LABORATORY WAVE GENERATION CORRECT TO SECOND ORDER

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ABSTRACT Through the eighties the theory for second order irregular wave generation was developed within the framework of Stokes wave theory. This pioneering work, however, is not fully consistent. Furthermore, due to the extensive algebra involved, the derived transfer functions appear in an unnecessarily complicated form. The present paper develops the full second order wavemaker theory (including superharmonics as well as subharmonics) valid for a variety of different types of wave board motion. In addition to the well known transfer functions some new terms evolve. These are related to the first order local disturbances (evanescent modes) and accordingly they are significant when the wave board motion makes a poor fit to the velocity profile of the desired progressive wave component. This is typically the case for the high-frequency part of a primary wave spectrum when using a piston type wavemaker. The transfer functions are given in a relatively simple form by which the computational effort is reduced substantially. This enhances the practical computation of second order wavemaker control signals for irregular waves, and no narrow band assumption is needed. The software is conveniently included in a PC-based wave generation system - the DHI Wave Synthesizer. The validity of the theory is analysed in a number of laboratory wave tests, covering the superharmonic generation for regular waves.

1. INTRODUCTION

First order wavemaker theory corresponding to linearized Stokes theory has long been well established (Havelock, 1929, Biesel, 1951, Ursell et al., 1960, and others, cf. the review by Svendsen, 1985) and we shall devote this introduction to second order theories of wave generation.

The first step towards the development of wavemaker theory is of cause the knowledge of the underlying wave theory. Already in 1847 Stokes gave results for regular waves in terms of a perturbation series using the wave steepness as the small ordering parameter. For regular waves only the sum frequencies appear (since the difference frequencies vanish) and Stokes found the resulting superharmonics.

Presumably the first approach to second order wavemaker theory was given by Fontanet (1961) for regular waves. Using a Lagrangian description he found the spurious superharmonics generated by a purely sinusoidal oscillation of the

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wave board and gave directions as how to suppress these by adding a superharmonic component to the wavemaker control signal.

Recently Hudspeth and Sulisz (1991) derived the complete second order lagrangian theory for regular waves with special emphasis on Stokes drift and return flow in wave flumes. The theories of Fontanet (1961) and Hudspeth and Sulisz (1991) appear to be the most complete theoretical developments yet.

Madsen (1971) developed an approximate theory for the suppression of spurious superharmonics in regular waves generated in fairly shallow water.

Buhr Hansen et al. (1975) chose an emperical approach to persue the second order control signal for regular waves. The second order regular wave field generated by a first order control signal has further been studied by Flick and Guza (1980).

For irregular waves both sum and difference frequencies appear in the interaction terms at second order. Longuet-Higgins and Stewart (1962,1964) derived results for the subharmonics with the restriction of only slightly different frequencies (narrow band restriction) in the interacting wavelets. Without this restriction Ottesen-Hansen (1978) gave similar results in a more suitable form i.e. using a transfer function giving the second order contribution in terms of the interacting first order wavelets. A generalization including both subharmonics and superharmonics for directional waves was given by Sharma and Dean (1979), see also Dean and Sharma (1981).

Neglecting the local disturbance terms in the first order solution Flick and Guza (1980) gave an approximate theory for the generation of spurious long waves by a first order bichromatic control signal under the narrow band assumption.

Without this assumption Sand (1982) calculated the second order subharmonic control signal for a piston type wavemaker needed to suppress spurious long wave generation. A more detailed description of the theory was given by Barthel et al. (1983), who also extended the theory to include a rotating wave board motion, restricting the centre of rotation to a point at or below the bottom. This is, however, an inconvenient restriction, since many wavemakers are equipped with a hinge situated above the bottom. Sand and Donslund (1985) gave the required theoretical extension.

For irregular waves a technique for the synthesis of wave maker control signals based on the narrow band assumption for the first order carrier waves has further been used by Klopman and Leeuwen (1990) and by Bowers (1988) (discussed by Mansard et al., 1989) for subharmonic frequencies. This approximation allows for an efficient time–domain computation of the second order control signal.

For irregular waves Mansard and Sand (1986a,b) derived the wavemaker theory for the superharmonics valid for translatory as well as rotating wave boards. For a directional wavemaker Suh and Dalrymple (1987) developed part of the theory for the spurious superharmonic and subharmonic second order waves generated by a first order wavemaker control signal.

In the present paper the full second order wavemaker theory is rederived in a unifying and compact form that includes both superharmonics and subharmonics and covers a variety of different types of wave board motion †. No narrow band approximation is applied.

† Inconsistensies in the previous derivations are pointed out concurrently using footnotes.

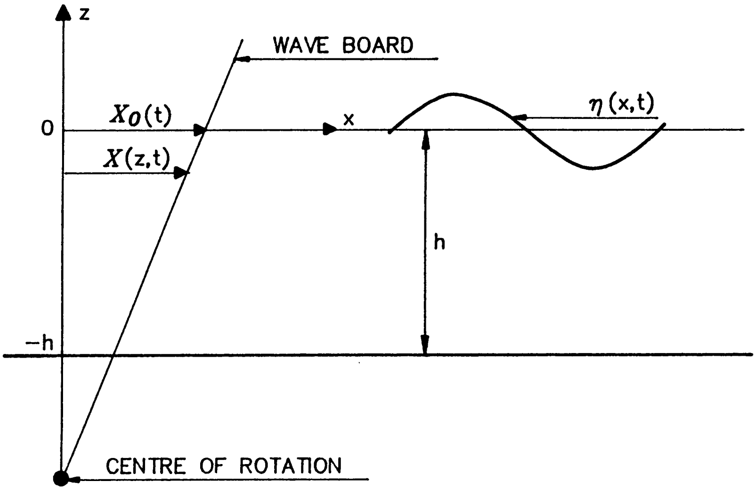

Figure 1 Definition sketch (the waveboard motion indicated corresponds to type (b) in Fig. 2).

In order to facilitate and reduce the theoretical calculations a complex representation is chosen although the resulting transfer functions are also given in real form.

2. GENERAL THEORY

2.1. Governing equations

The classical method of perturbation theory in combination with Taylor expansions of the boundary conditions at the free surface and at the wave board leads to a boundary value problem for the first and second order wave contributions, respectively (Stokes, 1947, Laitone, 1961, Flick and Guza, 1980, and others). Let $(u, w) = (\Phi_x, \Phi_z)$ define the velocity potential $\Phi = \Phi(x, z, t)$ in a Cartesian coordinate system (x,z), cf. Fig. 1, and let $\eta = \eta(x, t)$, $X = X(z, t)$, g, h, and t denote surface elevation, wave board position, acceleration of gravity, still water depth, and time, respectively, then the relevant boundary value problems may be written

$$\Delta\Phi = 0 \qquad \text{everywhere} \qquad (1a)$$

$$\Phi_{tt} + g\Phi_z = R \qquad \text{for } z = 0 \qquad (1b)$$

$$\Phi_x = Q \qquad \text{for } x = 0 \qquad (1c)$$

$$\eta = -\frac{1}{g}(\Phi_t + P) \qquad \text{for } z = 0 \qquad (1d)$$

$$\Phi_z = 0 \qquad \text{for } z = -h \qquad (1e)$$

where R, Q, and P are given below. The elevation, potential, and wave board position (see (5)) correct to second order are given by

$$\eta = \epsilon\eta^{(1)} + \epsilon^2\eta^{(2)} \tag{2a}$$

$$\Phi = \epsilon\Phi^{(1)} + \epsilon^2\Phi^{(2)} \tag{2b}$$

$$X_0 = \epsilon X_0^{(1)} + \epsilon^2 X_0^{(2)} \tag{2c}$$

where ϵ is a small ordering parameter, and (1) covers the first order problem for $(\Phi, \eta) = (\Phi^{(1)}, \eta^{(1)})$, where

$$R^{(1)} = 0 \tag{3a}$$

$$Q^{(1)} = f(z)X_{0t}^{(1)} \tag{3b}$$

$$P^{(1)} = 0 \tag{3c}$$

and the second order problem for $(\Phi, \eta) = (\Phi^{(2)}, \eta^{(2)})$, where †

$$R^{(2)} = -\left\{(\Phi_x^{(1)2} + \Phi_z^{(1)2})_t + \eta^{(1)}(\Phi_{tt}^{(1)} + g\Phi_z^{(1)})_z\right\} \tag{4a}$$

$$Q^{(2)} = \begin{cases} -X_0^{(1)}\{f(z)\Phi_{xx}^{(1)} - \dfrac{1}{h+\ell}\Phi_z^{(1)}\} + f(z)X_{0t}^{(2)} & \text{for } -(h-d) \le z \le 0 \\ 0 & \text{for } -h \le z < -(h-d) \end{cases} \tag{4b}$$

$$P^{(2)} = \frac{1}{2}\left(\Phi_x^{(1)2} + \Phi_z^{(1)2}\right) + \eta^{(1)}\Phi_{zt}^{(1)} \tag{4c}$$

The position of the wave board is

$$X(z,t) = f(z)X_0(t) \tag{5}$$

where $f(z)$ describes the type of wavemaker:

$$f(z) = \begin{cases} 1 + \dfrac{z}{h+\ell} & \text{for } -(h-d) \le z \le 0 \\ 0 & \text{for } -h \le z < -(h-d) \end{cases} \tag{6}$$

Here $z = -(h+\ell)$ gives the centre of rotation $(-h < \ell \le \infty)$ and $d \ge 0$ describes elevated wavemakers i.e. the last case in (6) is only relevant for $d > 0$. For $\ell = -d$ or $(d = 0, \ell \ge 0)$ equation (6) reduces to types of wavemakers considered in previous references. The present extension is made mainly because it covers wavemakers of the piston type (i.e. $\ell = \infty$), where the piston does not reach the bottom (i.e. $d > 0$), cf. Fig. 2, which shows the possible combinations. It is emphasized that solutions for such discontinuous $f(z)$ should be used with

† For wave boards hinged above the bottom $Q = 0$ must be required specifically below the hinge. The consequence of this fact was not fully drawn in Sand and Donslund (1985) and Sand and Mansard (1986a,b).

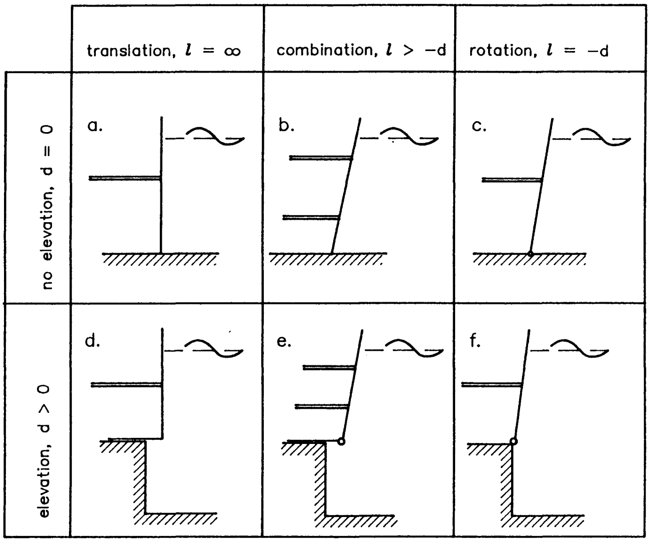

Figure 2 Different types of waveboard motions: (a) translatory (piston type) (b) combined (c) rotational (d) translatory, elevated (e) combined, elevated (f) rotational, elevated.

extreme care, since separated flow (which is not considered in the present theory) is inevitable around angular points.

As described in detail by Barthel et al. (1983), the second order problem is conveniently separated into three contributions

$$\Phi^{(2)} = \Phi^{(21)} + \Phi^{(22)} + \Phi^{(23)} \tag{7a}$$

$$\eta^{(2)} = \eta^{(21)} + \eta^{(22)} + \eta^{(23)} \tag{7b}$$

and the respective boundary value problems are given by (1), where for $(\Phi, \eta) = (\Phi^{(21)}, \eta^{(21)})$:

$$R^{(21)} = -\left\{ \left(\Phi_x^{(1)2} + \Phi_z^{(1)2}\right)_t + \eta^{(1)}(\Phi_{tt}^{(1)} + g\Phi_z^{(1)})_z \right\} \tag{8a}$$

$$Q^{(21)} = \text{arbitrary (i.e. condition disappears)} \tag{8b}$$

$$P^{(21)} = \frac{1}{2}\left(\Phi_x^{(1)2} + \Phi_z^{(1)2}\right) + \eta^{(1)}\Phi_{zt}^{(1)} \tag{8c}$$

for $(\Phi, \eta) = (\Phi^{(22)}, \eta^{(22)})$:

$$R^{(22)} = 0 \tag{9a}$$

$$Q^{(22)} = \begin{cases} -X_0^{(1)}\{f(z)\Phi_{xx}^{(1)} - \dfrac{1}{h+\ell}\Phi_z^{(1)}\} - \Phi_x^{(21)} & \text{for } -(h-d) \le z \le 0 \\ -\Phi_x^{(21)} & \text{for } -h \le z < -(h-d) \end{cases} \tag{9b}$$

$$P^{(22)} = 0 \tag{9c}$$

and for $(\Phi, \eta) = (\Phi^{(23)}, \eta^{(23)})$:

$$R^{(23)} = 0 \tag{10a}$$

$$Q^{(23)} = f(z)X_{0t}^{(2)} \tag{10b}$$

$$P^{(23)} = 0 \tag{10c}$$

Here $\Phi^{(21)}$ gives the bound waves due to the interaction between first order wavelets, $\Phi^{(22)}$ describes the free waves due to the wavemaker leaving its mean position and due to $\Phi^{(21)}$ mismatching the boundary condition at the wavemaker, and $\Phi^{(23)}$ gives the free waves generated by the second order wave board motion. If the control signal for the wavemaker is based on first order theory alone then the resulting second order waves are given by $\Phi^{(2)} = \Phi^{(21)} + \Phi^{(22)}$ i.e. the spurious free waves from $\Phi^{(22)}$ are not eliminated. Let subscript 0 (on Φ and η) denote the progressive part of a wave field then the objective for second order wavemaker theory is to determine $X_0^{(2)}(t)$ as to produce free waves $\eta_0^{(23)}$ which eliminate these spurious free waves $\eta_0^{(22)}$ by requiring

$$\eta_0^{(22)} + \eta_0^{(23)} = 0 \tag{11}$$

or equivalently

$$\Phi_0^{(22)} + \Phi_0^{(23)} = 0 \tag{12}$$

2.2. First order solution

The first order solution was obtained by Biesel (1951), Ursell et al. (1960), Flick and Guza (1980), Sand and Donslund (1985), and others. Here the solution will be given in a slightly more general version as regards the type of wave board motion, and in the compact notation provided by a complex representation.

Let the first order paddle position for each of the wavelets constituting the first order spectrum be given by

$$X_0^{(1)} = \frac{1}{2}\{-iX_a e^{i\omega t} + \text{c.c.}\} \tag{13}$$

where X_a is the constant complex first order wave board amplitude at still water level and where c.c. denotes the complex conjugate of the preceding term, then the solution to the first order problem (1) and (3) may be expressed as

$$\Phi^{(1)} = \frac{1}{2}\left\{\frac{igX_a}{\omega}\sum_{j=0}^{\infty} c_j \frac{\cosh k_j(z+h)}{\cosh k_j h} e^{i(\omega t - k_j x)} + \text{c.c.}\right\} \tag{14a}$$

$$\eta^{(1)} = \frac{1}{2}\left\{X_a \sum_{j=0}^{\infty} c_j e^{i(\omega t - k_j x)} + \text{c.c.}\right\} \tag{14b}$$

which includes both the wanted progressive-wave term and the local disturbances which are due to the mismatch between the shape of the progressive-wave velocity profile and the shape function $f(z)$. This solution obviously satisfies the bottom boundary condition (1e), and the free surface boundary condition (1b) is easily shown to require

$$\omega^2 = gk_j \tanh k_j h \tag{15}$$

This is the linear dispersion relation generalized to complex wavenumbers, and it has one real solution, say k_0, and an infinity of purely imaginary solutions $(k_1, k_2, \ldots)$, where $ik_j > 0,\ j = 1, 2, \ldots$. In order not to confuse k_0 with the deep water wave number we shall omit the subscribt on k when $j = 0$ appears explicitly (and only when it is the only index).

It may seem artificial to retain imaginary wavenumbers instead of letting $k_j,\ j = 1, 2, \ldots$ be the real solutions to $-\omega^2 = gk_j \tan k_j h$. However, this choice gives a considerable reduction of the algebra involved at a later stage, and it assures the analogy between the treatment of the progressive-wave term ($j = 0$) and the local disturbances ($j = 1, 2, \ldots$).

The coefficients c_j are determined by requiring the solution to satisfy the boundary condition at the wavemaker. This is done by inserting (3b) and (14a) in (1c), multiplying the equation by $\cosh k_l(z+h)$ and integrating the result over the depth. Due to the orthogonality relation

$$\int_{-h}^{0} \cosh k_j(z+h) \cosh k_l(z+h) dz = \begin{cases} \dfrac{1}{2k_j}(k_j h + \sinh k_j h \cosh k_j h) & \text{for } l = j \\ 0 & \text{for } l \neq j \end{cases} \tag{16}$$

which may readily be verified, only the term $j = l$ remains in the infinite series, and the resulting equation is solved for c_j to obtain

$$c_j = \sinh k_j h \frac{\Lambda_1(k_j)}{\Lambda_2(k_j)} \tag{17a}$$

where

$$\begin{aligned}\Lambda_1(k_j) &\equiv k_j \int_{-h}^{0} f(z) \cosh k_j(z+h) dz \\ &= \sinh k_j h - \frac{d+\ell}{h+\ell} \sinh k_j d + \frac{1}{h+\ell} \frac{\cosh k_j d - \cosh k_j h}{k_j}\end{aligned} \tag{17b}$$

$$\Lambda_2(k_j) \equiv k_j \int_{-h}^{0} \cosh^2 k_j(z+h) dz = \frac{1}{2}(k_j h + \sinh k_j h \cosh k_j h) \tag{17c}$$

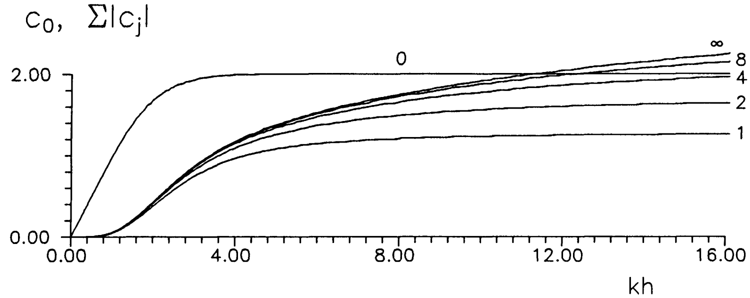

Figure 3 The Biesel transfer function c_0 with the sum of the local disturbance transfer functions c_j from $j = 1$ retaining 1,2,4,8, and "∞" terms in the series, respectively. The results shown are valid for a piston type wavemaker.

For $j = 0$ this gives the real quantity c_0 which is known as the Biesel transfer function in terms of which the complex amplitude A of the progressive part of the first order wavefield in (14) may be related to the complex amplitude X_a of the first order paddle position through

$$A = c_0 X_a \tag{18}$$

the elevation for the progressive part of the first order waves being

$$\eta_0^{(1)} = \frac{1}{2}\{Ae^{i(\omega t - kx)} + \text{c.c.}\} \tag{19}$$

With $A = a - ib$ (13) and (19) may be written $X_0^{(1)} = c_0(a \sin \omega t - b \cos \omega t)$ and $\eta_0^{(1)} = a \cos(\omega t - kx) + b \sin(\omega t - kx)$.

For $j = 1, 2, \ldots$ c_j is purely imaginary and $ic_j > 0$, $j = 1, 2, \ldots$.

Before we proceed to the second order solution we shall illustrate the magnitude of the local disturbances at the wave board. This is relevant for the importance of the interaction terms at second order and for special applications of first order theory like active absorption relying on wave gauges in conjunction with the wave board. For a piston type wave wavemaker Fig. 3 compares the Biesel transfer function c_0 with the sum of the local disturbance transfer functions c_j from $j = 1$ retaining 1,2,4,8, and "∞" terms in the series, respectively. While c_0 practically reaches its asymptotic value of 2 at the traditional deep water limit $kh = \pi$ the sum of the local disturbance coefficients increase continuously with increasing kh. It appears that for $kh > 11.5$ the amplitude of the local disturbance at the wave board exceeds the amplitude of the emitted progressive wave. This demonstrates the problems of generating deep water waves with a piston type wavemaker, since pronounced local disturbances will significantly increase the nonlinear wave interaction.

It turns out that the maximum slope of the local disturbance at the wave board is somewhat smaller than the maximum slope of the emitted progressive wave. In fact it can be shown theoretically that

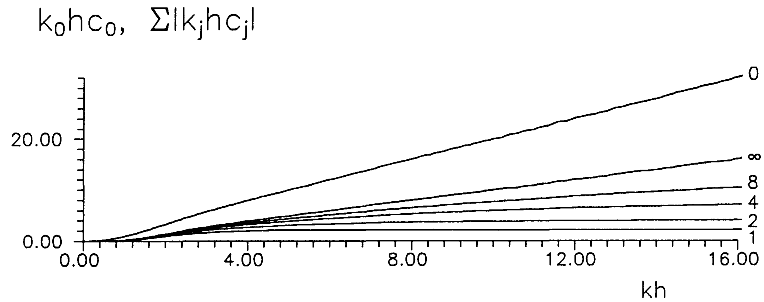

Figure 4 The maximum surface slopes equivalent to the representation of surface elevations in Fig. 3.

$$\sum_{j=0}^{\infty} k_j h c_j = \omega^2 \frac{h}{g} \tag{20}$$

by which the ratio of maximum slopes for a piston type wavemaker becomes

$$\frac{1}{khc_0} \sum_{j=1}^{\infty} |k_j h c_j| = \frac{1}{2}\left(1 - \frac{kh}{\sinh kh \cosh kh}\right) \tag{21}$$

approaching 1/2 in in the deep water limit. Fig. 4 shows the maximum surface slopes equivalent to the representation of surface elevations in Fig. 3. The reason for showing the result of retaining 1,2,4, and 8 terms, respectively in summation in Fig. 4 is to illustrate the slow rate of convergence compared with the curves in Fig. 3. It can be shown that (piston type wavemaker)

$$k_j h c_j \propto \frac{1}{j^2} \quad \text{for} \quad j \gg 1 \tag{22}$$

while

$$c_j \propto \frac{1}{j^3} \quad \text{for} \quad j \gg 1 \tag{23}$$

This gives an immense difference of convergence rate as can be seen by computing $\sum_{j=1}^{\infty} 1/j^N$, which to single precision requires $2^{24/N}$ terms i.e. 4096, 256, 64, and 28 terms for $N = 2, 3, 4$, and 5, respectively. The inconveniently slow rate of convergence charaterized by $N = 2$ reappears in the new second order terms found below, whereas the well known series term exhibits the the faster convergence rate corresponding to $N = 3$. The problem of slow convergence requires special attention in the computation of the second order transfer functions; It can be shown numerically that the accuracy obtained by straight forward summation of well over a hundred terms can be obtained using an order of magnitude fewer terms together with the knowledge of the assymptotic rate of convergence.

The significance of the local disturbances in the nonlinear interaction will be treated later.

2.3. Second order solution

Interactions between two wavelets of generally different frequencies ω_n and ω_m constitute the basis of the second order spectrum. The summation indices for the series (14) giving the first order solution for ω_n and ω_m are denoted j and l, respectively, and if $S^{(21)}$ stands for each of the quantities $R^{(21)}, \Phi^{(21)}, P^{(21)}$, and $\eta^{(21)}$, then $S^{(21)+}_{jnlm}$ and $S^{(21)-}_{jnlm}$ denote the complex superharmonic and subharmonic contributions, respectively, for the interaction between each pair of terms drawn from the two series, i.e.

$$S^{(21)} = \frac{1}{2}\left\{\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\left[S^{(21)+}_{jnlm} + S^{(21)-}_{jnlm}\right] + \text{c.c.}\right\}; \qquad S^{(21)} = \begin{cases} R^{(21)} \\ \Phi^{(21)} \\ P^{(21)} \\ \eta^{(21)} \end{cases} \tag{24}$$

Using (15) to eliminate any appearing $\tanh k_j h$-term (8a) and (14) yield after straight forward but lengthy algebra

$$\begin{aligned} R^{(21)\pm}_{jnlm} &= \delta_{nm}\left\{ iX_n X_m^{-:*} c_{jn} c_{lm}^{-:*} e^{i(\theta_{jn} \pm \theta_{lm}^{-:*})} \right. \\ &\qquad \left. \left[\omega_n\omega_m^2 \mp \frac{g^2 k_{jn} k_{lm}^{-:*}}{\omega_n} \pm \frac{1}{2}\left(\omega_m^3 - \frac{g^2 k_{lm}^2}{\omega_m}\right)\right] + \widehat{lmjn}^{\,-:*} \right\} \\ &= \delta_{nm} H^{\pm}_{jnlm} i X_n X_m^{-:*} c_{jn} c_{lm}^{-:*} e^{i(\theta_{jn} \pm \theta_{lm}^{-:*})} \end{aligned} \tag{25a}$$

where

$$\delta_{nm} \equiv \begin{cases} \frac{1}{2} & \text{for } n = m \\ 1 & \text{for } n \neq m \end{cases} \tag{25b}$$

$$\theta_{jn} \equiv \omega_n t - k_{jn} x; \qquad \theta_{lm} \equiv \omega_m t - k_{lm} x \tag{25c}$$

and $\widehat{lmjn}$ means "the preceding term permuting l and j as well as m and n". This symbol is used throughout even in cases where l or j are explicitly given as zero. Furthermore

$$H^{\pm}_{jnlm} \equiv (\omega_n \pm \omega_m)\left(\pm\omega_n\omega_m - \frac{g^2 k_{jn} k_{lm}^{-:*}}{\omega_n\omega_m}\right) + \frac{\omega_n^3 \pm \omega_m^3}{2} - \frac{g^2}{2}\left(\frac{k_{jn}^2}{\omega_n} \pm \frac{k_{lm}^2}{\omega_m}\right) \tag{25d}$$

and the symbol $^{-:*}$ introduced for brevity is defined by

$$Z^{-:*} = \begin{cases} Z & \text{for superharmonics} \\ Z^* & \text{for subharmonics} \end{cases} \tag{26}$$

where $*$ denotes complex conjugation, i.e. $^{-:*}$ is to be interpreted as complex conjugation in case of subharmonics, while being ignored for superharmonics. Furthermore the complex identity

$$\{Z_1 e^{iz_1} + c.c.\}\{Z_2 e^{iz_2} + c.c.\} = \left\{Z_1 Z_2 e^{i(z_1+z_2)} + Z_1 Z_2^* e^{i(z_1 - z_2^*)}\right\} + c.c. \tag{27}$$

where z_1, z_2, Z_1, Z_2 are complex numbers has been used. This identity reveals the origin of the destinction (26) between the superharmonics and the subharmonics used in (25). (Note that originally the term $\widehat{lmjn}$ in (25a) would appear without the conjugation symbol. However, it is seen that (24) still holds irrespective of whether or not the complex conjugation is imposed on a term in $R^{(21)\pm}_{jnlm}$, and accordingly $\widehat{lmjn}^{-:*}$ may well be used instead of $\widehat{lmjn}$.)

It follows that the solution to (1abe) and (8a) is †

$$\Phi^{(21)\ \pm}_{jnlm} = \frac{R^{(21)\pm}_{jnlm}}{D^{\pm}_{jnlm}} \frac{\cosh(k_{jn} \pm k_{lm}^{-:*})(z+h)}{\cosh(k_{jn} \pm k_{lm}^{-:*})h} \tag{28a}$$

where

$$D^{\pm}_{jnlm} \equiv g(k_{jn} \pm k_{lm}^{-:*}) \tanh(k_{jn} \pm k_{lm}^{-:*})h - (\omega_n \pm \omega_m)^2 \tag{28b}$$

For $(j,l) = (0,0)$ equation (28) is real and it is consistent with Dean and Sharma (1981).

We now turn to the corresponding surface elevation. From (8c) and (14) we obtain

$$\begin{aligned} P^{(21)\pm}_{jnlm} &= \delta_{nm}\Big\{X_n X_m^{-:*} c_{jn} c_{lm}^{-:*} e^{i(\theta_{jn} \pm \theta_{lm}^{-:*})} \\ &\qquad \left[\frac{g^2 k_{jn} k_{lm}^{-:*}}{4\omega_n \omega_m} \mp \frac{1}{4}\omega_n \omega_m - \frac{1}{2}\omega_m^2\right] + \widehat{lmjn}^{-:*}\Big\} \\ &= \delta_{nm} L^{\pm}_{jnlm} X_n X_m^{-:*} c_{jn} c_{lm}^{-:*} e^{i(\theta_{jn} \pm \theta_{lm}^{-:*})} \end{aligned} \tag{29a}$$

where

$$L^{\pm}_{jnlm} \equiv \frac{1}{2}\left\{\frac{g^2 k_{jn} k_{lm}^{-:*}}{\omega_n \omega_m} \mp \omega_n \omega_m - (\omega_n^2 + \omega_m^2)\right\} \tag{29b}$$

which by (1d) yields

$$\eta^{(21)\pm}_{jnlm} = G^{\pm}_{jnlm} X_n X_m^{-:*} c_{jn} c_{lm}^{-:*} e^{i(\theta_{jn} \pm \theta_{lm}^{-:*})} \tag{30a}$$

† In the previous papers on the subject involving more than one primary frequency only the contribution from the progressive first order waves i.e. $(j,l) = (0,0)$ has been considered. It appears that all contributions (j,l) enter the transfer functions used for second order wave generation.

where

$$G^{\pm}_{jnlm} \equiv \frac{\delta_{nm}}{g}\left\{(\omega_n \pm \omega_m)\frac{H^{\pm}_{jnlm}}{D^{\pm}_{jnlm}} - L^{\pm}_{jnlm}\right\} \tag{30b}$$

For $(j,l) = (0,0)$ this gives the progressive part of the second order bound wave, and $G^{\pm}_{0n0m}$ can be shown to equal the transfer functions $G^{\pm}_{nm}$ given by Ottesen-Hansen (1978) and Sand and Mansard (1986a,b), and we have †

$$\eta_0^{(21)\pm} = \frac{1}{2}\{G^{\pm}_{nm}A_nA_m^{-:*}e^{i(\theta_{0n}\pm\theta_{0m})} + \text{c.c.}\} \tag{31}$$

With $A_n = a_n - ib_n$ and $A_m^{-:*} = a_m \mp ib_m$ (31) may be written $\eta_0^{(21)\pm} = G^{\pm}_{nm}\{(a_na_m \mp b_nb_m)\cos(\theta_{0n} \pm \theta_{0m}) + (\pm a_nb_m + a_mb_n)\sin(\theta_{0n} \pm \theta_{0m})\}$.

It should be emphasized that in the present formulation of second order wavemaker theory the above calculations of $\eta^{(21)}$ and $G^{\pm}_{jnlm}$ are not needed.

The remaining contributions at second order, $\Phi^{(22)}$ and $\Phi^{(23)}$, represent free waves, and the analogy to the first order solution is evident, since the function Q cf. (3*b*), (9*b*), (10*b*), gives the only deviation from the first order problem. Thus, following the same procedure as for the first order waves only now for superharmonic and subharmonic frequencies, we have

$$\Phi^{(22)} = \Phi^{(22)+} + \Phi^{(22)-} \tag{32a}$$

$$\eta^{(22)} = \eta^{(22)+} + \eta^{(22)-} \tag{32b}$$

$$\Phi^{(22)\pm} = \frac{1}{2}\left\{\frac{igA_nA_m^{-:*}}{h(\omega_n \pm \omega_m)}\sum_{j=0}^{\infty} c_j^{(22)\pm}\frac{\cosh K_j^{\pm}(z+h)}{\cosh K_j^{\pm}h}e^{i((\omega_n\pm\omega_m)t-K_j^{\pm}x)} + \text{c.c.}\right\} \tag{32c}$$

$$\eta^{(22)\pm} = \frac{1}{2}\left\{\frac{A_nA_m^{-:*}}{h}\sum_{j=0}^{\infty} c_j^{(22)\pm}e^{i((\omega_n\pm\omega_m)t-K_j^{\pm}x)} + \text{c.c.}\right\} \tag{32d}$$

† In the papers by Sand (1982), Barthel et al. (1983), and Sand and Donslund (1985), the transfer function G^{-}_{nm} for the second order surface elevation as found by Ottesen-Hansen (1978) was used in connection with $\Phi^{(21)-}$ as if $\eta^{(21)-}$ was a free wave i.e. (1*d*) was solved for $\Phi^{(21)-}$ neglecting that $P^{(21)-} \neq 0$, cf. (8*c*). In terms of the quantities defined above, the term L^{-}_{0n0m} (L^{-}_{nm} for short) was retained in the transfer function for $\Phi^{(21)-}$. This was corrected for the superharmonics cf. Sand and Mansard (1986a,b) and Mansard, Sand, and Klinting (1987) by introducing a so-called $U^{\pm}$-factor, which may be shown to satisfy

$$U^{\pm} = \frac{g(k_n \pm k_m)\tanh(k_n \pm k_m)h}{(\omega_n \pm \omega_m)^2}\;\frac{G^{\pm}_{nm} + \delta_{nm}L^{\pm}_{nm}/g}{G^{\pm}_{nm}}$$

In the present formulation $H^{\pm}_{jnlm}/D^{\pm}_{jnlm}$ is used directly in the resulting transfer function without using the more complicated $G^{\pm}_{nm}$ and $U^{\pm}$-factors.

which includes the progressive-wave term as well as the local disturbances. Here $K_j^{\pm}$ is the solution to

$$(\omega_n \pm \omega_m)^2 = gK_j^{\pm} \tanh K_j^{\pm} h \tag{33}$$

which is the linear dispersion relation generalized to complex wavenumbers, cf. (15) and the discussion there. Using (32*c*) and (9*b*) in (1*c*) and multiplying the resulting equation with $\cosh K_l^{\pm}(z+h)$, integration from $z=-h$ to $z=0$ gives the coefficients $c_j^{(22)\pm}$ by virtue of orthogonality, exactly as in the first order case. Let index p take the place of index j then we get

$$c_p^{(22)\pm} = \delta_{nm} \frac{h(\omega_n \pm \omega_m)\cosh K_p^{\pm} h}{g c_{0n} c_{0m} \Lambda_2(K_p^{\pm})} \Big\{ \pm \frac{g}{2\omega_n} \sum_{j=0}^{\infty} \frac{c_{jn} k_{jn}}{\cosh k_{jn} h} \Gamma_4(k_{jn}, K_p^{\pm}) + \widehat{lmjn}$$
$$- \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} c_{jn} c_{lm}^{-:*} \frac{k_{jn} \pm k_{lm}^{-:*}}{\cosh(k_{jn} \pm k_{lm}^{-:*})h} \frac{H_{jnlm}^{\pm}}{D_{jnlm}^{\pm}} \Gamma_1(k_{jn} \pm k_{lm}^{-:*}, K_p^{\pm}) \Big\} \tag{34}$$

where

$$\Gamma_1(\kappa_1, \kappa_2) \equiv \int_{-h}^{0} \cosh \kappa_1(z+h) \cosh \kappa_2(z+h) dz \tag{35a}$$

and

$$\Gamma_4(\kappa_1, \kappa_2) \equiv \kappa_1 \Gamma_2(\kappa_1, \kappa_2) + \frac{1}{h+\ell} \Gamma_3(\kappa_1, \kappa_2) \tag{35b}$$

where

$$\Gamma_2(\kappa_1, \kappa_2) \equiv \int_{-h+d}^{0} f(z) \cosh \kappa_1(z+h) \cosh \kappa_2(z+h) dz \tag{35c}$$

and †

$$\Gamma_3(\kappa_1, \kappa_2) \equiv \int_{-h+d}^{0} \sinh \kappa_1(z+h) \cosh \kappa_2(z+h) dz \tag{35d}$$

The functions $\Gamma_{1\ldots 4}$ are evaluated in the appendix. Using $\Gamma_1(k_{jn} \pm k_{lm}^{-:*}, K_p^{\pm})$ as given in (A9) eq. (34) reduces to

$$c_p^{(22)\pm} = \delta_{nm} \frac{h(\omega_n \pm \omega_m)\cosh K_p^{\pm} h}{g c_{0n} c_{0m} \Lambda_2(K_p^{\pm})} \Big\{ \pm \frac{g}{2\omega_n} \sum_{j=0}^{\infty} \frac{c_{jn} k_{jn}}{\cosh k_{jn} h} \Gamma_4(k_{jn}, K_p^{\pm}) + \widehat{lmjn}$$
$$- \frac{\cosh K_p^{\pm} h}{g} \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} c_{jn} c_{lm}^{-:*} \frac{k_{jn} \pm k_{lm}^{-:*}}{(k_{jn} \pm k_{lm}^{-:*})^2 - K_0^{\pm 2}} H_{jnlm}^{\pm} \Big\} \tag{36}$$

† Here lower bound of the integral for Γ_3 must be taken as $-h+d$ (instead of just $-h$), cf. (9*b*) and the footnote in connection with (4*b*). In the integral for Γ_2, however, either lower bound may be used since $f(z) \equiv 0$ for $-h \leq z < -h+d$.

The denominator $D^{\pm}_{jnlm}$ and thereby one explicit shallow water singularity has now vanished from the expression.

We now turn to $\Phi^{(23)}$. Let the second order paddle position be given by

$$X_0^{(2)\pm} = \frac{1}{2}\left\{ -i\mathcal{F}^{\pm}\frac{A_n A_m^{-:*}}{h} e^{i(\omega_n \pm \omega_m)t} + \text{c.c.}\right\} \tag{37}$$

and let

$$\Phi^{(23)} = \Phi^{(23)+} + \Phi^{(23)-} \tag{38a}$$

$$\eta^{(23)} = \eta^{(23)+} + \eta^{(23)-} \tag{38b}$$

then

$$\Phi^{(23)\pm} = \frac{1}{2}\left\{ \frac{ig\mathcal{F}^{\pm}A_n A_m^{-:*}}{h(\omega_n \pm \omega_m)} \sum_{j=0}^{\infty} c_j^{(23)\pm} \frac{\cosh K_j^{\pm}(z+h)}{\cosh K_j^{\pm}h} e^{i((\omega_n \pm \omega_m)t - K_j^{\pm}x)} + \text{c.c.}\right\} \tag{38c}$$

$$\eta^{(23)\pm} = \frac{1}{2}\left\{ \mathcal{F}^{\pm}\frac{A_n A_m^{-:*}}{h} \sum_{j=0}^{\infty} c_j^{(23)\pm} e^{i((\omega_n \pm \omega_m)t - K_j^{\pm}x)} + \text{c.c.}\right\} \tag{38d}$$

where

$$c_j^{(23)\pm} = \sinh K_j^{\pm}h \frac{\Lambda_1(K_j^{\pm})}{\Lambda_2(K_j^{\pm})} \tag{39}$$

cf. (17). From (11) or (12) we obtain the transfer function

$$\mathcal{F}^{\pm} = -\frac{c_0^{(22)\pm}}{c_0^{(23)\pm}} \tag{40}$$

Note that this only assures that the progressive–wave terms ($j = 0$) in (32) and (38) cancel, i.e. the local disturbances ($j = 1, 2, \ldots$) still remain. From (36) and (39) we further get

$$\mathcal{F}^{\pm} = \frac{\delta_{nm} K_0^{\pm} h}{c_{0n} c_{0m} \Lambda_1(K_0^{\pm})(\omega_n \pm \omega_m)} \left\{ \mp \frac{g}{2\omega_n} \sum_{j=0}^{\infty} \frac{c_{jn} k_{jn}}{\cosh k_{jn} h} \Gamma_4(k_{jn}, K_0^{\pm}) + \widehat{lmjn} \right.$$
$$\left. - \frac{\cosh K_p^{\pm} h}{g} \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} c_{jn} c_{lm}^{-:*} \frac{k_{jn} \pm k_{lm}^{-:*}}{(k_{jn} \pm k_{lm}^{-:*})^2 - K_0^{\pm 2}} H^{\pm}_{jnlm} \right\} \tag{41}$$

which is the main result of this theory.

Sand (1982), Barthel et al. (1983), Sand and Donslund (1985) and Mansard and Sand (1986a,b) used three transfer functions denoted F_{11}, F_{12}, and F_{23}. These functions all eliminate free waves which otherwise would have been emitted from the wave paddle due to the interaction between the following combinations of two first order terms (the terms referenced are in (41))

(i): progressive wavelet and progressive wavelet (F_{11} or the first term in the double summation, $j = l = 0$)

(ii): component of paddle position and progressive wavelet (F_{12} or the first term in the single summation, $j = 0$)

(iii): component of paddle position and local disturbance wavelet (F_{23} or the rest of the single summation, $j = 1, 2, \ldots$)

In addition to these terms (41) comprises two additional qualitatively different terms both related to the local first order disturbances. These terms are present in order to eliminate the emission of free waves due to the interaction between the first order terms

(iv): progressive wavelet and local disturbance wavelet (F_{13} and F_{24} defined below or $j = 0, l = 1, 2, \ldots$ and vice versa in the double summation)

(v): local disturbance wavelet and local disturbance wavelet (F_{22} defined below or the rest of the double summation, $j = 1, 2, \ldots, l = j = 1, 2, \ldots$)

Except for the corrections given previously as footnotes the function $\mathcal{F}^{\pm}$ (excluding the new terms) is consistent with the findings in the above referenced papers.

The terms mentioned under (iv) and (v) are included in the analysis for regular waves given by Fontanet (1961) and Hudspeth and Sulisz (1991) and also recognized although not included by Suh and Dalrymple (1987).

With $A_n = a_n - ib_n$ and $A_m^{-:*} = a_m \mp ib_m$ we have in a real representation

$$
\begin{aligned}
X_0^{(2)\pm} = &\frac{\Re\{\mathcal{F}^{\pm}\}}{h}\{(a_n a_m \mp b_n b_m)\sin(\omega_n \pm \omega_m)t + (\mp a_n b_m - a_m b_n)\cos(\omega_n \pm \omega_m)t\} \\
&- \frac{\Im\{\mathcal{F}^{\pm}\}}{h}\{(\mp a_n b_m - a_m b_n)\sin(\omega_n \pm \omega_m)t - (a_n a_m \mp b_n b_m)\cos(\omega_n \pm \omega_m)t\}
\end{aligned} \tag{42}
$$

The new term (iv) mentioned above makes it tedious to derive the real and imaginary parts of the transfer function (41), since the complex term $k_{jn} \pm k_{lm}^{-:*}$ in this case is neither real nor purely imaginary. It is recommended to use the relatively simple complex formulation for practical application and to let a computer separate the complex result whenever appropriate in the actual implementation. However, we shall derive $\Re\{\mathcal{F}^{\pm}\}$ and $\Im\{\mathcal{F}^{\pm}\}$ for the sake of comparison with F_{11}, F_{12}, and F_{23}.

$\Re\{\mathcal{F}^{\pm}\}$ and $\Im\{\mathcal{F}^{\pm}\}$ may be split up into three contributions

$$
\Re\{\mathcal{F}^{\pm}\} = \mp \left(F_{11}^{\pm} + F_{12}^{\pm} + F_{13}^{\pm}\right) h \tag{43}
$$

and

$$
\Im\{\mathcal{F}^{\pm}\} = \left(F_{22}^{\pm} + F_{23}^{\pm} + F_{24}^{\pm}\right) h \tag{44}
$$

where the factor h is included for historical reasons.

In addition to the well known transfer functions $F_{11}^{\pm}$, $F_{12}^{\pm}$, and $F_{23}^{\pm}$ three new functions $F_{13}^{\pm}$, $F_{22}^{\pm}$, and $F_{24}^{\pm}$ appear (The succession of indices as well as the sign convention has been chosen to match the definitions of the old functions $F_{11}^{\pm}$, $F_{12}^{\pm}$, and $F_{23}^{\pm}$). Remembering that c_{jn} (c_{lm}) and k_{jn} (k_{lm}) are real for $j = 0$ ($l = 0$) and purely imaginary for $j = 1, 2, \ldots$ ($l = 1, 2, \ldots$) we get

$$F_{11}^{\pm} = B^{\pm} \frac{k_{0n} \pm k_{0m}}{(k_{0n} \pm k_{0m})^2 - K_0^{\pm 2}} H_{0n0m}^{\pm} \tag{45}$$

$$F_{12}^{\pm} = \frac{\delta_{nm} K_0^{\pm} h}{c_{0m} \Lambda_1(K_0^{\pm})(\omega_n \pm \omega_m)} \left\{ \mp \frac{g}{2\omega_n} \frac{k_{0n}}{\cosh k_{0n} h} \Gamma_4(k_{0n}, K_0^{\pm}) + \widehat{lm0n} \right\} \tag{46}$$

$$\begin{aligned} F_{13}^{\pm} = B^{\pm} \sum_{j=1}^{\infty} \frac{\frac{c_{jn}}{c_{0n}}}{\left(k_{jn}^2 + k_{0n}^2 - K_0^{\pm 2}\right)^2 - 4k_{jn}^2 k_{om}^2} & \Big\{ k_{jn} \left(k_{jn}^2 - k_{0m}^2 - K_0^{\pm 2}\right) \Re\{H_{jn0m}\} \\ & \pm k_{0m} \left(-k_{jn}^2 + k_{0m}^2 - K_0^{\pm 2}\right) i\Im\{H_{jn0m}\} \Big\} \\ & + \widehat{lm0n} \end{aligned} \tag{47}$$

and

$$iF_{22}^{\pm} = B^{\pm} \frac{1}{c_{0n} c_{0m}} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} c_{jn} c_{lm}^{-:*} \frac{k_{jn} \pm k_{lm}^{-:*}}{(k_{jn} \pm k_{lm}^{-:*})^2 - K_0^{\pm 2}} H_{jnlm}^{\pm} \tag{48}$$

$$iF_{23}^{\pm} = \frac{\delta_{nm} K_0^{\pm} h}{c_{0n} c_{0m} \Lambda_1(K_0^{\pm})(\omega_n \pm \omega_m)} \left\{ \mp \frac{g}{2\omega_n} \sum_{j=1}^{\infty} \frac{c_{jn} k_{jn}}{\cosh k_{jn} h} \Gamma_4(k_{jn}, K_0^{\pm}) + \widehat{lmjn} \right\} \tag{49}$$

$$\begin{aligned} iF_{24}^{\pm} = B^{\pm} \sum_{j=1}^{\infty} \frac{\frac{c_{jn}}{c_{0n}}}{\left(k_{jn}^2 + k_{0n}^2 - K_0^{\pm 2}\right)^2 - 4k_{jn}^2 k_{om}^2} & \Big\{ k_{jn} \left(k_{jn}^2 - k_{0m}^2 - K_0^{\pm 2}\right) i\Im\{H_{jn0m}\} \\ & \pm k_{0m} \left(-k_{jn}^2 + k_{0m}^2 - K_0^{\pm 2}\right) \Re\{H_{jn0m}\} \Big\} \\ & \pm \widehat{lm0n} \end{aligned} \tag{50}$$

where

$$B^{\pm} \equiv \frac{\delta_{nm} K_0^{\pm 2}}{(\omega_n \pm \omega_m)^3} \tag{51}$$

and from $(25d)$

$$\Re\{H_{jn0m}\} = \pm(\omega_n \pm \omega_m)\omega_n \omega_m + \frac{\omega_n^3 \pm \omega_m^3}{2} - \frac{g^2}{2} \left(\frac{k_{jn}^2}{\omega_n} \pm \frac{k_{0m}^2}{\omega_m} \right) \tag{52a}$$

$$i\Im\{H_{jn0m}\} = -(\omega_n \pm \omega_m) \frac{g^2 k_{jn} k_{0m}}{\omega_n \omega_m} \tag{52b}$$

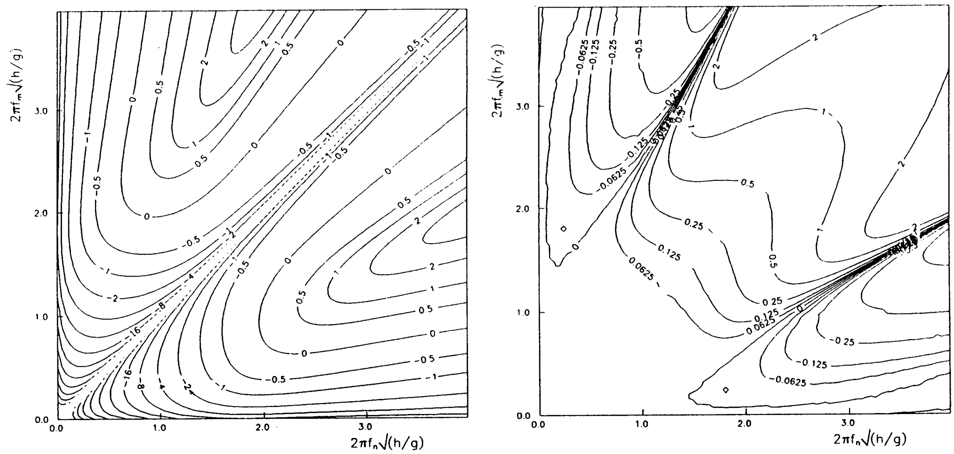

Figure 5 Components of subharmonic complex transfer function $\mathcal{F}^-/\delta_{nm}$ versus dimensionless first order frequencies $2\pi f_n\sqrt{h/g}$ and $2\pi f_n\sqrt{h/g}$ for a piston type wavemaker. (a): $\Re\{\mathcal{F}^-\}/\delta_{nm}$; Contour lines are a subset of (0,±1/2,±1,±2,±4,...) (b): $\Im\{\mathcal{F}^-\}/\delta_{nm}$; Contour lines are a subset of (0,±1/16,±1/8,±1/4, ±1/2,±1,±2,±4, ...)

3. PISTON TYPE WAVEMAKER

Using the results for Γ_4 from the appendix in (37b) gives

$$\mathcal{F}^{\pm} = \frac{\delta_{nm} K_0^{\pm 2} h}{c_{0n} c_{0m} (\omega_n \pm \omega_m)^3} \Big\{ \mp g \frac{\omega_n^2 - (\omega_n \pm \omega_m)^2}{2\omega_n} \sum_{j=0}^{\infty} c_{jn} \frac{k_{jn}^2}{k_{jn}^2 - K_0^{\pm 2}} + \widehat{lmjn}$$
$$+ \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} c_{jn} c_{lm}^{-:*} \frac{k_{jn} \pm k_{lm}^{-:*}}{(k_{jn} \pm k_{lm}^{-:*})^2 - K_0^{\pm 2}} H^{\pm}_{jnlm} \Big\} \quad (53)$$

For a piston wavemaker Fig. 5a,b show the real and imaginary parts of the subharmonic complex transfer function $\mathcal{F}^-/\delta_{nm}$ versus dimensionless first order frequencies $2\pi f_n\sqrt{h/g}$ and $2\pi f_n\sqrt{h/g}$. Fig. 6a,b show the equivalent for the superharmonics.

The simplification of $F_{11}^{\pm}$, $F_{13}^{\pm}$, $F_{22}^{\pm}$, and $F_{24}^{\pm}$ lies implicitly in the first order transfer functions, see (17). $F_{12}^{\pm}$ and $F_{23}^{\pm}$ reduce to

$$F_{12}^{\pm} = \mp B^{\pm} \Big\{ \frac{g}{c_{0m}} \frac{\omega_n^2 - (\omega_n \pm \omega_m)^2}{2\omega_n} \frac{k_{0n}^2}{k_{0n}^2 - K_0^{\pm 2}} + \widehat{lm0n} \Big\} \quad (54)$$

and

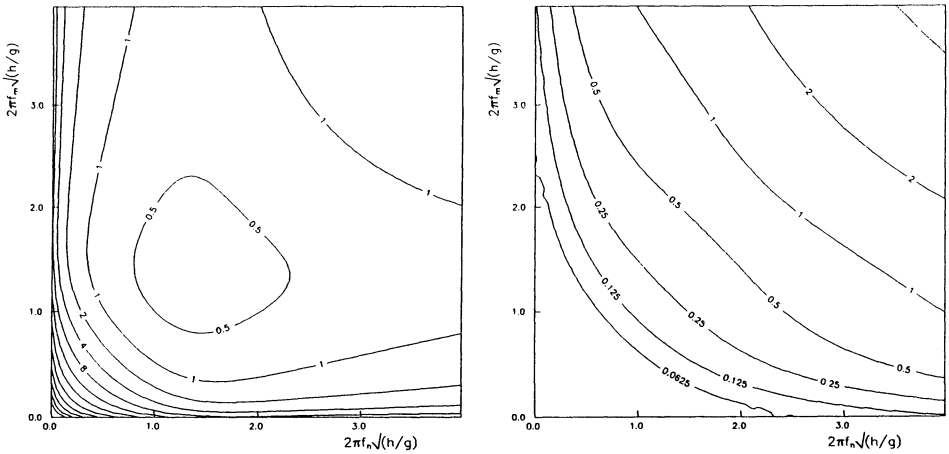

Figure 6 Components of superharmonic complex transfer function $\mathcal{F}^+/\delta_{nm}$ versus dimensionless first order frequencies $2\pi f_n\sqrt{h/g}$ and $2\pi f_n\sqrt{h/g}$ for a piston type wavemaker. (a): $\Re\{\mathcal{F}^+\}/\delta_{nm}$; Contour lines are a subset of (0,±1/2,±1,±2,±4,...) (b): $\Im\{\mathcal{F}^+\}/\delta_{nm}$; Contour lines are a subset of (0,±1/16,±1/8,±1/4, ±1/2,±1,±2,±4,...)

$$iF_{23}^{\pm} = \mp B^{\pm}\frac{1}{c_{0n}c_{0m}}\left\{g\frac{\omega_n^2 - (\omega_n \pm \omega_m)^2}{2\omega_n}\sum_{j=1}^{\infty} c_{jn}\frac{k_{jn}^2}{k_{jn}^2 - K_0^{\pm 2}} \pm \widehat{lmjn}\right\} \tag{55}$$

Note that for the subharmonics the factor $\omega_m^2 - (\omega_m - \omega_n)^2$ in the $\widehat{lmjn}$-term in $F_{12}^{\pm}$ and $F_{23}^{\pm}$ vanishes for $\omega_n = 2\omega_m$ (if $\omega_n < \omega_m$ is chosen this appears for $\omega_m = 2\omega_n$ in the previous term) and the series-term vanishes except for the first term ($j = 0$) where $k_{0m} - K_0^- = 0$. In this special case the Γ_4-part of the expression (cf. (37b)) should be replaced by $\Lambda_1(k_{0m})$ (cf. (A11) in the appendix) or equivalently $(\omega_m^2 - (\omega_m - \omega_n)^2)/(k_{0m}^2 - K_0^2)$ should be recognized as cc_g (where we have locally defined c and c_g as the phase velocity and group velocity for $\omega = \omega_m$) using the identity $cc_g = \omega^4/(gk^3c_0)$ (where c_0 is the Biesel transfer function). This way it may easily be shown that the contribution $g(\omega_m^2 - (\omega_m - \omega_n)^2)/(2\omega_m)\sum_{l=0}^{\infty} c_{lm}k_{lm}^2/(k_{lm}^2 - K_0^2)$ should be replaced by $\omega_m^3/(2k_{om})$ for $\omega_m = \omega_n/2$.

4. EXPERIMENTAL VERIFICATION

A variety of experiments have been made with a piston type wavemaker. At present only the tests for regular waves have been analysed and experimental results for wave groups and irregular waves will be published later.

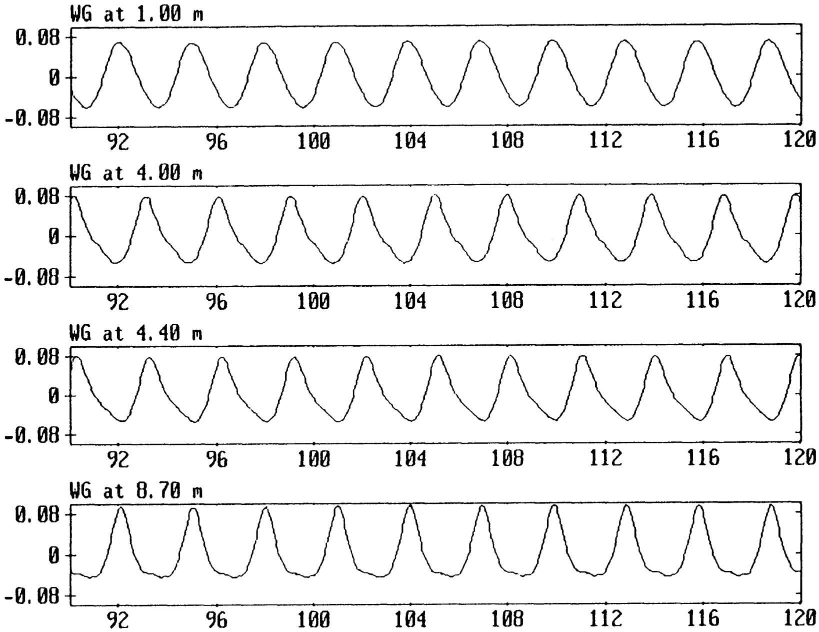

Figure 7 Time series of surface elevation at different positions in the flume ($x = 0$ at wavemaker) using a FIRST ORDER control signal ($h = 0.70m$, $T = 3.0s$, and $H = 0.14m$).

The experimental facility used is a $20m$ flume, $1.0m$ deep and $0.60m$ wide, equipped with a piston type wavemaker in one end ($x = 0$) and an efficient absorber in the other end. For a depth of $h = 0.70m$, a period of $T = 3.0s$ and a height of $H = 0.14m$ Figs. 7 and 8 show time series of surface elevation at different positions in the flume using first order and second order control signals, respectively. The use of first order wavemaker theory (Fig. 7) gives a spurious, free superharmonic, progressive wave generated in addition to the natural, forced superharmonic. These two superharmonic wave components progress at different celerities creating an interference pattern resulting in different wave shapes throughout the flume, the repetition length (in this case) theoretically predicted to be $19.1m$. Close to the wavemaker (WG at $1.00m$ in Fig. 7) the two superharmonics almost cancel and the elevation becomes nearly sinusoidal, which is not a stable form for this rather nonlinear wave. Further down the flume this results in vertical assymetry or secondary peaks in the troughs. The use of second order wavemaker theory solves these problems, see Fig. 8.

This test was used by Sand and Mansard to succesfully verify their version of second order wavemaker theory. The difference between the two theories is small for this rather shallow water example, but it is indeed significant in deeper water.

A detailed analysis was made using the following two steps for the surface elevation time series measured at each wave gauge (positions 1.00, 4.00, 4.40, and

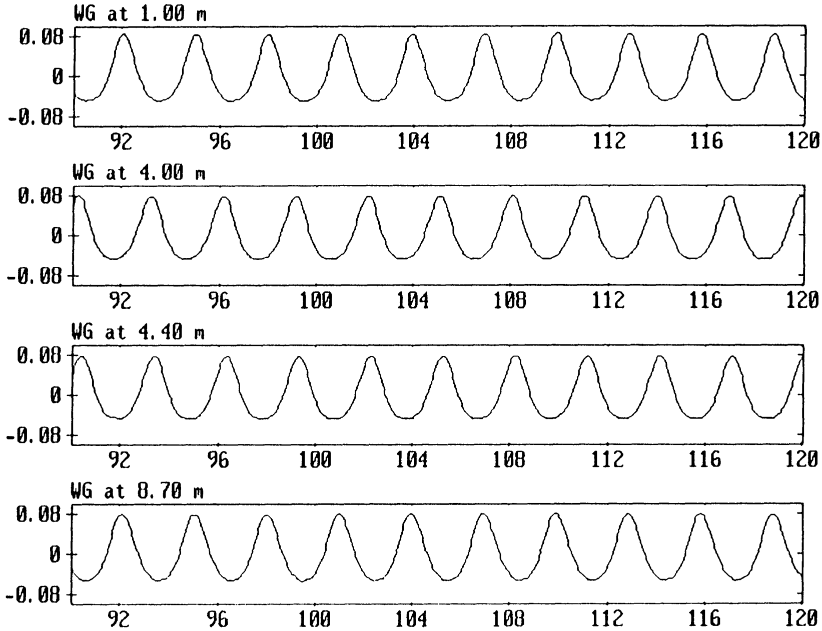

Figure 8 Time series of surface elevation at different positions in the flume ($x = 0$ at wavemaker) using a SECOND ORDER control signal ($h = 0.70m$, $T = 3.0s$, and $H = 0.14m$).

$8.70m$ from the wavemaker): 1) Band pass filtering around the primary frequency and the superharmonic frequency, respectively and 2) zero-crossing of the filtered time series in order to obtain the respective wave heights.

For a wide range of periods, $T = 3.0$, 2.0, 1.2, and 0.8 s corresponding to water depth to wave length ratios, $h/L = 0.09$, 0.15, 0.32, and 0.7. The values of the complex transfer function $\mathcal{F}^+$ for these periods are $(1.53, 0.00)$, $(0.40, 0.03)$, $(0.18, 0.15)$, and $(0.29, 0.41)$. If the new terms corresponding to F_{13}^+, F_{22}^+ and F_{22}^+ are neglected these values become $(1.46, -0.02)$, $(0.25, -0.07)$, $(-0.17, -0.13)$, and $(-0.44, -0.20)$. As expected the new terms are important except near the shallow water limit. Even in the intermediate water depth (represented here by $T = 1.2s$) the new terms change the original values by roughly -100%.

Fig. 9 compares the experiments with theory. The solid curves are theoretical results and the experiments are given by white squares in case of first order generation and black squares in case of second order generation.

For the primary wave the constant wave height given by the solid line gives the specified wave height and the experimental values are seen to be somewhat smaller.

For the superharmonic the constant bound-wave height given by the straight lines are the theoretical values based on the specified primary wave height and

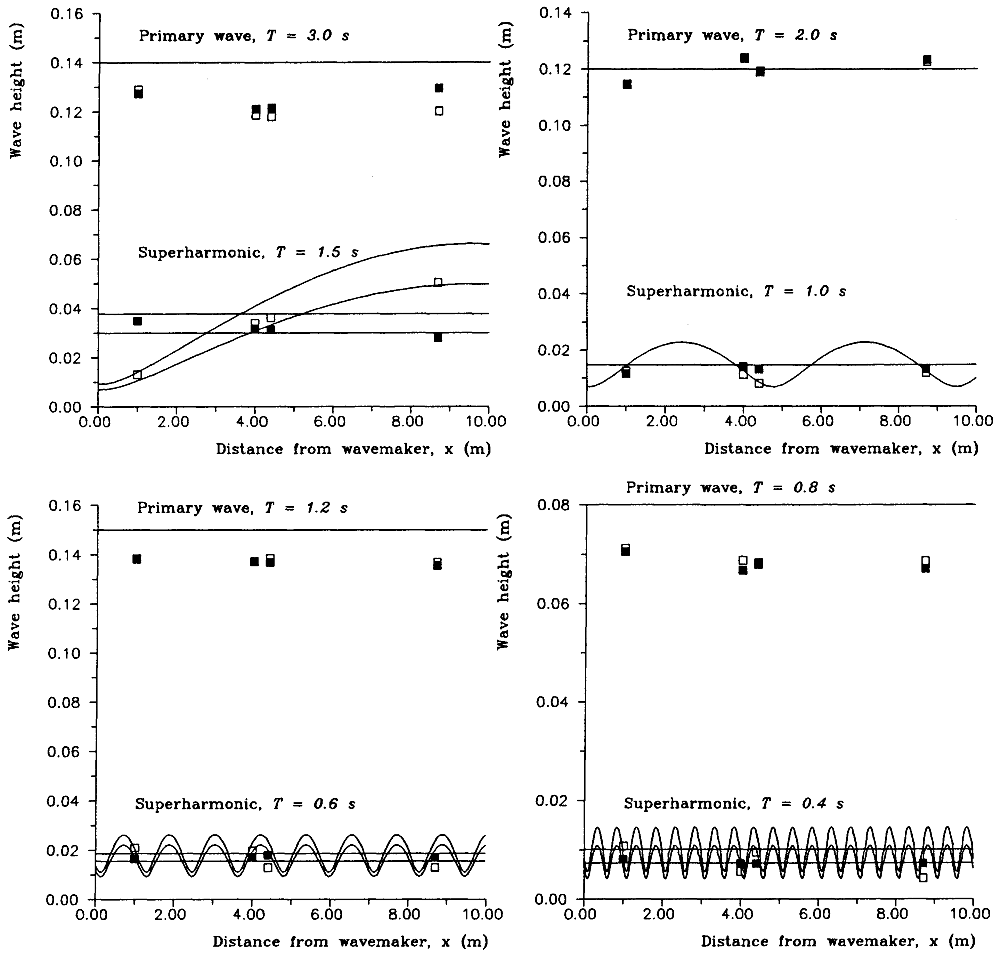

Figure 9 Wave heights for the primary wave and the superharmonic using first order control signal (white squares) and second order control signal (black squares), respectively. Theory is indicated by the curves and the straight lines. For the superharmonic these give the second order theory for generation correct to first order (curves) and to second order (lines), respectively. The theory of the superharmonic is shown both for the specified and for the actually measured primary wave height (spatial mean). Water depth is $h = 0.70m$.

the spatial mean of the measured primary wave heights, respectively. The black squares are expected to lie between these two lines and the agreement is satisfactory.

The oscillating curves give the second order theoretical height of the interference wave pattern of bound and free superharmonic components resulting from first order wave generation. Again, the upper curve is based on the specified primary wave height whereas the lower one is based on the spatial mean of the measured primary wave heights. The white squares are expected to lie between the two oscillating curves. The agreement is generally good.

The experiments clearly show how second order wave generation gives waves of constant form as opposed to first order generation (the black squares are far closer to being constant than the white squares). This conclusion holds even in deep water.

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Appendix

The functions $\Gamma_{1\ldots4}$ (see (35)) are evaluated in this appendix.

In order to reduce the computational cost for the case of irregular waves it is important to avoid terms like $\sinh(k_{0n} \pm k_{0m})h$, since these would have to be computed $\propto N^2$ times, where N is the number of frequencies representing the first order spectrum, whereas terms like $\sinh k_{0n}h$ need only be computed $\propto N$ times, provided the sufficient computer memory is available. Three forms of each of the functions $\gamma_{1\ldots3}$ (see below) are given. The first one is the standard form for the relevant integral, the second form is suited for the general wave board type, whereas the last form is preferable for $d = 0$.

For $\kappa_1 \neq \kappa_2$ we get

$$\Gamma_1(\kappa_1,\kappa_2) \equiv \int_{-h}^{0} \cosh\kappa_1(z+h)\cosh\kappa_2(z+h)dz$$
$$= \gamma_1(\kappa_1,\kappa_2;h) \tag{A1}$$

where

$$\begin{aligned}\gamma_1(\kappa_1,\kappa_2;r) \equiv& \frac{\sinh(\kappa_1-\kappa_2)r}{2(\kappa_1-\kappa_2)} + \frac{\sinh(\kappa_1+\kappa_2)r}{2(\kappa_1+\kappa_2)}\\ =&\frac{1}{\kappa_1^2-\kappa_2^2}\Big\{\kappa_1\sinh\kappa_1 r\cosh\kappa_2 r - \kappa_2\cosh\kappa_1 r\sinh\kappa_2 r\Big\}\\ =&\frac{\cosh\kappa_1 r\cosh\kappa_2 r}{\kappa_1^2-\kappa_2^2}\Big\{\kappa_1\tanh\kappa_1 r - \kappa_2\tanh\kappa_2 r\Big\}\end{aligned} \tag{A2}$$

and

$$\begin{aligned}\Gamma_2(\kappa_1,\kappa_2) \equiv& \int_{-h+d}^{0} f(z)\cosh\kappa_1(z+h)\cosh\kappa_2(z+h)dz\\ =&\gamma_1(\kappa_1,\kappa_2;h) - \frac{d+\ell}{h+\ell}\gamma_1(\kappa_1,\kappa_2;d)\\ &-\frac{1}{h+\ell}\Big\{\gamma_2(\kappa_1,\kappa_2;h)-\gamma_2(\kappa_1,\kappa_2;d)\Big\}\end{aligned} \tag{A3}$$

where

$$\begin{aligned}\gamma_2(\kappa_1,\kappa_2;r) \equiv& \frac{\cosh(\kappa_1-\kappa_2)r}{2(\kappa_1-\kappa_2)^2} + \frac{\cosh(\kappa_1+\kappa_2)r}{2(\kappa_1+\kappa_2)^2}\\ =&\frac{1}{(\kappa_1^2-\kappa_2^2)^2}\Big\{(\kappa_1^2+\kappa_2^2)\cosh\kappa_1 r\cosh\kappa_2 r - 2\kappa_1\kappa_2\sinh\kappa_1 r\sinh\kappa_2 r\Big\}\\ =&\frac{\cosh\kappa_1 r\cosh\kappa_2 r}{(\kappa_1^2-\kappa_2^2)^2}\Big\{(\kappa_1^2+\kappa_2^2) - 2\kappa_1\kappa_2\tanh\kappa_1 r\tanh\kappa_2 r\Big\}\end{aligned} \tag{A4}$$

and

$$\begin{aligned}\Gamma_3(\kappa_1,\kappa_2) \equiv& \int_{-h+d}^{0} \sinh\kappa_1(z+h)\cosh\kappa_2(z+h)dz\\ =&\gamma_3(\kappa_1,\kappa_2;h) - \gamma_3(\kappa_1,\kappa_2;d)\end{aligned} \tag{A5}$$

where

$$\begin{aligned}\gamma_3(\kappa_1,\kappa_2;r) \equiv& \frac{\cosh(\kappa_1-\kappa_2)r}{2(\kappa_1-\kappa_2)} + \frac{\cosh(\kappa_1+\kappa_2)r}{2(\kappa_1+\kappa_2)}\\ =&\frac{1}{\kappa_1^2-\kappa_2^2}\Big\{\kappa_1\cosh\kappa_1 r\cosh\kappa_2 r - \kappa_2\sinh\kappa_1 r\sinh\kappa_2 r\Big\}\\ =&\frac{\cosh\kappa_1 r\cosh\kappa_2 r}{\kappa_1^2-\kappa_2^2}\Big\{\kappa_1 - \kappa_2\tanh\kappa_1 r\tanh\kappa_2 r\Big\}\end{aligned} \tag{A6}$$

and

$$\begin{aligned}\Gamma_4(\kappa_1,\kappa_2) \equiv& \kappa_1\Gamma_2(\kappa_1,\kappa_2) + \frac{1}{h+\ell}\Gamma_3(\kappa_1,\kappa_2)\\ =&\kappa_1\Big\{\gamma_1(\kappa_1,\kappa_2;h) - \frac{d+\ell}{h+\ell}\gamma_1(\kappa_1,\kappa_2;d)\Big\}\\ &-\frac{1}{h+\ell}\frac{\kappa_2}{(\kappa_1^2-\kappa_2^2)^2}\Big(2\kappa_1\kappa_2(\cosh\kappa_1 h\cosh\kappa_2 h - \cosh\kappa_1 d\cosh\kappa_2 d)\end{aligned}$$

$$- (\kappa_1^2 + \kappa_2^2)(\sinh\kappa_1 h \sinh\kappa_2 h - \sinh\kappa_1 d \sinh\kappa_2 d)\Big) \tag{A7}$$

The case of $\kappa_1 = \kappa_2 = \kappa$ may occur in Γ_4 and we get

$$\Gamma_4(\kappa,\kappa) = \frac{1}{2}\Big\{\kappa h + \sinh\kappa h\cosh\kappa h - \frac{d+\ell}{h+\ell}(\kappa d + \sinh\kappa d\cosh\kappa d) - \frac{\kappa}{h+\ell}\frac{h^2-d^2}{2}\Big\} \tag{A8}$$

With reference to (34a) we only need Γ_1 for $(\kappa_1,\kappa_2) = (k_{jn} \pm k_{lm}^{-:*}, K_p^{\pm})$. In terms of $D_{jnlm}^{\pm}$ defined in (28b) we get

$$\Gamma_1(k_{jn} \pm k_{lm}^{-:*}, K_p^{\pm}) = \frac{\cosh(k_{jn} \pm k_{lm}^{-:*})h\cosh K_p^{\pm}h}{g\left((k_{jn} \pm k_{lm}^{-:*})^2 - K_p^{\pm 2}\right)} D_{jnlm}^{\pm} \tag{A9}$$

In the special case of a piston type wavemaker ($d = 0$, $\ell = \infty$) application of the dispersion relations (29) and (33) brings $\Gamma_4(k_{jn}, K_p^{\pm})$ in a very simple form:

$$\Gamma_4(k_{jn}, K_p^{\pm}) = k_{jn}\frac{\cosh k_{jn}h\cosh K_p^{\pm}h}{g(k_{jn}^2 - K_p^{\pm 2})}\left(\omega_n^2 - (\omega_n \pm \omega_m)^2\right) \tag{A10}$$

Finally we note that for the piston type wavemaker

$$\Gamma_4(\kappa,\kappa) = \Lambda_2(\kappa) \tag{A11}$$

where Λ_2 is given by (17*b*).