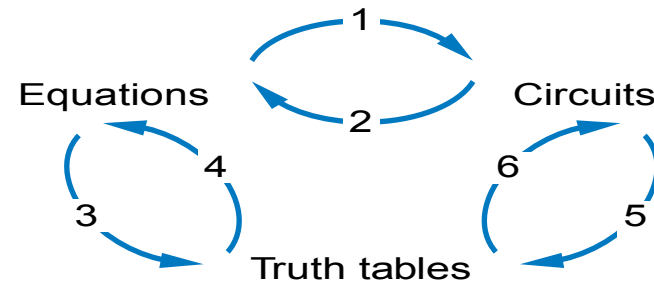


Converting among Representations

- Can convert from any representation to another
- Common conversions
 - Equation to circuit (we did this earlier)
 - Circuit to equation
 - Start at inputs, write expression of each gate output



Converting among Representations



- More common conversions
 - Truth table to equation (which we can then convert to circuit)
 - Easy—just OR each input term that should output 1
 - Equation to truth table
 - Easy—just evaluate equation for each input combination (row)
 - Creating intermediate columns helps

Inputs		Outputs	Term
a	b	F	F = sum of
0	0	1	$a'b'$
0	1	1	$a'b$
1	0	0	
1	1	0	

$$F = a'b' + a'b$$

Q: Convert to equation

a	b	c	F	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	1	$ab'c$
1	1	0	1	abc'
1	1	1	1	abc

$$F = ab'c + abc' + abc$$

Q: Convert to truth table: $F = a'b' + a'b$

Inputs				Output
a	b	$a'b'$	$a'b$	F
0	0	1	0	1
0	1	0	1	1
1	0	0	0	0
1	1	0	0	0



Example: Converting from Truth Table to Equation

- Parity bit: Extra bit added to data, intended to enable detection of error (a bit changed unintentionally)
 - e.g., errors can occur on wires due to electrical interference
- Even parity: Set parity bit so total number of 1s (data + parity) is even
 - e.g., if data is 001, parity bit is 1
→ 0011 has even number of 1s
- Want equation, but easiest to start from truth table for this example

a	b	c	P
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

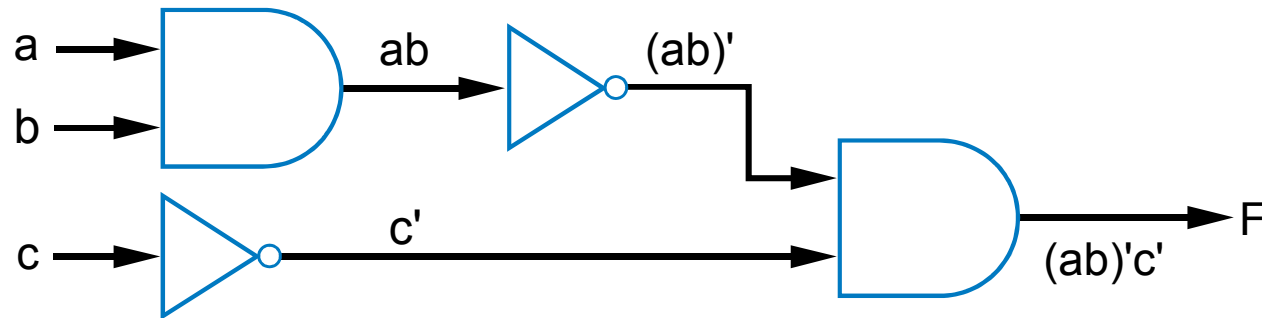
Convert to eqn.

$$P = a'b'c + a'bc' + ab'c' + abc$$



Example: Converting from Circuit to Truth Table

- First convert to circuit to equation, then equation to table



Inputs						Outputs
a	b	c	ab	$(ab)'$	c'	F
0	0	0	0	1	1	1
0	0	1	0	1	0	0
0	1	0	0	1	1	1
0	1	1	0	1	0	0
1	0	0	0	1	1	1
1	0	1	0	1	0	0
1	1	0	1	0	1	0
1	1	1	1	0	0	0



Standard Representation: Truth Table

- How can we determine if two functions are the same?
 - Recall automatic door example
 - Same as $f = hc' + h'pc'$?
 - Used algebraic methods
 - But if we failed, does that prove *not* equal? No.
- Solution: Convert to truth tables
 - Only ONE truth table representation of a given function
 - Standard** representation—for given function, only one version in standard form exists

$$f = c'hp + c'hp' + c'h'$$

$$f = c'h(p + p') + c'h'p$$

$$f = c'h(1) + c'h'p$$

$$f = c'h + c'h'p$$

(what if we stopped here?)

$$f = hc' + h'pc'$$

Q: Determine if $F=ab+a'$ is same function as $F=a'b'+a'b+ab$, by converting each to truth table first

F = ab + a'			F = a'b' + a'b + ab		
a	b	F	a	b	F
0	0	1	0	0	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	1	1	1

Same



Truth Table Canonical Form

- Q: Determine via truth tables whether $ab + a'$ and $(a+b)'$ are equivalent

$F = ab + a'$			$F = (a+b)'$		
a	b	F	a	b	F
0	0	1	0	0	1
0	1	1	0	1	0
1	0	0	1	0	0
1	1	1	1	1	0

Not equivalent



Canonical Form – Sum of Minterms

- Truth tables too big for numerous inputs
- Use standard form of equation instead
 - Known as **canonical form**
 - Regular algebra: group terms of polynomial by power
 - $ax^2 + bx + c$ ($3x^2 + 4x + 2x^2 + 3 + 1 \rightarrow 5x^2 + 4x + 4$)
 - Boolean algebra: create sum of minterms
 - **Minterm**: product term with every function literal appearing exactly once, in true or complemented form
 - Just multiply-out equation until sum of product terms
 - Then expand each term until all terms are minterms

Q: Determine if $F(a,b)=ab+a'$ is equivalent to $F(a,b)=a'b'+a'b+ab$, by converting first equation to canonical form (second already is)

$$F = ab+a' \text{ (already sum of products)}$$

$$a \quad F = ab + a'(b+b') \text{ (expanding term)}$$

$$F = ab + a'b + a'b' \text{ (Equivalent – same three terms as other equation)}$$



Canonical Form – Sum of Minterms

- Q: Determine whether the functions $G(a,b,c,d,e) = abcd + a'bcde$ and $H(a,b,c,d,e) = abcde + abcde' + a'bcde + a'bcde(a' + c)$ are equivalent.

$$G = abcd + a'bcde$$

$$G = abcd(e+e') + a'bcde$$

$$G = abcde + abcde' + a'bcde$$

$$G = a'bcde + abcde' + abcde \quad (\text{sum of minterms form})$$

Equivalent

$$H = abcde + abcde' + a'bcde + a'bcde(a' + c)$$

$$H = abcde + abcde' + a'bcde + a'bcdea' + a'bcdec$$

$$H = abcde + abcde' + a'bcde + a'bcde + a'bcde$$

$$H = abcde + abcde' + a'bcde$$

$$H = a'bcde + abcde' + abcde$$



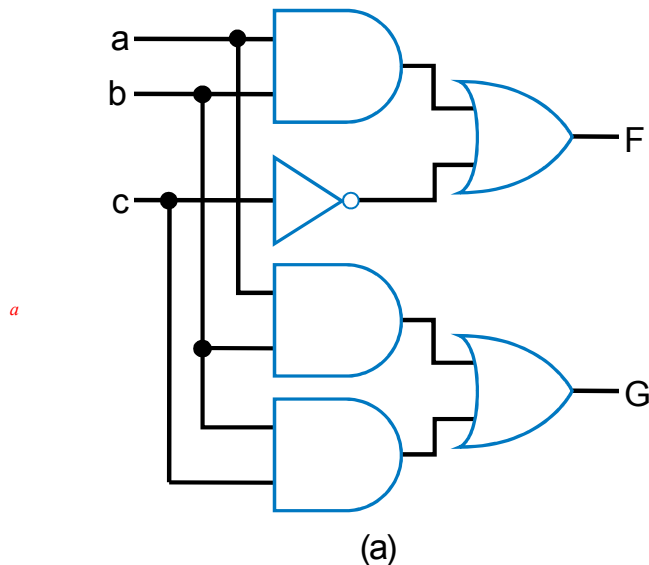
Compact Sum of Minterms Representation

- List each minterm as a number
- Number determined from the binary representation of its variables' values
 - $a'b c d e$ corresponds to 01111, or 15
 - $a b c d e'$ corresponds to 11110, or 30
 - $a b c d e$ corresponds to 11111, or 31
- Thus, $H = a'b c d e + a b c d e' + a b c d e$ can be written as:
 - $H = \sum m(15, 30, 31)$
 - "H is the sum of minterms 15, 30, and 31"

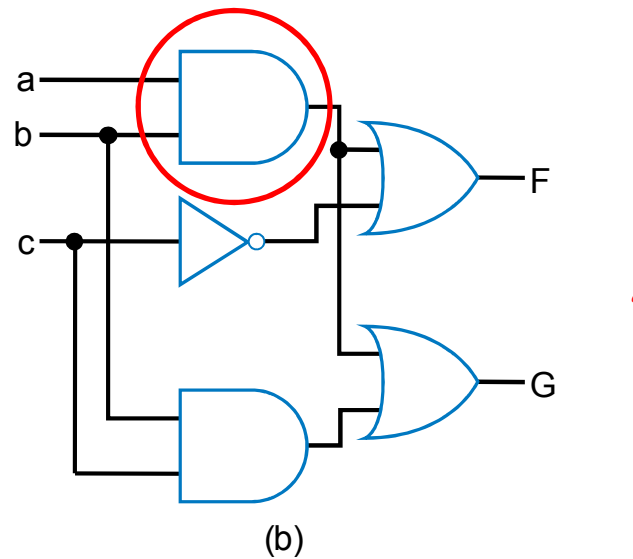


Multiple-Output Circuits

- Many circuits have more than one output
- Can give each a separate circuit, or can share gates
- Ex: $F = \underline{ab} + c'$, $G = \underline{ab} + bc$



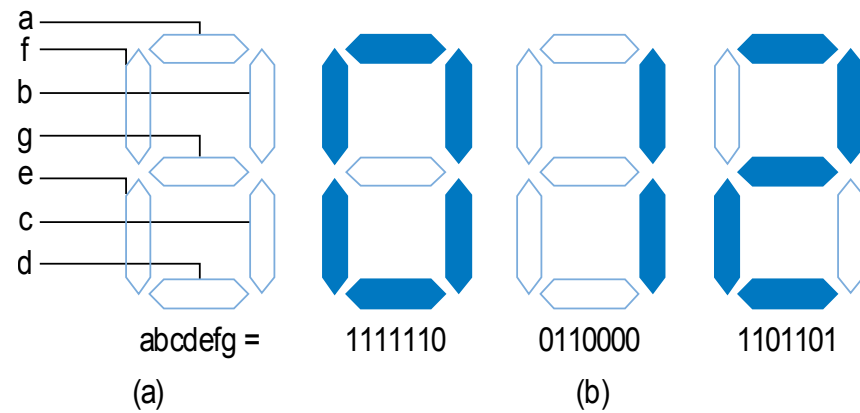
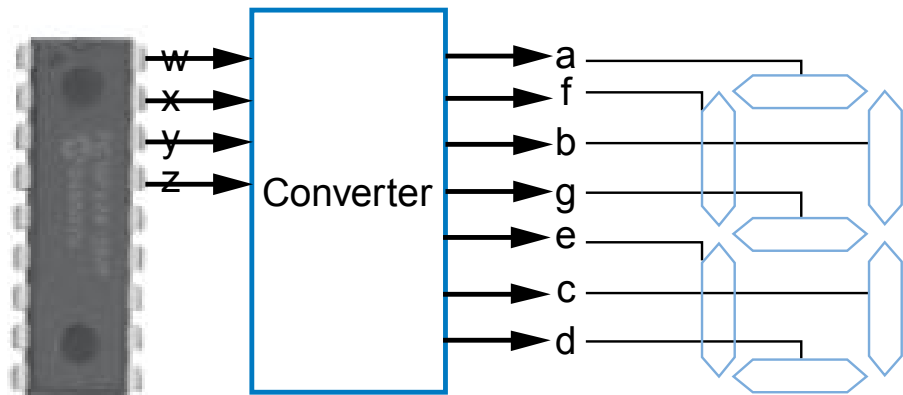
Option 1: Separate circuits



Option 2: Shared gates



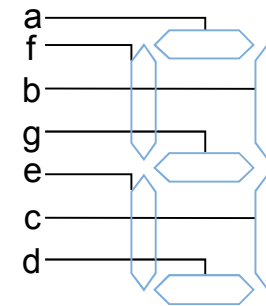
Multiple-Output Example: BCD to 7-Segment Converter



Multiple-Output Example: BCD to 7-Segment Converter

TABLE 2-4 4-bit binary number to seven-segment display truth table

w	x	y	z	a	b	c	d	e	f	g
0	0	0	0	1	1	1	1	1	1	0
0	0	0	1	0	1	1	0	0	0	0
0	0	1	0	1	1	0	1	1	0	1
0	0	1	1	1	1	1	1	0	0	1
0	1	0	0	0	1	1	0	0	1	1
0	1	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	0	0	0	0
1	0	0	0	1	1	1	1	1	1	1
1	0	0	1	1	1	1	1	0	1	1
1	0	1	0	0	0	0	0	0	0	0
1	0	1	1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	0	0	0



$$a = w'x'y'z' + w'x'yz' + w'x'yz + w'xy'z + w'xyz' + w'xyz + wx'y'z' + wx'y'z$$

$$b = w'x'y'z' + w'x'y'z + w'x'yz' + w'x'yz + w'xy'z' + w'xyz + wx'y'z' + wx'y'z$$

...



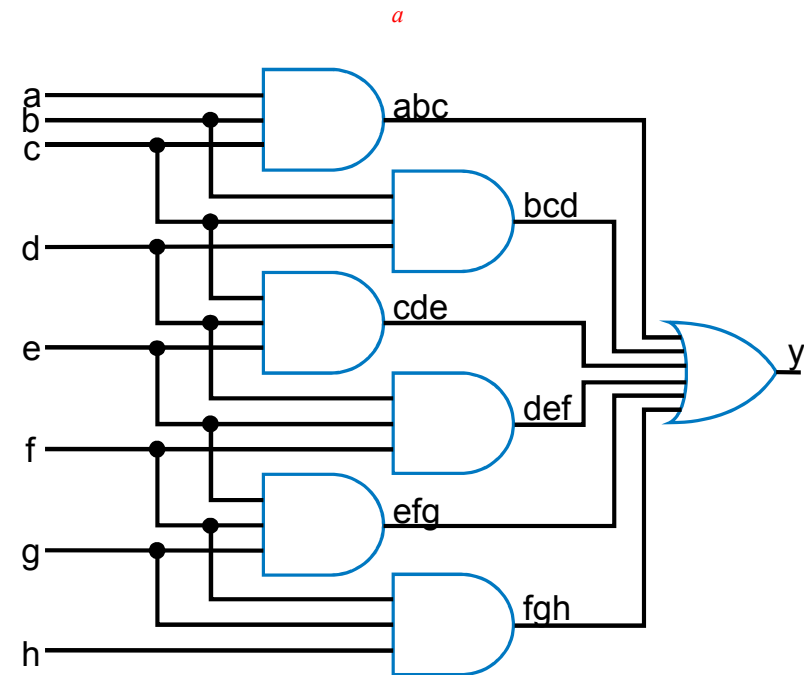
Combinational Logic Design Process

Step	Description
Step 1: Capture behavior Capture the function	Create a truth table or equations, <i>whichever is most natural for the given problem</i> , to describe the desired behavior of each output of the combinational logic.
Step 2: Convert to circuit 2A: Create equations 2B: Implement as a gate-based circuit	<p>This substep is only necessary if you captured the function using a truth table instead of equations. Create an equation for each output by ORing all the minterms for that output. Simplify the equations if desired.</p> <p>For each output, create a circuit corresponding to the output's equation. (Sharing gates among multiple outputs is OK optionally.)</p>



Example: Three 1s Pattern Detector

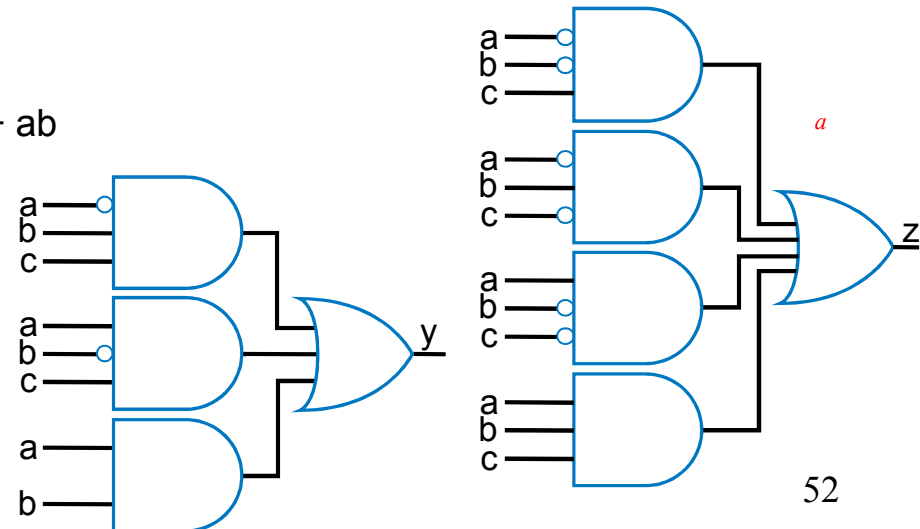
- Problem: Detect three consecutive 1s in 8-bit input: abcdefgh
 - 00011101 → 1
 - 10101011 → 0
 - 11110000 → 1
- **Step 1: Capture** the function
 - Truth table or equation?
 - Truth table too big: $2^8=256$ rows
 - Equation: create terms for each possible case of three consecutive 1s
 - **$y = abc + bcd + cde + def + efg + fgh$**
- **Step 2a: Create** equation -- already done
- **Step 2b: Implement** as a gate-based circuit



Example: Number of 1s Counter

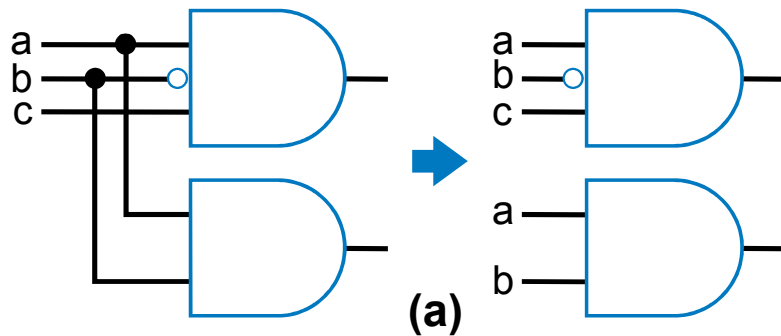
- Problem: Output in binary on two outputs yz the # of 1s on three inputs
 - 010 → 01
 - 101 → 10
 - 000 → 00
- **Step 1: Capture** the function
 - Truth table or equation?
 - Truth table is straightforward
- **Step 2a: Create** equations
 - $y = a'bc + ab'c + abc' + abc$
 - $z = a'b'c + a'bc' + ab'c' + abc$
 - Optional: Let's simplify y:
 - $y = a'bc + ab'c + ab(c' + c) = a'bc + ab'c + ab$
- **Step 2b: Implement** as a gate-based circuit

Inputs			(# of 1s)	Outputs	
a	b	c		y	z
0	0	0	(0)	0	0
0	0	1	(1)	0	1
0	1	0	(1)	0	1
0	1	1	(2)	1	0
1	0	0	(1)	0	1
1	0	1	(2)	1	0
1	1	0	(2)	1	0
1	1	1	(3)	1	1

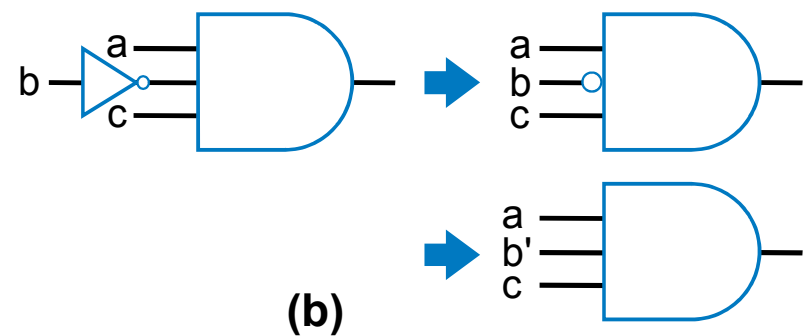


Simplifying Notations

- Used in previous circuit



List inputs multiple times
→ Less wiring in drawing

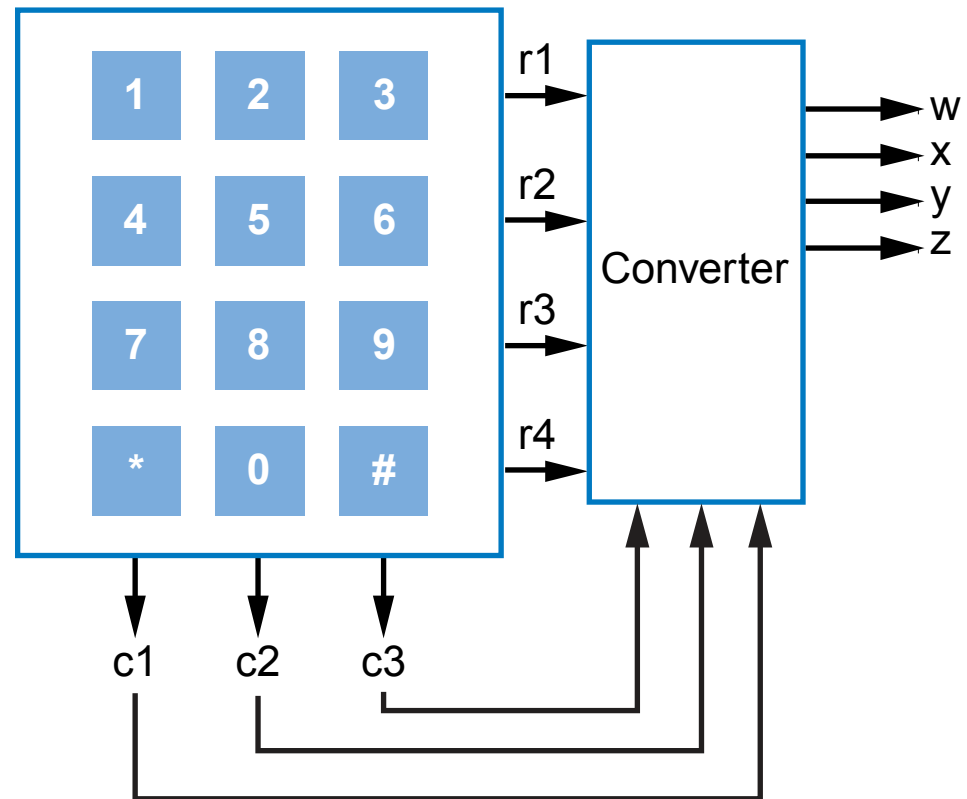


Draw inversion bubble
rather than inverter. Or list
input as complemented.



Example: Keypad Converter

- Keypad has 7 outputs
 - One per row
 - One per column
- Key press sets one row and one column output to 1
 - Press "5" \rightarrow r2=1, c2=1
- Goal: Convert keypad outputs into 4-bit binary number
 - 0-9 \rightarrow 0000 to 1001
 - * \rightarrow 1010, # \rightarrow 1011
 - nothing pressed: 1111



Example: Keypad Converter

- Step 1: Capture behavior
 - Truth table too big (2^7 rows); equations not clear either
 - Informal table can help

TABLE 2.7 Informal table for the 12-button keypad to 4-bit code converter.

Button	Signals	4-bit code outputs			
		w	x	y	z
1	r1 c1	0	0	0	1
2	r1 c2	0	0	1	0
3	r1 c3	0	0	1	1
4	r2 c1	0	1	0	0
5	r2 c2	0	1	0	1
6	r2 c3	0	1	1	0
7	r3 c1	0	1	1	1

Button	Signals	4-bit code outputs			
		w	x	y	z
8	r3 c2	1	0	0	0
9	r3 c3	1	0	0	1
*	r4 c1	1	0	1	0
0	r4 c2	0	0	0	0
#	r4 c3	1	0	1	1
(none)		1	1	1	1

Step 2b: Implement^a as circuit (note sharable gates) ...

$$w = r3c2 + r3c3 + r4c1 + r4c3 + r1'r2'r3'r4'c1'c2'c3'$$

$$x = r2c1 + r2c2 + r2c3 + r3c1 + r1'r2'r3'r4'c1'c2'c3'$$

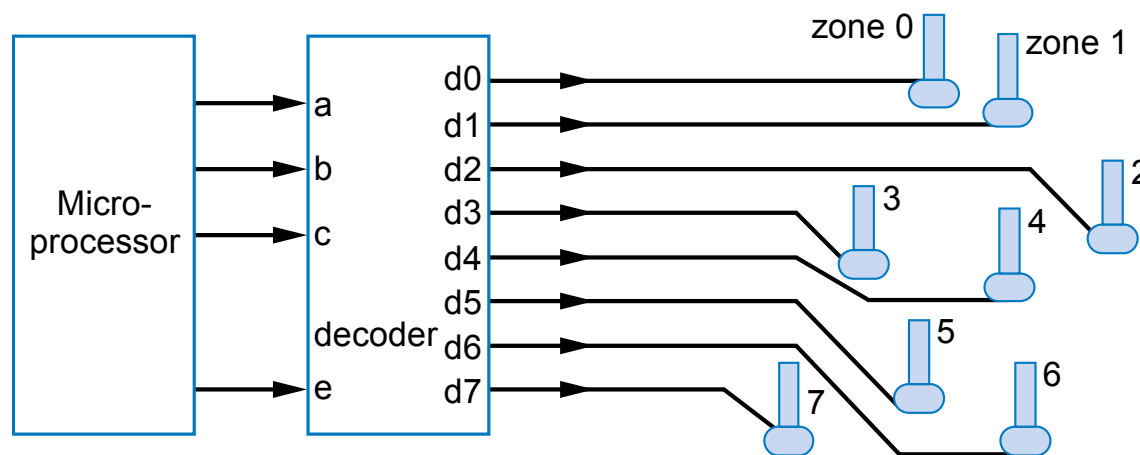
$$y = r1c2 + r1c3 + r2c3 + r3c1 + r4c1 + r4c3 + r1'r2'r3'r4'c1'c2'c3'$$

$$z = r1c1 + r1c3 + r2c2 + r3c1 + r3c3 + r4c3 + r1'r2'r3'r4'c1'c2'c3'$$



Example: Sprinkler Controller

- Microprocessor outputs which zone to water (e.g., cba=110 means zone 6) and enables watering (e=1)
- Decoder should set appropriate valve to 1



Step 1: Capture behavior

$$d0 = a'b'c'e$$

$$d1 = a'b'ce$$

$$d2 = a'bc'e$$

$$d3 = a'bce$$

$$d4 = ab'c'e$$

$$d5 = ab'ce$$

$$d6 = abc'e$$

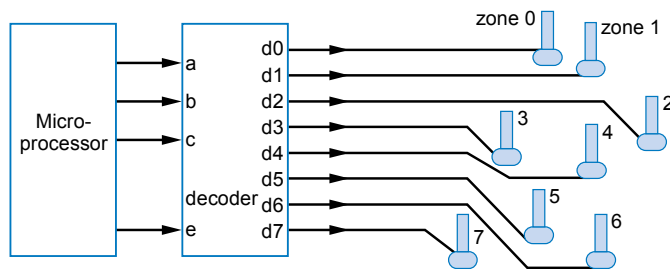
$$d7 = abce$$

*Equations seem like
a natural fit*



Example: Sprinkler Controller

- Step 2b: Implement as circuit



$$d0 = a'b'c'e$$

$$d1 = a'b'ce$$

$$d2 = a'bc'e$$

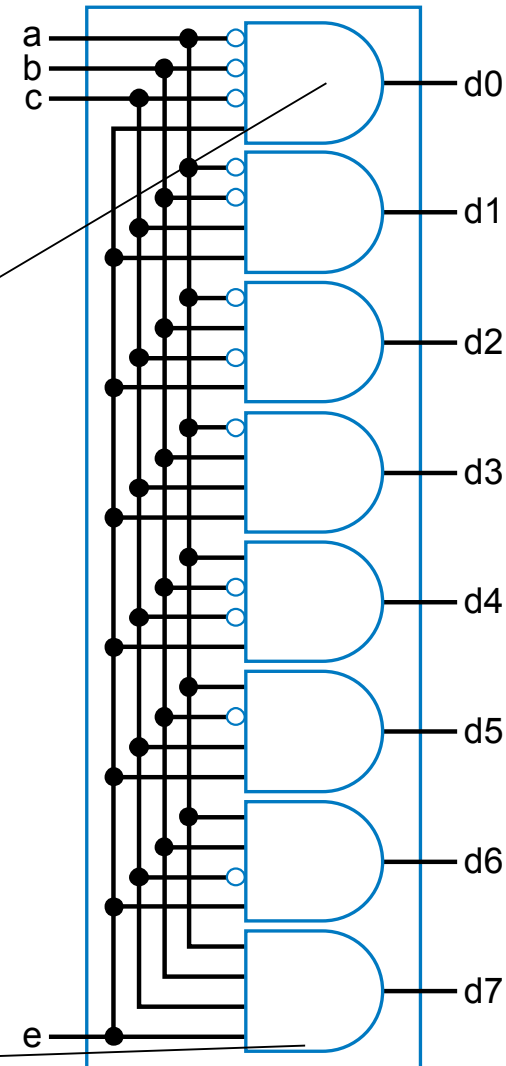
$$d3 = a'bce$$

$$d4 = ab'c'e$$

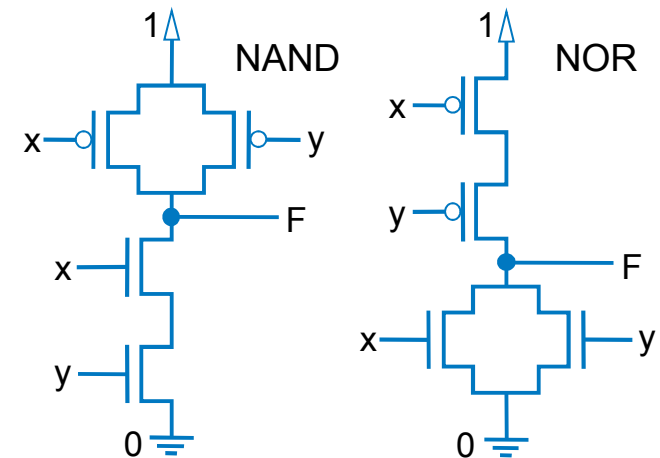
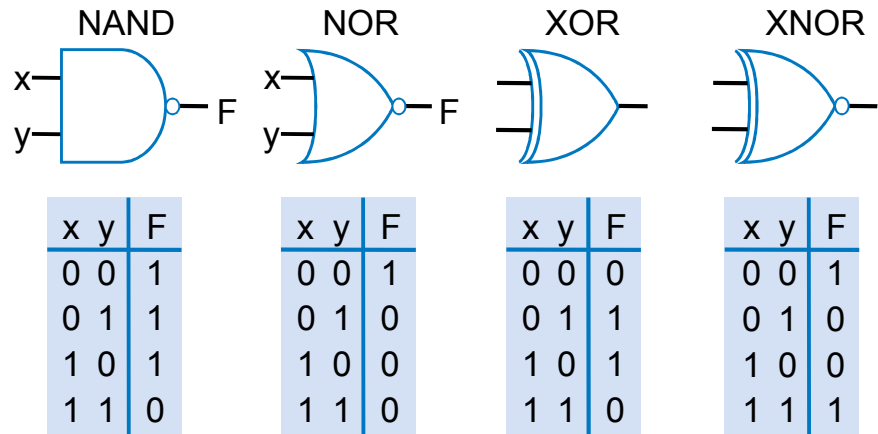
$$d5 = ab'ce$$

$$d6 = abc'e$$

$$d7 = abce$$



More Gates



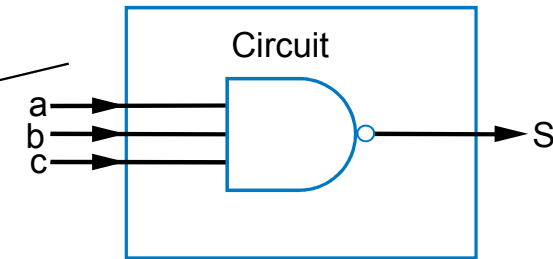
- NAND: Opposite of AND (“NOT AND”)
- NOR: Opposite of OR (“NOT OR”)
- XOR: Exactly 1 input is 1, for 2-input XOR. (For more inputs -- odd number of 1s)
- XNOR: Opposite of XOR (“NOT XOR”)
- NAND same as AND with power & ground switched
 - nMOS conducts 0s well, but not 1s (reasons beyond our scope) – so NAND is more efficient
- Likewise, NOR same as OR with power/ground switched
- NAND/NOR more common
- AND in CMOS: NAND with NOT
- OR in CMOS: NOR with NOT



More Gates: Example Uses

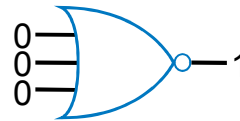
- Aircraft lavatory sign example

- $S = (abc)'$



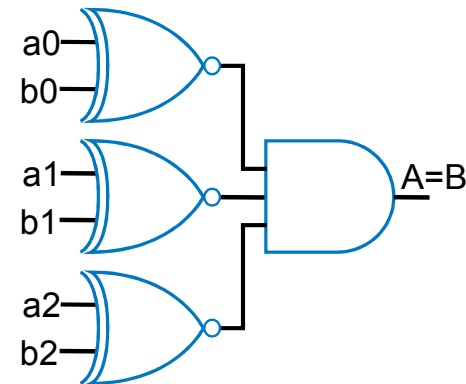
- Detecting all 0s

- Use NOR



- Detecting equality

- Use XNOR



- Detecting odd # of 1s

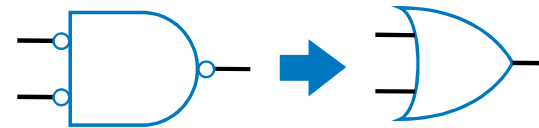
- Use XOR

- Useful for generating “parity” bit common for detecting errors



Completeness of NAND

- Any Boolean function can be implemented *using just NAND gates*. Why?
 - Need AND, OR, and NOT
 - NOT: 1-input NAND (or 2-input NAND with inputs tied together)
 - AND: NAND followed by NOT
 - OR: NAND preceded by NOTs
 - Thus, NAND is a universal gate
 - Can implement any circuit using just NAND gates
- Likewise for NOR



Number of Possible Boolean Functions

- How many possible functions of 2 variables?
 - 2^2 rows in truth table, 2 choices for each
 - $2^{(2^2)} = 2^4 = 16$ possible functions
- N variables
 - 2^N rows
 - $2^{(2^N)}$ possible functions

a	b	F
0	0	0 or 1 2 choices
0	1	0 or 1 2 choices
1	0	0 or 1 2 choices
1	1	0 or 1 2 choices

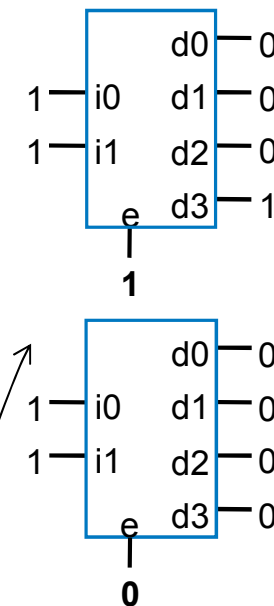
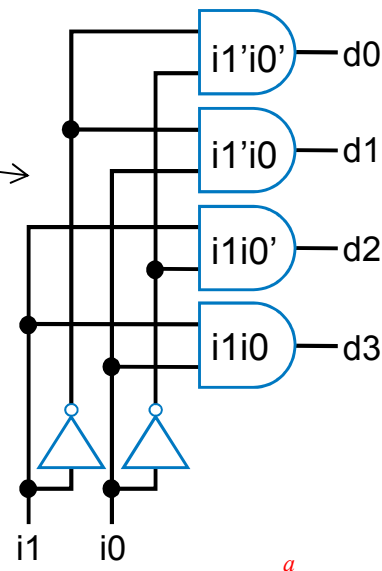
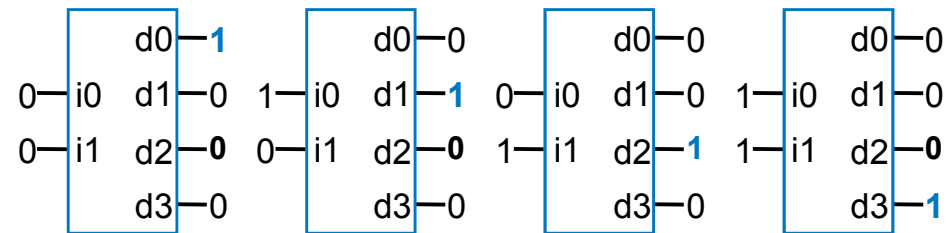
$2^4 = 16$
possible functions

a	b	f0	f1	f2	f3	f4	f5	f6	f7	f8	f9	f10	f11	f12	f13	f14	f15
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
		0	a AND b		a		b	a XOR b	a OR b	a NOR b	a XNOR b	b'		a'		a NAND b	1



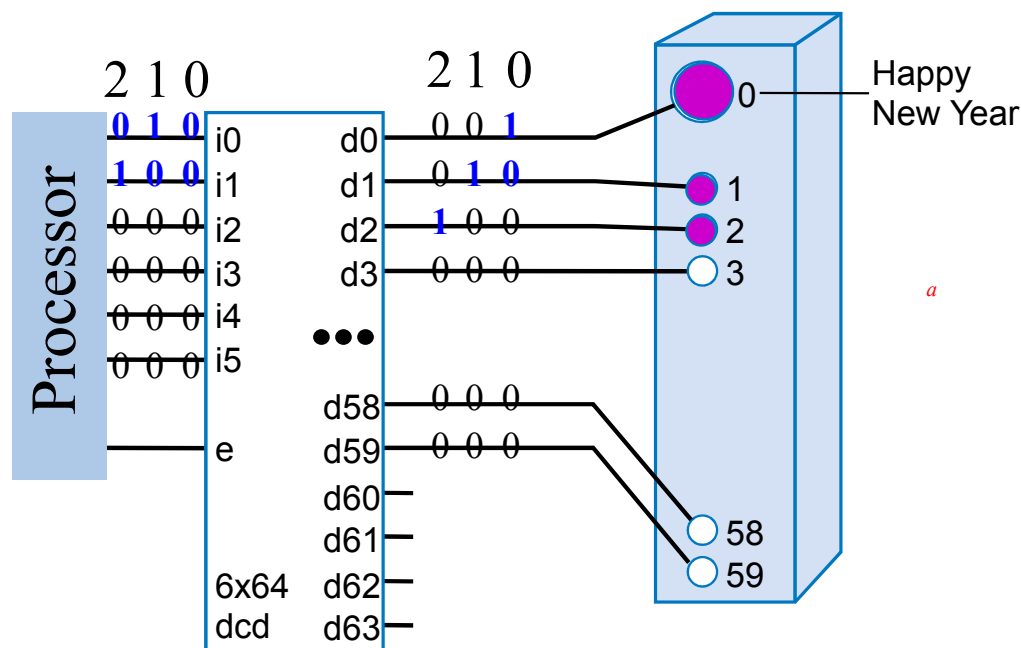
Decoders and Muxes

- **Decoder:** Popular combinational logic building block, in addition to logic gates
 - Converts input binary number to one high output
- 2-input decoder: four possible input binary numbers
 - So has four outputs, one for each possible input binary number
- Internal design
 - AND gate for each output to detect input combination
- Decoder with enable e
 - Outputs all 0 if $e=0$
 - Regular behavior if $e=1$
- n -input decoder: 2^n outputs



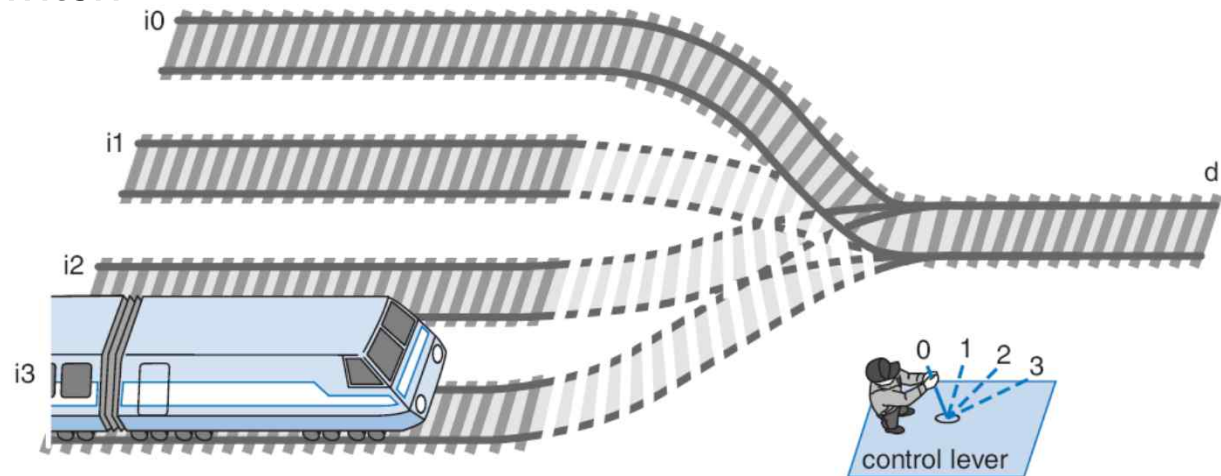
Decoder Example

- New Year's Eve Countdown Display
 - Microprocessor counts from 59 down to 0 in binary on 6-bit output
 - Want illuminate one of 60 lights for each binary number
 - Use 6x64 decoder
 - 4 outputs unused

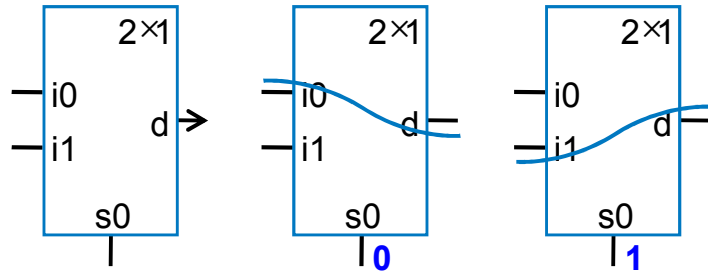


Multiplexor (Mux)

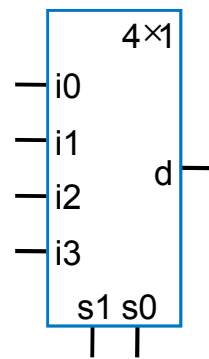
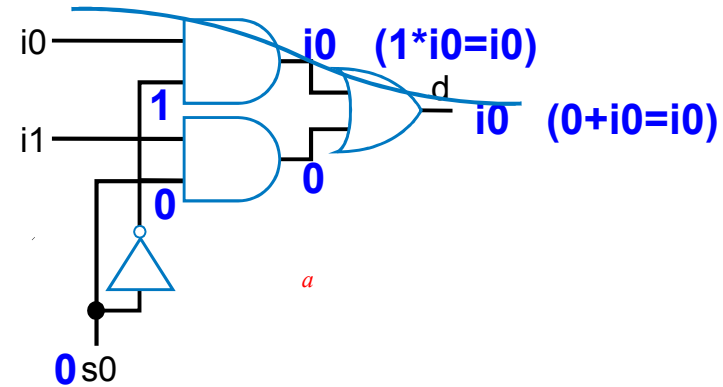
- Mux: Another popular combinational building block
 - Routes one of its N data inputs to its one output, based on binary value of select inputs
 - 4 input mux \rightarrow needs 2 select inputs to indicate which input to route through
 - 8 input mux \rightarrow 3 select inputs
 - N inputs $\rightarrow \log_2(N)$ selects
 - Like a rail yard switch



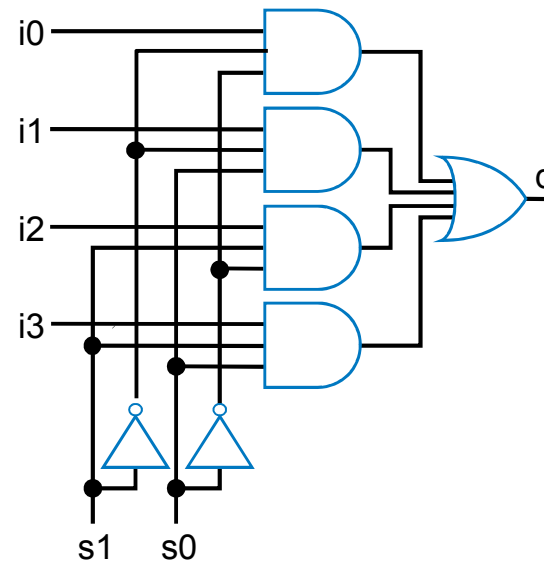
Mux Internal Design



2x1 mux

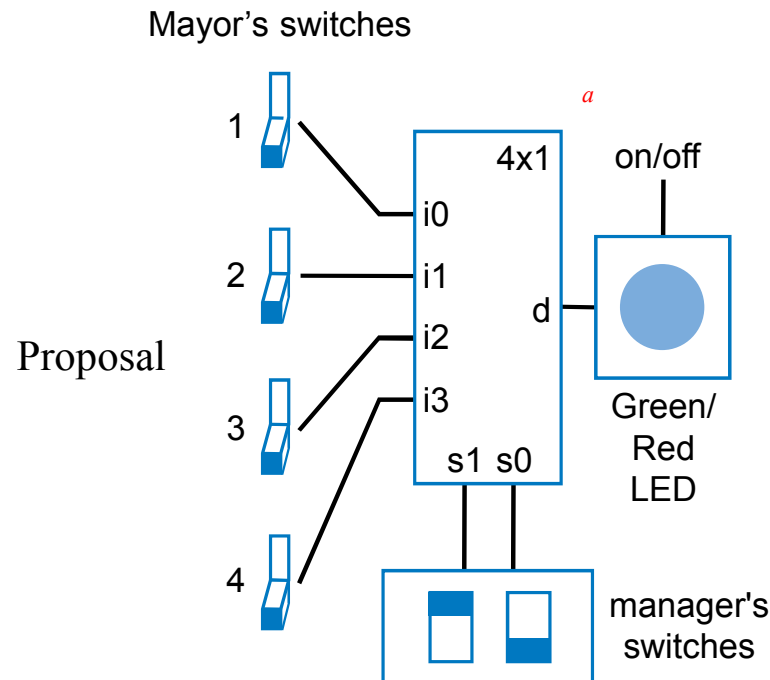


4x1 mux

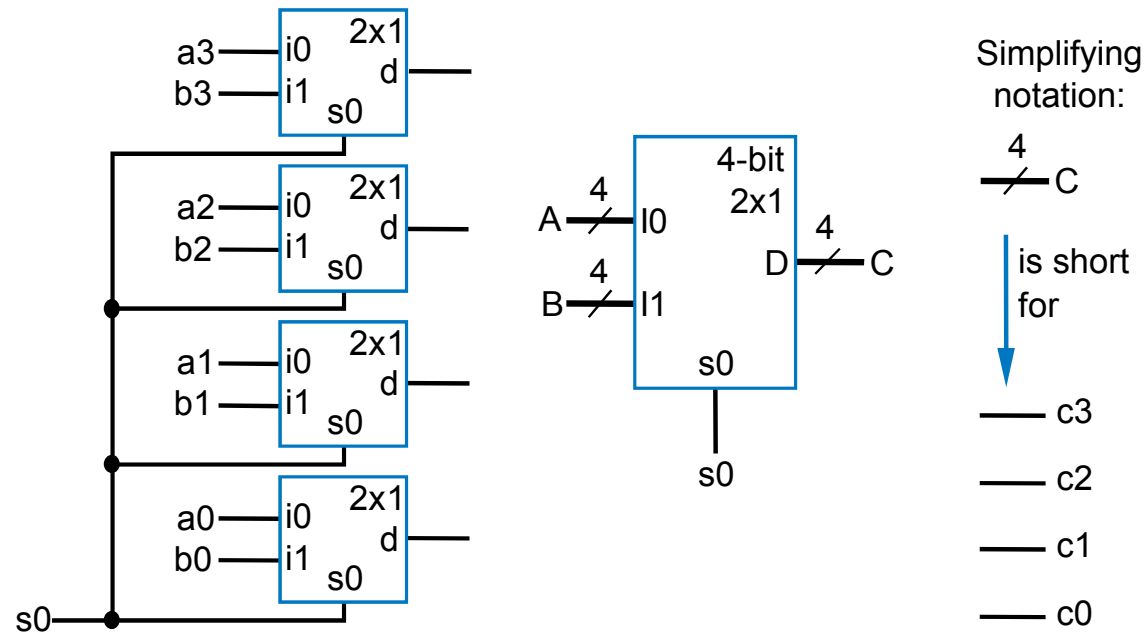


Mux Example

- City mayor can set four switches up or down, representing his/her vote on each of four proposals, numbered 0, 1, 2, 3
- City manager can display any such vote on large green/red LED (light) by setting two switches to represent binary 0, 1, 2, or 3
- Use 4x1 mux



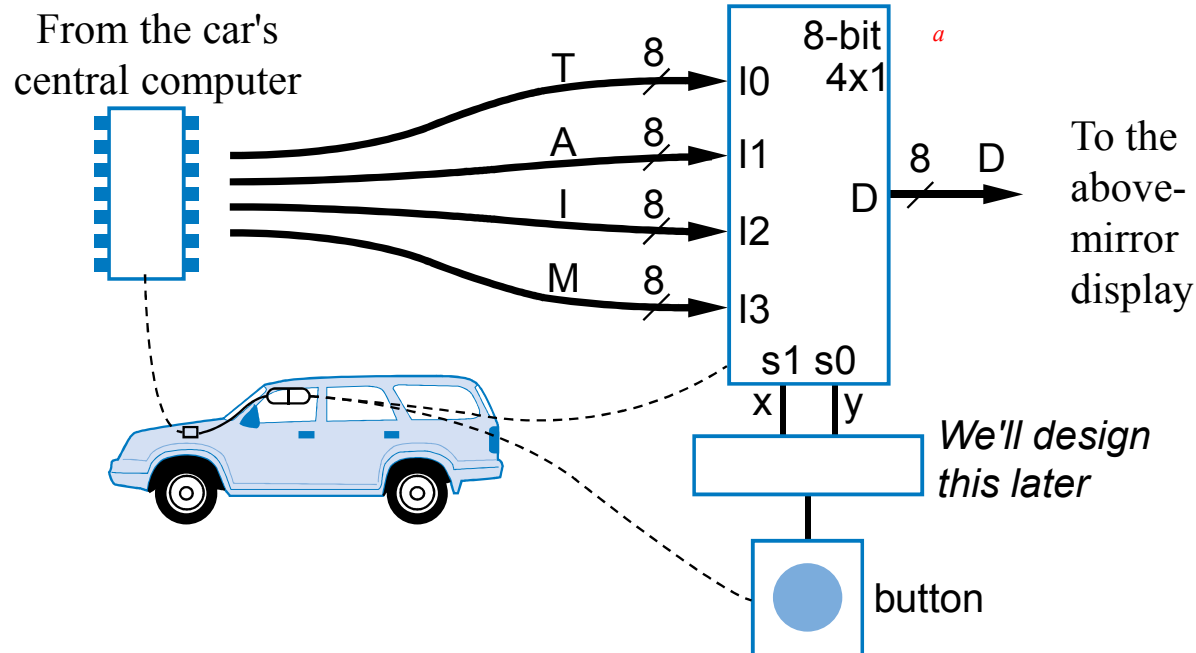
Muxes Commonly Together – N-bit Mux



- Ex: Two 4-bit inputs, A ($a_3 a_2 a_1 a_0$), and B ($b_3 b_2 b_1 b_0$)
 - 4-bit 2x1 mux (just four 2x1 muxes sharing a select line) can select between A or B



N-bit Mux Example

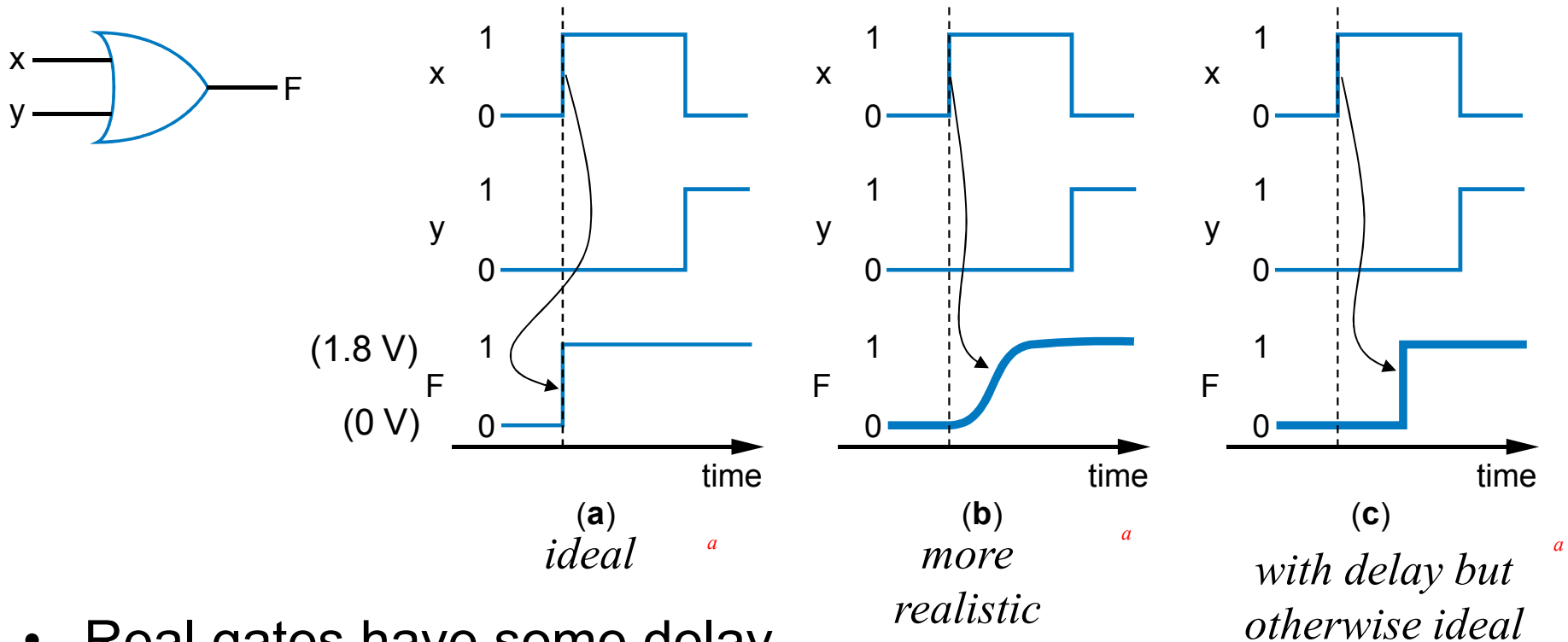


- Four possible display items
 - Temperature (T), Average miles-per-gallon (A), Instantaneous mpg (I), and Miles remaining (M) – each is 8-bits wide
 - Choose which to display on D using two inputs x and y
 - Pushing button sequences to the next item
 - Use 8-bit 4x1 mux



Additional Considerations

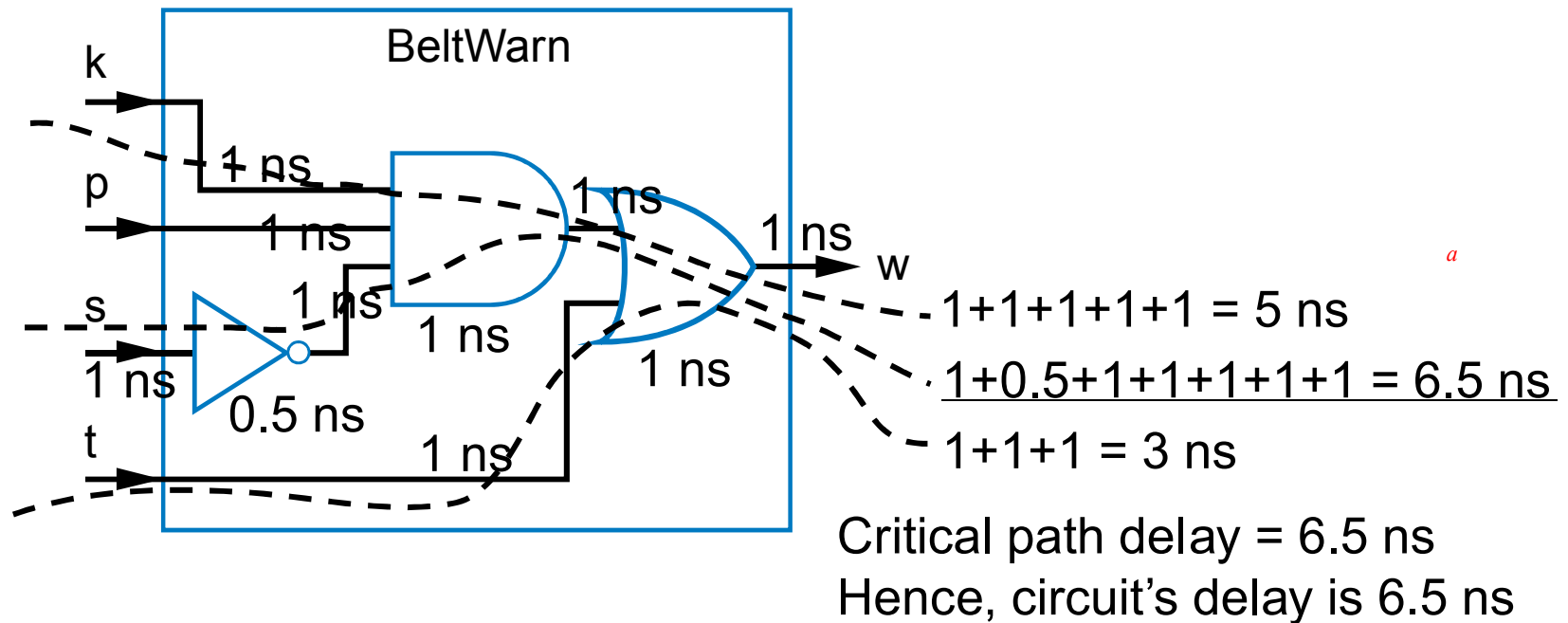
Non-Ideal Gate Behavior -- Delay



- Real gates have some delay
 - Outputs don't change immediately after inputs change



Circuit Delay and Critical Path

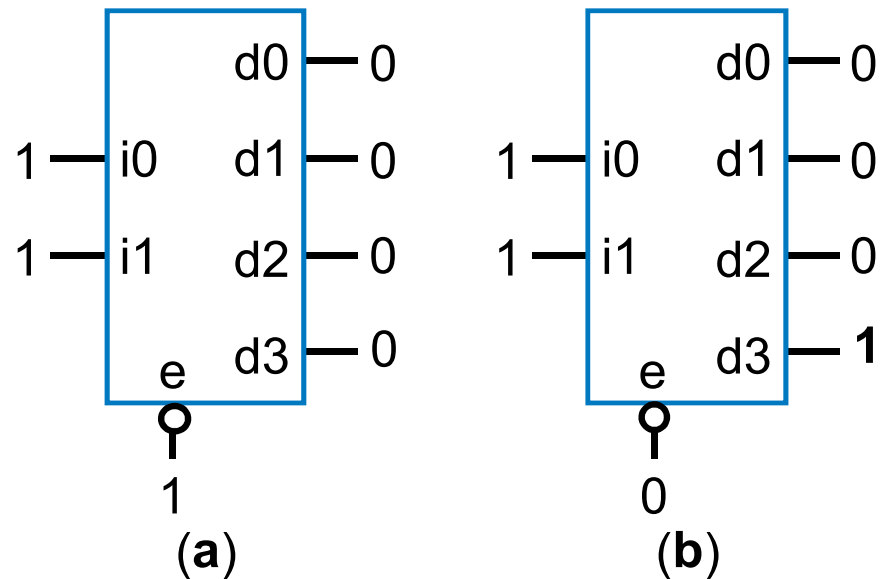


- Wires also have delay
- Assume gates and wires have delays as shown
- Path delay – time for input to affect output
- Critical path – path with longest path delay
- Circuit delay – delay of critical path

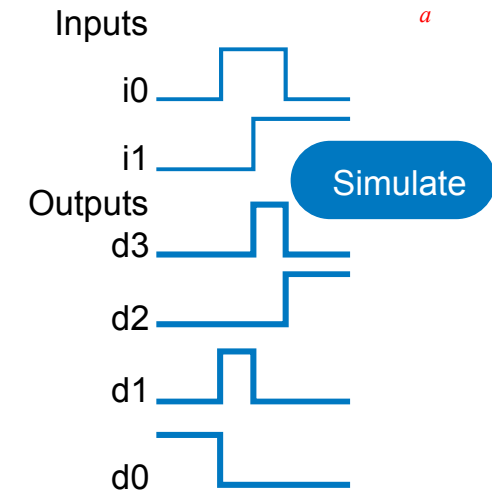
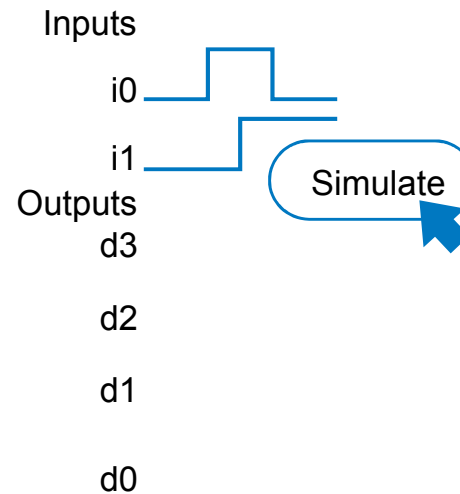
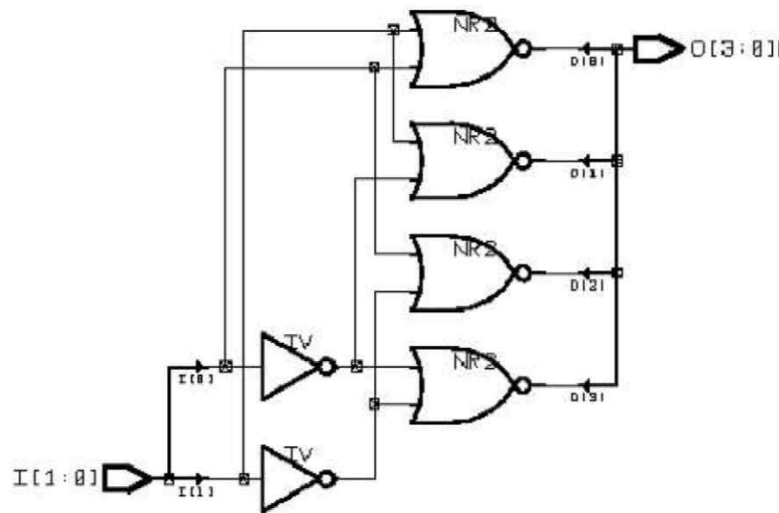


Active Low Inputs

- Data inputs: flow through component (e.g., mux data input)
- Control input: influence component behavior
 - Normally active high – 1 causes input to carry out its purpose
 - Active low – Instead, 0 causes input to carry out its purpose
 - Example: 2x4 decoder with active low enable
 - 1 disables decoder, 0 enables
 - Drawn using inversion bubble



Schematic Capture and Simulation



- **Schematic capture**
 - Computer tool for user to capture logic circuit graphically
- **Simulator**
 - Computer tool to show what circuit outputs would be for given inputs
 - Outputs commonly displayed as **waveform**



Chapter Summary

- Combinational circuits
 - Circuit whose outputs are function of present inputs
 - No “state”
- Switches: Basic component in digital circuits
- Boolean logic gates: AND, OR, NOT – Better building block than switches
 - Enables use of Boolean algebra to design circuits
- Boolean algebra: Uses true/false variables/operators
- Representations of Boolean functions: Can translate among
- Combinational design process: Translate from equation (or table) to circuit through well-defined steps
- More gates: NAND, NOR, XOR, XNOR also useful
- Muxes and decoders: Additional useful combinational building blocks

