Lens spaces via Heegaard Diagrams

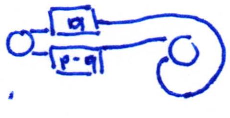
Proposition: Let $0 < q < p$ with p, q coprime & consider the 3-manifold:

$$L(p,q) := S^3/\Gamma, \quad \text{w/ } S^3 \cong \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$$

$$\Gamma = \langle e \rangle \subset SU(2), \quad e = \begin{bmatrix} e^{2\pi i/p} & 0 \\ 0 & e^{2\pi i q/p} \end{bmatrix}$$

$$\cong \mathbb{Z}_p$$

Then $L(p,q)$ is give by the Heegard diagram:



w/ $-\boxed{q}-$, $-\boxed{p-q}-$ groups of q & $p-q$ strands respectively

i.e $L(p,q)$ is gotten by doing p,q surgery on trivial knot.

Proof: We find a Heegaard splitting of $L(p,q)$ into $\tilde{H}_1 \cong \tilde{H}_2 \cong$ genus 1 handle w/ the map $\partial \tilde{H}_2 \to \partial \tilde{H}_1$ ~~being~~ taking the meridian class $\tilde{\mu}_2 \in H^1(\partial \tilde{H}_2)$ to $p\tilde{\ell}_1 + q\tilde{\mu}_2$, where $\tilde{\ell}_1, \tilde{\mu}_1$ are longitudinal class of $H^1(\partial \tilde{H}_1)$ & the meridian class of $H^1(\partial \tilde{H}_1)$ respectively.

Thus, let:

$$S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \text{ w/ Heegaard splitting}$$

$$\Sigma = \{(z,w) \in S^3 \mid |z|^2 = |w|^2\}$$

$$H_1 = \{(z,w) \in S^3 \mid |z|^2 > |w|^2\}; \; \mu_1 = \{(z,w) \in \Sigma \mid w = \tfrac{1}{\sqrt{2}}\}, \; \ell_1 = \{(z,w) \in \Sigma \mid z = \tfrac{1}{\sqrt{2}}\}$$

$$H_2 = \{(z,w) \in S^3 \mid |z|^2 < |w|^2\}; \; \mu_2 = \{(z,w) \in \Sigma \mid z = \tfrac{1}{\sqrt{2}}\}, \; \ell_2 = \{(z,w) \in \Sigma \mid w = \tfrac{1}{\sqrt{2}}\}$$

This is the standard 1-~~handle~~ genus splitting of S^3.

~~First observe that~~:

$$\Gamma = \langle e \rangle \curvearrowright S^3, \quad e = \begin{bmatrix} e^{2\pi i/p} & 0 \\ 0 & e^{2\pi i q/p} \end{bmatrix}, \quad e(z,w) = (e^{2\pi i/p} z, e^{2\pi i q/p} w)$$

$$\Gamma = \Gamma' = \langle e' \rangle, \quad e' = e^b = \begin{bmatrix} e^{2\pi i b/p} & 0 \\ 0 & e^{2\pi i/p} \end{bmatrix}, \quad b \text{ s.t. } 1 = ap + bq$$

Now observe that if:

$$L(p,q) := S^3/\Gamma, \quad \tilde{H}_i := H_i/\Gamma, \quad \tilde{\Sigma} := \Sigma/\Gamma$$

then $\tilde{H}_i, \tilde{\Sigma}$ is a genus 1 Heegaard splitting for $L(p,q)$. Indeed;

$F_1 = \{(z,w) \in H_1 \mid \arg(z) \in [0, 1/p]\}$ is a fundamental domain of the action $\Gamma \curvearrowright H_1$, so it is ~~homeom~~ diffeomorphic to $D^2 \times [0, 1/p]$, ~~w/ the~~ $D^2 \times 1/p$ is identified w/ $D^2 \times 0$ via an orientation preserving diffeomorphism. Thus $\tilde{H}_1 \cong$ ~~#~~ genus 1 handle body & $\hat{T} = T/\Gamma \cong \partial\tilde{H}_1 \cong T^2$.

The same argument can be made w/ the fundamental domain by Γ:

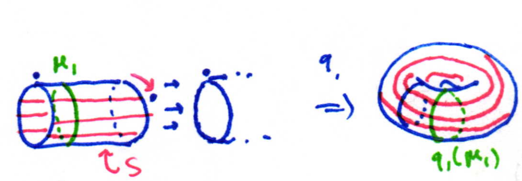
$$F_2 = \{(z,w) \in H_2 \mid \arg(w) \in [0, 1/p]\}.$$

To see where the meridian $\tilde{\mu}_2$ of $\tilde{H}_2$ goes under the map $\partial\tilde{H}_2 \to \partial\tilde{H}_1$, we need to calculate $(q_1)_*(\gamma)$ w/:

$$q_i : \Sigma \to \partial\tilde{H}_i, \quad \gamma \text{ a curve s.t. } q_2(\gamma) = \tilde{m}_2.$$

First, we calculate the image of μ_1 & l_1 under $q_1 : \Sigma \to \partial\tilde{H}_1$.

~~If we pick~~ a meridian $\mu_1 \subset F_1$ is clearly sent to $\tilde{\mu}_1$ under the map $\partial F_1 \to \partial\tilde{H}_1$ (which is a local diffeomorphism).



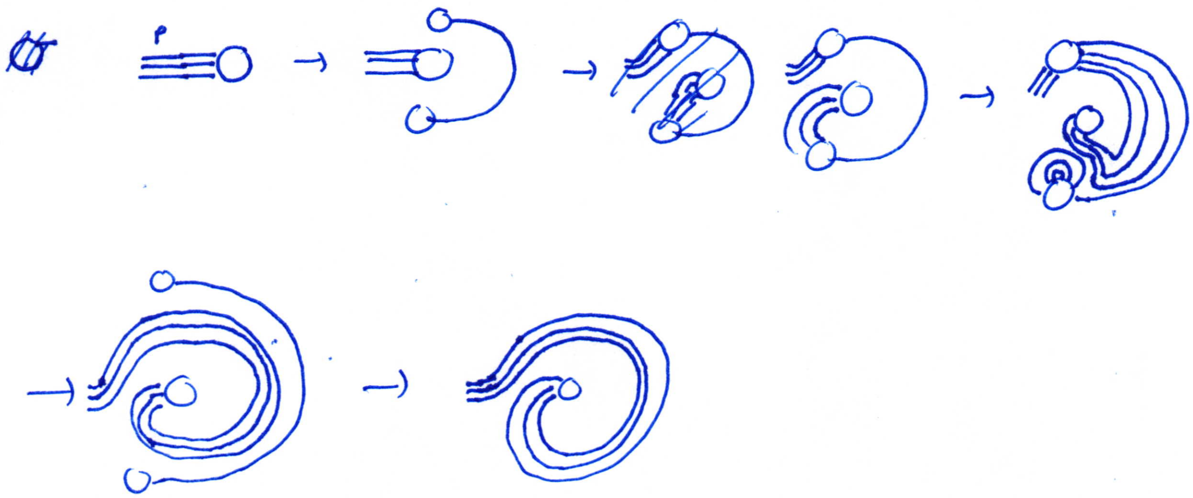
In the fundamental domain, l_1 is given by the set of points in its orbit, i.e $\bigcup_{k=0}^{p-1} (e^{2\pi i k/p}, e^{2\pi i k/q}) = S$. Under the quotient, the image is a curve circling the longitude p times & the meridian q times, i.e

$$(q_1)_*([l_1]) = p[\tilde{l}_1] + q[\tilde{\mu}_1], \quad (q_1)_*([\mu_1]) = [\tilde{\mu}_1].$$

Same argument $\Rightarrow$ $(q_2)_*([l_2]) = p[\check{l}_2] + q[\check{\mu}_2]$, $(q_2)_*([\mu_2]) = \tilde{\mu}_2$.

$\Rightarrow$ pick $\gamma = m_2 = l_1$, so then:

$(q^*_1)(l_1) = p\check{l}_1 + q\check{\mu}_1$. ✓.



$\Rightarrow$ O p,q curve $\cong$ O $\cong$ O $\cong$ p,q#p curve  $=$ O $\cong$ p,p+q curve $\cong$ O

↑p strands p strands