Kirby Calculus: Moves & Examples

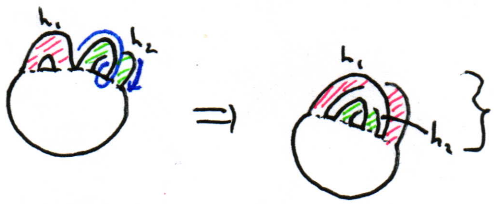
Agenda:

- Handle Slides
  - 1-handles
  - 2-handles
- Linking Form & 2-handle slides
- Handle Cancellation
- Blow-ups & Examples
- Circle-Dot ~~1~~-handle notation.

Ref: Gompf & Stipsicz, ch. 5 [GS]

Kirby Moves

- 3 types of moves; relating $\underbrace{\text{any two handle decompositions.}}$ Thm 4.2.12, [GS]
- Handle slides (most non-trivial one)



} Note twist, change in framing.

- Handle cancellation (of pairs):

- Handle creation (of pairs):

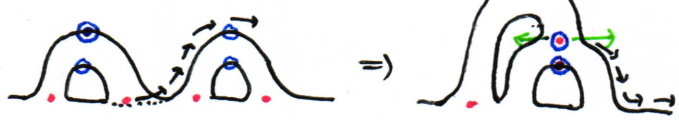
Handle Slides:

Def: Given 2 k-handles h_1 & h_2 attached to ∂X, a handle slide is given by the following procedure:

- Isotope the attaching sphere $\underline{A}$ of h_1 across the boundary ~~sec~~ section corresponding to h_2 on ∂X
- At some point, attaching sphere will intersect belt sphere B of h_2. Perturb so intersections ~~are~~ is transverse.
- Intersection is at 1pt and we have $\dim T_pB \oplus T_pA = \dim X - 1 \Rightarrow$ 'two' normal directions to push A back off ∂h_2.
- one direction $\Rightarrow$ undoes isotopy
- other direction $\Rightarrow$ handle slide result

(can use explicit vector-fields on $h_i \cong D^k \times D^{n-k}$ to be explicit near the handles).

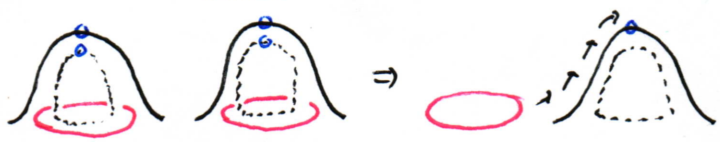
1-handles in 2-d.

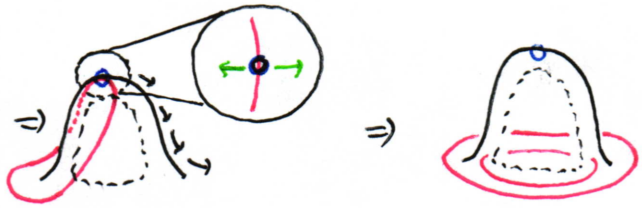


o = belt spheres

• = attaching spheres.

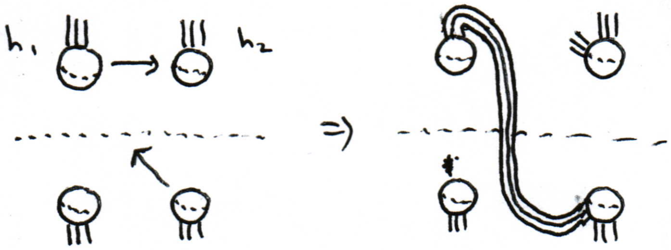
1-handles in 3-d





Handle Slides in Kirby diagrams

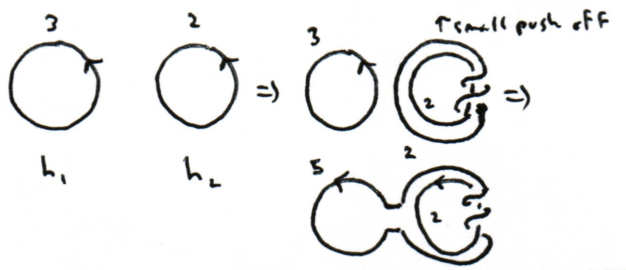
1-handles:



Note: Double strand notation for framings is helpful for keeping track of framings.

2-handles:

- Has effect of band summing h_1 with small push-off of h_2 by framing:



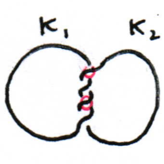
Example Application: Change of Basis on Link Form

Many facts that I don't have time to explain:

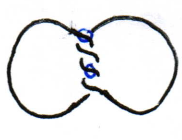
- If X is a 4-handle body w/ no 1-handles, & it's closed, then its Kirby diagram is a framed link, $L = \{K_i\}$.
- $H_2(X;\mathbb{Z})$ is generated by the 2-handles, w/ no relations. Say h_i is attached at K_i.
- Furthermore, intersection form Q_X ~~can~~ is given by linking form Lk.
- Lk is defined as:
  - $Lk(K_i, K_j)$ = signed count of overcrossings of K_i over K_j if $K_i \neq K_j$.
  - $Lk(K_i, K_i)$ = framing coeff w.r.t Seifert surface framing

Proofs: [GS], Ch. 4.5.

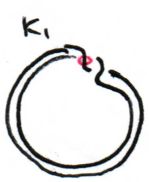
Pictures of Lk:



$Lk(K_1, K_2) = 2.$



$Lk(K_1, K_2) = -2$



$Lk(K_1, K_1) = 1$

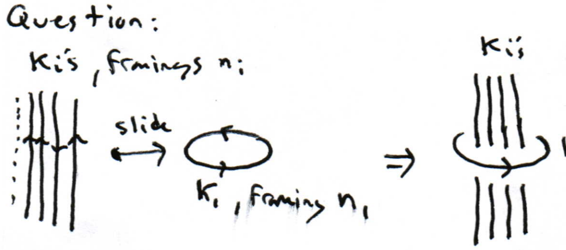
$Lk(K_1, K_1) = 3.$

Summary: ~~$Q_X(K_i, K_j) =$~~

$Q_X([h_i],[h_j]) = Lk(K_i, K_j)$

Example Continued:

Question:



~~Question~~: What happens to Lk? Answers:

- New framings n_i' given by:

$$Lk(K_i', K_i') = n_i' = n_i + k_i^2 n_1 + 2k_i \cdot Lk(K_i, K_1)$$

$$Lk(K_i', K_j') = Lk(K_i, K_j) + k_i k_j n_1$$

$i, j \neq 1$

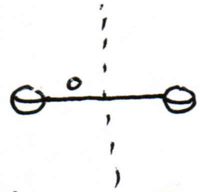
Proofs: Draw one slide, derive formula, use induction.

N.B.: Really useful formulae!

Handle cancellation:

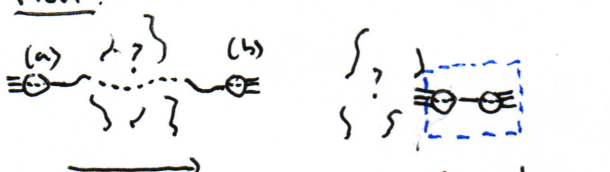
Recall: A k+1 handle can cancel a k handle if the belt sphere of k+1 handle intersects attaching sphere of k-handle at <u>1 pt</u>.

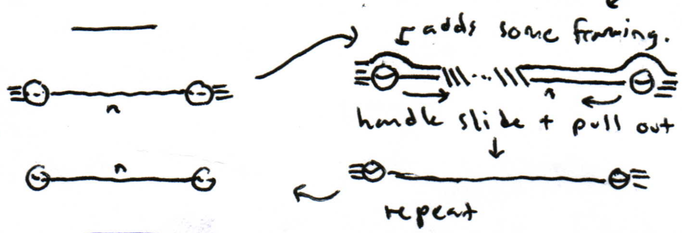
1-handles: model pair



Basically all cancelling pairs look like this.

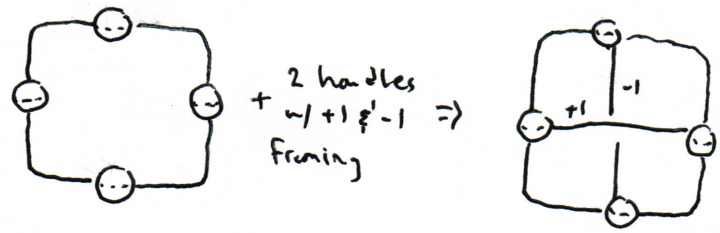
"Proof":



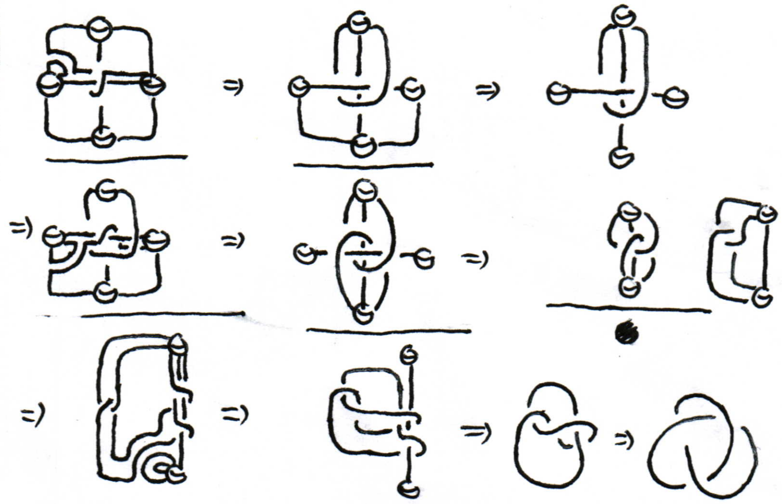


Ready to Calculate!

Proof that:



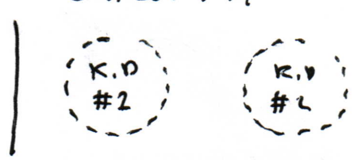
is actually framed trefoil!



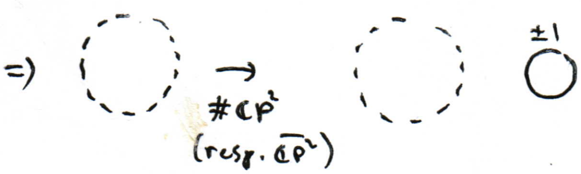
Blow ups (skipping 2-handle cancellations)

$\mathbb{C}P^2$ / $\overline{\mathbb{C}P}^2$ diagrams

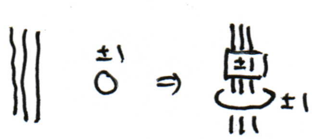
connect sum



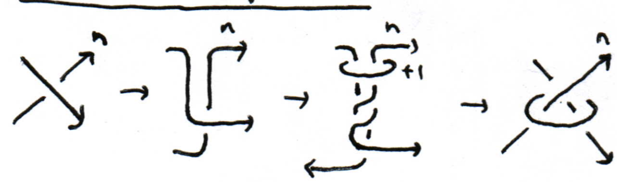
place diagrams next to each other.



Using slide rule:



Other blow up moves



(Exercise: do other crossing/direction combos).