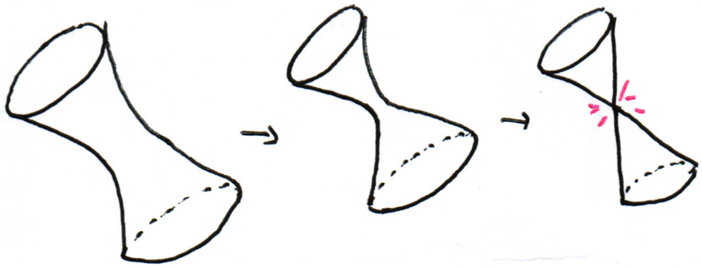
A Crash Course In J-curves

Agenda

- What is a J-holomorphic curve?
- Why do we like them?
- Local / Analytic Properties
- Topological Properties
- Properties of the Moduli



$xy = 1$ $\quad$ $xy = \frac{1}{10}$ $\quad$ $xy = 0$

What is a j-curve?

Def: An Almost-Complex structure J on (M, ω) (symplectic) an endomorphism of TM s.t $J^2 = -\mathrm{Id}$. i.e, fiberwise complex structure (multiplication by i).

Def: A J-holomorphic curve (or J-curve) is a map $u: \Sigma \to M$ (w/ Σ a Riemann surface) s.t $\boxed{J \circ du = du \circ j}$ (j is cmplx mult by i in Σ).

With ω around, we usually impose 1 or 2 more conditions.

Def: J is tame if $\omega(v, Jv) > 0 \ \forall v \neq 0 \in T_pM$.
J is compatible if it is tame and $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$

$\Rightarrow \omega(\cdot, J\cdot)$ is a metric (all norms are w.r.t this).

Def: By $\mathcal{J}^k(M, \omega)$ we denote either space of tame or compatible J. w/ C^k regularity.

Why do we like J-curves? (preaching to choir)

They act like actual holomorphic curves...

- Ellipticity $\Rightarrow$ smoothness of u, mfld w/ singularities in M.
- Geometric intersection controls homological intersection (i.e intersection positivity).
- Adjunction (chern classes)
- Families of curves degenerate in controlled ways (for nice J) $\Rightarrow$ allows compactification of moduli space (later).

And sometimes they're better!

- Almost complex structures are Flexible.
- Generic J produce moduli of curves with similar properties.
- Sometimes honest complex structures aren't generic.

Naming the moduli spaces

Fix:

- $g \in \mathbb{Z}^+$ used as genus of Σ_g.
- $p_1, \ldots, p_n \in \Sigma_g$
- $J \in \mathcal{J}^k(M, \omega)$
- $A \in H_2(M, \mathbb{Z})$.

Define:

G_{Σ_g} = group of endomorphisms (reparameterizations) of Riemann surface of genus g.

$\mathcal{M}_{g,\vec{p}}(J, A) := \{ u : \Sigma_g - \{p_i\} \to M \mid u \text{ J-holomorphic and } u_*[\Sigma] = A \}$

$\widetilde{\mathcal{M}}_{g,\vec{p}}(J, A) = \mathcal{M}_{g,\vec{p}}(J, A) / G_{\Sigma_g}$

under $u \mapsto u \circ \phi^{-1}$ for $\phi \in G_{\Sigma_g}$.

Local / Analytic Properties

• **Regularity**: Fix $\ell \geq 2$, $p > 2$.
$J \in \mathcal{J}^\ell(M,\omega)$, $u : \Sigma \to M$ s.t $u \in W^{1,p}$
and u J-holomorphic $\Rightarrow u \in W^{\ell,p}$. [M-S B.4.1]

• "**Compactness**": If:

- $J_\nu \to J$ in $\mathcal{J}^\ell(M,\omega)$
- $j_\nu \to j$ in C^∞ ← cmplx structures on Σ.
- u_ν is a sequence of maps $(\Sigma, j_\nu) \to (M, J_\nu, \omega)$ s.t $\forall\, Q \subset\subset \Sigma$, $\exists\, K \subset M$ and c. w/

$\|du_\nu\|_{L^p} \leq c$, $u_\nu \in K$ $\forall\, \nu >> 0$.

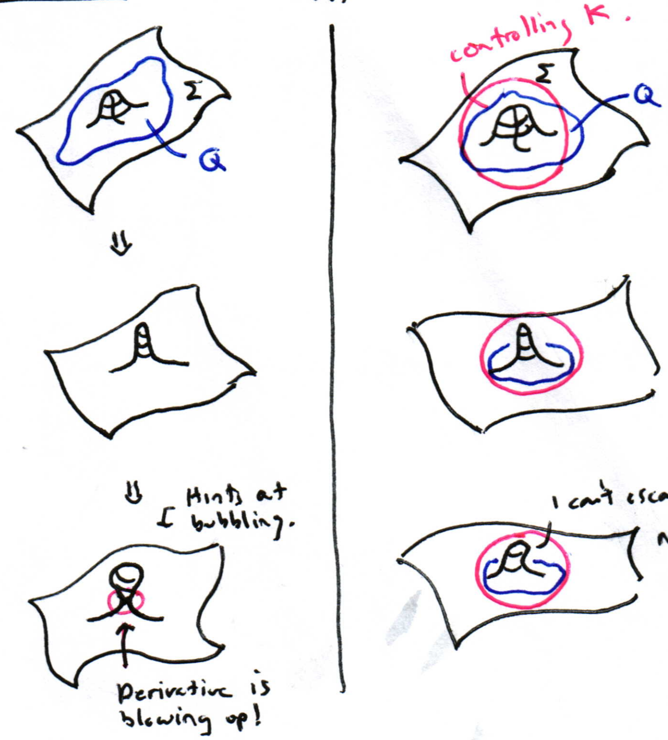
Then!

$u_\nu \to u \in C^{\ell-1}$ topology on compact sets of Σ.

[M-S B.4.2]

Pictures

Bad vs. Good



Simple & Injective Curves

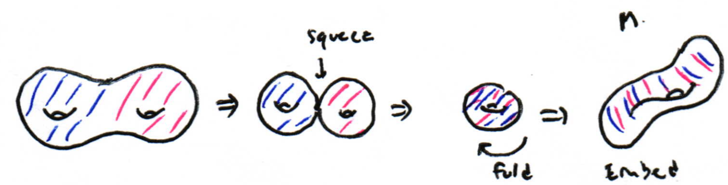
Def: A multiply covered curve u is a J-curve $u : \Sigma \to M$ s.t $\exists$ branched $\phi : \Sigma \to \Sigma'$ w/

$$\begin{tikzcd} \Sigma \arrow[r, "u"] \arrow[dr, "\phi"'] & M \\ & \Sigma' \arrow[u, "u'"'] \end{tikzcd}$$

Def: Simple curves u are not multiply covered.

Def: Somewhere injective u satisfy:
$\exists\, z$ s.t $du(z) \neq 0$ and $u^{-1}(u(z)) = \{z\}$.

Thm: Σ compact, $\partial\Sigma = \emptyset$, $u : \Sigma \to M$ simple $\Rightarrow$ u is somewhere inj. and non-injective pts are countable, only accumulate at $du = 0$ pts. [MS, 2.5.1]



2 main pts

1) J-curves are a lot like Riemann surface maps: If they have same image, basically only differ by factorization by branch cover. Also "branch"/"singular" pts are countable/Finite.

2) branched/multi-covered maps have bad limiting behavior + bad moduli behavior
$\Rightarrow$ want to discard them.

Topological Properties

- Like algebraic curves in $\mathbb{C}P^n$, J-holomorphic submanifolds intersect <u>positively</u>.

Thm: (Intersection positivity): Assume $\dim M = 4$. Let $u_1 : \Sigma_1 \to M$, $u_2 : \Sigma_2 \to M$ be simple, J-holomorphic, w/ $u_1(U_1) \neq u_2(U_2)$ for nonempty, open $U_i \subset \Sigma_i$. Also let:

$\delta(u_0, u_1) =$ # of intersection pts of u_0 and u_1

Then:

$$\delta(u_0,u_1) \leq u_1^*[\Sigma_1] \underbrace{\cap}_{\text{homological cap product.}} u_2^*[\Sigma_2]$$

w/ $=$ if all intersections are transverse. [M-S 2.6.3]

$p = u_1(z_1) = u_2(z_2)$

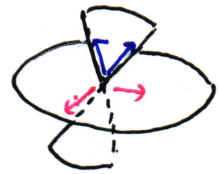
Proof when transverse is easy. At each point of intersection, orientation of $T\Sigma_{1,z_1} \oplus T\Sigma_{2,z_2}$ induced by orientation of Σ_i is:

$dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$, x_i, y_i cmplx coords on Σ_i at z_i

while pullback orientation is

$(u_1^* \oplus u_2^*)(d\tilde{x}_1 \wedge d\tilde{y}_1 \wedge d\tilde{x}_2 \wedge d\tilde{y}_2)$ for $(\tilde{x}_i, \tilde{y}_i)$ cmplx coords on M at p.

u_i J-holomorphic $\Rightarrow$ orientations agree b/c pullbacks of $d\tilde{x}_i$ and $d\tilde{y}_i$ will be cmplx basis of $T\Sigma_1 \oplus T\Sigma_2$.



blue basis + red basis yields cmplx basis, w/ orientation agreeing.

Adjunction:

- Also have adjunction as with cmplx mflds (at least at a topological level) due to presence of <u>Chern classes</u>.

Thm: Let $u : \Sigma \to M$ be J-holomorphic, then:

$$2\delta(u) - \chi(\Sigma) \leq u^*[\Sigma] \cap u^*[\Sigma] - c_1(u^*[\Sigma])$$

w/ equality iff u is an immersion w/ transverse self intersections.

Proof (when transverse):

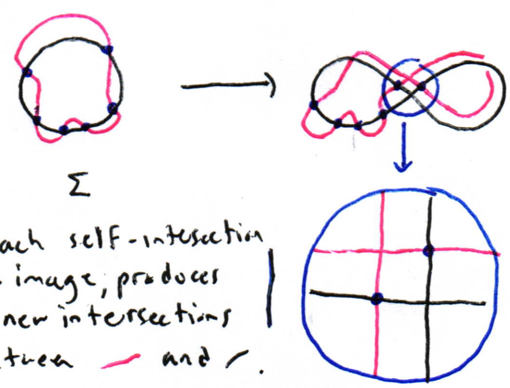
$c_1(u^*[\Sigma]) := \langle c_1(TM), u^*[\Sigma]\rangle = \langle c_1(u_*TM), [\Sigma]\rangle$

$= \langle c_1(T\Sigma \oplus \nu\Sigma), [\Sigma]\rangle = \langle c_1(T\Sigma) + c_1(\nu\Sigma), [\Sigma]\rangle$

$= \langle c_1(T\Sigma), [\Sigma]\rangle + \langle c_1(\nu\Sigma), [\Sigma]\rangle$

$= \chi(\Sigma) +$ Euler # of $\nu\Sigma$

Euler # of $\nu\Sigma$ = # of pts in $\sigma \cap 0 \subset \nu\Sigma$ for generic section $\sigma : \Sigma \to \nu\Sigma$.



Each self-intersection in image, produces 2 new intersections between and .

$\Rightarrow$

Euler class

$\langle c_1(\nu\Sigma), [\Sigma]\rangle + 2\delta(u) = u^*[\Sigma] \cap u^*[\Sigma]$

Mention contradiction argument in Dani's talk. In particular, <u>intersection</u> <u>positivity</u> is used to show

can't happen for $A \cdot A = 1$

Properties of the Moduli

- 2 main properties of interest
  1) $\mathcal{M}$ is a manifold / dim $\mathcal{M}$.
  2) $\mathcal{M}$ has a compactification that isn't awful.

Both properties are not always true, in particular transversality can ruin our party. But ignore for now...

1) Apply Sard-Smale. & Banach mfld implicit function thm.

Thm: If $F: X \to Y$ is a Fredholm C^k map (Fredholm meaning dF_p is Fredholm) ~~at a~~ between Banach mflds and $p \in Y$ is a regular value then:

$F^{-1}(p)$ is a submanifold of dim $\mathrm{Ind}(dF_p)$

Thm: (Sard-Smale) Such regular values are generic.

Idea: for $\mathcal{M}$'s.

- Start w/ universal bundle:

$$\left.\begin{matrix} \mathrm{Map}(\Sigma, M) \times \mathcal{J}^k(M,\omega) \\ \cup \\ \underbrace{\mathcal{M}(J) \xrightarrow[\pi]{} \mathcal{J}^k(M,\omega)} \end{matrix}\right\} \text{Big-ass bundles}$$

Fiber-bundle over Banach mfld $\mathcal{J}^k(M,\omega)$ w/ F.d Fiber.

Apply Sard-Smale to π.

What is dimension?

Calculate derivative of j-curve eqn.

Get ~~non-linear~~ low-order perturbation of $\bar\partial$ operator on cmplx vector-bundle over Σ.

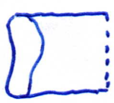
$\Rightarrow$ Use Riemann-Roch (see App. C of [M-S])

$\Rightarrow$ $\mathrm{Index}\, \underbrace{D_u}_{J} = n\chi(\Sigma) + 2c_1(u^*TM)$

linearization

Note: Different when Σ has boundary components.

Compactification.



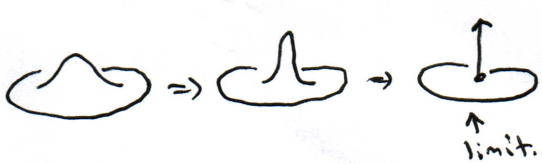
2 reasons to compactify:

1) Want to say $\mathcal{M}$ is independent of J modulo cobordism, not meaningful if $\mathcal{M}$ isn't compact.
2) Want to use image of $\mathcal{M}$ under $ev: \mathcal{M} \to M$ (ev is in Morgan's talk), also hard if $\mathcal{M}$ isn't compact.

$\Rightarrow$ Need to know what limits look like.

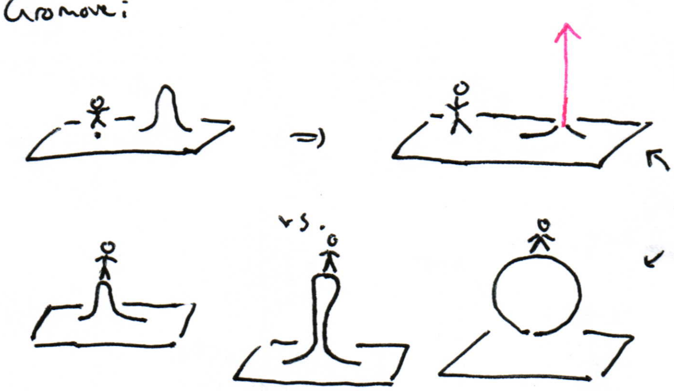
Note:

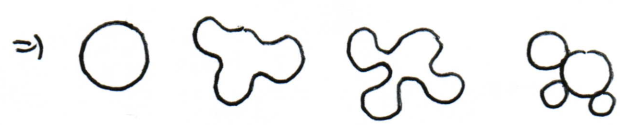
Compactness thm from before implies that a sequence u_ν converges in $\mathcal{M}$ unless focusing of energy occurs.



What is happening there?

Gromov:





Gromov compactness: A sequence $u_\nu: S^2 \to M$ of J-spheres w/ $E(u_\nu) < \infty$ (for instance, fixed $u_\nu^*[S^2] \in H_2^{\circ}$) converges to a stable map of J curves $\{v_i: S^2 \to M\}_{i=1}^n$ w/ $\sum E(v_i) \le E(u)$ and nodal pts z_{ij} [M-S 5.3.1]

Mention cobordism equivalence!