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Second-order wavemaker theory for multidirectional waves

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Abstract

The present paper develops the complete second-order wavemaker theory for the generation of multidirectional waves in a semi-infinite basin. The theory includes superharmonics and subharmonics and is valid for a rotational as well as a translatory serpent-type wave-board motion. The primary goal is to obtain the second-order motion of the wave paddles required to get a prescribed multidirectional irregular wave field correct to second order, i.e. to suppress spurious free-wave generation. The wavemaker theory is a 3D extension of the full second-order wavemaker theory for wave flumes by Schäffer (1996).

Keywords: Wave generation; Wavemaker; Second order; Stokes theory; Evanescent modes; Multidirectional waves; Short-crested waves; Superharmonics; Subharmonics

1. Introduction

Waves at frequencies outside the usual primary-wave spectrum are important for several phenomena along the coast and in the ocean. A few examples with engineering significance are those of long waves causing problems for ships at berth and high-frequency waves exciting offshore structures. One of the sources of the long waves is the non-linear subharmonic interaction between primary wave components at difference frequencies. Similar primary-wave interactions that occur at sum-frequencies (superharmonics) affect the high-frequency tail of the spectrum. Second-order wavemaker theory is required for the correct laboratory generation of both subharmonic and superharmonic waves. This paper derives the relevant theory under

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the assumption of a serpent-type wavemaker of infinite length. Some steps in this direction were taken by Suh and Dalrymple (1987). The underlying wave theory complies with a second-order Stokes type development (Sharma and Dean, 1979, 1981; Dalrymple, 1989).

First-order wavemaker theory for unidirectional regular waves corresponding to linearized Stokes theory has been known for almost a century and reference is made to the review by Svendsen (1985) and the book by Hughes (1993, Chapter 7). Numerous authors have contributed to the derivation of second-order wavemaker theory for unidirectional regular and irregular waves. Schäffer (1993a, 1996) used the compact notation provided by complex notation (Gilbert, 1976) and re-derived the theory for irregular waves considering both superharmonic and subharmonic interactions. Several errors and omissions in previous theories were found. For example, only the regular-wave theories by Fontanet (1961) and Hudspeth and Sulisz (1991) (see also Sulisz and Hudspeth, 1993), and the irregular-wave extension by Mobayed and Williams (1994) were found to include the complete evanescent-mode interaction potential.

First-order theory for the generation of oblique waves by two wavemaker segments between lateral walls was studied by Madsen (1974), while Takayama (1984), Takayma and Hiraishi (1987) and Dalrymple and Greenberg (1985) used different methods for analysing waves generated by a single finite-width wavemaker. Dalrymple and Greenberg used their result as a starting point for application to several wavemaker arrangements via the superposition principle. Gilbert (1976) solved the wavemaker problem for an infinitely long serpent-type wavemaker (see also Dean and Dalrymple, 1991), and found that spurious waves were generated in case of finite segment width of a multidirectional wavemaker. Further analyses of the spurious waves were made by Sand (1979) and Schäffer (1998).

The boundary conditions from reflecting lateral sidewalls can be included directly in the wavemaker theory limiting the wave modes to the possible eigenmodes in the wave basin. This approach was taken by Madsen (1974) and Dalrymple (1989) and recently carried to second order for regular waves by Li and Williams (1998). At first order, Dalrymple further combined the theory with an analytical wave propagation model by which sidewalls would help in providing a plane wave at a desired distance from the wavemaker. This theory was verified by use of a numerical model by Mandsard et al. (1992). As a predecessor of Dalrymple's theory, Funke and Miles (1987) had introduced the so-called corner reflection method which has a similar purpose but neglects diffraction.

As an alternative to directly accounting for sidewalls in the generation theory, sidewall conditions can be met by constructing the appropriate solution from the theory of an infinitely long wavemaker. This could be carried out in conjunction with a wave model backtracking a desired wave field at some cross-section of the wave basin to the wavemaker. This perspective gives a wider scope of the present paper, which assumes an infinitely long wavemaker.

Active absorption is a highly desirable component of any wave generation system exposed to backscattered energy, see Schäffer and Klopman (2000) for a review. Specifically, in the context of second-order wave generation, there is little gained in a correct reproduction of long waves if unwanted re-reflections at the wavemaker ruin the incident wave field anyway. Our long-term goal is to combine active absorption and second-order wave generation for multidirectional waves.

The present paper develops the full second-order wavemaker theory for an infinitely long serpent-type wavemaker in a semi-infinite basin. The theory is derived for bichromatic and bidirectional waves and thus it forms the full basis for second-order generation of multidirectional irregular waves.

The theory of this paper was derived by hand and later checked independently by a symbolic calculation software package (Mathematica). The numerical results were computed using Mathematica and double checked by an independent Fortran code.

2. General theory

2.1. Governing equations

A three-dimensional irrotational flow in a homogeneous, incompressible and inviscid fluid is considered. Let the particle velocity component in the x-, y- and z-direction, respectively, $(u,v,w) = (\phi_x,\phi_y,\phi_z)$, define the velocity potential, $\phi = \phi(x,y,z,t)$, where t denotes the time, in a Cartesian coordinate system (x,y,z), cf. Fig. 1. By mass conservation, the governing equation for the velocity potential in the fluid domain becomes the Laplace equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \text{ in the fluid} \tag{1}$$

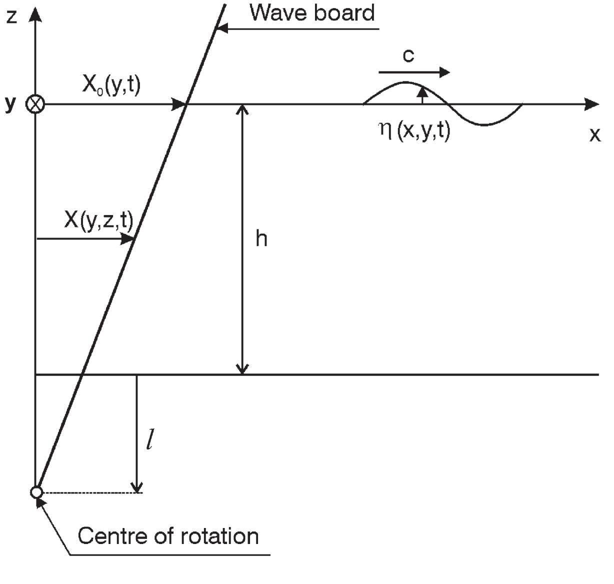


Fig. 1. Definition sketch.

The fluid domain is bounded by a free surface, a horizontal bottom, and a wavemaker. On the surface the nonlinear kinematic and dynamic boundary conditions are

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 \text{ for } z = \eta \tag{2}$$

$$g\eta + \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0 \text{ for } z = \eta \tag{3}$$

where $\eta = \eta(x,y,t)$ is the surface elevation, g is the acceleration due to gravity and the subscripts denote partial derivatives. The pressure at the surface is taken as zero. Eliminating the surface elevation by combining the kinematic and the dynamic surface boundary conditions gives

$$\begin{aligned} g\phi_z + \phi_{tt} = & -\frac{1}{2}\phi_x(\phi_x^2 + \phi_y^2 + \phi_z^2)_x - \frac{1}{2}\phi_y(\phi_x^2 + \phi_y^2 + \phi_z^2)_y \\ & -\frac{1}{2}\phi_z(\phi_x^2 + \phi_y^2 + \phi_z^2)_z - (\phi_x^2 + \phi_y^2 + \phi_z^2)_t \end{aligned} \quad \text{for } z = n \tag{4}$$

Note that this involves the chain rule, since the surface conditions are given along the surface and thus ϕ varies with time both directly and through the dependence on $z = \eta(x,y,t)$.

The horizontal bottom is impermeable and thus

$$\phi_z = 0 \text{ for } z = -h \tag{5}$$

where h is the still water depth. The kinematic boundary condition at the wavemaker is

$$\phi_x - X_z\phi_z - X_y\phi_y = X_t \text{ for } x = X(y,z,t) \tag{6}$$

The classic method of perturbation theory in combination with Taylor expansion of the boundary conditions at the free surface and at the wave board leads to a boundary value problem for the first- and second-order wave contributions, respectively. Thus the surface elevation, the velocity potential and the wave board position correct to second order (see eq. (21)) are given by the perturbation expansions

$$\eta = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} \tag{7}$$

$$\phi = \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} \tag{8}$$

$$X_0 = \varepsilon X_0^{(1)} + \varepsilon^2 X_0^{(2)} \tag{9}$$

where ε is a small ordering parameter relating to the wave steepness, and the superscripts (1) and (2) denote the order of the quantity. Introducing the perturbed quantities into the Taylor expanded boundary conditions and collecting terms of first and second order, the governing equations may be written

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \text{ in fluid} \tag{10}$$

$$g\phi_z + \phi_{tt} = R \text{ for } z = 0 \tag{11}$$

$$\phi_z = 0 \text{ for } z = -h \tag{12}$$

$$\phi_x = Q \text{ for } x = 0 \tag{13}$$

$$\eta = -\frac{1}{g}(\phi_t + P) \text{ for } z = 0 \tag{14}$$

where R, Q, P are given below. Eqs. (10)–(14) cover the first-order problem for $(\phi,\eta) = (\phi^{(1)},\eta^{(1)})$, where

$$R^{(1)} = 0 \tag{15}$$

$$Q^{(1)} = f(z)X_{0t}^{(1)} \tag{16}$$

$$P^{(1)} = 0 \tag{17}$$

and the second-order problem for $(\phi,\eta) = (\phi^{(2)},\eta^{(2)})$, where

$$R^{(2)} = -\{\eta^{(1)}(g\phi_z^{(1)} + \phi_{tt}^{(1)})_z + (\phi_x^{(1)2} + \phi_y^{(1)2} + \phi_z^{(1)2})_t\} \tag{18}$$

$$Q^{(2)} = -f(z)\{X_0^{(1)}\phi_{xx}^{(1)} - X_{0y}^{(1)}\phi_y^{(1)}\} + f'(z)X_0^{(1)}\phi_z^{(1)} + f(z)X_{0t}^{(2)} \tag{19}$$

$$P^{(2)} = \eta^{(1)}\phi_{zt}^{(1)} + \frac{1}{2}(\phi_x^{(1)2} + \phi_y^{(1)2} + \phi_z^{(1)2}) \tag{20}$$

The position of the wave board is given as

$$X(y,z,t) = f(z)X_0(y,t) \tag{21}$$

where $f(z)$ describes the type of wave board motion

$$f(z) = \begin{cases} 1 + \dfrac{z}{h + l} & \text{for } -(h-d) \leq z \leq 0 \\ 0 & \text{for } -h \leq z < -(h-d) \end{cases} \tag{22}$$

Here $z = -(h + l)$gives the centre of rotation $(-h < l \leq \infty)$, see Fig. 1. If the centre of rotation is at or above the bottom then $d \geq 0$ is the elevation of the hinge above the bottom, i.e. $d = -l$. If the centre of rotation is at or below the bottom, then d must be set to zero and the last case in eq. (22) becomes irrelevant. Fig. 2 shows

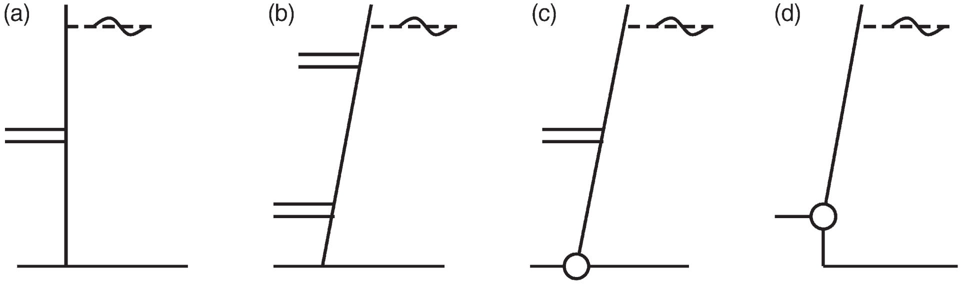


Fig. 2. The theory is developed for these types of wave board motions: (a) translatory (piston type) and (b)–(d) rotational with the centre of rotation (b) below, (c) at, (d) above the bottom.

the types of wavemakers considered. Thus we have chosen that either $d = -l$ or $d = 0$ (for $l \geq 0$). This ensures that only continuous shape functions $f(z)$ are considered, and it can be shown to be a necessary condition for the convergence of the infinite series that appear in the second-order wavemaker theory. The solution to the 1st order problem is given below.

Due to the linearity in $\eta^{(2)}$and $\phi^{(2)}$ of the second-order eqs. (10)–(14) with eqs. (18)–(20), it is permissible to divide the boundary value problem into a wave problem and problems taking the wavemaker into account. Thus for convenience, the second-order problem is written as a superposition of three terms (see also Barthel et al., 1983).

$$\phi^{(2)} = \phi^{(21)} + \phi^{(22)} + \phi^{(23)} \tag{23}$$

$$\eta^{(2)} = \eta^{(21)} + \eta^{(22)} + \eta^{(23)} \tag{24}$$

These three contributions have to satisfy the following equations, which ensure that $\phi^{(2)}$ and $\eta^{(2)}$ satisfy the original second-order boundary value problem given by eqs. (10)–(14). We have for $(\phi,\eta) = (\phi^{(21)},\eta^{(21)})$:

$$R^{(21)} = -\{\eta^{(1)}(g\phi_z^{(1)} + \phi_{tt}^{(1)})_z + (\phi_x^{(1)2} + \phi_y^{(1)2} + \phi_z^{(1)2})_t\} \tag{25}$$

$$Q^{(21)} = \text{arbitrary} \tag{26}$$

$$P^{(21)} = \eta^{(1)}\phi_{zt}^{(1)} + \frac{1}{2}(\phi_x^{(1)2} + \phi_y^{(1)2} + \phi_z^{(1)2}) \tag{27}$$

for $(\phi,\eta) = (\phi^{(22)},\eta^{(22)})$:

$$R^{(22)} = 0 \tag{28}$$

$$Q^{(22)} = -f(z)\{X_0^{(1)}\phi_{xx}^{(1)} - X_{0y}^{(1)}\phi_y^{(1)}\} + f'(z)X_0^{(1)}\phi_z^{(1)} - \phi_x^{(21)} \tag{29}$$

$$P^{(22)} = 0 \tag{30}$$

and for $(\phi,\eta) = (\phi^{(23)},\eta^{(23)})$:

$$R^{(23)} = 0 \tag{31}$$

$$Q^{(23)} = f(z)X_{0t}^{(2)} \tag{32}$$

$$P^{(23)} = 0 \tag{33}$$

Here $\phi^{(21)}$ gives the bound waves due to the interaction between first-order wave components including mutual evanescent mode interactions from the first-order wavemaker problem. $\phi^{(22)}$ describes the free waves due to the wavemaker leaving its mean position and due to $\phi^{(21)}$ mismatching the boundary condition at the wavemaker. $\phi^{(23)}$ gives the free waves generated by the second-order wave board motion as tailored to suppress the generation of free sub- and superharmonics.

If the control signal for the wavemaker is based on first-order theory alone, then the resulting second-order waves are given by $\phi^{(2)} = \phi^{(21)} + \phi^{(22)}$ and second-order spurious free waves are not eliminated. Let subscript 0 (on ϕ and η) denote the progressive part of the wave field then the objective for the second-order wavemaker theory is to determine $X_0^{(2)}(y,t)$ as to produce free waves $\eta_0^{(23)}$ which eliminate the spurious, free waves $\eta_0^{(22)}$ by requiring

$$\eta_0^{(22)} + \eta_0^{(23)} = 0 \tag{34}$$

or equivalently

$$\phi_0^{(22)} + \phi_0^{(23)} = 0 \tag{35}$$

2.2. First-order solution

Using the fact that the first-order problem is linear in $\phi^{(1)}$, it is possible to superpose wave components to make irregular waves. Therefore it is only necessary to analyse monochromatic waves in the frequency domain and afterwards the inverse FFT can be applied to obtain time series for the wave board position for irregular waves. The first-order solution was obtained by Madsen (1974); Gilbert (1976) and others. Here the solution will be given in the compact notation provided by a complex representation.

We consider an infinitely long continuous wavemaker and let the first-order paddle position at still water level for each of the wave components constituting the first-order spectrum be given by

$$X_0^{(1)} = \frac{1}{2}\{-iX_a e^{i(\omega t - k_y y)} + c.c.\} \tag{36}$$

where X_a is the constant first-order paddle amplitude at still water level, k_y is the y-component of the wave number vector, $\vec{k}_0$, and c.c. denotes the complex conjugate of the preceding term. The solution to the first-order problem eqs. (10)–(14)- with eqs. (15)–(17) may be expressed as

$$\phi^{(1)} = \frac{1}{2}\left\{\frac{igX_a}{\omega}\sum_{j=0}^{\infty} e_j \frac{\cosh k_j(z+h)}{\cosh k_j h} e^{i(\omega t - \vec{k}_j \cdot \vec{x})} + c.c.\right\} \tag{37}$$

$$\eta^{(1)} = \frac{1}{2}\left\{X_a \sum_{j=0}^{\infty} e_j e^{i(\omega t - \vec{k}_j \cdot \vec{x})} + c.c.\right\} \tag{38}$$

provided that

$$k_{xj}^2 = k_j^2 - k_y^2 \tag{39}$$

where $\vec{k}_j = (k_{xj},k_y)$ is the wave number vector, $k_j \equiv |\vec{k}_j|$ denotes the length of the wave number vector, and $\vec{x} = (x,y)$ is the horizontal position vector. Eqs. (37) and (38) include both the wanted progressive part of the wave and the evanescent modes that are due to the mismatch between the shape of the progressive-wave velocity profile and the shape function $f(z)$, eq. (22). This solution obviously satisfies the bottom boundary condition, and the combined free surface boundary condition eq. (11) with eq. (15) is easily shown to require

$$\omega^2 = gk_j \tanh k_j h \tag{40}$$

This is the linear dispersion relation generalized to complex wave numbers and it has one real solution, k_0, and an infinity of purely imaginary solutions $(k_1,k_2, \ldots)$, where $ik_j > 0$, $j = 1,2, \ldots$. In the following, subscript j is sometimes omitted for $j = 0$.

The coefficients e_j are often called transfer functions and they are determined by requiring the velocity potential, $\phi^{(1)}$, to satisfy the boundary condition at the wavemaker. This is achieved by substituting eqs. (16) and (37) into eq. (13) multiplying the equation by $\cosh k_l(z+h)$ and integrating the result over the depth. Following the principle of orthogonality

$$\int_{-h}^{0} \cosh k_j(z+h)\cosh k_l(z+h)dz \tag{41}$$

$$= \begin{cases} \dfrac{1}{2k_j}(k_jh + \sinh k_jh\cosh k_jh) & \text{for } l = j \\ 0 & \text{for } l \neq j \end{cases}$$

only terms for $j = l$ remain in the infinite sums and the resulting equation is solved for e_j to obtain

$$e_j = \frac{k_j}{k_{xj}}c_j \tag{42}$$

Here c_j are the transfer functions for normally emitted waves ($k_y = 0$) given by

$$c_j = \sinh k_jh\frac{\Lambda_1(k_j)}{\Lambda_2(k_j)} \tag{43}$$

where

$$\Lambda_1(k_j) = k_j\int_{-h}^{0} f(z)\cosh k_j(z+h)dz = \sinh k_jh - \frac{d+l}{h+l}\sinh k_jd + \frac{1}{h+l}\frac{\cosh k_jd - \cosh k_jh}{k_j} \tag{44}$$

$$\Lambda_2(k_j) = k_j\int_{-h}^{0} \cosh^2 k_j(z+h)dz = \frac{1}{2}(k_jh + \sinh k_jh\cosh k_jh) \tag{45}$$

For $j = 0$, eq. (43) gives the real quantity c_0, and for $j = 1,2, \ldots, c_j$ is imaginary. Often, c_0 is referred to as the Biésel transfer function due to the extensive work on the topic by Biésel and Suquet (1951).

Depending on the choice of k_y, e_0 is either real or imaginary, while e_j is always imaginary for $j = 1,2, \ldots$.

For $|k_y| \leq k$ (serpent wave length longer than or equal to length of progressive wave), k_x is real and the wave field generated has a progressive part

$$\eta_0 = \frac{1}{2}\{Ae^{i(\omega t - \vec{k}\cdot\vec{x})} + c.c.\} \tag{46}$$

where A is a complex amplitude given by

$$A = e_0 X_a \tag{47}$$

and where e_0 is real. In this case

$$(k_x,k_y) = k(\cos\alpha,\sin\alpha) \tag{48}$$

and

$$e_0 = \frac{1}{\cos\alpha}c_0 \tag{49}$$

where α is the wave propagation direction (Fig. 4). For $j \neq 0$, k_{xj} is imaginary corresponding to evanescent modes. Generalizing to complex α_j as defined by

$$k_y = k\sin\alpha = k_j\sin\alpha_j \tag{50}$$

we obtain

$$k_{xj} = k_j\cos\alpha_j \tag{51}$$

in order to satisfy eq. (49), and thus eq. (42) becomes

$$e_j = \frac{1}{\cos\alpha_j}c_j \tag{52}$$

where

$$\cos\alpha_j = \sqrt{1 - \frac{k^2\sin^2\alpha}{k_j^2}} = \begin{cases} \cos\alpha \leq 1 & \text{for } j = 0 \\ \sqrt{1 + \dfrac{k^2\sin^2\alpha}{|k_j|^2}} \geq 1 & \text{for } j \geq 1 \end{cases} \tag{53}$$

This shows that while the amplitude of the progressive wave is amplified by $1/\cos\alpha$ for a given stroke then the evanescent mode amplitudes are reduced. This reduction vanishes for large j, since $k_jh \propto \pi\, i\, j$ for $j \gg 1$ (see e.g. Schäffer, 1993b). Fig. 3 shows the progressive wave transfer function for several wave propagation directions.

If $|k_y| > k$ (serpent wave length shorter than length of progressive wave) even the mode for $j = 0$ is evanescent and no progressive component is generated.

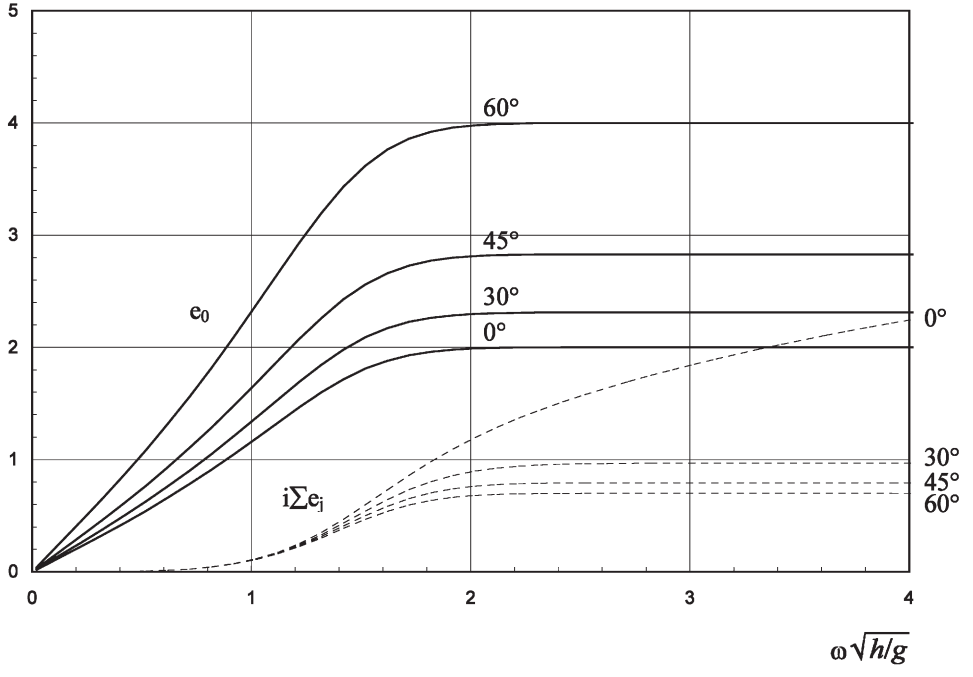


Fig. 3. The transfer function, e_0, and the total evanescent mode transfer function, $i\Sigma e_j$ for the directions 0°, 30°, 45° and 60° for a piston-type wavemaker.

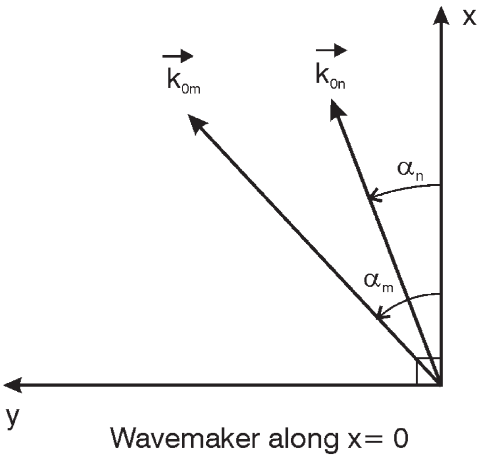


Fig. 4. Definition of wave number vectors.

2.3. Second-order solution

At second order, interactions between two wave components of generally different angular frequencies, ω_n and ω_m, and directions, $\alpha_n$and α_m, constitute the basis of the second-order spectrum (see Fig. 4). The interaction terms appear as subharmonics (difference frequencies) and superharmonics (sum frequencies), and the second-order theory calls for a bichromatic analysis in the frequency domain. Superposition and the inverse FFT can then be applied to obtain second-order time series of the paddle position for irregular waves to be added to the corresponding first-order time series.

The aim of the second-order wave generation is to get a second-order bound surface elevation $\eta_0^{(21)\,\pm}$ without any spurious free-wave generation.

2.3.1. Solution of the $\phi^{(21)}$-problem—the bound waves

The summation indices for the series eqs. (37) and (38) giving the first-order solution for ω_n and ω_m are denoted j and l. If $S^{(21)}$ stands for each of the quantities $R^{(21)}$, $\phi^{(21)}$, $P^{(21)}$, and $\eta^{(21)}$, and $S_{jnlm}^{(21)\,+}$ and $S_{jnlm}^{(21)-}$ denote the complex superharmonic and complex subharmonic contributions, respectively, for the interaction between each pair of terms drawn from the two series, then[1]

$$S^{(21)} = \frac{1}{2}\left\{\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}[S_{jnlm}^{(21)-} + S_{jnlm}^{(21)+} + S_{jnln}^{(21)+} + S_{jmlm}^{(21)+}] + c.c.\right\}; \tag{54}$$

$$S^{(21)} = \begin{cases} R^{(21)} \\ \phi^{(21)} \\ P^{(21)} \\ \eta^{(21)} \end{cases}$$

Using eq. (37) the first-order velocity potentials for the two frequencies, ω_n and ω_m, are inserted into $R^{(21)}$, eq. (25), and eq. (40) is used for eliminating $\tanh k_{jn}h$- and $\tanh k_{lm}^{-:*}h$-terms (see eq. (59) for a definition of the symbol $-:*$). This yields after straightforward but lengthy algebra

$$R_{jnlm}^{(21)\pm} = i\delta_{nm}X_nX_m^{-:*}e_{jn}e_{lm}^{-:*}H_{jnlm}^{\pm}e^{i(\theta_{jn}\pm\theta_{lm}^{-:*})} \tag{55}$$

where

$$\delta_{nm} = \begin{cases} \frac{1}{2} & \text{for } \ \omega_n = \omega_m \ \text{ and } \ \alpha_n = \alpha_m \\ 1 & \text{otherwise} \end{cases} \tag{56}$$

[1] Eq. (54) equivalent to the formulation of Schäffer (1996), although he did not explicitly list the double frequency terms.

$$\theta_{jn} = \omega_n t - \vec{k}_{jn} \cdot \vec{x} \tag{57}$$

$$\theta_{lm} = \omega_m t - \vec{k}_{lm} \cdot \vec{x}$$

Furthermore

$$H^{\pm}_{jnlm} = (\omega_n \pm \omega_m)\left(\pm \omega_n\omega_m - \frac{g^2 \vec{k}_{jn} \cdot \vec{k}^{-:*}_{lm}}{\omega_n \omega_m} \right) + \frac{\omega_n^3 \pm \omega_m^3}{2} - \frac{g^2}{2}\left(\frac{k^2_{jn}}{\omega_n} \pm \frac{k^2_{lm}}{\omega_m} \right) \tag{58}$$

and the symbol $-:*$ introduced for brevity is defined by

$$Z^{-:*} = \begin{cases} Z & \text{for superharmonics} \\ Z^* & \text{for subharmonics} \end{cases} \tag{59}$$

where $*$ denotes complex conjugation and $-:*$ is complex conjugation in case of subharmonics while being ignored for superharmonics. The complex identity

$$(Z_1 e^{iz_1} + c.c.)\ (Z_2 e^{iz_2} + c.c.) = (Z_1 Z_2 e^{i(z_1+z_2)} + Z_1 Z_2^* e^{i(z_1 - z_2^*)}) + c.c. \tag{60}$$

where z_1, z_2, Z_1, Z_2 are complex numbers, reveals the origin of the distinction in eq. (59) between the superharmonics and the subharmonics used in eqs. (55) and (58). It follows that the solution to eqs. (10)–(12) with (25) is

$$\phi^{(21)\pm}_{jnlm} = \frac{R^{(21)\pm}_{jnlm}}{D^{\pm}_{jnlm}} \frac{\cosh k^{\pm}_{jnlm}(z + h)}{\cosh k^{\pm}_{jnlm} h} \tag{61}$$

where

$$k^{\pm}_{jnlm} = \sqrt{(k_{xjn} \pm k^{-:*}_{xlm})^2 + (k_{yn} \pm k_{ym})^2} \tag{62}$$

$$D^{\pm}_{jnlm} = g k^{\pm}_{jnlm} \tanh k^{\pm}_{jnlm} h - (\omega_n \pm \omega_m)^2 \tag{63}$$

For $(j,l) = (0,0)$, eqs. (61)–(63) are real and consistent with Sharma and Dean (1981) and Suh and Dalrymple (1987).

We now turn to the corresponding surface elevation. From eqs. (27), (37) and (38) we find

$$P^{(21)\pm}_{jnlm} = \delta_{nm} X_n X_m^{-:*} e_{jn} e_{lm}^{-:*} L^{\pm}_{jnlm} e^{i(\theta_{jn} \pm \theta^{-:*}_{lm})} \tag{64}$$

where

$$L^{\pm}_{jnlm} = \frac{1}{2}\left\{ \frac{g^2 \vec{k}_{jn} \cdot \vec{k}^{-:*}_{lm}}{\omega_n \omega_m} \mp \omega_n \omega_m - (\omega_n^2 \pm \omega_m^2) \right\} \tag{65}$$

by which eq. (14) yields

$$\eta^{(21)\pm}_{jnlm} = G^{\pm}_{jnlm} X_n X_m^{-:*} e_{jn} e_{lm}^{-:*} e^{i(\theta_{jn} \pm \theta^{-:*}_{lm})} \tag{66}$$

where

$$G^{\pm}_{jnlm} = \frac{\delta_{nm}}{g}\left\{(\omega_n \pm \omega_n)\frac{H^{\pm}_{jnlm}}{D^{\pm}_{jnlm}} - L^{\pm}_{jnlm}\right\} \tag{67}$$

For $(j,l) = (0,0)$ this gives the progressive part of the second-order bound wave and we have

$$\eta^{(21)\pm}_{0n0m} = \frac{1}{2}\{G^{(21)\pm}_{nm}A_nA^{-:*}_m e^{i(\theta_{0n}\pm\theta^{-:*}_{0m})} + c.c.\} \quad \text{with} \quad G^{(21)\pm}_{nm} \equiv G^{(21)\pm}_{0n0m} \tag{68}$$

which is real and consistent with Sharma and Dean (1981).[2]

2.3.2. Solution of the $\phi^{(22)}$-problem—the unwanted free waves

This resembles the first-order problem, since $R^{(22)} = P^{(22)} = 0$, and the only difference is due to the forcing in the wavemaker boundary condition, which is now given by $Q^{(22)}$. Thus following the same procedure as for the first-order waves only now for super- and subharmonic frequencies, we have (see footnote 1)

$$\phi^{(22)} = \phi^{(22)-}_{nm} + \phi^{(22)+}_{nm} + \phi^{(22)+}_{nn} + \phi^{(22)+}_{mm} \tag{69}$$

$$\eta^{(22)} = \eta^{(22)-}_{nm} + \eta^{(22)+}_{nm} + \eta^{(22)+}_{nn} + \eta^{(22)+}_{mm} \tag{70}$$

$$\phi^{(22)\pm}_{nm} = \frac{1}{2}\left\{\frac{igA_nA^{-:*}_m}{h(\omega_n \pm \omega_m)}\sum_{p=0}^{\infty} e^{(22)\pm}_p \frac{\cosh K^{\pm}_p(z+h)}{\cosh K^{\pm}_p h} e^{i((\omega_n\pm\omega_m)t - \vec{K}^{\pm}_p\cdot\vec{x})} + c.c.\right\} \tag{71}$$

$$\eta^{(22)\pm}_{nm} = \frac{1}{2}\left\{\frac{A_nA^{-:*}_m}{h}\sum_{p=0}^{\infty} e^{(22)\pm}_p e^{i((\omega_n\pm\omega_m)t - \vec{K}^{\pm}_p\cdot\vec{x})} + c.c.\right\} \tag{72}$$

where $\vec{K}^{\pm}_p = (K^{\pm}_{xp}, K^{\pm}_y)$ is the wave number vector for these free waves, and the length of this wave number vector, $K^{\pm}_p \equiv |\vec{K}^{\pm}_p|$, is a solution to the linear dispersion relation generalized to complex wave numbers

$$(\omega_n \pm \omega_m)^2 = gK^{\pm}_p \tanh K^{\pm}_p h \tag{73}$$

As $K^{\pm}_p$ and $K^{\pm}_y$ are known quantities (see eq. (77)), then $K^{\pm}_{xp}$ can be found from

$$K^{\pm 2}_{xp} = K^{\pm 2}_p - K^{\pm 2}_y \tag{74}$$

The first-order velocity potentials for the two frequencies ω_n and ω_m are now substituted into the forcing term (29) in the wavemaker boundary condition (13). Using eq. (40) to eliminate $\tanh k_{jn}h$- and $\tanh k^{-:*}_{lm}h$-terms, we further obtain

$$Q^{(22)} = \frac{1}{2}\delta_{nm}\left\{ \pm \frac{g}{2\omega_n}\sum_{j=0}^{\infty}\frac{e_{jn}}{\cosh k_{jn}h}(f(z)(k^2_{xjn} \mp k_{yn}k_{ym})\cosh k_{jn}(z+h) + f'(z)k_{jn}\sinh k_{jn}(z+h))\right.$$

[2] As opposed to the present formulation, there is no factor δ_{nm} in Sharma and Dean's results for the second-order surface elevation and velocity potential. The reason for this difference is that they regard the interaction between ω_n and ω_m as two contributions permuting n and m, while we collect these in one interaction term. Thus, for N frequencies their summation reads $\Sigma^N_{n=1}\Sigma^N_{m=1}$ while ours equivalently reads $2\delta_{nm}\Sigma^N_{n=1}\Sigma^N_{m=n}$.

$$\pm \frac{g}{2\omega_m} \sum_{l=0}^{\infty} \frac{e_{lm}^{-;*}}{\cosh k_{lm}^{-;*} h} (f(z)(k_{xlm}^{-;*\,2} \mp k_{yn} k_{ym}) \cosh k_{lm}^{-;*}(z+h) + f'(z) k_{lm}^{-;*} \sinh k_{lm}^{-;*}(z+h)) \tag{75}$$

$$- \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} e_{jn} e_{lm}^{-;*} (k_{xjn} \pm k_{xlm}^{-;*}) \frac{H_{jnlm}^{\pm}}{D_{jnlm}^{\pm}} \frac{\cosh k_{jnlm}^{\pm}(z+h)}{\cosh k_{jnlm}^{\pm} h} \Bigg\} X_n X_m^{-;*} e^{i((\omega_n \pm \omega_m)t - (k_{yn} \pm k_{ym})y)}$$

Using eqs. (71) and (75) in (13) and multiplying the resulting equation by $\cosh K_l^{\pm}(z+h)$, integration from $z = -h$ to $z = 0$ gives the coefficients $e_p^{(22)\pm}$ by virtue of orthogonality, exactly as in the first-order case.[3] We obtain

$$e_p^{(22)\pm} = \delta_{nm} \frac{h(\omega_n \pm \omega_m) K_p^{\pm} \cosh K_p^{\pm} h}{g e_{0n} e_{0m} K_{xp}^{\pm} \Lambda_2(K_p^{\pm})} \Bigg\{ \pm \frac{g}{2\omega_n} \sum_{j=0}^{\infty} \frac{e_{jn}}{\cosh k_{jn} h} \Gamma_4(k_{jn}, K_p^{\pm}, k_{xjn}^2 \mp k_{yn} k_{ym})$$

$$\pm \frac{g}{2\omega_m} \sum_{l=0}^{\infty} \frac{e_{lm}^{-,*}}{\cosh k_{lm}^{-,*} h} \Gamma_4(k_{lm}^{-,*}, K_p^{\pm}, k_{xlm}^{-,*2} \mp k_{yn} k_{ym}) \tag{76}$$

$$- \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} e_{jn} e_{lm}^{-,*} \frac{k_{xjn} \pm k_{xlm}^{-,*}}{\cosh k_{jnlm}^{\pm} h} \frac{H_{jnlm}^{\pm}}{D_{jnlm}^{\pm}} \Gamma_1(k_{jnlm}^{\pm}, K_p^{\pm}) \Bigg\}$$

where that

$$K_y^{\pm} = k_{yn} \pm k_{ym} \tag{77}$$

Here $\Lambda_2(K_p^{\pm})$, e_{jn}, and e_{lm} are quantities known from the first-order solution, and they are defined in eqs. (45) and (42), respectively. The functions $\Gamma_{1\ldots4}$ are defined as

$$\Gamma_1(\kappa_1,\kappa_2) \equiv \int_{-h}^{0} \cosh\kappa_1(z+h)\cosh\kappa_2(z+h)dz \tag{78}$$

$$\Gamma_4(\kappa_1,\kappa_2,\tilde{\kappa}) \equiv \tilde{\kappa}\,\Gamma_2(\kappa_1,\kappa_2) + \kappa_1\Gamma_3(\kappa_1,\kappa_2) \tag{79}$$

where

[3] Van Leeuwen and Klopman (1996) used the perturbation method of multiple scales, to derive expressions for the wave-board control signal for the generation of second-order waves in a channel based on a narrow band assumption for the first-order waves. Solving the first-order problem they used the principle of orthogonality, thus satisfying infinitely many conditions at the wave board. The infinity of evanescent modes ensured that the boundary condition at the wave board was satisfied locally. However, when solving the second-order problem, they eliminated the z-variation by simple integration of the wavemaker boundary condition over the depth, thus satisfying only one condition in the flume, which is that the flux becomes correct. While this is a necessary condition, it does not ensure that the boundary condition at the board is satisfied locally. The problem can be illustrated by applying this method already at first order. Instead of the correct Biésel transfer function (43) it would result in $c_0 = kh$; $c_j = 0$; $j > 0$, which is identical to the result from shallow water theory. At second order the problem would be more pronounced for superharmonics than for subharmonics, since the method essentially rests on an assumption of a uniform velocity profile.

$$\Gamma_2(\kappa_1,\kappa_2) \equiv \int_{-h+d}^{0} f(z)\cosh\kappa_1(z+h)\cosh\kappa_2(z+h)dz \tag{80}$$

and

$$\Gamma_3(\kappa_1,\kappa_2) \equiv \int_{-h+d}^{0} f'(z)\sinh\kappa_1(z+h)\cosh\kappa_2(z+h)dz \tag{81}$$

The functions $\Gamma_{1\ldots4}$ are evaluated in the Appendix.[4] Using $\Gamma_1(k^{\pm}_{jnlm},K^{\pm}_p)$ and $\Gamma_4(k_{jn},K^{\pm}_p,k^2_{xjn}\mp k_{yn}k_{ym})$ as given in eqs. (A9) and (A10), (76) becomes[5]

$$\begin{aligned} e_p^{(22)\pm} = \delta_{nm}\frac{h(\omega_n \pm \omega_m)K^{\pm}_p\cosh^2K^{\pm}_p h}{g^2 e_{0n}e_{0m}K^{\pm}_{xp}\Lambda_2(K^{\pm}_p)}\Bigg\{ &\pm \frac{g}{2\omega_n}\sum_{j=0}^{\infty}\frac{e_{jn}}{k^2_{jn}-K^{\pm 2}_p}(k^2_{xjn}\mp k_{yn}k_{ym})(\omega_n^2-(\omega_n \pm \omega_m)^2 \\ &+ M_2(k_{jn},K^{\pm}_p,k^2_{xjn}\mp k_{yn}k_{ym})) \pm \frac{g}{2\omega_m}\sum_{j=0}^{\infty}\frac{e^{-:*}_{lm}}{k^{-:*2}_{lm}-K^{\pm 2}_p}(k^{-:*2}_{xlm}\mp k_{yn}k_{ym})(\omega_m^2-(\omega_n \pm \omega_m)^2 \\ &+ M_2(k^{-:*}_{lm},K^{\pm}_p,k^{-:*2}_{xlm}\mp k_{yn}k_{ym})) - \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} e_{jn}e^{-:*}_{lm}\frac{k_{xjn} \pm k^{-:*}_{xlm}}{k^{\pm 2}_{jnlm}-K^{\pm 2}_p}H^{\pm}_{jnlm}\Bigg\} \end{aligned} \tag{82}$$

where

$$\begin{aligned} M_2(k_{jn}, K^{\pm}_p,\tilde{\kappa}) = -\frac{g}{h+l}\frac{1}{k^2_{jn}-K^{\pm 2}_p}\Bigg[&\left((k^2_{jn}+K^{\pm 2}_p)-\frac{k^2_{jn}(k^2_{jn}-K^{\pm 2}_p)}{\tilde{\kappa}}\right) \\ &\left(1-\frac{\cosh k_{jn}d\cosh K^{\pm}_p d}{\cosh k_{jn}h\cosh K^{\pm}_p h}\right)-k_{jn}K^{\pm}_p\left(2-\frac{k^2_{jn}-K^{\pm 2}_p}{\tilde{\kappa}}\right) \\ &\left(\frac{\omega_n^2(\omega_n \pm \omega_m)^2}{g^2 k_{jn}K^{\pm}_p}-\frac{\sinh k_{jn}d\sinh K^{\pm}_p d}{\cosh k_{jn}h\cosh K^{\pm}_p h}\right)\Bigg] \end{aligned} \tag{83}$$

vanishes for a piston-type wavemaker.

The denominator $D^{\pm}_{jnlm}$ in eq. (76), which is singular when the waves propagate in the same direction in shallow water, has vanished from the expression.

2.3.3. Solution of the $\phi^{(23)}$—problem (the compensating free waves) and obtaining the second-order control signal

This problem concerns the wave field generated by a second-order wavemaker control signal in order to eliminate spurious free-wave generation. The solution of

[4] Note that Γ_3 and Γ_4 differ by the factors κ_1 and $h + l$, respectively, from the definitions chosen by Schäffer (1996).

[5] For $p = j$=0 and $\omega_m = 2\omega_n$ there is a removable singularity in the single summation in case of subharmonics, cf. eqs. (A8) and (A10) in the Appendix. (If $\omega_m < \omega_n$ is chosen then this appears for $p = j$=0 and $2\omega_m = \omega_n$).

$\phi^{(23)\pm}$ is in practice determined the same way as $\phi^{(22)\pm}$ only now the forcing (this time given by the second-order paddle position, see eq. (32)) is unknown.

In terms of a second-order transfer function $\mathbf{F}^{\pm}$, the second-order paddle position may be written

$$X_0^{(2)} = X_{nm}^{(2)-} + X_{nm}^{(2)+} + X_{nn}^{(2)+} + X_{mm}^{(2)+} \tag{84}$$

$$X_{nm}^{(2)\pm} = \frac{1}{2}\left\{-i\mathbf{F}^{\pm}\frac{A_n A_m^{-:*}}{h}e^{i(\omega_n \pm \omega_m)t-(k_{yn} \pm k_{ym})y} + c.c.\right\} \tag{85}$$

and with

$$\phi^{(23)} = \phi_{nm}^{(23)-} + \phi_{nm}^{(23)+} + \phi_{nn}^{(23)+} + \phi_{mm}^{(23)+} \tag{86}$$

$$\eta^{(23)} = \eta_{nm}^{(23)-} + \eta_{nm}^{(23)+} + \eta_{nn}^{(23)+} + \eta_{mm}^{(23)+} \tag{87}$$

the solution to eq. (10) satisfying eq. (12) yields

$$\phi_{nm}^{(23)\pm} = \frac{1}{2}\left\{\frac{ig\mathbf{F}^{\pm}A_n A_m^{-:*}}{h(\omega_n \pm \omega_m)}\sum_{p=0}^{\infty} e_p^{(23)\pm}\frac{\cosh K_p^{\pm}(z+h)}{\cosh K_p^{\pm}h}e^{i((\omega_n \pm \omega_m)t-\vec{K}_p^{\pm}\cdot\vec{x})} + c.c.\right\} \tag{88}$$

$$\eta_{nm}^{(23)\pm} = \frac{1}{2}\left\{\mathbf{F}^{\pm}\frac{A_n A_m^{-:*}}{h}\sum_{p=0}^{\infty} e_p^{(23)\pm}e^{i((\omega_n \pm \omega_m)t-\vec{K}_p^{\pm}\cdot\vec{x})} + c.c.\right\} \tag{89}$$

where

$$e_p^{(23)\pm} = \frac{K_p^{\pm}}{K_{xp}^{\pm}}c_p^{(23)\pm} = \frac{K_p^{\pm}}{K_{xp}^{\pm}}\sinh K_p^{\pm}h\frac{\Lambda_1(K_p^{\pm})}{\Lambda_2(K_p^{\pm})} \tag{90}$$

The second-order transfer function, $\mathbf{F}^{\pm}$, can now be determined from the requirement that the free spurious waves must be eliminated, see eq. (35). This leads to

$$\mathbf{F}^{\pm} = -\frac{e_0^{(22)\pm}}{e_0^{(23)\pm}} \tag{91}$$

Notice that only the propagating free waves cancel, while the evanescent modes still remain. Using eqs. (82) and (90) we find for $p = 0$

$$\begin{aligned}\mathbf{F}^{\pm} = E^{\pm}\Bigg\{ &\mp\frac{g}{2\omega_n}\sum_{j=0}^{\infty} e_{jn}\frac{k_{xjn}^2 \mp k_{yn}k_{ym}}{k_{jn}^2 - K_0^{\pm 2}}(\omega_n^2 - (\omega_n \pm \omega_m)^2 + M_2(k_{jn},K_0^{\pm},k_{xjn}^2 \mp k_{yn}k_{ym})) \\ &\mp\frac{g}{2\omega_m}\sum_{l=0}^{\infty} e_{lm}^{-:*}\frac{k_{xlm}^{-:*2} \mp k_{yn}k_{ym}}{k_{lm}^{-:*2} - K_0^{\pm 2}}(\omega_m^2 - (\omega_n \pm \omega_m)^2 + M_2(k_{lm}^{-:*},K_0^{\pm},k_{xlm}^{-:*2} \mp k_{yn}k_{ym})) \\ &+ \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} e_{jn}e_{lm}^{-:*}\frac{k_{xjn} \pm k_{xlm}^{-:*}}{k_{jnlm}^{\pm 2} - K_0^{\pm 2}}H_{jnlm}^{\pm}\Bigg\}\end{aligned} \tag{92}$$

where

$$E^{\pm} = \frac{\delta_{nm}K_0^{\pm 2}h}{e_{0n}e_{0m}(\omega_n \pm \omega_m)^3(1 + M_1(K_0^{\pm}))} \tag{93}$$

and where

$$M_1(K_0^{\pm}) = \frac{1}{h+l}\frac{g}{(\omega_n \pm \omega_m)^2}\left(\frac{\cosh K_0^{\pm}d}{\cosh K_0^{\pm}h} - 1\right) \quad (94)$$

vanishes for a piston-type wavemaker. The second-order transfer function, $\mathbf{F}^{\pm}$ is the main result of the theory.

For the 2D-case of normally generated long-crested waves, where $(k_{xjn}, k_{yn}) = (k_{jn}, 0)$ and $(k_{xlm}, k_{ym}) = (k_{lm}, 0)$ the results are identical to those of Schäffer (1996).

3. Numerical evaluation of second-order transfer function

The double infinite series of in the second-order solution are characterised by an extremely slow rate of convergence. This is not a problem as long as only a few primary-wave frequencies are considered, because straightforward summation for a sufficiently large number of evanescent modes can be included without excessive computational demands. However, for practical application to a full spectrum of irregular primary waves, the computational effort becomes prohibitive. Schäffer (1993b) provided a solution to this problem by developing a special asymptotic summation method reducing the computational cost by orders of magnitude. Although this method was developed for 2D wavemaker theory, it can also be applied to the present case of 3D waves. The numerical results given in the following figures are based on this technique and computed using Mathematica. Results without the asymptotic summation method were thoroughly checked by an independent Fortran code.

Although the theory is valid for a range of wavemakers only results for a piston-type wavemaker will be shown. In the following figures contour plots of the real and imaginary parts of the superharmonic and subharmonic complex transfer functions $\mathbf{F}^{+}/\delta_{nm}$ and $\mathbf{F}^{-}/\delta_{nm}$ are shown as functions of non-dimensional first order frequencies $\omega_n\sqrt{h/g}$ and $\omega_m\sqrt{h/g}$. Results are shown for $(\alpha_n, \alpha_m) = (0^{\circ}, 0^{\circ})$, $(30^{\circ}, 30^{\circ})$, $(30^{\circ}, -30^{\circ})$, and $(60^{\circ}, 0^{\circ})$. The direction of the two first-order wave trains are shown by arrows in the upper right corner. When the directions coincide only one arrow is shown. Figs. 5–8 show the real and imaginary part of $\mathbf{F}^{+}/\delta_{nm}$, while Figs. 9–12 show the corresponding results for $\mathbf{F}^{-}/\delta_{nm}$.

For $(\alpha_n, \alpha_m) = (0^{\circ}, 0^{\circ})$, Figs. 5 and 9, the plots are identical to those of Fig. 6(a) and 7(a) in Schäffer (1996), with the exception that Schäffer (1996) did not recognise the *anti*-symmetry of the subharmonic transfer function and therefore mirrored the lower part of the figure ($\omega_n > \omega_m$) into the upper part ($\omega_n < \omega_m$). This, however, did not affect his further applications.

As an example, let the water depth be 1.00 m and the two periods be $(T_n, T_m) = (2\text{s}, 3\text{s})$, by which $(\omega_n\sqrt{h/g}, \omega_m\sqrt{h/g}) = (1.00, 0.67)$. The subharmonic is a 6 s wave, while the three superharmonic periods are 1, 1.2 and 1.5 s. Table 1 lists the second order transfer functions for $(\alpha_n, \alpha_m) = (0°, 0°)$, $(0°, 15°)$, $(0°, 60°)$ and $(60°, 0°)$. The magnitude of the subharmonic and the three superharmonic transfer functions are

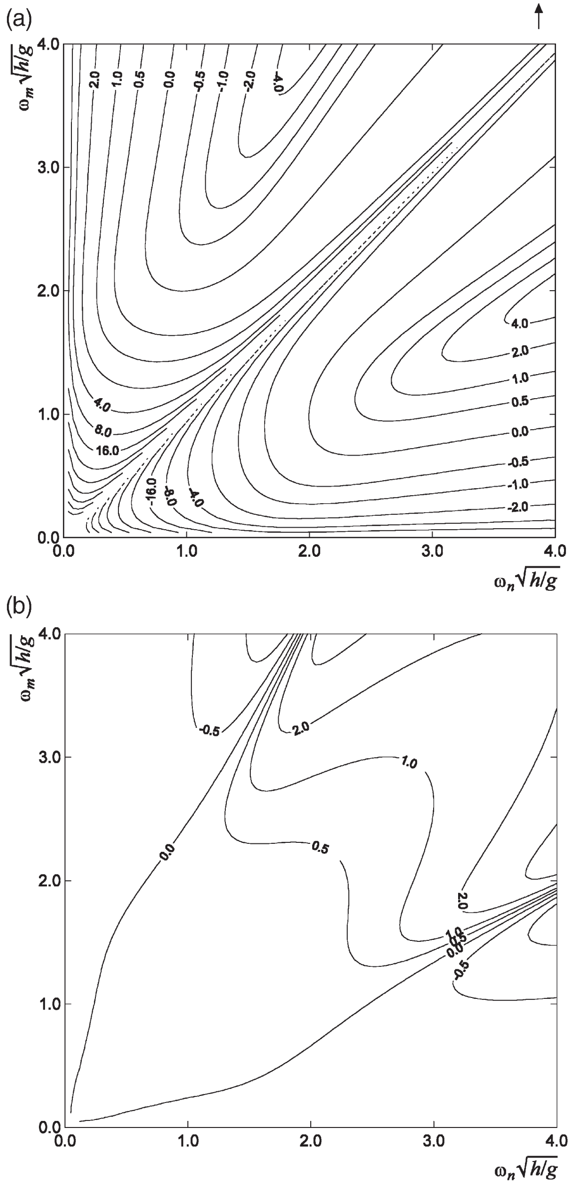


Fig. 5. Real (top) and imaginary (bottom) parts of subharmonic complex transfer functions $\mathbf{F}^{-}/\delta_{nm}$ vs dimensionless first-order frequencies $\omega_n\sqrt{h/g}$ and $\omega_m\sqrt{h/g}$ for a piston type wavemaker and $\alpha_n = 0^\circ$ and $\alpha_m = 0^\circ$. Contour lines are a subset of 0, $\pm 2^n$, $n = -1,0,1,2,....$

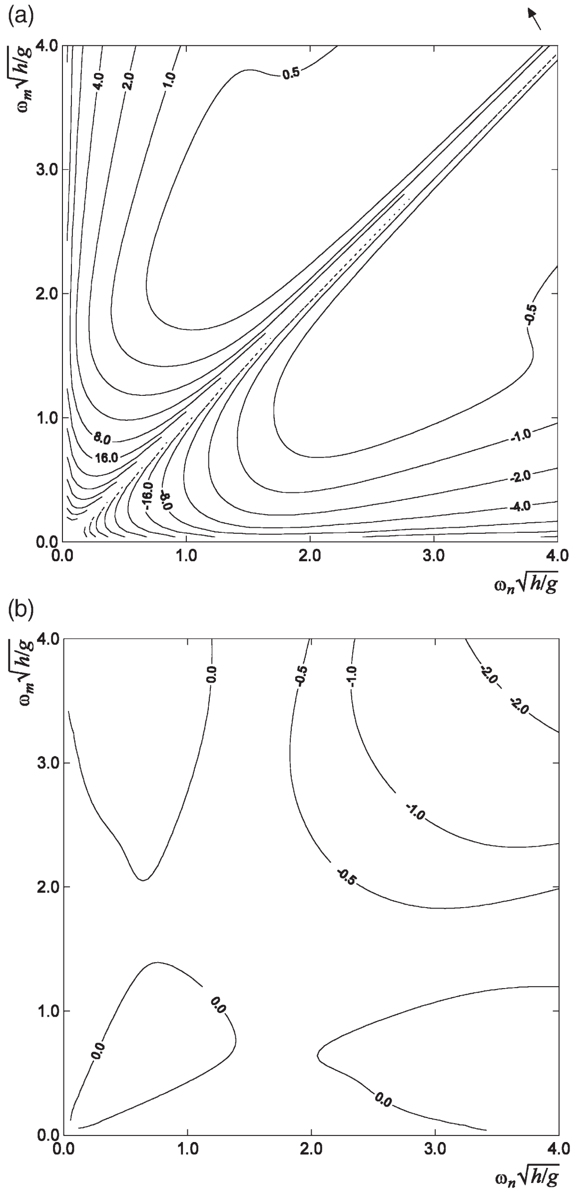


Fig. 6. As Fig. 5, but for $\alpha_n = 30^\circ$ and $\alpha_m = 30^\circ$.

given for second order paddle position ($\mathbf{F}^{\pm}$, listed as $\mathbf{F}^{\pm}_{nm}$ with indices referring to the two interacting components) as well as for second-order surface elevation of the bound waves ($G^{(21)\pm}_{nm}$). To arrive at the second-order amplitudes for paddle position and bound surface elevation let, for example, $(A_n,A_m) = (0.10\text{m},0.10\text{m})$ resulting in

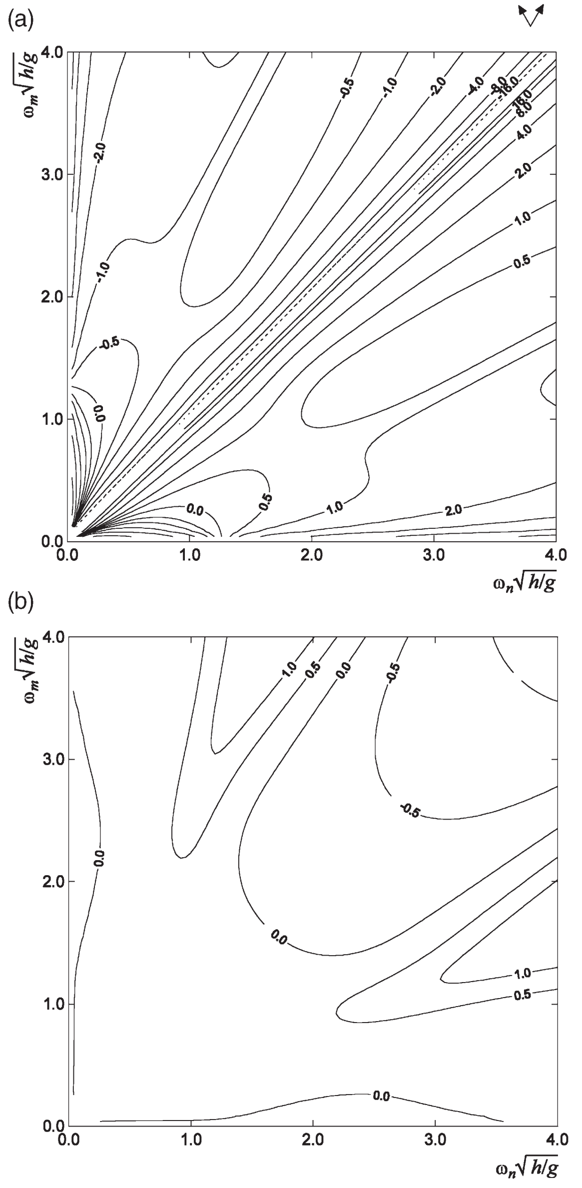


Fig. 7. As Fig. 5, but for $\alpha_n = 30^{\circ}$ and $\alpha_m = -30^{\circ}$.

$A_nA_m/h = 0.01$m (see eqs. (68) and (85)), by which the numbers in Table 1 provide the surface elevation and paddle position amplitudes in cm. Note the well known decrease in the amplitudes for increasing difference angles.

As discussed by Suh and Dalrymple (1987) the propagation directions of the spurious free waves are different from the directions of the bound waves. The directions

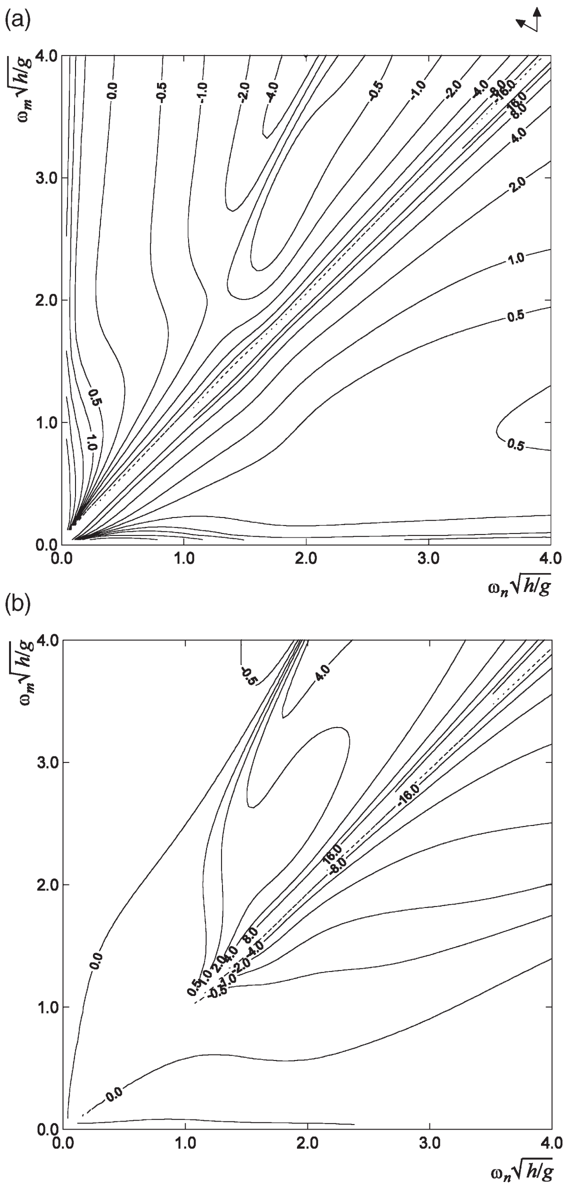
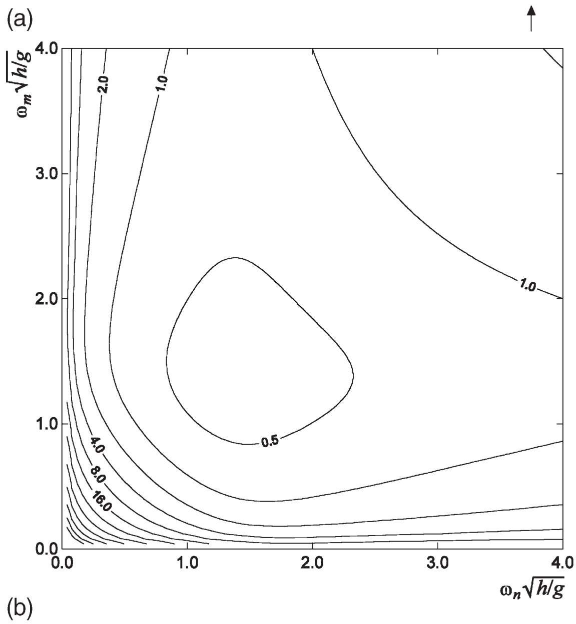


Fig. 8. As Fig. 5, but for $\alpha_n = 60^\circ$ and $\alpha_m = 0^\circ$.

of the bound waves are given by the vector $\vec{k}_n - \vec{k}_m$ if $\omega_n - \omega_m > 0$ (and $-(\vec{k}_n - \vec{k}_m)$ if $\omega_n - \omega_m < 0$ is chosen). Both the spurious free waves and the free waves generated to eliminate them are directed by $\vec{K}_0^{\pm}$, see eqs. (73), (74) and (77) Table 2 gives the free and bound wave directions for the situation in Table 1. These examples confirm the general result that for subharmonics the free waves are more oblique



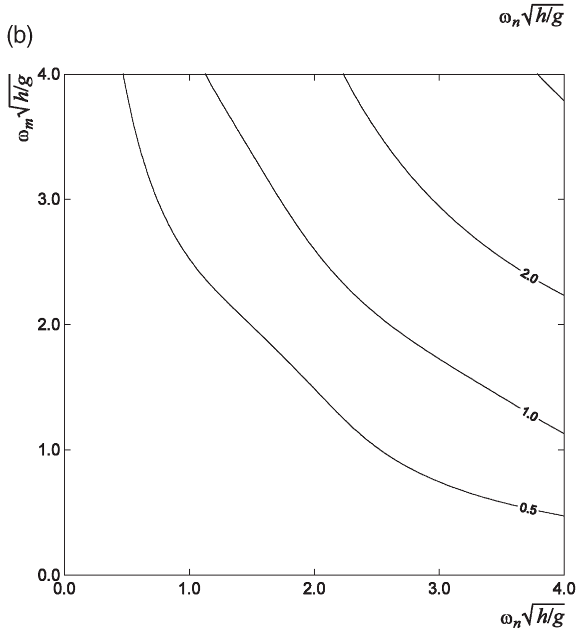


Fig. 9. Real (top) and imaginary (bottom) parts of superharmonic complex transfer functions $\mathbf{F}^+/\delta_{nm}$ vs dimensionless first-order frequencies $\omega_n\sqrt{h/g}$ and $\omega_m\sqrt{h/g}$ for a piston type wavemaker and $\alpha_n = 0^\circ$ and $\alpha_m = 0^\circ$. Contour lines are a subset of 0, $\pm 2^n$, $n = -1,0,1,2,\ldots$.

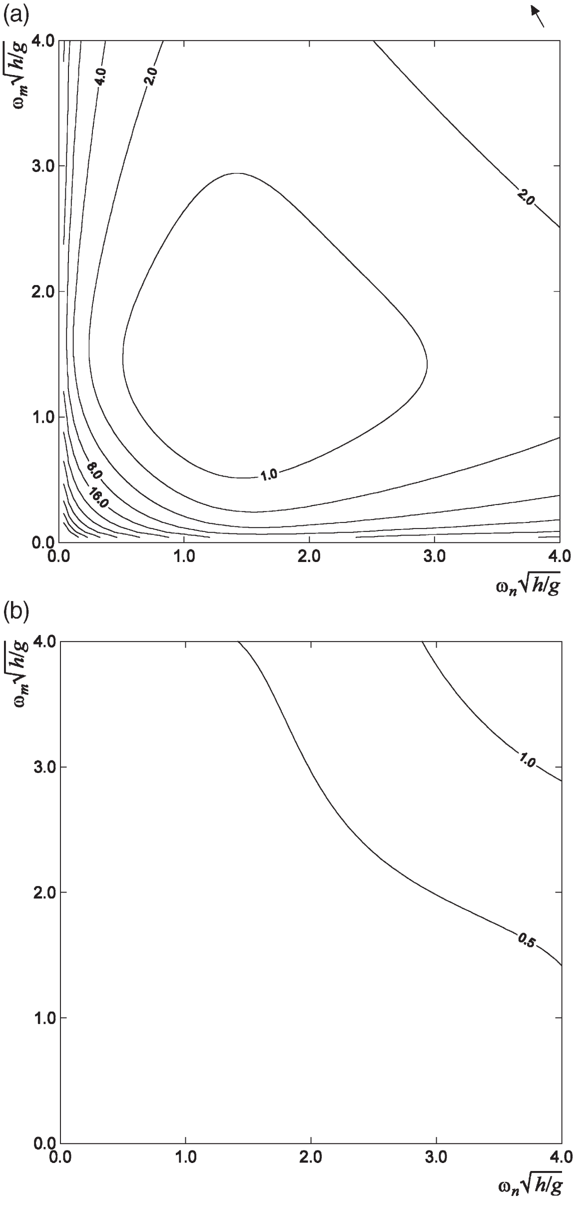


Fig. 10. As Fig. 9, but for $\alpha_n = 30^\circ$ and $\alpha_m = 30^\circ$.

than the bound waves, while it is the other way around for the superharmonics. The bound subharmonics may have directions beyond the range $\pm 90^\circ$ i.e. propagating towards the wavemaker, not away from it. This is the situation for the last case of Table 2, $(\alpha_n,\alpha_m) = (60^\circ,0^\circ)$. As opposed to the bound waves, the free waves can

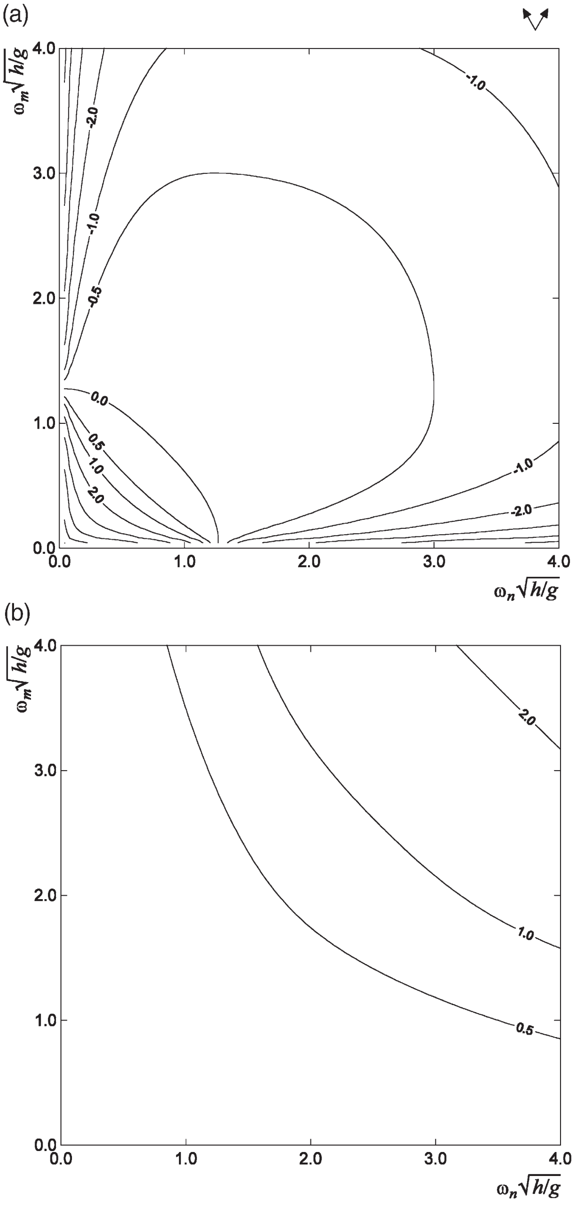


Fig. 11. As Fig. 9, but for $\alpha_n = 30^\circ$ and $\alpha_m = -30^\circ$.

only be directed away from the wavemaker. Attempts to break this limit will result in evanescent modes as in the two last cases of Table 2. For the second last case, $(\alpha_n,\alpha_m) = (0°,60°)$, the bound subharmonic does propagate away from the wavemaker, but the free subharmonic is still evanescent. In general the directional

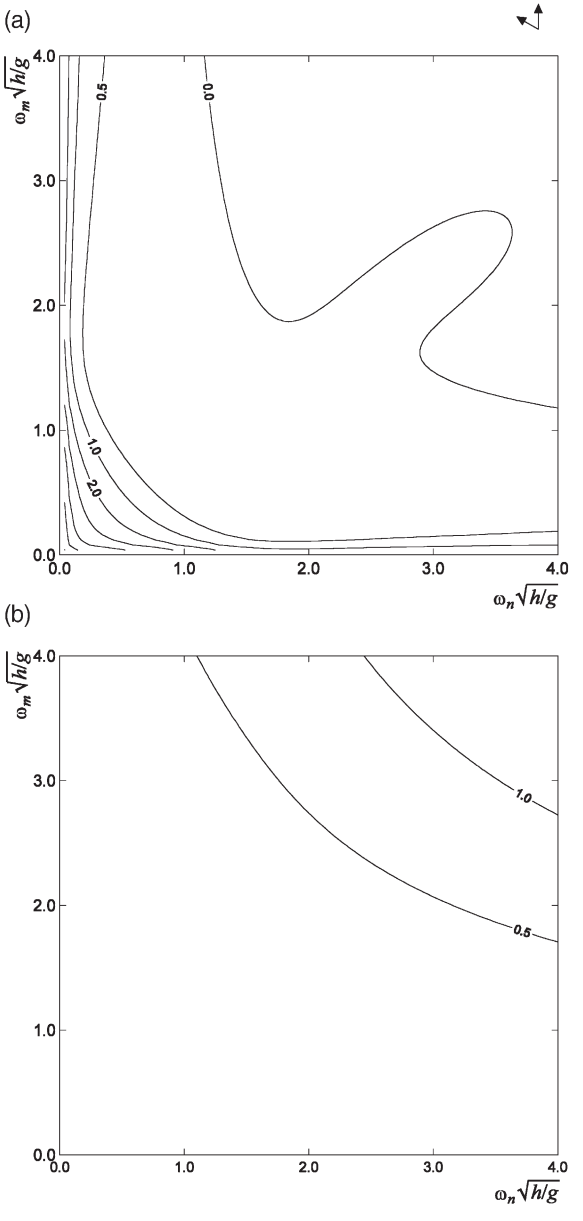


Fig. 12. As Fig. 9, but for $\alpha_n = 60^\circ$ and $\alpha_m = 0^\circ$.

spreading of bound subharmonics is somewhat larger than the spreading of the primary-wave spectrum (see e.g. Sand, 1982). With an even larger spreading for the free subharmonics these will often turn out as evanescent modes.

Table 1
Example: Second-order transfer functions for $(\omega_n\sqrt{h/g},\omega_m\sqrt{h/g}) = (1.00,0.67)$

α_n	α_m	$G_{nm}^{(21)-}$	$G_{nm}^{(21)+}$	$G_{nn}^{(21)+}$	$G_{mm}^{(21)+}$	$\lvert\mathbf{F}_{nm}^{-}\rvert$	$\lvert\mathbf{F}_{nm}^{+}\rvert$	$\lvert\mathbf{F}_{nn}^{+}\rvert$	$\lvert\mathbf{F}_{mm}^{+}\rvert$
0°	0°	−2.30	3.08	1.19	2.00	4.49	0.85	0.28	0.82
0°	15°	−1.58	2.92	1.19	2.00	1.99	0.79	0.28	0.84
0°	60°	−0.13	1.32	1.19	2.00	0.84	0.22	0.28	0.68
60°	0°	−0.13	1.32	1.19	2.00	1.63	0.20	0.31	0.82

Table 2
Propagation directions for bound ($\alpha_{nm}^{(21)\pm}$) and free ($\alpha_{nm}^{(22)\pm}$) waves. Example for $(\omega_n\sqrt{h/g},\omega_m\sqrt{h/g}) = (1.00,0.67)$. Missing values indicate evanescent modes

α_n	α_m	$\alpha_{nm}^{(21)-}$	$\alpha_{nm}^{(21)+}$	$\alpha_{nn}^{(21)+}$	$\alpha_{mm}^{(21)+}$	$\alpha_{nm}^{(22)-}$	$\alpha_{nm}^{(22)+}$	$\alpha_{nn}^{(22)+}$	$\alpha_{mm}^{(22)+}$
0°	0°	0°	0°	0°	0°	0°	0°	0°	0°
0°	15°	−20.3°	5.6°	0°	15°	−33.3°	3.8°	0°	11.5°
0°	60°	−36.6°	21.8°	0°	60°	–	12.9°	0°	41.9°
60°	0°	96.6°	38.2°	60°	0°	–	21.8°	31.2°	0°

4. Summary and concluding remarks

A full second-order theory for the generation of waves by an infinitely long, serpent-type wavemaker has been developed. The motion perpendicular to the wavemaker may be rotational, translatory or a combination of the two. Apart from being restricted by the usual limitations of second-order Stokes theory, no additional assumptions were made such as shallow water, small evanescent-mode interactions, or narrow-band spectra. The primary result of the theory is a second-order transfer function relating the second-order paddle motion to the first-order waves by which spurious, free superharmonics and subharmonics can be avoided.

The theory may also be used for the second-order prediction of spurious free-wave generation when a first-order control signal is applied.

Tasks for future work include practical experiments for validation of the theory and coupling with simultaneous active wave absorption.

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Appendix

The functions $\Gamma_{1\ldots4}$ (eqs. (78)–(81)) are evaluated in this Appendix. For $\kappa_1 \neq \kappa_2$ we obtain

$$\Gamma_1(\kappa_1,\kappa_2) \equiv \int_{-h}^{0} \cosh\kappa_1(z+h)\cosh\kappa_2(z+h)dz = \gamma_1(\kappa_1,\kappa_2;h) \tag{A1}$$

$$\gamma_1(\kappa_1,\kappa_2;r) \equiv \frac{\sinh(\kappa_1-\kappa_2)r}{2(\kappa_1-\kappa_2)} + \frac{\sinh(\kappa_1+\kappa_2)r}{2(\kappa_1+\kappa_2)} \tag{A2}$$

$$= \frac{\cosh\kappa_1 r\cosh\kappa_2 r}{\kappa_1^2-\kappa_2^2}\{\kappa_1\tanh\kappa_1 r-\kappa_2\tanh\kappa_2 r\}$$

Furthermore

$$\Gamma_2(\kappa_1,\kappa_2) \equiv \int_{-h+d}^{0} f(z)\cosh\kappa_1(z+h)\cosh\kappa_2(z+h)dz = \gamma_1(\kappa_1,\kappa_2;h) \tag{A3}$$

$$-\frac{1}{h+l}\{\gamma_2(\kappa_1,\kappa_2;h)-\gamma_2(\kappa_1,\kappa_2;d)\}$$

where

$$\gamma_2(\kappa_1,\kappa_2;r) \equiv \frac{\cosh(\kappa_1-\kappa_2)r}{2(\kappa_1-\kappa_2)^2} + \frac{\cosh(\kappa_1+\kappa_2)r}{2(\kappa_1+\kappa_2)^2} \tag{A4}$$

$$= \frac{\cosh\kappa_1 r\cosh\kappa_2 r}{(\kappa_1^2-\kappa_2^2)^2}\{\kappa_1^2+\kappa_2^2-2\kappa_1\kappa_2\tanh\kappa_1 r\tanh\kappa_2 r\}$$

Finally,

$$\Gamma_3(\kappa_1,\kappa_2) \equiv \int_{-h+d}^{0} f'(z)\sinh\kappa_1(z+h)\cosh\kappa_2(z+h)dz \tag{A5}$$

$$= \frac{1}{h+l} \quad (\gamma_3(\kappa_1,\kappa_2;h)-\gamma_3(\kappa_1,\kappa_2;d))$$

where

$$\gamma_3(\kappa_1,\kappa_2;r) \equiv \frac{\cosh(\kappa_1-\kappa_2)r}{2(\kappa_1-\kappa_2)} + \frac{\cosh(\kappa_1+\kappa_2)r}{2(\kappa_1+\kappa_2)} = \tag{A6}$$

$$\frac{\cosh\kappa_1 r\cosh\kappa_2 r}{\kappa_1^2-\kappa_2^2}\{\kappa_1-\kappa_2\tanh\kappa_1 r\tanh\kappa_2 r\}$$

by which

$$\Gamma_4(\kappa_1,\kappa_2,\tilde{\kappa}) \equiv \tilde{\kappa}\,\Gamma_2(\kappa_1,\kappa_2) + \kappa_1\Gamma_3(\kappa_1,\kappa_2)$$

$$= \tilde{\kappa}\frac{\cosh\kappa_1 h\cosh\kappa_2 h}{\kappa_1^2-\kappa_2^2}\Bigg\{\kappa_1\tanh\kappa_1 h-\kappa_2\tanh\kappa_2 h$$

$$-\frac{1}{h+l}\frac{1}{\kappa_1^2-\kappa_2^2}\left[(\kappa_1^2+\kappa_2^2)\left(1-\frac{\cosh\kappa_1 d\cosh\kappa_2 d}{\cosh\kappa_1 h\cosh\kappa_2 h}\right)\right. \tag{A7}$$

$$\left.\left.-2\kappa_1\kappa_2\left(\tanh\kappa_1 h\tanh\kappa_2 h-\frac{\sinh\kappa_1 d\sinh\kappa_2 d}{\cosh\kappa_1 h\cosh\kappa_2 h}\right)\right]\right\}$$

$$+\kappa_1\frac{1}{h+l}\frac{\cosh\kappa_1 h\cosh\kappa_2 h}{\kappa_1^2-\kappa_2^2}\left[\kappa_1\left(1-\frac{\cosh\kappa_1 d\cosh\kappa_2 d}{\cosh\kappa_1 h\cosh\kappa_2 h}\right)\right.$$

$$\left.-\kappa_2\left(\tanh\kappa_1 h\tanh\kappa_2 h-\frac{\sinh\kappa_1 d\sinh\kappa_2 d}{\cosh\kappa_1 h\cosh\kappa_2 h}\right)\right]$$

The case of $\kappa_1 = \kappa_2$may occur in Γ_4, and we obtain

$$\Gamma_4(\kappa,\kappa,\tilde{\kappa}) \equiv \tilde{\kappa}\,\Gamma_2(\kappa,\kappa)+\kappa\Gamma_3(\kappa,\kappa)=\tilde{\kappa}\left\{\frac{1}{2}h+\frac{1}{4\kappa}\sinh 2\kappa h+\frac{1}{h+l}\left[\right.\right. \tag{A8}$$

$$\left.\left.-\frac{1}{4}(h^2-d^2)+\frac{2\kappa^2-\tilde{\kappa}}{8\kappa^2\tilde{\kappa}}(\cosh 2\kappa h-\cosh 2\kappa d)\right]\right\}$$

With reference to eq. (76) we only need Γ_1 for $(\kappa_1,\kappa_2) = (k_{jnlm}^{\pm},K_p^{\pm})$. In terms of $D_{jnlm}^{\pm}$ defined in eq. (63), we obtain

$$\Gamma_1(k_{jnlm}^{\pm},K_p^{\pm}) = \frac{\cosh k_{jnlm}^{\pm}h\cosh K_p^{\pm}h}{g((k_{jnlm}^{\pm})^2-(K_p^{\pm})^2)}D_{jnlm}^{\pm} \tag{A9}$$

Furthermore, Γ_4 appears as a function of $(\kappa_1,\kappa_2,\tilde{\kappa}) = (k_{jn},K_p^{\pm},k_{xjn}^2\mp k_{yn}k_{ym})$ for which we have

$$\Gamma_4(k_{jn},K_p^{\pm},k_{xjn}^2\mp k_{yn}k_{ym}) = (k_{xjn}^2\mp k_{yn}k_{ym})\frac{\cosh k_{jn}h\cosh K_p^{\pm}h}{g(k_{jn}^2-K_p^{\pm 2})}\{\omega_n^2-(\omega_n \tag{A10}$$

$$\pm\,\omega_m)^2+M_2(k_{jn},K_p^{\pm},k_{xjn}^2\mp k_{yn}k_{ym})\}$$

where M_2 is given by eq. (83), and where the dispersion relations (40) and (73) have been applied to reduce the number of hyperbolic functions.

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