Essential tori in spaces of symplectic embeddings

Julian Chaidez, Mihai Munteanu

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Abstract

Given two n–dimensional symplectic ellipsoids whose symplectic sizes satisfy certain inequalities, we show that a certain map from the n–torus to the space of symplectic embeddings from one ellipsoid to the other induces an injective map on singular homology with mod 2 coefficients. The proof uses parametrized moduli spaces of J–holomorphic cylinders in completed symplectic cobordisms, provided by cobordism maps of full contact homology. This is a pre-Arxiv draft. Please do not download or distribute.

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1 Introduction

The study of symplectic embeddings is a major area of focus in symplectic geometry. Remarkably, the space of such embeddings can have a rich and complex structure, even when the domain and target manifolds are relatively simple.

Symplectic embeddings between ellipsoids are a well–studied instance of this phenomenon. For a nondecreasing sequence of positive real numbers $a = (a_1, a_2, \ldots, a_n)$ define the *symplectic ellipsoid* $E(a)$ by

$$E(a) = E(a_1, a_2, \ldots, a_n) = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} \leq 1 \right\}. \tag{1.1}$$

The space $E(a)$ carries the structure of an exact symplectic manifold with boundary endowed with the restriction of the standard Liouville form λ on $\mathbb{C}^n$, given by

$$\lambda = \frac{1}{2} \sum_{i=1}^{n} (x_i dy_i - y_i dx_i). \tag{1.2}$$

A special case is the *symplectic ball* $B^{2n}(r)$, which is simply $E(a)$ for $a = (r, \ldots, r)$.

The types of results that one can prove about symplectic embeddings, together with the tools used to do so, are surveyed at length by Schlenk in [21]. Most research has thus far sought to address the existence problem. Let us recall some of the more striking progress in this direction. The first nontrivial result was Gromov's eponymous *nonsqueezing theorem*, proven in the seminal paper [8].

Theorem 1.1 ([8]). *There exists a symplectic embedding*

$$B^{2n}(r) \to B^2(R) \times \mathbb{C}^{2n-2}$$

if and only if $r \leq R$.

This result demonstrated that there are obstructions to symplectic embeddings beyond the volume and initiated the study of quantitative symplectic geometry. Note that Theorem 1.1 can be seen as a result about ellipsoid embeddings, since $B^2(R) \times \mathbb{C}^{2n-2}$ can be viewed as the degenerate ellipsoid $E(R, \infty, \ldots, \infty)$.

In dimension 4, the question of when the ellipsoid $E(a, b)$ symplectically embeds into the ellipsoid $E(a', b')$ was answered by McDuff in [12]. Let $\{N_k(a, b)\}_{k \geq 0}$ denote the sequence of nonnegative integer linear combinations of a and b, ordered nondecreasingly with repetitions.

Theorem 1.2 ([12]). *There exists a symplectic embedding*

$$\operatorname{int}(E(a, b)) \to E(a', b')$$

if and only if $N_k(a, b) \leq N_k(a', b')$ *for every nonnegative integer* k.

A special case of this embedding problem, where the target ellipsoid is the ball $B^4(\lambda)$, was studied by McDuff and Schlenk in an earlier paper [13] using methods different from [12]. In that paper, McDuff and Schlenk give a remarkable calculation of the function $c_0 : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$c_0(a) = \inf \left\{ \lambda \;\middle|\; E(1,a) \text{ symplectically embeds into } B^4(\lambda) \right\}.$$

In particular, they show that for $a \in [1, (\frac{1+\sqrt{5}}{2})^4]$, the function c_0 is given by a piecewise linear function involving the Fibonacci numbers, which they call the *Fibonacci staircase*. Some higher dimensional cases of the existence problem for symplectic embeddings have been studied in a similar manner. For instance, a family of stabilized analogues of the function c_0, which are defined as

$$c_n(a) = \inf \left\{ \lambda \;\middle|\; E(1,a) \times \mathbb{C}^n \text{ symplectically embeds into } B^4(\lambda) \times \mathbb{C}^n \right\},$$

are studied in the more recent papers [5] and [6].

Beyond problems of existence, one can ask about the algebraic topology of the space of symplectic embeddings $\mathrm{SympEmb}(U, V)$ between two symplectic manifolds U and V, with respect to the C^∞ topology. Again, most results have been proven in dimensions 2 and 4. For instance, in [11], McDuff demonstrated that the space of embeddings between 4–dimensional symplectic ellipsoids is connected whenever it is nonempty. Other results in dimension 4 can be found in [1] and [9].

More recently, in [16], the second author developed methods to show that the contractibility of certain loops of symplectic embeddings of ellipsoids depends on the relative sizes of the two ellipsoids.

1.1 Main result

In this paper, we build upon the methods developed in [16] to tackle the question of describing the higher homology groups of spaces of symplectic embeddings between ellipsoids in any dimension.

More precisely, we will be studying families of symplectic embeddings that are restrictions of the following unitary maps. For $\theta = (\theta_1, \ldots, \theta_n) \in T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, let U_θ denote the unitary transformation

$$U_\theta(z_1, \ldots, z_n) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n). \tag{1.3}$$

Given symplectic ellipsoids $E(a)$ and $E(b)$ such that $a_i < b_i$ for every $i \in \{1 \ldots, n\}$, we may define the family of ellipsoid embeddings

$$\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b)), \qquad \Phi(\theta) = U_\theta|_{E(a)} \tag{1.4}$$

by restricting the domain of the maps U_θ. The following theorem about the family Φ is the main result of this paper.

Theorem 1.3 (Main theorem)**.** *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences of real numbers satisfying*

$$a_i < b_i < a_{i+1} \quad \textit{for all} \quad i \in \{1, \ldots, n-1\} \quad \textit{and} \quad a_n < b_n < 2a_1.$$

Furthermore, let $\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b))$ be the family of symplectic embeddings (1.4). Then the induced map

$$\Phi_* : H_*(T^n; \mathbb{Z}/2) \to H_*(\mathrm{SympEmb}(E(a), E(b)); \mathbb{Z}/2)$$

on homology with $\mathbb{Z}/2$–coefficients is injective.

In order to demonstrate the nontriviality of Theorem 1.3, we note that the map induced by $\Phi : T^n \to \mathrm{Symp}(E(a), E(b))$ on $\mathbb{Z}/2$–homology has a sizeable kernel when $E(a)$ is very small relative to $E(b)$. More precisely, we have the following.

Proposition 1.4. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two nondecreasing sequences of real numbers satisfying $a_n < b_1$. Furthermore, let $\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b))$ be as in (1.4). Then the induced map Φ_* on $\mathbb{Z}/2$–homology has rank 1 in degree ≤ 1 and rank 0 otherwise.*

Unlike the proof of Theorem 1.3, the proof of Proposition 1.4 is an elementary calculation in algebraic topology which we defer to §2.

Remark 1.5 (Comparison to [16])**.** In dimension 4, the fact that Φ_* is injective in degree 1 was proven by the second author, Munteanu, in [16]. Specifically, this is equivalent to [16, Theorem 1.4] which states that the loop

$$\Psi : S^1 \to \mathrm{Symp}(E(a), E(b))$$

defined by

$$\Psi(t)(z_1, z_2) = \begin{cases} (e^{4\pi i t} z_1, z_2) & t \in \left[0, \frac{1}{2}\right] \\ (e^{-4\pi i t} z_1, z_2) & t \in \left(\frac{1}{2}, 1\right] \end{cases}$$

is noncontractible. In fact, [16] actually addresses the more general 4–dimensional case where $E(a)$ and $E(b)$ are replaced with convex toric domains in $\mathbb{C}^2$. We expect Theorem 1.3 to hold at this level of generality, and we hope to address this in future work using somewhat different methods (see Remark 1.7).

Remark 1.6 ($\mathbb{Z}$ vs $\mathbb{Z}/2$ coefficients)**.** Our use of $\mathbb{Z}/2$ coefficients, instead of $\mathbb{Z}$ coefficients, allows us to use the methods of §4 to work entirely with smooth manifolds with boundary as opposed to cochains. While the contents of §4 provide a nice technical work around, we expect Theorem 1.3 to hold at the level of $\mathbb{Z}$ coefficients as well. We plan to develop the methods needed to work over $\mathbb{Z}$ in forthcoming work (see Remark 1.7).

Remark 1.7 (Lagrangian analogues)**.** In forthcoming work, we hope to demonstrate results analogous to Theorem 1.3 for families of Lagrangian torus embeddings in toric domains. We anticipate that these results will be useful for demonstrating the various generalizations of Theorem 1.3 discussed in Remarks 1.5 and 1.6 above.

Organization. The rest of the paper is organized as so. In §2, we give the proof of Theorem 1.3. The final two section are dedicated to demonstrating some technical results needed to deduce the steps of the proof. Namely, in §3 we recall the definition of the contact dg–algebra (as constructed in full generality by Pardon in [17]) together with the computations for symplectic ellipsoids that are relevant to Theorem 1.3. In §4, we prove some useful technical results about the topology of all symplectic embeddings spaces.

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2 Proof of the main result

In this section, we prove Theorem 1.3 assuming a small number of technical results discussed in §3–4. Here is a brief overview of the proof to help guide the reader.

We assume for contradiction that the map Φ_* induced by the family Φ of (1.4) is not injective in degree k. Using this assumption and the results in §4, we find that a certain family of symplectic embeddings, parametrized by a union of an odd number of k–tori $\sqcup_1^m T^k$ and built from Φ, is null–bordant in the space $\mathrm{SympEmp}(E(a), E(b))$. By this, we mean that the family extends to a smooth $(k+1)$–dimensional family of symplectic embeddings $\Psi : P \to \mathrm{SympEmp}(E(a), E(b))$ where P is smooth, compact, $(k+1)$–dimensional manifold with boundary $\partial P \simeq \sqcup_1^m T^k$.

Using Ψ, we construct a moduli space of holomorphic curves $\mathcal{M}_I(\mathfrak{J})$ in cobordisms parametrized by P. Moreover, we construct an associated evaluation map $\mathrm{ev}_I : \mathcal{M}_I(\mathfrak{J}) \to T^k$ to a k–torus T^k. We then show that the degree of this evaluation map is $1 \mod 2$ when restricted to any of the torus components of $\partial\mathcal{M}_I(\mathfrak{J})$. This is the contradiction, since the evaluation map extends to the bounding manifold $\mathcal{M}_I(\mathfrak{J})$ and so must have degree 0.

2.1 Review of contact geometry

We now provide a quick review of basic contact geometry, and in the process establish notation for §2 and §3. We also discuss the Reeb dynamics on the boundary of a symplectic ellipsoid with rationally independent parameters.

Review 2.1 (Contact manifolds)**.** Recall that a *contact manifold* (Y,ξ) is a smooth $(2n-1)$–manifold Y together with a rank $2n-2$ sub–bundle $\xi \subset TY$ that is given fiberwise by the kernel $\xi = \ker(\alpha)$ of a contact 1–form $\alpha \in \Omega^1(Y)$. A *contact form* α is a 1–form on Y satisfying $\alpha \wedge d\alpha^{n-1} \neq 0$ everywhere.

Every contact form α on Y has a naturally associated *Reeb vector field* R_α defined implicitly from α via the equations

$$\iota_{R_\alpha}\alpha = 1, \qquad \iota_{R_\alpha}d\alpha = 0. \tag{2.1}$$

The *Reeb flow* $\Phi_\alpha : Y \times \mathbb{R} \to Y$ is the flow of the vector field R_α, i.e. the family of diffeomorphisms satisfying

$$\left.\frac{d\Phi_\alpha^t(y)}{dt}\right|_{t=s} = R_\alpha \circ \Phi_\alpha^s(y). \tag{2.2}$$

A *Reeb orbit* is a closed orbit of the flow Φ_α, i.e. a curve $\gamma : S^1 = \mathbb{R}/L\mathbb{Z} \to Y$ satisfying $\frac{d\gamma}{dt} = R_\alpha \circ \gamma$ for some positive number L which is called the *period.* Note that L coincides with the *action* $\mathcal{A}_\alpha(\gamma)$ of γ, which is defined as

$$\mathcal{A}_\alpha(\gamma) = \int_{S^1} \gamma^*\alpha. \tag{2.3}$$

A Reeb orbit γ is called *nondegenerate* if the differential $T\Phi^L_{\gamma(0)}$ of the time L flow satisfies

$$\det(T\Phi^L_{\gamma(0)}|_\xi - \mathrm{Id}_\xi) \neq 0. \tag{2.4}$$

A contact form α is called *nondegenerate* if every Reeb orbit of α is nondegenerate.

Review 2.2 (Conley–Zehnder indices)**.** Any nondegenerate Reeb orbit γ posseses a fundamental numerical invariant called the *Conley–Zehnder index* $\mathrm{CZ}(\gamma,\tau)$, whose definition and computation we now review.

The Conley–Zehnder index $\mathrm{CZ}(\gamma,\tau)$ depends on a choice of symplectic trivialization $\tau : \gamma^*\xi \simeq S^1 \times \mathbb{C}^{n-1}$.

The invariant is defined by $\mathrm{CZ}(\gamma,\tau) := \mu_{\mathrm{RS}}(\phi)$ were μ_{RS} denotes the Robbin–Salamon index (see [20]) and ϕ is the path of symplectic matrices defined as follows.

$$\phi : [0,L] \to \mathrm{Sp}(2n-2), \qquad \phi(t) := \tau_{\gamma(t)} \circ T\Phi^t_{\gamma(0)}|_\xi \circ \tau^{-1}_{\gamma(0)}.$$

In the case where $c_1(\xi) = 0 \in H^2(Y;\mathbb{Z})$ and $[\gamma] = 0 \in H_1(Y;\mathbb{Z})$, a canonical Conley–Zehnder index $\mathrm{CZ}(\gamma)$ (which does not depend on a choice of trivialization) can be associated to γ via the following procedure. Extend γ to a map $u : \Sigma \to Y$

from an oriented surface Σ with boundary $\partial\Sigma = S^1$ satisfying $u|_{\partial\Sigma} = \gamma$. Pick a symplectic trivialization $\sigma : u^*\xi \simeq \Sigma \times \mathbb{C}^{n-1}$ and define $\mathrm{CZ}(\gamma)$ by the formula

$$\mathrm{CZ}(\gamma) := \mathrm{CZ}(\gamma, \sigma|_{\partial\Sigma}). \tag{2.5}$$

The fact that $\mathrm{CZ}(\gamma)$ is independent of Σ and σ follows from the vanishing of the first Chern class. The index $\mathrm{CZ}(\gamma)$ can be related to the index $\mathrm{CZ}(\gamma,\tau)$ with respect to a trivialization τ by the formula

$$\mathrm{CZ}(\gamma) = \mathrm{CZ}(\gamma,\tau) + 2c_1(\gamma,\tau). \tag{2.6}$$

Here $c_1(\gamma,\tau)$ is the relative first Chern number with respect to τ of the pullback $u^*\xi$ of ξ to a capping surface u of γ.

For the purposes of this paper, we are interested in a specific family of examples of contact manifolds, namely boundaries $(\partial E(a), \alpha)$ of irrational symplectic ellipsoids.

Example 2.3 (Ellipsoids)**.** Let $E(a)$ be a symplectic ellipsoid with parameters $a = (a_1, \dots, a_n) \in (0,\infty)^n$. Consider the boundary of the ellipsoid $(\partial E(a), \alpha)$ as a contact manifold with contact form $\alpha = \lambda|_{\partial E(a)}$, induced by the standard Liouville form λ on $\mathbb{C}^n$ defined by (1.2). Assume that the parameters a_i satisfy $a_i/a_j \notin \mathbb{Q}$ for each $i \neq j$. The Reeb vector field R_α is given by

$$R_\alpha = 2\pi \sum_i a_i^{-1} \frac{\partial}{\partial\theta_i}. \tag{2.7}$$

Here θ_i is the angular coordinate in the ith $\mathbb{C}$ factor of $\mathbb{C}^n$, which we denote by $\mathbb{C}_i$. The Reeb flow Φ_α on $\partial E(a)$ is given by:

$$\Phi_\alpha : \partial E(a) \times \mathbb{R} \to Y, \qquad \Phi_\alpha^t(z_1, \dots, z_n) = (e^{2\pi t/a_1} z_1, \dots, e^{2\pi t/a_n}). \tag{2.8}$$

Due to our assumption that $a_i/a_j \notin \mathbb{Q}$ for each $i \neq j$, there are precisely n simple orbits γ_i for $1 \leq i \leq n$. Each curve γ_i is a parametrization of the curve of points in Y with $z_j = 0$ for all $j \neq i$. The iterates γ_i^m (for any $m \geq 1$ and $1 \leq i \leq n$) are all nondegenerate, as we will show below by computing the linearized flow. The action of γ_i^m is given by $\mathcal{A}_\alpha(\gamma_i^m) = ma_i$ by (2.8).

To compute the Conley–Zehnder indices of the Reeb orbits γ_i^m, we proceed as follows. Note that along γ_i^m the fiber $\xi_{\gamma_i^m(t)}$ agrees at each t with the orthogonal complex subspace to the ith component $\mathbb{C}_i^\perp = \oplus_{j \neq i} \mathbb{C}_j \subset \mathbb{C}^n \simeq T\mathbb{C}^n_{\gamma_i^m(t)}$. The linearized flow in this trivialization is a direct sum of loops $t \mapsto e^{2\pi i m a_i t/a_j}$ for $j \neq i$ for $t \in [0,1]$. Thus, in this trivialization, the Conley–Zehnder index is given by

$$\mathrm{CZ}(\gamma_i^m, \tau) = \sum_{j \neq i} \left(2 \left\lfloor \frac{ma_i}{a_j} \right\rfloor + 1 \right).$$

On the other hand, the relative Chern number $c_1(\gamma_i^m, \tau)$ with respect to τ is

$$c_1(\gamma_i^m, \tau) = m.$$

Thus we have the following formula for the canonical CZ index of γ_i^m.

$$\mathrm{CZ}(\gamma_i^m) = \sum_{j \neq i} \left(2 \left\lfloor \frac{ma_i}{a_j} \right\rfloor + 1 \right) + 2m, \tag{2.9}$$

which after some smart rewriting becomes

$$\mathrm{CZ}(\gamma_i^m) = n - 1 + 2 \left| \{ L \in \mathrm{Spec}(Y, \alpha) \mid L \leq ma_i \} \right|. \tag{2.10}$$

Next we review the basic terminology of exact symplectic cobordisms and associated structures. Throughout the discussion for the rest of the section, let $(Y_\pm, \alpha_\pm)$ be closed contact $(2n-1)$–manifolds with contact forms $\alpha_\pm$.

Review 2.4 (Exact symplectic cobordisms)**.** Recall that an *exact symplectic cobordism* (W, λ, ι) from (Y_+, α_+) to (Y_-, α_-) consists of the following data.

- A compact, exact symplectic manifold (W, λ) with boundary ∂W such that the *Liouville vector field* Z (defined by the equation $d\lambda(Z, \cdot) = \lambda$) is transverse to ∂W everywhere. In this situation, $\partial W = \partial_+ W \sqcup \partial_- W$ where Z points outward along $\partial_+ W$ and inward along $\partial_- W$.
- A pair of boundary inclusion maps ι_+ and ι_-, which are strict contactomorphisms of the form

$$\iota_+ : (Y_+, \alpha_+) \simeq (\partial_+ W, \lambda|_{\partial_+ W}) \qquad \iota_- : (Y_-, \alpha_-) \simeq (\partial_- W, \lambda|_{\partial_- W}) \tag{2.11}$$

We will generally suppress the inclusions in the notation, using ι_+ and ι_- when needed. The maps ι_+ and ι_- extend, via flow along Z or $-Z$, to collar coordinates

$$([0, \epsilon) \times Y_-, e^s \lambda_-) \simeq (N_-, \lambda|_{N_-}), \qquad ((-\epsilon, 0] \times Y_+, e^s \lambda_+) \simeq (N_+, \lambda|_{N_+}). \tag{2.12}$$

Here N_- and N_+ are collar neighborhoods of Y_- and Y_+ respectively, the maps preserve the 1–forms above and s denotes the coordinate on $[0, \epsilon)$ and $(-\epsilon, 0]$.

Given exact symplectic cobordisms (W, λ, ι) from (Y_0, α_0) to (Y_1, α_1) and (W', λ', ι') from (Y_1, α_1) to (Y_2, α_2), we can form the *composition* $(W \# W', \lambda \# \lambda', \iota \# \iota')$ by gluing W and W' via the identification $(\iota'_+)^{-1} \circ \iota_-$ of $\partial_- W$ and $\partial_+ W'$. The Liouville forms and inclusions extend in the obvious way to the glued manifold.

Using these identifications (2.12), we can complete the exact symplectic cobordism (W, λ) by adding cylindrical ends $(-\infty, 0] \times Y_-$ and $[0, \infty) \times Y_+$ to obtain the *completed exact symplectic cobordism* $(\widehat{W}, \widehat{\lambda})$, given by

$$\widehat{W} = (-\infty, 0] \times Y_- \sqcup_{\iota_-} W \sqcup_{\iota_+} [0, \infty) \times Y_+. \tag{2.13}$$

The Liouville 1–forms λ, $e^s\alpha_-$ and $e^s\alpha_+$ glue together to a Liouville form $\widehat{\lambda}$ on $\widehat{W}$. An important special caase of completed cobordisms is given by the *symplectization* of a contact manifold $(\mathbb{R} \times Y, e^s\alpha)$, which is denoted by $\widehat{Y}$.

Given a manifold P (with or without boundary), a *P–parametrized family of exact symplectic cobordisms* $(W_p, \lambda_p)_{p\in P}$ from Y_+ to Y_- is a fiber bundle $W \to P$ over P with a 1–form λ on W and a bundle map $\iota^\pm : P \times Y^\pm \to W$ such that $(W_p, \lambda_p, \iota_p)$ is an exact symplectic cobordism for each $p \in P$. A pair of exact symplectic cobordisms $(V, \eta, \jmath)$ and $(V, \eta', \jmath')$ are called *deformation equivalent* if there is a $[0, 1]$–parametrized family of exact symplectic cobordisms such that $(V, \eta, \jmath) \simeq (W_0, \lambda_0, \iota_0)$ and $(V', \eta', \iota') \simeq (W_1, \lambda_1, \iota_1)$.

Review 2.5. (Almost-Complex Structures) Recall that a compatible almst complex structure J on the symplectic vector bundle ξ gives rise to an $\mathbb{R}$-invariant compatible almost complex structure $\widehat{J}$ on the symplectization $\widehat{Y} = \mathbb{R} \times Y$, defined by

$$\widehat{J}(\partial_s) = R_\alpha \qquad \widehat{J}(R_\alpha) = -\partial_s \qquad \widehat{J}|_\xi = J$$

We denote the set of compatible almost complex structures on Y by $\mathcal{J}(Y)$, and the $\mathbb{R}$-invariant almost complex structures arising from these as $\mathcal{J}(\widehat{Y})$.

An almost complex structure J on a completed exact symplectic cobordism $\widehat{W}$ as above is called *compatible* if it has the following properties.

- On the ends $[0, \infty) \times Y_+$ and $(-\infty, 0] \times Y_-$, J restricts to $\mathbb{R}$-invariant complex structures arising from $J_+ \in \mathcal{J}(Y_+)$ and $J_- \in \mathcal{J}(Y_-)$, respectively.
- The almost complex structure J is compatible with the symplectic form $d\lambda$.

Such an almost complex structure extends to an almost complex structure on $\widehat{W}$ in the obvious way. We let $\mathcal{J}(W)$ denote the set of all such compatible almost complex structures on a given exact symplectic cobordism W.

As with contact manifolds, we are interested in a particular family of examples of exact symplectic cobordisms related to ellipsoid embeddings.

Notation 2.6 (Cobordisms of embeddings)**.** Let $E(a)$ and $E(b)$ be irrational ellipsoids. Given a symplectic embedding $\varphi : E(a) \to \mathrm{int}(E(b))$, we denote by W_φ the exact symplectic cobordism given by

$$W_\varphi := E(b) \setminus \mathrm{int}(\varphi(E(a))), \qquad \iota_+ := \mathrm{Id}\,|_{\partial E(b)}, \qquad \iota_- := \varphi|_{\partial E(a)}. \tag{2.14}$$

In this context, we label the simple Reeb orbits of $\partial E(b)$ by γ_i^+ and the simple Reeb orbits of $\partial E(a)$ by γ_i^-. The simple Reeb orbits of the negative boundary of W_φ are, of course, the images $\varphi(\gamma_i^-)$ and will be denoted as such.

More generally, let P ve a compact manifold with boundary and $\Psi : P \times E(a) \to \mathrm{int}(E(b))$ be a P–parametrized family of symplectic embeddings such that $\mathrm{Im}(\Psi_p)$ is independent of p for p near ∂P. We then acquire a family of cobordisms $(W_{\Psi_p}, \lambda_{\Psi_p})$ with fiber given by (2.14). We let $W_{\partial P} = E(b) \setminus \Psi_p(E(a))$ for $p \in \partial P$ and $\lambda_{\partial P}$ be the Liouville form. Note that in this case, the cobordisms $(W_{\Psi_p}, \lambda_{\Psi_p}, \iota_{\Psi_p})$ for $p \in \partial P$ differ only by the boundary inclusion ι_{Ψ_p}. In situations where ι_{Ψ_p} plays no role, we will often not distinguish between $(W_{\Psi_p}, \lambda_{\Psi_p}, \iota_{\Psi_p})$ for different $p \in \partial P$.

In this setting, we let $\mathcal{J}(\Psi)$ denote the set of P–parametrized families $\mathfrak{J} = \{J_p \mid p \in P\}$ of almost complex structures with the following properties.

- J_p is compatible with $\widehat{W}_{\Psi_p}$ for each $p \in P$, i.e. $J_p \in \mathcal{J}(\widehat{W}_{\Psi_p})$.
- J_p is equal to some p-independent $J_{\partial P} \in \mathcal{J}(\widehat{W}_{\partial P})$ for p near ∂P.

We note that $\mathcal{J}(\widehat{W})$ is contractible for any W (see for instance [14, Proposition 4.11]). This implies that the space of families $\mathcal{J}(\Psi)$ is also contractible, and that any family $\{J_p \mid p \in \partial P\}$ over ∂P extends to a family $\{J_p \mid p \in P\}$ over all of P.

2.2 Moduli spaces in cobordisms

We now introduce the spaces of holomorphic curves that are relevant to our proof and we derive the salient properties of these spaces, namely generic transversality (Lemma 2.10) and compactness (Lemma 2.12). We also state a point count result for one of the moduli spaces of interest, whose proof we defer to §3.

Notation 2.7 (Curve domains)**.** Fix a subset $I \subset \{1, \dots, n\}$ and denote by $|I|$ the size of I. For the remainder of §2, we adopt the following notation.

For each $i \in I$, let Σ_i denote a copy of the twice punctured Riemann sphere $\mathbb{R} \times S^1 \simeq \mathbb{CP}^1 \setminus \{0, \infty\}$ with the usual complex structure $j_{\mathbb{CP}^1}$ and let $\overline{\Sigma}_i$ denote the corresponding copy of $\mathbb{C}P^1$ itself. Let p_i^+ and p_i^- denote the points ∞ and 0 in the copy $\overline{\Sigma}_i$ of $\mathbb{C}P^1$. We refer to p_i^+ and p_i^- as the positive and negative punctures of Σ_i, respectively. Denote by Σ_I the disjoint union $\sqcup_{i \in I} \Sigma_i$.

Definition 2.8 (Unparametrized moduli space)**.** Given any symplectic embedding $\varphi \in \mathrm{Symp}(E(a), E(b))$ and any admissible almost complex structure $J_\varphi \in \mathcal{J}(\varphi)$ on $\widehat{W}_\varphi$ as above, we denote by $\mathcal{M}_I(\widehat{W}_\varphi; J_\varphi)$ the moduli space defined as

$$\mathcal{M}_I(\widehat{W}_\varphi; J_\varphi) := \left\{ u : \Sigma_I \to \widehat{W}_\varphi \;\middle|\; \begin{array}{c} (du)^{0,1}_{J_\varphi} = 0 \\ u \to \gamma_i^\pm \text{ at } p_i^\pm \end{array} \right\} \Big/ (\mathbb{C}^\times)^{|I|} \tag{2.15}$$

That is, $u : \Sigma_I \to \widehat{W}_\varphi$ is a J_φ–holomorphic curve such that u is asymptotic to the trivial cylinder over γ_i^+ in $[0,\infty) \times \partial_+ W_\varphi \simeq [0,\infty) \times \partial E(b)$ at the puncture p_i^+ and u is asymptotic to the trivial cylinder over $\varphi(\gamma_i^-)$ in $(-\infty, 0] \times \partial_- W_\varphi \simeq (-\infty, 0] \times \varphi(\partial E(a))$ at the puncture p_i^-, for each $i \in I$. We quotient the space of such maps by the group of domain reparametrizations, which is the product $(\mathbb{C}^\times)^{|I|}$ of the biholomorphism groups $\mathbb{C}^\times$ of each component cylinder $\Sigma_i \simeq \mathbb{R} \times S^1$.

Definition 2.9 (Parametrized moduli space over P)**.** Given a compact manifold with boundary P, a P–parametrized family of symplectic embeddings $\Psi : P \times E(a) \to E(b)$ and a P–parametrized family of complex structure $\mathfrak{J} \in \mathcal{J}(\Psi)$, let $\mathcal{M}_I(\mathfrak{J})$ denote the moduli space of pairs

$$\mathcal{M}_I(\mathfrak{J}) := \left\{ (p, u) \;\middle|\; p \in P,\ u \in \mathcal{M}_I(\widehat{W}_{\Psi_p}; J_p) \right\}. \tag{2.16}$$

Lemma 2.10 (Transversality)**.** *Let $E(a)$ and $E(b)$ be irrational symplectic ellipsoids with parameters $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ satisfying*

$$a_i < b_i, \qquad a_i < 2a_1, \quad \text{and} \quad b_i < 2b_1 \qquad \text{for all } i \text{ with } 1 \leq i \leq n. \tag{2.17}$$

Then there exists a comeager $\mathcal{J}^{\mathrm{reg}}(\Psi) \subset \mathcal{J}(\Psi)$ with $J|_{\partial P}$

(a) Every $u \in \mathcal{M}_I(\widehat{W}_{\partial P}, J_{\partial P})$ is Fredholm regular (see [23, Definition 7.14]), and thus the moduli space $\mathcal{M}_I(\widehat{W}_{\partial P}, J_{\partial P})$ is a 0–dimensional manifold.

(b) Every $(p, u) \in \mathcal{M}_I(\mathfrak{J})$ is parametrically Fredholm regular (see [23, Remark 7.4] and [22, Definition 4.5.5]) and thus $\mathcal{M}_I(\mathfrak{J})$ is a $(|I|+1)$–dimensional manifold with boundary $\partial\mathcal{M}_I(\mathfrak{J}) \simeq \partial P \times \mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$.

Proof. This essentially follows from the general transversality results of [22] and [23, §7], which we now discuss in some detail.

First, observe that every curve $u \in \mathcal{M}_I(\widehat{W}_{\partial P}, J_{\partial P})$ must be somewhere injective (see [23, p. 123]) for any choice of $J_{\partial P}$. Indeed, note that all of the orbits γ_i^- and γ_i^+ are simple. This means that none of them can be factored as $\eta \circ \varphi$ where η is a closed Reeb orbit and $\varphi : S^1 \to S^1$ is a k–fold cover with $k \geq 2$. This implies that u is simple as well, i.e. that u cannot factor as $v \circ \phi$ where $v : \Sigma' \to \widehat{W}_{\partial P}$ is a J-holomorphic curve and $\phi : \Sigma_I \to \Sigma'$ is a holomorphic branched cover. Simple curves are somewhere injective. In fact, these conditions are equivalent in our setting, see [23, Theorem 6.19]. The same reasoning shows that any curve u appearing as a factor in a point $(p, u) \in \mathcal{M}_I(\mathfrak{J})$ is somewhere injective for any choice of $\mathfrak{J}$.

To see (a), we now note that by [23, Theorems 7.1–7.2] there exists a comeager subset $\mathcal{J}^{\mathrm{reg}}(\widehat{W}_{\partial P}) \subset \mathcal{J}(\widehat{W}_{\partial P})$ with the property that for any $J_{\partial P} \in \mathcal{J}^{\mathrm{reg}}(\widehat{W}_{\partial P})$, every somewhere injective curve $u \in \mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ is Fredholm regular. Furthermore, [23, Theorems 7.1] states that the moduli space $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ is a manifold near these points with dimension given by the index formula

$$\operatorname{ind}(u) = \sum_{i \in I} \left((n-3)\chi(\Sigma_i) + 2c_1(u|_{\Sigma_i}, \tau) + \mathrm{CZ}(\gamma_i^+, \tau) - \mathrm{CZ}(\gamma_i^-, \tau) \right). \tag{2.18}$$

Here the Conley–Zehnder indices $\mathrm{CZ}(\gamma^\pm, \tau)$ and relative Chern numbers $c_1(u|_{\Sigma_i}, \tau)$ are as in Review 2.2, and τ denotes a trivialization of ξ over $\sqcup_i(\gamma_i^+ \sqcup \gamma_i^-)$.

Note that $\partial E(a)$ and $\partial E(b)$ are simply connected and $W_{\partial P}$ is diffeomorphic to a product. Thus we may choose τ by taking capping disks D_i for γ_i^-, thus inducing trivializations of ξ along γ_i^-, and then extending τ to a trivialization along Σ_i to induce trivializations of ξ along γ_i^+. The resulting trivialization has $c_1(u|_{\Sigma_i}, \tau) = 0$, $\mathrm{CZ}(\gamma_i^+, \tau) = \mathrm{CZ}(\gamma_i^+)$ and $\mathrm{CZ}(\gamma_i^-, \tau) = \mathrm{CZ}(\gamma_i^-)$. Here $\mathrm{CZ}(\gamma_i^+)$ and $\mathrm{CZ}(\gamma_i^-)$ denote the canonical indices described in Review 2.2. Thus, using this special choice of τ and noting that $\chi(\Sigma_i) = 0$, the formula (2.18) simplifies to

$$\dim(\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})) = \sum_{i \in I} \left(\mathrm{CZ}(\gamma_i^+) - \mathrm{CZ}(\gamma_i^-) \right) \tag{2.19}$$

Finally, we observe that the hypotheses (2.17) and the Conley–Zehnder index formula (2.10) imply that $\mathrm{CZ}(\gamma_i^+) = \mathrm{CZ}(\gamma_i^-) = n - 1 + 2i$. Therefore, the moduli space $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ is 0–dimensional, and we have proven (a).

To see (b), we apply the appropriate parametric version of transversality (see [23, Remark 7.4] and [22, §4.5]), which states that there exists a family $\mathfrak{J} \in \mathcal{J}(\Psi)$, such that $\mathfrak{J}|_{\partial P} \equiv J_{\partial P}$ and $\mathcal{M}_I(\mathfrak{J})$ is a manifold with boundary. Since $\mathfrak{J}$ is independent of $p \in \partial P$ on the boundary, the boundary of the moduli space is simply the product $\partial \mathcal{M}_I(\mathfrak{J}) = \partial P \times \mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$. The dimension is given by

$$\dim(\mathcal{M}_I(\mathfrak{J})) = \dim(P) + \dim(\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})) = k + 1.$$

This concludes the proof of (b), and also the whole proof of Lemma 2.10. □

Before continuing on to the proof of compactness in Lemma 2.12, let us give a brief, very simplified review of a version of SFT compactness. We refer the reader to [2, §10] for the original proof and to [22, §9.4] for a detailed overview.

Review 2.11 (SFT Compactness)**.** Let P be a compact manifold with boundary, and let (Y_*, α_*) for $* \in \{+, -\}$ be closed, nondegenerate contact manifolds. Let (W_p, λ_p, J_p) be a P–paramaterized family of exact symplectic cobordisms from Y_+ to Y_- equipped with a P–parametrized family of compatible almost complex structures on $\widehat{W}_p$ such that $J_p|_{[0,\infty)\times Y_+} = J_+$ and $J_p|_{(-\infty,0]\times Y_-} = J_-$ for some fixed almost complex structures $J_\pm$. Fix a surface Σ, acquired by taking a closed surface $\overline{\Sigma}$ and removing a finite set of punctures. Finally, consider a sequence $p_i \in P$ and $u^i : \Sigma \to (\widehat{W}_{p_i}, J_{p_i})$ of J_{p_i}–holomorphic curves asymptoting to collections of Reeb orbits Γ^+ (at the positive end of $\widehat{W}_{p_i}$) and Γ^- (at the negative end of $\widehat{W}_{p_i}$) independent of i.

The SFT compactness theorem states that, after passing to a subsequence, $p_i \to p \in P$ and u^i converges to a J_p*–holomorphic building*, which is a tuple of the form.

$$v = (u_1^+, \ldots, u_M^+, u^W, u_1^-, \ldots, u_N^-). \tag{2.20}$$

Here $M, N \in \mathbb{Z}^{\geq 0}$ are integers and the elements of the tuple (called *levels*) are holomorphic maps from punctured surfaces of the form

$$u_j^* : S_j^* \to (\mathbb{R} \times Y_*, J_*) \text{ for } * \in \{+, -\} \quad \text{and} \quad u^W : S^W \to (\widehat{W}_p, J_p).$$

The maps u_j^* and the map u^W are considered modulo domain reparametrization, and modulo translation when the target manifold is a symplectization. The surfaces S_j can be glued together along the boundary punctures asymptotic to matching Reeb orbits, and this glued surface $\#_j S_j$ is homeomorphic to Σ.

All of the curves u_j^* and u^W must be asymptotic to a Reeb orbit at each positive and negative puncture. We denote the collections of positive and negative limit Reeb orbits of u^W (with multiplicity) by $\Gamma^+(u^W)$ and $\Gamma^-(u^W)$, respectively, and we adopt similar notation for u_j^*. The asymptotics of the u_j^* and u^W must be compatible, in the sense that the negative ends of u_j^* and the positive ends of u_{j+1}^* must agree (and likewise for u_M^+ and u^W, etc.). Furthermore, we must have $\Gamma^+(u_1^+) = \Gamma^+$ and $\Gamma^-(u_N^-) = \Gamma^-$. Finally, every symplectization level u_j^* must have at least one component that is not a trivial cylinder $\mathbb{R} \times \gamma$.

Since (W_p, λ_p) is an exact symplectic cobordism, one may apply Stoke's theorem to derive the following expression for the energies of the levels of v:

$$\mathcal{E}(u^W) := \int_{S^W} [u^W]^* d\lambda_p = \sum_{\eta^+ \in \Gamma^+(u^W)} \mathcal{A}(\eta^+) - \sum_{\eta^- \in \Gamma^-(u^W)} \mathcal{A}(\eta^-) \tag{2.21}$$

and

$$\mathcal{E}(u_j^\pm) := \int_{S_j} [u_j^\pm]^* d(e^t \alpha_\pm) = \sum_{\eta^+ \in \Gamma^+(u_j^\pm)} \mathcal{A}(\eta^+) - \sum_{\eta^- \in \Gamma^-(u_j^\pm)} \mathcal{A}(\eta^-). \tag{2.22}$$

The positivity of the energy of any holomorphic curve implies that the right hand sides of (2.21) and (2.22) are nonnegative. More generally, if we let $\mathcal{A}[\Gamma]$ denote the total action of a collection of Reeb orbits (with multiplicity, then we have the string of inequalities

$$\begin{aligned} \mathcal{A}[\Gamma^-] = \mathcal{A}[\Gamma(u_N^-)] \leq \cdots \leq \mathcal{A}[\Gamma(u_1^-)] \leq \mathcal{A}[\Gamma(u^W)] \leq \\ \leq \mathcal{A}[\Gamma(u_M^+)] \leq \cdots \leq \mathcal{A}[\Gamma(u_1^+)] = \mathcal{A}[\Gamma^+]. \end{aligned} \tag{2.23}$$

There is some of additional data, beyond the holomorphic curves themselves, associated to a holomorphic building. However, we suppress this data since it will play no role in any of our arguments below.

With the above review of SFT compactness finished, we are ready to move on to the statement and proof of Lemma 2.12.

Lemma 2.12 (Compactness)**.** *Let $E(a)$ and $E(b)$ be irrational symplectic ellipsoids with parameters $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ satisfying*

$$a_i < b_i < a_{i+1} \text{ for all } i \in \{1, \dots, n-1\} \quad \text{and} \quad b_n < 2a_1. \tag{2.24}$$

Choose an almost complex structure $J_{\partial P} \in \mathcal{J}(\widehat{W}_{\partial P})$ and a family $\mathfrak{J} \in \mathcal{J}(\Psi)$ as in Lemma 2.10. Then the moduli spaces $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ and $\mathcal{M}_I(\mathfrak{J})$ are compact.

Proof. Let (p_i, u^i) be a sequence in $\mathcal{M}_I(\mathfrak{J})$. By SFT compactness, after passing to a subsequence u^i converges to a limit building v. We use the notation of Review 2.11 for this building. We must show that v has no symplectization levels. By considering components of Σ_I and $\#_j S_j$, we can assume that $|I| = 1$, i.e. that Σ_I has one component and each u^i is positively asymptotic to a single $\gamma^+_{l_i}$ where $1 \le l_i \le n$.

Now consider a positive symplectization level u^+_j of v. Due to action monotonicity (2.23), the collections $\Gamma^+(u^+_j)$ and $\Gamma^-(u^+_j)$ of positive and negative limit Reeb orbits of $(Y_+, \alpha_+) \simeq (\partial E(b), \lambda|_{\partial E(b)})$ must satisfy

$$a_{l_i} = \mathcal{A}(\gamma^-_{l_i}) \le \mathcal{A}[\Gamma^-(u^+_j)] \le \mathcal{A}[\Gamma^+(u^+_j)] \le \mathcal{A}(\gamma^+_{l_i}) = b_{l_i}.$$

Consider $\Gamma^+(u^+_j)$ only. Due to the hypotheses (2.24), $\Gamma^+(u^+_j)$ cannot contain either a copy of γ^+_r for $r > l_i$ or a copy of an iterate $(\gamma^+_r)^m$ for any $m \ge 2$ and any r. Otherwise, we would have $\mathcal{A}[\Gamma^+(u^+_j)] > b_{l_i}$. This implies that $\Gamma^+(u^+_j)$ can only contain Reeb orbits γ^+_r for $r \le l_i$. Moreover, since $\mathcal{A}[\Gamma^+(u^+_j)] \ge a_{l_i}$ and $\mathcal{A}(\gamma^+_r) = b_r < a_{l_i}$ for $r < l_i$ (again by (2.24)), we must have $\Gamma^+(u^+_j) = \{\gamma^+_{l_i}\}$. The same reasoning shows that $\Gamma^-(u^+_j) = \{\gamma^+_{l_i}\}$. Thus the energy $\mathcal{E}(u^+_j)$ of the level u^+_j is 0 by (2.22) and the level u^+_j must be a branched cover of a trivial cylinder (see [22, Lemma 9.9]). Since the ends are embedded, u^+_j must be simple and thus a trivial cylinder. This is disallowed by the SFT compactness statement, so u^+_j cannot exist.

The same reasoning implies that negative levels u^-_j of v cannot exist. Thus the building v consists of a single level u^W, whose domain is a cylinder and which is asymptotic to $\gamma^+_{l_i}$ and $\gamma^-_{l_i}$ at the positive and negative ends. We have found a limit curve $(p, u^W) \in \mathcal{M}_I(\mathfrak{J})$ for a subsequence of (p_i, u^i) and thus we have proven the compactness of $\mathcal{M}_I(\mathfrak{J})$. The compactness of $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ follows from that of $\mathcal{M}_I(\mathfrak{J})$. $\square$

Finally, we state the following curve count lemma. The proof is an application of (full) contact homology, and we defer it to §3.2.

Lemma 2.13 (Curve count)**.** *Let $E(a)$ and $E(b)$ be irrational symplectic ellipsoids with parameters $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ satisfying*

$$a_i < b_i < a_{i+1} \text{ for all } i \in \{1, \dots, n-1\} \quad \text{and} \quad b_n < 2a_1. \tag{2.25}$$

Then there is a comeager $\mathcal{J}^{\text{reg}} \subset \mathcal{J}(\widehat{W}_{\partial P})$ such that, for any $J_{\partial P} \in \mathcal{J}^{\text{reg}}$, the compact 0–dimensional manifold $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ has an odd number of points.

2.3 Proofs of Theorem 1.3 and Proposition 1.4

In this section, we use the moduli spaces constructed in §2.2 to prove our main result, Theorem 1.3. We also provide a proof of Proposition 1.4. The following small piece of notation will be helpful for both proofs.

Notation 2.14. For any $n \in \mathbb{Z}^+$ and any $I \subset \{1, \dots, n\}$, define the $|I|$–torus by

$$T_I = \{(\theta_1, \dots, \theta_n) \in T^n \mid \theta_j = 0, \ \forall j \notin I\}. \tag{2.26}$$

Note that the kth homology group $H_k(T^n; \mathbb{Z}/2)$ of the n-torus T^n is generated by the fundamental classes $[T_I]$, where I runs over all subsets of size $|I| = k$.

For Theorem 1.3, we also require the following result, whose proof is the purpose of §4.

Lemma 2.15. *Let U and V be compact symplectic manifolds with boundary. Let Z be a closed manifold with total Stieffel–Whitney class $w(Z) = 1 \in H^*(Z; \mathbb{Z}/2)$ and let Φ be a smooth family of symplectic embeddings*

$$\Phi : Z \to \text{SympEmb}(U, V) \quad \textit{with} \quad \Phi_*[Z] = 0.$$

Then there exists a compact manifold P with boundary Z and an extension of Φ to a smooth family Ψ of symplectic embeddings

$$\Psi : P \to \text{SympEmb}(U, V) \quad \textit{with} \quad \Psi|_{\partial P} = \Phi.$$

Given the above preparation, we are now ready for the proof of Theorem 1.3.

Proof. (of Theorem 1.3) We pursue the argument by contradiction outlined at the begining of §2. Fix an integer k with $1 \le k \le n$ and suppose that there were a nonzero $\mathbb{Z}/2$–homology class of the n–torus of the form

$$[A] = \sum_L c_L[T_L] \quad \text{with} \quad \Phi_*[A] = 0 \in H_k(\text{SympEmb}(E(a), E(b)); \mathbb{Z}/2). \tag{2.27}$$

Let $Z = \sqcup_{c_L \neq 0} T_L$. Then Lemma 2.15 states that there exists a smooth $(k+1)$–dimensional manifold P with boundary $\partial P = Z$ and a smooth family of embeddings

$$\Psi : P \to \mathrm{SympEmb}(E(a), E(b)) \quad \text{with} \quad \Psi|_{T_L} = \Phi|_{T_L} \text{ for each } T_L \subset Z. \tag{2.28}$$

By passing to subellipsoids, we may assume that $E(a)$ and $E(b)$ are irrational. In this setting, Lemmas 2.10 and 2.12 state that there exist choices of $J_{\partial P} \in \mathcal{J}(\widehat{W}_{\partial P})$ and $\mathfrak{J} \in \mathcal{J}(\Psi)$ such that the parametrized moduli space $\mathcal{M}_I(\mathfrak{J})$ is a compact $(k+1)$–dimensional manifold with boundary $\partial\mathcal{M}(\mathfrak{J}) \simeq Z \times \mathcal{M}(\widehat{W}_{\partial P}; J_{\partial P})$. On the parametrized moduli space $\mathcal{M}_I(\mathfrak{J})$, we can define an evaluation map

$$\mathrm{Ev}_I : \prod_{i\in I} \gamma_i^+ \times \mathcal{M}_I(\mathfrak{J}) \to \prod_{i \in I} \gamma_i^- \simeq T^k \tag{2.29}$$

via the following procedure. Let $q = (q_i)_{i\in I}$ be a point in $\times_{i\in I}\gamma_i^+$ and let $(p,u) \in \mathcal{M}_I(\mathfrak{J})$. According to (2.16), (p,u) is a pair of a point $p \in P$ and an equivalence class of holomorphic maps $u : \Sigma_I \to \widehat{W}_{\Psi_p}$ up to reparametrization. Pick a representative holomorphic curve $\widetilde{u}$ of u, which consists of k maps $\widetilde{u}_i : \Sigma_i \to \widehat{W}_{\Psi_p}$ for each $i \in I$. We have limit parametrizations of γ_i^+ and γ_i^- induced by $\widetilde{u}_i$, defined by

$$\lim_+ \widetilde{u}_i : S^1 \to \gamma_i^+ \subset \partial E(b), \qquad \lim_+ \widetilde{u}_i(t) := \lim_{s\to+\infty} \pi_{\partial_+ W_{\Psi_p}} \widetilde{u}_i(s,t)$$

$$\lim_- \widetilde{u}_i : S^1 \to \Psi_p(\gamma_i^-) \subset \Psi_p(\partial E(a)), \qquad \lim_- \widetilde{u}_i(t) := \lim_{s\to-\infty} \pi_{\partial_- W_{\Psi_p}} \widetilde{u}_i(s,t)$$

Here $\pi_{\partial_+ W_{\Psi_p}}$ and $\pi_{\partial_- W_{\Psi_p}}$ denote projection to the positive and negative boundaries of W_{Ψ_p}. Note that these projections are only defined in the limit as $s \to \pm\infty$. In terms of these parametrizations, we define the evaluation map Ev_I by the formula

$$\mathrm{Ev}_I(q; p, u) := \Big([\Psi_p^{-1} \circ \lim_- \widetilde{u}_i \circ (\lim_+ \widetilde{u}_i)^{-1}](q_i) \Big)_{i\in I} \in \prod_{i\in I} \gamma_i^-. \tag{2.30}$$

This definition is independent of the choice of representative $\widetilde{u}$. Finally, fix an arbitrary $\underline{q}$ in the product $\prod_{i\in I}\gamma_i^+$ and define

$$\mathrm{ev}_I : \mathcal{M}_I(\mathfrak{J}) \to \prod_{i\in I} \gamma_i^-, \qquad \mathrm{ev}_I(p,u) := \mathrm{Ev}_I(\underline{q}; p, u). \tag{2.31}$$

Now consider the restriction of ev_I to each component $T_L \times \{u\}$ of the boundary $Z \times \mathcal{M}(\widehat{W}_{\partial P}, J_{\partial P})$ of $\mathcal{M}(\mathfrak{J})$. Since the equivalence class of curve u is independent of $\theta \in T_L \subset T^n$, we can use (2.30) and (2.31) to write

$$\mathrm{ev}_I(\theta, u) = (\Psi_\theta^{-1}(\underline{r}_i))_{i\in I} \qquad \text{with} \quad \underline{r} := \Big([\lim_- \widetilde{u}_i \circ (\lim_+ \widetilde{u}_i)^{-1}](\underline{q}_i) \Big)_{i\in I} \in \prod_{i\in I}\gamma_i^-.$$

Here $\underline{r}$ is independent of θ. Using the fact that $\Psi^{-1}|_Z = \Phi^{-1}$ and the formula (1.4) for the family of embeddings Φ, we have the formula

$$\mathrm{ev}_I(\theta, u) = (\Phi_\theta^{-1}(\underline{r}_i))_{i\in I} = (e^{-2\pi i\theta_i} \cdot \underline{r}_i)_{i\in I}. \tag{2.32}$$

In the right–most expression of (2.32), we identify $\underline{r}_i$ with an element of $\mathbb{C}^n$ via the inclusion $\gamma_i^- \subset E(a) \subset \mathbb{C}^n$.

The expression (2.32) allows us to compute the degree of ev_I on each component $T_L \times \{u\}$. There are two cases. If $L = I$, then (2.32) shows that $\mathrm{ev}_I|_{T_L\times\{u\}}$ is degree 1. If $I \neq L$, then for any $j \in L\setminus(L\cap I)$, θ_j is constant for every $\theta \in T_L$ and it follows from (2.32) that the degree of $\mathrm{ev}_I|_{T_L\times\{u\}}$ is 0. To derive our final contradictiction, we now observe that the total degree mod 2 of ev_I restricted to the boundary is

$$\begin{aligned} \deg(\mathrm{ev}_I|_{\partial\mathcal{M}_I(\mathfrak{J})}) = &\sum_{T_L\times\{u\}\subset\partial\mathcal{M}_I(\mathfrak{J})} \deg(\mathrm{ev}_I|_{T_L\times\{u\}}) = \\ &= |\mathcal{M}_I(\widehat{W}_{\partial P}, J_{\partial P})| \equiv 1 \mod 2. \end{aligned} \tag{2.33}$$

The right–most equality in (2.33) crucially uses the point count of Lemma 2.13. The equality (2.33) also provides the contradiction, since the degree of the restricton of a map to a boundary must be 0 mod 2. This concludes the proof. □

Having concluded the proof of Theorem 1.3, we now move on to Proposition 1.4. The proof is much less involved than that of Theorem 1.3, and does not use any of the machinery from §2.1–2.2. We begin with a lemma about the homology groups of the unitary group $U(n)$.

Lemma 2.16. *Consider the map $U : T^n \to U(n)$ given by $\theta \mapsto U_\theta$. Then the induced map $U_* : H_*(T^n; \mathbb{Z}/2) \to H_*(U(n); \mathbb{Z}/2)$ on $\mathbb{Z}/2$–homology is:*

(a) surjective if $* = 0$ or $* = 1$.

(b) identically 0 if $* \geq 2$.

Proof. To show (a), we first note that T^n and $U(n)$ are connected so $U_*|_{H_0} = \mathrm{Id}$. Furthermore, if we consider the loop $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to T^n$ given by $\theta \mapsto (\theta, 0, \ldots, 0)$, we see that the composition:

$$\det_{\mathbb{C}} \circ U \circ \gamma : \mathbb{R}/2\pi\mathbb{Z} \to U(1) \simeq \mathbb{R}/2\pi\mathbb{Z}$$

is the identity. Since $\det_{\mathbb{C}} : U(n) \to U(1)$ induces an isomorphism on H_1, the induced map of U must be surjective on H_1.

To show (b) we proceed as follows. It suffices to show that $U_*[T_L] = 0$ for all L with $|L| \geq 2$. We can factorize $T_L = T_J \times T_K$ for $J \sqcup K = L$ and $|J| = 2$. Then we have a factorization:

$$T_L = T_J \times T_K \xrightarrow{\iota_J \times \iota_K} U(2) \times U(n-2) \xrightarrow{j} U(n)$$

Here j is the inclusion of a product of unitary subgroups, and ι_J and ι_K are inclusions of the tori into these unitary subgroups. It suffices to show that $(\iota_J \times \iota_K)_*[T_L] = [\iota_J]_*[T_J] \otimes [\iota_K]_*[T_K] = 0$, or simply that $[\iota_J]_*[T_J] = 0 \in H_2(U(2); \mathbb{Z}/2)$.

Now we simply note that $\dim(U(2)) = 4$ and $H^*(U(2); \mathbb{Z}/2) \simeq \mathbb{Z}/2[c_1, c_3]$ where c_i is a generator of index i. In particular, $H_2(U(2); \mathbb{Z}/2) \simeq H^2(U(2); \mathbb{Z}/2) = 0$. $\square$

Using Lemma 2.16, we can now prove Propositon 1.4. The point is that the entire unitary group $U(n)$ embeds into $\mathrm{SympEmb}(E(a), E(b))$ via domain restriction when $a_i < b_j$ for all i and j (which is equivalent to $a_n < b_1$ by our ordering convention).

Proof. (of Proposition 1.4) Let $D : \mathrm{SympEmb}(E(a), E(b)) \to U(n)$ denote the map $\varphi \mapsto r(d\varphi|_0)$, given by taking derivatives $d\varphi|_0 \in \mathrm{Sp}(2n)$ at the origin and composing with a retraction $r : \mathrm{Sp}(2n) \to U(n)$. Under the hypotheses on a and b, we can factor the identity $\mathrm{Id} : U(n) \to U(n)$ and $\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b))$ as

$$\mathrm{Id} : U(n) \xrightarrow{\mathrm{res}} \mathrm{Symp}(E(a), E(b)) \xrightarrow{D} U(n),$$

$$\Phi : T^n \xrightarrow{U} U(n) \xrightarrow{\mathrm{res}} \mathrm{SympEmb}(E(a), E(b)).$$

Here $\mathrm{res}(\varphi) := \varphi|_{E(a)}$ denotes restriction of domain. In particular, $\mathrm{res} : U(n) \to \mathrm{Symp}(E(a), E(b))$ is injective on homology and $\mathrm{Im}(\Phi_*) \simeq \mathrm{Im}(U_*)$ as $\mathbb{Z}$–graded $\mathbb{Z}/2$–vector spaces. The result thus follows from Lemma 2.16. $\square$

3 Contact homology

In this section, we discuss the main Floer–theoretic tool in this paper, the full contact homology $CH(Y, \xi)$ of a closed contact manifold (Y, ξ). The goal is to extract the point count result, Lemma 2.13 in §2.2, from the basic properties of this invariant.

3.1 Contact dg–algebra

We now review the contact dg–algebra of a contact manifold and the cobordism dg–algebra maps induced by exact symplectic cobordisms. This invariant package was originally introduced by Eliashberg-Givental-Hofer [7] without foundations. Here we use the construction by Pardon [17] using virtual fundamental cycles.

Remark 3.1 (Assumptions)**.** We restrict our discussion to contact manifolds Y and exact symplectic cobordisms W satisfying the following assumptions.

$$H_1(Y;\mathbb{Z}) = H_2(Y;\mathbb{Z}) = 0 \qquad \text{and} \qquad c_1(\xi) = 0 \tag{3.1}$$

$$H_1(W;\mathbb{Z}) = H_2(W;\mathbb{Z}) = 0 \qquad \text{and} \qquad c_1(TW) = 0 \tag{3.2}$$

The hypotheses (3.1) and (3.2) are sufficient for our applications. Furthermore, they allow us to simplify various definitions, formulas and notations from [17] by suppressing the homology classes of holomorphic curves and using $\mathbb{Z}$-gradings.

We begin by fixing notation for the choices of Floer data that are needed to define the relevant chain groups and cobordism maps.

Setup 3.2 (Contact manifold data)**.** Fix the following setup and notation.

(a) (Y, ξ) is a closed contact $(2n-1)$–manifold satisfying (3.1), with nondegenerate contact form α and associated Reeb vector-field R_α.

(b) $b(\underline{\gamma})$ is a basepoint for each simple orbit $\underline{\gamma}$ of (Y, α). Denote the set of orbits of (Y, α) by $\mathcal{P}(Y, \alpha)$. Given any orbit $\gamma \in \mathcal{P}(Y, \alpha)$, denote the underlying simple orbit by $\underline{\gamma}$ and the degree of the covering map $\gamma \to \underline{\gamma}$ by $d(\gamma)$.

(c) J is a $d\alpha$–compatible complex structure on ξ and $\widehat{J}$ is the associated $\mathbb{R}$–invariant almost complex structure on the symplectization $\widehat{Y} := \mathbb{R} \times Y$.

(d) $\theta \in \Theta(Y, \alpha, J)$ is a choice of virtual perturbation data (in the sense of [17] §1.1) for the compactified moduli spaces of holomorphic curves $\overline{\mathcal{M}}_\partial(\widehat{Y}; \gamma^+, \Gamma^-)$ (see Definition 3.5) for all good orbits γ^+ and collections of good orbits Γ^-.

We use $\mathsf{Data}(Y, \xi)$ to denote the set of choices of data associated to a fixed contact manifold (Y, ξ). Note that $\mathsf{Data}(Y, \xi)$ is natural: any contactomorphism $\Phi : (Y, \xi) \to (Y', \xi')$ induces an obvious map $\Phi_* : \mathsf{Data}(Y, \xi) \to \mathsf{Data}(Y', \xi')$ acquired by pushing forward the contact forms, markers, complex structures and VFC data.

Setup 3.3 (Symplectic cobordism data)**.** Fix the following setup and notation.

(a) For each $* \in \{+, -\}$, let Y_*, ξ_* and α_* be contact data, J_* and $\widehat{J}_*$ be complex data, and θ_* be virtual perturbation data, all as in Setup 3.2.

Furthermore, fix the following setup and notation for corresponding cobordism data.

(b) (W, λ) is an exact symplectic $2n$–cobordism from $\partial_+ W \simeq Y_+$ to $\partial_- W \simeq Y_-$ satisfying (3.2), with completion $(\widehat{W}, \widehat{\lambda})$ as in Review 2.4.

(c) J is a $d\lambda$ compatible complex structure on W agreeing with J_* on symplectic collar neighborhoods of Y_*, and $\widehat{J}$ is the associated complex structure on $\widehat{W}$.

(d) $\theta \in \Theta(W, \lambda, \widehat{J})$ is a choice of virtual perturbation data (in the sense of [17] §1.3) for the compactified moduli spaces of holomorphic curves $\overline{\mathcal{M}}_c(\widehat{W}; \gamma^+, \Gamma^-)$ (see Definition 3.5) for all good orbits γ^+ and collections of good orbits Γ^-.

We use $\mathsf{Data}[W, \lambda]$ to denote the set of choices of data as above for a fixed deformation class $[W, \lambda]$ of exact symplectic cobordism. For each $* \in \{+, -\}$, there is a projection map of Floer data:

$$\pi_* : \mathsf{Data}[W, \lambda] \to \mathsf{Data}(Y_*, \xi_*) \qquad (\lambda, J, \theta) \mapsto (\alpha_*, J_*, \pi_*\theta).$$

Here we use the projection map of VFC data $\pi_* : \Theta(W, \lambda, J) \to \Theta(Y_*, \alpha_*, J_*)$ described in [17], §1.3. Note that the following product map is surjective.

$$\pi_+ \times \pi_- : \Theta(W, \lambda, J) \to \Theta(Y_+, \alpha_+, J_+) \times \Theta(Y_-, \alpha_-, J_-)$$

Before discussing the moduli spaces involved in contact homology, we recall the definition of asymptotic markers, which have not appeared yet in this paper.

Definition 3.4 (Asymptotic markers)**.** Let (Σ, j) be a closed Riemann surface with punctures. Let $\overline{\Sigma}$ denote the surface Σ with punctures added back in. An *asymptotic marker* $m(p)$ at a puncture p of Σ is a point in the projectivized tangent space $S_p\overline{\Sigma}$.

Let $u : (\Sigma, j) \to (\widehat{W}, \widehat{J})$ asymptotic at the positive or negative end to an orbit γ at p, where W and J are as in Setup 3.3(a)-(c). A natural map $Su_p : S_p\overline{\Sigma} \to \underline{\gamma}$ may be defined in this setting as the limit

$$S_pu(v) = \lim_{\epsilon \to 0} \pi_{Y_+} u(\eta(\epsilon)) \quad \text{ or } \quad S_pu(v) = \lim_{\epsilon \to 0} \pi_{Y_-} u(\eta(\epsilon))$$

for any arc $\eta : [0, 1] \to \overline{\Sigma}$ with $\eta(0) = p$ and $\frac{d\eta}{d\epsilon} = v$. In either of these cases, we say that $m(p)$ is asymptotic $x \in \underline{\gamma}$ under u if $S_pu(m(p)) = x$.

Here are precise descriptions of the moduli spaces referenced in Setups 3.2 and 3.3 above. The reader should reference [17, §2.3] for Pardon's definitions.

Definition 3.5 (Moduli spaces for CH)**.** Let $(\widehat{W}, \widehat{\lambda})$ and $\widehat{J}$ be as in Setup 3.3(a)-(b). Let γ^+ be a Reeb orbit of (Y_+, α_+) and let Γ^- be a collection of k Reeb orbits in (Y_-, α_-) (allowing repitition of orbits). Consider holomorphic curves $\mathbf{u}$ consisting of the following data.

(a) A punctured, genus 0 Riemann surface (Σ, j) with a positive puncture p^+ and k negative punctures P^-.

(b) A holomorphic map $u : (\Sigma, j) \to (\widehat{W}, \widehat{J})$ which is asymptotic to γ^+ at the puncture p^+ and the orbits Γ^- at the punctures P^-.

(c) Asymptotic markers $m(p^+)$ at p^+ and $m(p^-)$ at each $p^- \in P^-$. These markers must have the property that $m(p^+)$ is asymptotic to $b(\underline{\gamma})$ under u and p^- is asymptotic to $b(\underline{\gamma}^-)$, for any $p^- \in P^-$ and corresponding $\gamma^- \in \Gamma^-$.

A pair of holomorphic curves $\mathbf{u}_* = (\Sigma_*, j_*, u_*, p^+_*, P^-_*, m_*)$ for $* \in \{0, 1\}$ are equivalent, denoted by $\mathbf{u}_0 \sim \mathbf{u}_1$, if there is a smooth map $\varphi : \Sigma_0 \to \Sigma_1$ such that

(d) φ is holomorphic, i.e. $T\varphi \circ j_0 = j_1 \circ T\varphi$

(e) φ sends punctures to punctures, i.e. $\varphi(p^+_0) = p^-_1$ and $\varphi(P^-_0) = P^-_1$.

(f) φ sends markers to markers, i.e. the induced map $S_{p^+}\varphi : S_{p^+_0}\Sigma_0 \to S_{p^+_1}\Sigma_1$ satisfies $S_{p^+}\varphi(m_0(p^+_0)) = m_1(p^+_1)$ and similarly for negative punctures.

Let $\mathcal{M}(\widehat{W}; \gamma^+, \Gamma^-)$ denote the moduli space of such holomorphic curves $\mathbf{u}$ modulo equivalence. When the points of this moduli-space are Fredholm regular (in the sense of [17, Definition 2.39]), the moduli space is a smooth manifold of dimension

$$\dim(\mathcal{M}(\widehat{W}; \gamma^+, \Gamma^-)) = |\gamma^+| - |\Gamma^-| \in \mathbb{Z} \tag{3.3}$$

If the completed cobordism $(\widehat{W}, \widehat{\lambda})$ and $\widehat{J}$ arises via a symplectization $(\widehat{Y}, e^t\alpha)$ as in Setup 3.2(a)-(b), then this moduli space posseses a natural $\mathbb{R}$ action given by $\mathbb{R}$-translation on $\widehat{Y}$. We adopt following notation for the quotient.

$$\mathcal{M}_\partial(\widehat{Y}; \gamma^+, \Gamma^-) := \mathcal{M}(\widehat{Y}; \gamma^+, \Gamma^-)/\mathbb{R}. \tag{3.4}$$

In the general case of a completed cobordism $(\widehat{W}, \widehat{\lambda})$ as above, we write

$$\mathcal{M}_c(\widehat{W}; \gamma^+, \Gamma^-) := \mathcal{M}(\widehat{W}; \gamma^+, \Gamma^-). \tag{3.5}$$

We let $\overline{\mathcal{M}}_\partial(\widehat{Y}; \gamma^+, \Gamma^-)$ and $\overline{\mathcal{M}}_c(\widehat{W}; \gamma^+, \Gamma^-)$ denote the compactifications of (3.4) and (3.5) defined in [17]. A detailed understanding of this compactificiation is unnecesary for this paper, although it is similar to the SFT compactification described in [2].

Given the setup discussed above, we now give an overview of the construction of the contact dg–algebra and cobordism maps.

Construction 3.6. (Contact DG–Algebra) Consider a contact manifold (Y, ξ) along with a choice of data $\mathcal{D} \in \mathsf{Data}(Y, \xi)$, all as in Setup 3.2.

We now give the construction of the $\mathbb{Z}$–graded differential algebra $CC(Y,\xi)_{\mathcal{D}}$ with differential $\partial_{\mathcal{D}}$ of degree -1, called the *contact dg–algebra* or the *full contact homology algebra*. We denote the homology of this dg–algebra by:

$$CH(Y,\xi)_{\mathcal{D}} := H(CC(Y,\xi)_{\mathcal{D}}, \partial_{\mathcal{D}})$$

(Algebra) To define the algebra $CC(Y,\xi)_{\mathcal{D}}$, consider the set $\mathcal{P}(Y,\alpha)$ of unparametrized Reeb orbits of (Y,α). We can divide $\mathcal{P}(Y,\alpha)$ into good orbits $\mathcal{P}_g(Y,\alpha)$ and bad orbits $\mathcal{P}_b(Y,\alpha)$. An orbit γ is *bad* if it is a cover of an orbit γ' with $\mathrm{CZ}(\gamma) - \mathrm{CZ}(\gamma') \equiv 1 \mod 2$. An orbit is *good* if it is not bad. To each good orbit we can associate an $\mathbb{Z}$–graded orientation line $\mathfrak{o}_\gamma$ supported in grading $|\gamma| = CZ(\gamma) + n - 3$ (see for instance [3]). For a finite set of orbits $\Gamma \subset \mathcal{P}_g(Y,\alpha)$ of good orbits, we define $\mathfrak{o}_\Gamma := \otimes_{\gamma\in\Gamma}\mathfrak{o}_\gamma$ and $|\Gamma| := \sum_{\gamma\in\Gamma}|\gamma|$. We then define the contact dg–algebra as

$$CC(Y,\xi)_{\mathcal{D}} := \bigwedge\Big(\bigoplus_{\gamma\in\mathcal{P}_g(Y,\alpha)} \mathfrak{o}_\gamma \otimes_{\mathbb{Z}} \mathbb{Q}\Big). \tag{3.6}$$

That is, $CC(Y,\xi)_{\mathcal{D}}$ is the free, graded–commutative, unital $\mathbb{Z}$-graded $\mathbb{Q}$-algebra generated by the orientation lines $\mathfrak{o}_\gamma$ for $\gamma \in \mathcal{P}_g(Y,\alpha)$.

(Differential) We define the differential $\partial_{\mathcal{D}} : CC(Y,\xi)_{\mathcal{D}} \to CC(Y,\xi)_{\mathcal{D}}$ on any pure element $x \in \mathfrak{o}_{\gamma_+}$ of the algebra by the following formula.

$$\partial_{\mathcal{D}} x := \sum_{|\gamma^+| = |\Gamma^-|+1} \frac{\#_\theta \overline{\mathcal{M}}_\partial(\widehat{Y};\gamma^+,\Gamma^-)}{|\mathrm{Aut}(\gamma^+,\Gamma^-)|}\cdot x. \tag{3.7}$$

The symbol $\#_\theta$ denotes taking a virtual point count (with respect to the VFC data θ coming with the data $\mathcal{D}$) valued in $\mathfrak{o}_{\Gamma^-} \otimes \mathfrak{o}^\vee_{\gamma^+}$. See [17, §1.2, 2.3] for a full discussion. The differential is extended to the entire algebra by imposing the graded Leibniz rule $\partial_{\mathcal{D}}(xy) = \partial_{\mathcal{D}}(x)y + (-1)^{|x|}x\partial_{\mathcal{D}}(y)$ for any $x, y \in CC(Y,\xi)_{\mathcal{D}}$.

Construction 3.7 (Contactomorphism Maps)**.** Consider a contactomorphism $\Phi : (Y,\xi) \to (Y',\xi')$ between contact manifolds (Y,ξ) and (Y',ξ'), along with a choice of data $\mathcal{D} \in \mathsf{Data}(Y,\xi)$, all as in Setup 3.2. Then we can construct a morphism of $\mathbb{Z}$–graded dg-algebras

$$CC(\Phi)_{\mathcal{D}} : CC(Y,\xi)_{\mathcal{D}} \to CC(Y',\xi')_{\Phi_*\mathcal{D}}$$

To define this map, consider the bijection $\mathcal{P}_{\mathrm{g}}(Y,\alpha) \to \mathcal{P}_{\mathrm{g}}(Y',(\Phi^{-1})^*\alpha)$ given by $\gamma \mapsto \Phi\circ\gamma$. This map comes with isomorphisms of $\mathbb{Z}$-graded $\mathbb{Z}$-modules $\mathfrak{o}_\gamma \to \mathfrak{o}_{\Phi\circ\gamma}$ for each $\gamma \in \mathcal{P}(Y,\alpha)$ (see [3]). We therefore define $CH(\Phi)$ to be the unique algebra map where the restriction $CH(\Phi)|_{\mathfrak{o}_\gamma\otimes\mathbb{Q}}$ to $\mathfrak{o}_\gamma \otimes \mathbb{Q}$ is the induced map of orientation lines (tensored up to $\mathbb{Q}$).

Construction 3.8 (Cobordism Maps)**.** Consider an exact symplectic cobordism (W,λ) between contact manifolds (Y_+,ξ_+) and (Y_-,ξ_-), along with a choice of data $\mathcal{D} \in \mathsf{Data}(W,V,\lambda)$ as in Setup 3.3, where $\mathcal{A} = \pi_+\mathcal{D}$ and $\mathcal{B} = \pi_-\mathcal{D}$. Then we can construct a morphism of $\mathbb{Z}$–graded dg-algebras

$$CC(W,\lambda)_{\mathcal{D}} : CC(Y_+,\xi_+)_{\mathcal{A}} \to CC(Y_-,\xi_-)_{\mathcal{B}}.$$

To define this cobordism map, we declare its value on generators and extend it to an algebra map. On an $x \in \mathfrak{o}_{\gamma^+}$ for $\gamma^+ \in \mathcal{P}(Y_+,\alpha_+)$, it is given by the sum:

$$CC(W,\lambda)_{\mathcal{D}}(x) = \sum_{|\gamma^+|=|\Gamma^-|+1} \frac{\#_\theta \overline{\mathcal{M}}_{\mathrm{c}}(\widehat{W};\gamma^+,\Gamma^-)}{|\operatorname{Aut}(\gamma^+,\Gamma^-)|} \cdot x. \tag{3.8}$$

The symbol $\#_\theta$ denotes taking a virtual moduli count in $\mathfrak{o}_{\Gamma^-} \otimes \mathfrak{o}_{\gamma^+}^\vee$ with respect to the VFC data θ, as with the differential. See [17, §1.3, 2.3] for a full discussion.

The salient features of Constructions 3.6 and 3.8 can be summarized in the following Theorem, various parts of which are covered in [17, §1.3-1.8].

Theorem 3.9 ([17] Contact homology)**.** *The dg–algebra and maps described in Constructions 3.6 and 3.8 have the following properties.*

(a) (Homology) The map $\partial_{\mathcal{D}}$ of Construction 3.6 is a differential and the homology algebra $CH(Y,\xi) \simeq CH(Y,\xi)_{\mathcal{D}}$ is independent of the choice of data $\mathcal{D} \in \mathsf{Data}(Y,\xi)$ up to canonical isomorphism.

(b) (Contactomorphism) The contactomorphism map $CC(W,\lambda)_{\mathcal{D}}$ of Construction 3.7 induces a well-defined isomorphism

$$CH(\Phi) : CH(Y,\xi) \simeq CH(Y',\xi')$$

(c) (Cobordism) The cobordism map $CC(W,\lambda)_{\mathcal{D}}$ of Construction 3.8 is a map of dg-algebras, and induces a well-defined map

$$CH[W,\lambda] : CH(Y_+,\xi_+) \to CH(Y_-,\xi_-)$$

(d) (Deformation/Composition) The cobordism map $CH[W,\lambda]$ depends only on the deformation class of (W,λ) (see Review 2.4). Furthermore, if (W_{01},λ_{01}) and (W_{12},λ_{12}) are cobordisms from Y_0 to Y_1 and Y_1 to Y_2, respectively, then

$$CH[W_{12}\#_{Y_1}W_{01},\lambda_{12}\#_{Y_1}\lambda_{01}] = CH[W_{12},\lambda_{12}] \circ CH[W_{01},\lambda_{01}]$$

(e) (Transversality) Suppose that a 0*-dimensional moduli space used in either Constructions 3.6 and 3.8, i.e. one of the spaces*

$$\mathcal{M}_\partial(\widehat{Y};\gamma^+,\Gamma^-) \quad or \quad \mathcal{M}_c(\widehat{W};\gamma^+,\Gamma^-),$$

is Fredholm regular (see [17, §2.11]) and SFT compact. Then the corresponding virtual count, which is either

$$\#_\theta\mathcal{M}_\partial(\widehat{Y};\gamma^+,\Gamma^-) \quad or \quad \#_\theta\mathcal{M}_c(\widehat{W};\gamma^+,\Gamma^-),$$

is given by a signed point count (according to coherent orientations, see [23, §11]), after one identifies the orientation lines $\mathfrak{o}_\gamma$ *of all good orbits* γ *with* $\mathbb{Z}$*.*

3.2 Proof of point count

We now compute the examples of contact homology and cobordism maps that are relevant to this paper, and use these computations to prove Lemma 2.13.

Definition 3.10. A contact form α on a closed contact manifold (Y,ξ) is *lacunary* if, for every good orbit $\gamma \in \mathcal{P}_g(Y,\alpha)$, the grading $|\gamma| \in \mathbb{Z}$ is even.

The contact dg–algebra of a contact manifold with lacunary contact form has vanishing differential for grading reasons. Therefore we have the following lemma.

Lemma 3.11. *Let* (Y,ξ) *be a contact manifold with lacunary contact form* α*. Then*

$$CH(Y,\xi) \simeq \bigwedge\Big(\mathbb{Q}[\mathcal{P}_g(Y,\alpha)]\Big)$$

Example 3.12 (Ellipsoid CH)**.** Consider the boundary $(\partial E(a),\alpha)$ of an irrational ellipsoid $E(a)$ equipped with the standard contact structure ξ and contact form α, as in Example 2.3. Note that $(\partial E(a),\xi)$ satisfies the hypotheses (3.1).

The Conley-Zehnder index formula (2.10) implies that every orbit of $(\partial E(a),\alpha)$ is good, and that we have the grading formula

$$|\gamma_i^m| = 2n-4+2|\{L\in \mathrm{Spec}(Y,\alpha) | L \le ma_i\}| \in \mathbb{Z}. \tag{3.9}$$

Therefore the grading of $\mathfrak{o}_{\gamma_i^m}$ is even and the contact form is lacunary. The contact homology is thus computed by the formula in Lemma 3.11.

Next, we use the axioms of Theorem 3.9 to demonstrate some properties of the cobordism maps on contact homology induced by the cobordisms of Example 2.6.

Lemma 3.13. *Let $E(a)$ and $E(b)$ be irrational ellipsoids with $a_i < b_i$ for all i, and let $c \in (0,1)$ be a constant such that $c \cdot b_i < a_i$ for all i. Let $\iota : E(a) \to \operatorname{int}(E(b))$ and $\jmath : E(c \cdot b) \to E(a)$ be the standard inclusions, and let $\mathrm{sc} : \partial E(b) \to \partial E(c \cdot b)$ be the contactomorphism given by scaling, i.e. $\mathrm{sc}(x) := c \cdot x$.*

Then the cobordism maps $CH[W_\iota, \lambda_\iota]$ and $CH[W_\jmath, \lambda_\jmath]$ induced by the cobordism (W_ι, λ_ι) and $(W_\jmath, \lambda_\jmath)$ (see Example 2.6) are inverses to each other, i.e.

$$CH[W_\jmath, \lambda_\jmath] \circ CH[W_\iota, \lambda_\iota] = CH(\mathrm{sc}) \tag{3.10}$$

Proof. Notice that we have the following isomorphisms of exact symplectic cobordisms from $(\partial E(a), \lambda|_{\partial E(a)})$ to $(\partial E(c \cdot a), \lambda|_{\partial E(c \cdot a)})$.

$$(W_\jmath, \lambda_\jmath) \#_{\partial E(a)} (W_\iota, \lambda_\iota) \simeq (W_{\iota \circ \jmath}, \lambda_{\iota \circ \jmath}) \simeq (E(a) \times [\log(c), 0], e^s \lambda|_{\partial E(a)}) \tag{3.11}$$

Therefore by Theorem 3.9(c), we have the following identity of cobordism maps

$$CH[W_\jmath, \lambda_\jmath] \circ CH[W_\iota, \lambda_\iota] = CH[E(a) \times [\log(c), 0], e^s \lambda|_{\partial E(a)}] \tag{3.12}$$

The completion of the right-hand side of (3.10) is simply the symplectization of $\partial E(a)$. By [17, Lemma 1.2], the induced map is equal to $CH(\mathrm{sc})$. $\square$

With the above computations in hand, we now begin the proof of Lemma 2.13 from §2.2. We first verify that the moduli spaces of Definition 2.8 are, in our case of interest, given by a product of those provided by the cobordism maps of contact homology, described in Definition 3.5 and [17].

Lemma 3.14. *Let $(W_{\partial P}, \lambda_{\partial P})$ and $J_{\partial P}$ be as in Lemma 2.13. Pick basepoints $b(\gamma_i^+)$ and $b(\gamma_i^-)$ for all i, as in Setup 3.2(b). Then there is a natural bijection*

$$\mathcal{M}_I(\widehat{W}_{\partial P}; \widehat{J}_{\partial P}) \simeq \prod_{i \in I} \mathcal{M}_{\mathrm{c}}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-). \tag{3.13}$$

Proof. To prove this, we construct natural maps between the two moduli spaces that are inverses of each other.

($\to$) Let u be a point in $\mathcal{M}_I(\widehat{W}_{\partial P}; \widehat{J}_{\partial P})$. By Definition 2.8, u is a tuple $(u_i)_{i \in I}$ of reparametrization classes u_i of maps $\Sigma_i \to \widehat{W}_{\partial P}$ where $\Sigma_i := \mathbb{C}P^1 \setminus \{0, \infty\}$. Let $\widetilde{u}_i$ be a representative for each $i \in I$. Because the orbits γ_i^+ and γ_i^- are embedded, there is a unique choice of asymptotic markers $m_i(p_i^+)$ and $m_i(p_i^-)$ at the punctures p_i^+ and p_i^- of Σ_i which satisfy Definition 3.5(c). We thus define the map

$$\mathcal{M}_I(\widehat{W}_{\partial P}; \widehat{J}_{\partial P}) \to \prod_{i \in I} \mathcal{M}_{\mathrm{c}}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-) \qquad u \mapsto \prod_{i \in I} [\Sigma_i, j_{\mathbb{C}P^1}, \widetilde{u}_i, p_i^+, p_i^-, m_i] \tag{3.14}$$

The bracket $[-]$ within the product represents that we have taken the equivalence class of the curve up to the relation $\sim$ described in Definition 3.5(d)-(f). Any two choices of representative produce curves equivalent under $\sim$.

($\leftarrow$) In the other direction, let $\mathbf{u}$ be a point in the moduli space of Definition 3.5, which can be written as so.

$$\mathbf{u} = (\mathbf{u}_i)_{i\in I} \in \prod_{i\in I} \mathcal{M}_{\mathrm{c}}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-) \qquad \mathbf{u}_i = [S_i, j_i, \widetilde{u}_i, p_i^+, p_i^-, m_i]$$

As above, $\widetilde{u}_i$ is a holomorphic cylinder and u_i is the corresponding reparametrization class. Due to uniformization, for each $i \in I$ we can pick a biholomorphism $\varphi_i : (\Sigma_i, j_{\mathbb{C}P^1}) \simeq (S_i, j_i)$ preserving the marked points. Then we define the map

$$\prod_{i\in I} \mathcal{M}_{\mathrm{c}}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-) \to \mathcal{M}_I(\widehat{W}_{\partial P}; \widehat{J}_{\partial P}) \qquad \mathbf{u} \mapsto (\widetilde{u}_i \circ \varphi_i)_{i\in I} \tag{3.15}$$

By the right-hand side we mean the reprametrization class of $\widetilde{u}_i \circ \varphi_i$. Any two choices of φ_i evidently produce reparametrization equivalent tuples.

Verifying that the maps (3.14) and (3.15) are inverses of each other, modulo the equivalence relations involved, is straight forward and we leave it to the reader. $\square$

Next, we need a regularity and compactness result for the cobordism moduli space $\mathcal{M}_{\mathrm{c}}(\gamma_i^+; \gamma_i^-)$ under weaker hypotheses than Lemmas 2.12 and 2.12. The proof is very similar to the proofs of those Lemmas, so we will be terse in our exposition.

Lemma 3.15 (Compactness/transversality for $\mathcal{M}_{\mathrm{c}}$)**.** *Let $E(c)$ and $E(d)$ be irrational symplectic ellipsoids with parameters $c = (c_1, \dots, c_n)$ and $d = (d_1, \dots, d_n)$ satisfying*

$$c_i < d_i \quad \textit{for all } i \in \{1, \dots, n-1\} \qquad d_n < 2d_1 \quad \textit{and} \quad c_n < 2c_1 \tag{3.16}$$

Let $\jmath : E(c) \to E(d)$ be the inclusion and let $(W_\jmath, \lambda_\jmath)$ be the associated embedding.

Then there exists a comeager $\mathcal{J}^{\mathrm{reg}}(\jmath) \subset \mathcal{J}(W_\jmath)$ such that the space $\mathcal{M}_{\mathrm{c}}(\widehat{W}; \gamma_i^+, \gamma_i^-)$ of Definition 3.5 is Fredholm regular and SFT compact for any $J_\jmath \in \mathcal{J}^{\mathrm{reg}}(\jmath)$.

Proof. Let $Y_+ = \partial E(d)$ and $Y_- = \partial E(d)$, and consider the moduli spaces

$$\mathcal{M}_\partial(\widehat{Y}_+; \gamma_i^+, \gamma_j^+) \qquad \mathcal{M}_\partial(\widehat{Y}_-; \gamma_i^-, \gamma_j^-) \quad \text{and} \quad \mathcal{M}_c(\widehat{W}_\jmath; \gamma_i^+, \Gamma^-) \tag{3.17}$$

Here γ is any orbit in Y and Γ^- is any finite collection of orbits on $\partial_- W_\jmath \simeq \partial E(a)$. By energy monotonicity, only finitely many such moduli spaces can be non-empty. These spaces will be needed in the compactness argument.

(Regularity) Note that the orbits $\gamma_i^\pm$ are all embedded. Therefore, by the same discussion as in the proof of Lemma 2.10, every curve in $\mathcal{M}_c(\widehat{W}_\jmath; \gamma_i^+, \Gamma^-)$ is somewhere injective, and we can invoke [23, Theorems 7.1–7.2] to see that there is a comeager subset $\mathcal{J}^{\mathrm{reg}}(W_\jmath; \gamma_i^+, \Gamma^-) \subset \mathcal{J}(W_\jmath)$ such that the moduli space $\mathcal{M}_c(\widehat{W}_\jmath; \gamma_i^+, \Gamma^-)$ is transverse. An analogous argument (using [23, Theorem 8.1]) shows transversality for any of the $\mathcal{M}_\partial$ moduli spaces for $J_\pm$ in a comeager $\mathcal{J}^{\mathrm{reg}}(Y_\pm; \gamma_i^\pm, \gamma_j^\pm)$. Intersecting these (countably many) comeager sets yields the desired $\mathcal{J}^{\mathrm{reg}}(\jmath)$.

(Compactness) Pick a $J_\jmath \in \mathcal{J}^{\mathrm{reg}}(\jmath)$ as above and let (p_i, u^i) be a sequence in $\mathcal{M}_c(\widehat{W}_\jmath; \gamma_i^+, \gamma_i^-)$ converging to a building v. We use the notation of Review 2.11 for this building. As in Lemma 2.12, we show that v has no symplectization levels.

First, consider a positive symplectization level u_k^+. By action monotonicity, we know that $\mathcal{A}[\Gamma^\pm(u_k^+)] \leq \mathcal{A}(\gamma_i^+)$. It follows from (3.16) that there is a sequence $\{a_k\}_1^M$ such that

$$\Gamma^-(u_k^+) = \{\gamma_{a_k}^+\} \quad \text{and} \quad u_k^+ \in \mathcal{M}(\widehat{Y}_+; \gamma_{a_k}^+, \gamma_{a_k}^+) \quad \text{for all } k \in \{1, \dots, M\}.$$

Next, consider the cobordism level u^W. Since $\Gamma^+(u^W) = \Gamma^-(u_M^+) = \{\gamma_{a_M}^+\}$, we know by the above transversality argument above that the moduli space of u^W is Fredholm regular, with dimension given by

$$\dim(\mathcal{M}_c(\widehat{W}_\jmath; \gamma_{a_M}^+, \Gamma^-(u^W)) = |\gamma_{a_M}^+| - |\Gamma^-(u^W)| \geq 0$$

Using the grading formula (3.9), we see that if $\Gamma^-(u^W)$ contains an iterate γ_i^m for $m \geq 2$ or more than one orbit, then $|\Gamma^-(u^W)| \geq 4n - 2$. On the other hand, $|\gamma_{a_M}^+| \leq 4n - 4$. Therefore, $\Gamma^-[u^W] = \{\gamma_{b_0}^-\}$ with $1 \leq b_0 \leq n$. Finally, we can argue analogously to the positive symplectization case to show that there is a sequence $\{b_k\}_0^N$ such that

$$\Gamma^-(u_k^-) = \{\gamma_{b_k}^-\} \quad \text{and} \quad u_k^+ \in \mathcal{M}(\widehat{Y}_+; \gamma_{b_{k-1}}^+, \gamma_{b_k}^+) \quad \text{for all } k \in \{1, \dots, N\}.$$

We have thus shown that every level of v is Fredholm regular, and therefore

$$|\gamma_{a_k}^+| - |\gamma_{a_{k+1}}^+| \geq 1 \qquad |\gamma_{b_k}^-| - |\gamma_{b_{k+1}}^-| \geq 1 \qquad |\gamma_{a_M}^+| - |\gamma_{b_0}^-| \geq 0$$

The above equations contradict the fact that $|\gamma_i^+| - |\gamma_i^-| = 0$ unless $M = N = 0$, i.e. unless there are no symplectization levels. □

Finally, we proceed with the actual proof of Lemma 2.13.

Proof. (Lemma 2.13) Since $(W_{\partial P}, \lambda_{\partial P}) = (W_\iota, \lambda_\iota)$ (without considering the boundary inclusion maps), we know that the moduli spaces are equal.

$$\mathcal{M}_c(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-) = \mathcal{M}_c(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-)$$

This equality, along with Lemma 3.14, then implies that it suffices for us to show that $\mathcal{M}_c(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-)$ has an odd number of points for J in a comeager set.

To show this, we argue as so. By (3.9) in Example 3.12, we know that on any of the contact manifolds $\partial E(c \cdot a)$, $\partial E(a)$ or $\partial E(b)$, a non–simple orbit η satisfies $|\eta| \geq 4n-2$ and any set of 2 or more orbits Γ satisfies $|\Gamma| \geq 4n-2$. In particular, we have the following isomorphisms for $1 \leq i \leq n$.

$$CH_{2(n+i-2)}(\partial E(a)) \simeq \mathfrak{o}_{\gamma_i} \otimes \mathbb{Q} \qquad CH_{2(n+i-2)}(\partial E(c \cdot a)) \simeq \mathfrak{o}_{\mathrm{sc}(\gamma_i)} \otimes \mathbb{Q}$$

By applying Lemma 3.13, along with the definitions of $CH(W_\iota)$ and $CH(W_\jmath)$ in terms of the virtual moduli count, we find that

$$CH(\mathrm{sc}) = CH[W_\iota, \lambda_\iota] \circ CH[W_\jmath, \lambda_\jmath] = \frac{\#_\theta \mathcal{M}_{\mathrm{c}}(\widehat{W}_\jmath; \gamma_i^-, \mathrm{sc}(\gamma_i^+))}{|\operatorname{Aut}(\gamma_i^-, \mathrm{sc}(\gamma_i^+))} \circ \frac{\#_\theta \mathcal{M}_{\mathrm{c}}(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-)|}{|\operatorname{Aut}(\gamma_i^+, \gamma_i^-)|}$$

Now note that $\operatorname{Aut}(\gamma_i^-, \mathrm{sc}(\gamma_i^+))$ and $\operatorname{Aut}(\gamma_i^+, \gamma_i^-)$ are both trivial groups since the orbits γ_i^+, γ_i^- and $\mathrm{sc}(\gamma_i^+)$ are all embedded. Furthermore, by Lemma 3.15 there are comeager sets of almost complex structures such that the moduli spaces above are both Fredholm regular and compact. Thus, by applying Theorem 3.9(e) and choosing identifications of $\mathfrak{o}_{\gamma_i^+}, \mathfrak{o}_{\gamma_i^-}$ and $\mathfrak{o}_{\mathrm{sc}(\gamma_i^+)}$ with $\mathbb{Z}$ such that the map $\mathbb{Z} \to \mathbb{Z}$ induced by $CH(\mathrm{sc})$ is the identity, we acquire the following formula.

$$1 = \#\mathcal{M}_{\mathrm{c}}(\widehat{W}_\jmath; \gamma_i^-, \mathrm{sc}(\gamma_i^+)) \cdot \#\mathcal{M}_{\mathrm{c}}(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-) \in \mathbb{Z}$$

Here # denotes taking a $\mathbb{Z}$-valued, signed point count. We can conclude that

$$\#\mathcal{M}_{\mathrm{c}}(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-) = \pm 1 \quad \text{and} \quad |\mathcal{M}_{\mathrm{c}}(\widehat{W}_\iota; \gamma_i^+, \gamma_i^-)| \equiv 1 \mod 2$$

This proves the point count for the cobordism moduli-spaces, and ends the proof. □

4 Spaces of symplectic embeddings

In this section, we discuss some basic results about the Fréchet manifold of symplectic embeddings $\operatorname{SympEmb}(U, V)$ between symplectic manifolds with boundary. In §4.1, we construct the Fréchet manifold structure on $\operatorname{SympEmb}(U, V)$. In §4.2, we discuss the relationship between the bordism groups and homology groups of a Fréchet manifold. Last, we prove a version of the Weinstein neighborhood with boundary as Proposotion 4.13 in §4.3.

4.1 Fréchet manifold structure

Let (U, ω_U) and (V, ω_V) be n–dimensional compact symplectic manifolds with nonempty contact boundaries. We now give a proof of the folklore result that the space of symplectic embeddings from U to V is a Fréchet manifold.

Proposition 4.1. *The space* $\mathrm{SympEmb}(U,V)$ *of symplectic embeddings* $\varphi : U \to \mathrm{int}(V)$ *with the* C^∞ *compact open topology is a metrizable Fréchet manifold.*

Proof. Let $(U\times V, \omega_{U\times V})$, with $\omega_{U\times V} = \pi_U^*\omega_U - \pi_V^*\omega_V$, denote the product symplectic manifold with corners. Given a symplectic embedding $\varphi : U \to \mathrm{int}(V)$, we may associate the graph $\Gamma(\varphi) \subset U \times V$ given by:

$$\Gamma(\varphi) := \{(u, \varphi(u)) \in U \times V\}.$$

The graph is a Lagrangian submanifold with boundary transverse to the characteristic foliation $T(\partial U)^\omega$ on the contact hypersurface $\partial U \times \mathrm{int}(V)$. By the Weinstein neighborhood theorem with boundary, Proposition 4.13, there is a neighborhood A of U, a neighborhood B of $\Gamma(\varphi)$ and a symplectomorphism $\psi : A \simeq B$ with $\psi|_U : U \to \Gamma(\varphi)$ given by $u \mapsto (u, \varphi(u))$ and $\psi^*\omega_{U\times V} = \omega_{\mathrm{std}}$.

Let $\mathcal{A}(\varphi,\psi) \subset \ker(d : \Omega^1(L) \to \Omega^2(L))$ and $\mathcal{B}(\varphi,\psi) \subset \mathrm{SympEmb}(U,V)$ denote the open subsets given by:

$$\mathcal{A}(\varphi,\psi) := \{\alpha \in \Omega^1(L) | d\alpha = 0 \text{ and } \mathrm{Im}(\alpha) \subset A\}$$

$$\mathcal{B}(\varphi,\psi) := \{\phi \in \mathrm{SympEmb}(U,V) | \, \mathrm{Im}(\varphi) \subset B\}.$$

Then we have maps $\Phi : \mathcal{A}(\varphi,\psi) \to \mathcal{B}(\varphi,\psi)$ and $\Psi : \mathcal{A}(\varphi,\psi) \to \mathcal{B}(\varphi,\psi)$ given by:

$$\alpha \mapsto \Phi[\alpha] := (\pi_V \circ \psi \circ \alpha) \circ (\pi_U \circ \psi \circ \alpha)^{-1}$$

$$\phi \mapsto \Psi[\phi] := (\psi^{-1} \circ (\mathrm{Id} \times \phi)) \circ (\pi_L \circ \psi^{-1} \circ (\mathrm{Id} \times \phi))^{-1}.$$

It is a tedious but straight forward calculation to check that $\Phi \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Phi = \mathrm{Id}$. The fact that Φ and Ψ are continuous in the C^∞ compact open topologies on the domain and images follows from the fact that function composition defines a continuous map $C^\infty(M,N) \times C^\infty(N,O) \to C^\infty(M,O)$ for any compact manifolds M, N and O (in fact, smooth; see [19, Theorem 42.13]).

Since $C^\infty(U,V)$ is metrizable under the compact open C^∞–topology (see [19, Corollary 41.12]), the subspace $\mathrm{SympEmb}(U,V)$ is also metrizable. □

Lemma 4.2. *Let* L *be a compact manifold with boundary and let* $\sigma : L \to T^*L$ *be a section. Then* $\sigma(L)$ *is Lagrangian if and only if* σ *is closed.*

Proof. The same as the closed case, see [14, Proposition 3.4.2]. □

4.2 Bordism groups of Fréchet manifolds

We now discuss (unoriented) bordism groups and their structure in the case of Fréchet maifolds. We begin by defining the relevant notions of (continuous and smooth) bordism.

Definition 4.3 (Bordisms)**.** Let X be a topological space and $f : Z \to X$ be a map from a closed manifold. We say that the pair (Z, f) is *null–bordant* if there exists a pair (Y, g) of a compact manifold with boundary Y and a continuous map $g : Y \to X$ such that $\partial Y = Z$ and $g|_{\partial Y} = f$. Given a pair of manifold/map pairs (Z_i, f_i) for $i \in \{0, 1\}$, we say that (Z_0, f_0) and (Z_1, f_1) are *bordant* if $(Z_0 \sqcup Z_1, f_0 \sqcup f_1)$ is null–bordant.

Definition 4.4 (Smooth bordism)**.** Let X be a Fréchet manifold and $f : Z \to X$ be a smooth map from a smooth closed manifold. Then (Z, f) is *smoothly null–bordant* if it is null–bordant via a pair (Y, g) where $g : Y \to X$ be a smooth map of Banach manifolds with boundary. Similarly, a pair (Z_i, f_i) for $i\{0, 1\}$ is *smoothly bordant* if $(Z_0 \sqcup Z_1, f_0 \sqcup Z_1)$ is smoothly null-bordant.

The above notions come with accompanying versions of the bordism group.

Definition 4.5 (Bordism Group Of X)**.** The *n–th bordism group* $\Omega_n(X; \mathbb{Z}/2)$ of a topological space X is group generated by equivalence classes $[Z, f]$ of pairs (Z, f), where Z is a closed n–dimensional manifold and $f : Z \to X$ is a continuous map, modulo the relation that $(Z_0, f_0) \sim (Z_1, f_1)$ if the pair is bordant. Addition is defined by disjoint union:

$$[Z_0, f_0] + [Z_1, f_1] := [Z_0 \sqcup Z_1, f_0 \sqcup f_1].$$

Definition 4.6 (Smooth bordism group of X)**.** The *n–th smooth bordism group* $\Omega_n^\infty(X; \mathbb{Z}/2)$ of a Fréchet manifold X is group generated by equivalence classes $[Z, f]$ of pairs (Z, f), where Z is a closed n–dimensional manifold and $f : Z \to X$ is a smooth map, modulo the relation that $(Z_0, f_0) \sim (Z_1, f_1)$ if the pair is smoothly bordant. Addition in the group $\Omega_*^\infty(X; \mathbb{Z}/2)$ is defined by disjoint union as before.

Lemma 4.7. *The natural map $\Omega_*^\infty(X; \mathbb{Z}_2) \to \Omega_*(X; \mathbb{Z}_2)$ is an isomorphism.*

Proof. The argument uses smooth approximation and is identical to the case where X is a finite dimensional smooth manifold, which can be found in [4, Section I.9]. □

Given the above terminology, we can now prove the main result of this subsection, Proposition 4.8. It provides a class of submanifolds for which being null–bordant and being null–homologous are equivalent.

Proposition 4.8. *Let X be a metrizable Fréchet manifold, and let $f : Z \to X$ be a smooth map from a closed manifold Z with Stieffel–Whitney class $w(Z) = 1 \in H^*(Z; \mathbb{Z}/2)$. Then $f_*[Z] = 0 \in H_*(X; \mathbb{Z}/2)$ if and only $[Z, f] = 0 \in \Omega_*^\infty(X; \mathbb{Z}_2)$.*

Proof. Proposition 4.8 will follow immediately from the following results. First, by Lemma 4.7, it suffices to show $f_*[Z] = 0 \in H_*(X; \mathbb{Z}/2)$ if and only $[Z, f] = 0 \in \Omega_*(X; \mathbb{Z}_2)$. By Proposition 4.9, we can replace X with a CW complex. Lemma 4.10 proves the result in this context. □

Proposition 4.9 ([18, Theorem 14]). *A metrizable Fréchet manifold is homotopy equivalent to a CW complex.*

Lemma 4.10. *Let X homotopy equivalent to a CW complex, and let $f : Z \to X$ be a continuous map from a closed manifold Z with Stieffel–Whitney class $w(Z) = 1 \in H^*(Z; \mathbb{Z}/2)$. Then $f_*[Z] = 0 \in H_*(X; \mathbb{Z}/2)$ if and only $[Z, f] = 0 \in \Omega_*(X; \mathbb{Z}_2)$.*

Remark 4.11. Crucially, we make no finiteness assumptions on the CW structure.

Proof. ($\Rightarrow$) Suppose that $f_*[Z] = 0 \in H_2(Z; \mathbb{Z}/2)$. Pick a homotopy equivalence $\varphi : X \simeq X'$ with a CW complex X'. Such an equivalence induces an isomorphism of unoriented bordism groups $\Omega_*(X; \mathbb{Z}_2) \simeq \Omega_*(X'; \mathbb{Z}_2)$, so it suffices to show that the pair $(Z, \varphi \circ f)$ is null–bordant, or equivalently to assume that X is a CW complex to begin with.

So assume that X is a CW complex. By Lemma 4.12, we can find a finite sub–complex $A \subset X$ such that $f(Z) \subset A$ and $f_*[Z] = 0 \in H_*(A; \mathbb{Z}/2)$. By Theorem 17.2 of [4], $[Z, f] = 0 \in \Omega_*(A; \mathbb{Z}_2)$ if and only if the Stieffel–Whitney numbers $\mathrm{sw}_{\alpha,I}[Z, f]$ are identically 0. Recall that the Stieffel–Whitney number $\mathrm{sw}_{\alpha,I}[Z, f]$ associated to $[Z, f]$, a cohomology class $\alpha \in H_k(A; \mathbb{Z}_2)$ and a partition $I = (i_1, \ldots, i_k)$ of $\dim(Z) - k$ is defined to be:

$$\mathrm{sw}_{\alpha,I}[Z, f] = \langle w_{i_1}(Z) w_{i_2}(Z) \ldots w_{i_k}(Z) f^*\alpha, [Z] \rangle \in \mathbb{Z}_2.$$

Here $w_j(Z) \in H^j(Z; \mathbb{Z}_2)$ denotes the j–th Stieffel–Whitney class of Z. By assumption, $w(Z) = 1$ and so $w_j(Z) = 0$ for all $j \neq 0$. In particular, the only possible nonzero Stieffel–Whitney numbers have $I = (0)$. But we see that:

$$\mathrm{sw}_{\alpha,(0)}[Z, f] = \langle f^*\alpha, [Z] \rangle = \langle \alpha, f_*[Z] \rangle = 0.$$

Therefore, $\mathrm{sw}_{\alpha,I}[Z, f] \equiv 0$ and $[Z, f]$ must be null–bordant.

($\Leftarrow$) This direction is completely obvious, since the map $\Omega_*(X) \to H_*(X; \mathbb{Z}/2)$ given by $[Z, f] \mapsto f_*[Z]$ is well defined. $\square$

Lemma 4.12. *Let X be a CW complex, and let $f : Z \to X$ be a map from a closed manifold Z with $f_*[Z] = 0 \in H_*(X; \mathbb{Z}/2)$. Then there exists a finite sub–complex $A \subset X$ with $f(Z) \subset A$ and $f_*[Z] = 0 \in H_*(A; \mathbb{Z}/2)$.*

Proof. A very convenient tool for this is the stratifold homology theory of [10], which we now review briefly.

Given a space M, the n–th stratifold group $sH_n(M; \mathbb{Z}/2)$ with $\mathbb{Z}/2$–coefficients (see Proposition 4.4 in [10]) is generated by equivalence classes of pairs (S, g) of a compact, regular stratifold S and a continuous map $g : S \to M$. Two pairs (S_i, g_i) for $i \in \{0, 1\}$ are equivalent if they are bordant by a c–stratifold, i.e. if there is a pair (T, h) of a compact, regular c–stratifold and a continuous map $g : T \to M$ such that $(\partial T, h|_{\partial T}) = (S_0 \sqcup S_1, g_0 \sqcup g_1)$ (see Chapter 3 and Section 4.4 of [10]). Given a map $\varphi : M \to N$ of spaces, the pushforward map $\varphi_* : sH(M;\mathbb{Z}_2) \to sH(M;\mathbb{Z}_2)$ on stratifold homology is given (on generators) by $[S, g] \mapsto [S, \varphi \circ g] = \varphi_*[\Sigma, g]$.

Stratifold homology satisfies the Eilenberg–Steenrod axioms (see Chapter 20 of [10]), and thus if M is a CW complex then there is a natural isomorphism $sH_*(M;\mathbb{Z}_2) \simeq H_*(M;\mathbb{Z}_2)$. If M is a manifold of dimension n, the fundamental class $[M] \in sH_n(M;\mathbb{Z}_2)$ is given by the tautological equivalence class $[M] = [M, \mathrm{Id}]$.

The proof of the lemma is simple with the above machinery in place. Since $f_*[Z] = 0$, the pair (Z, f) must be null–bordant via some compact c–stratifold (Y, g). Since Y and its image $g(Y)$ are both compact, we can choose a sub–complex $A \subset X$ such that $g(T) \subset A \subset X$. Then the pair (Z, f) are null–bordant by (Y, g) in A as well, so that $[Z, f] = 0 \in sH_*(A;\mathbb{Z}_2)$ and thus $f_*[Z] = 0 \in H_*(A;\mathbb{Z}_2)$ via the isomorphism $sH_*(A;\mathbb{Z}_2) \simeq H_*(A;\mathbb{Z}_2)$. $\square$

4.3 Weinstein neighborhood theorem with boundary

In this section, we prove the analogue of the Weinstein neighborhood theorem for a Lagrangian L with boundary, within a symplectic manifold X with boundary. We could find no reference for this fact in the literature.

Proposition 4.13 (Weinstein neighborhood theorem with boundary)**.** *Let (X, ω) be a symplectic manifold with boundary ∂X and let $L \subset X$ be a properly embedded, Lagrangain submanifold with boundary $\partial L \subset \partial X$ transverse to $T(\partial X)^\omega$.*

Then there exists a neighborhood $U \subset T^*L$ of L (as the zero section), a neighborhood $V \subset X$ of L and a diffeomorphism $f : U \simeq V$ such that $\varphi^*(\omega|_V) = \omega_{\mathrm{std}}|_U$.

Proof. The proof has two steps. First, we construct neighborhoods $U \subset T^*L$ and $V \subset X$ of L, and a diffeomorphism $\varphi : U \simeq V$ such that:

$$\varphi|_L = \mathrm{Id} \qquad \varphi^*(\omega|_V)|_L = \omega_{\mathrm{std}}|_L \qquad T(\partial U)^{\omega_{\mathrm{std}}} = T(\partial U)^{\varphi^*\omega} \tag{4.1}$$

Here $T(\partial U)^{\omega_{\mathrm{std}}} \subset T(\partial U)$ is the symplectic perpendicular to $T(\partial U)$ with respect to ω_{std} (and similarly for $T(\partial U)^{\varphi^*\omega}$. Second, we apply Lemma 4.14 and a Moser type argument to conclude the result.

(Step 1) Let J be a compatible almost complex structure on X and g be the induced metric on L. Recall that the normal bundle $\nu_g L$ with respect to g is a bundle over L with Lagrangian fiber, and that $J : TL \to \nu_g L$ gives a natural isomorphism. Let $\Phi^g : T^*L \to TL$ denote the bundle isomorphism induced by the metric g and let $\exp^g$ denote the exponential map with respect to g.

Since L is compact, we can choose a tubular neighborhood U' of νL such that $\exp^g : U \to X$ is a diffeomorphism onto its image V. We then let $U := [J \circ \Phi^g]^{-1}(U') \subset T^*L$ and let:

$$\phi^g : U \simeq V \qquad (x, v) \mapsto \exp^g_x(J \circ \Phi_g(v)).$$

Note that $\phi^g|_L = \mathrm{Id}$ and $[\phi^g]^*\omega|_L = \omega_{\mathrm{std}}|_L$ by the same calculations as in [14, Theorem 3.4.13]. We now must modify U, V and ϕ^g to satisfy the last condition of (4.1).

To this end, we apply Lemma 4.15. Taking $\kappa_0 = T(\partial U)^{\omega_{\mathrm{std}}}$ and $\kappa_1 = T(\partial U)^{[\phi^g]^*\omega}$, we acquire a neighborhood $N \subset \partial(T^*L)$ of ∂L and a family of embeddings $\psi : N \times I \to \partial(T^*L)$ with the following four properties.

$$\psi_t|_{\partial L} = \mathrm{Id} \qquad d(\psi_t)_u = \mathrm{Id} \text{ for } u \in \partial L \qquad \psi_0 = \mathrm{Id}$$

$$[\psi_1]_*(T(\partial U)^{\omega_{\mathrm{std}}}) = T(\partial U)^{[\phi^g]^*\omega}.$$

Note here that we are using the fact that $T(\partial U)^{\omega_{\mathrm{std}}}|_L = T(\partial U)^{[\phi^g]^*\omega}|_L$ already by the construction of ϕ^g. By shrinking N and U, we can simply assume that $N = \partial U$. Let $\mathrm{tc} : [0,1) \times \partial U \simeq T \subset U$ be tubular neighborhood coordinates near boundary. By choosing the tubular neighborhood coordinates $\mathrm{tc} : [0,1) \times \partial U \simeq T$ appropriately, we can also assume that $\mathrm{tc}([0,1) \times \partial L) = L \cap T$. We define a map $\Phi : U \to T^*L$ by:

$$\Phi(u) = \begin{cases} (s, \psi_{1-s}(v)) & \text{if } u = (s, v) \in [0,1) \times \partial U \text{ via tc} \\ u & \text{otherwise.} \end{cases}$$

The map Φ has the following properties which are analogous to those of ψ_s.

$$\Phi|_L = \mathrm{Id} \quad d(\Phi)_u = \mathrm{Id} \text{ for } u \in L \qquad \Phi_*(T(\partial L)^{\omega_{\mathrm{std}}}) = T(\partial L)^{[\phi^g]^*\omega}$$

Also note that Φ is smooth since ψ_t is constant for t near 0 and 1. We thus define f as the composition $\varphi = \phi^g \circ \Phi$. It is immediate that f has the properties in (4.1).

(Step 2) We closely follows the Moser type argument of [14, Lemma 3.2.1]. By shrinking U, we may assume that it is an open disk bundle. Let $\omega_t = (1-t)\omega_{\mathrm{std}} + tf^*\omega$ and $\tau = \frac{d}{dt}(\omega_t) = f^*\omega - \omega_{\mathrm{std}}$. Let $\kappa = T(\partial U)^{\omega_t}$ (by the previous work, it does not depend on t). Note that τ satisfies all of the assumptions of Lemma 4.14(4.3). We prove that κ is invariant under the scaling map $\phi_t(x, u) = (x, tu)$ in Lemma 4.16. We can thus find a σ satisfying the properties listed in (4.2).

Let Z_t be the unique family of vector fields satisfying $\sigma = \iota(Z_t)\omega_t$. Due to the properties of σ, Z_t satisfies the following properties for each t.

$$Z_t|_L = 0 \qquad Z_t|_{\partial U} \in T(\partial U) \text{ for all } t$$

The first property is immediate, while the latter is a consequence of the following.

$$\omega_t(Z_t, \cdot)|_\kappa = \sigma|_\kappa = 0 \implies Z_t \in (\kappa)^{\omega_t} = T(\partial U)$$

These two properties imply that Z_t generates a map $\Psi : U' \times [0,1] \to U$ for some smaller tubular neighborhood $U' \subset U$ with the property that $\Psi_t|_L = \mathrm{Id}$ and $\Psi_t^*\omega_t = \omega_0$ (see [14, §3.2], as the reasoning is identical to the closed case). In particular, we get a map $\Psi_1 : U' \to U$ with $\Psi_1|_L = \mathrm{Id}$ and $\Psi_1^* f^*\omega$. By shrinking U, taking $\varphi = f \circ \Psi_1$ and taking $V = \varphi(U)$, we at last acquire the desired result. □

The remainder of this section is devoted to proving the various lemmas that we used in the proof above.

Lemma 4.14 (Fiber integration with boundary)**.** *Let X be a compact manifold with boundary, $\pi : E \to X$ be a rank k vector bundle with metric and $\pi : U \to X$ be the (open) disk bundle of E with closure $\overline{U}$. Let $\kappa \subset T(\partial U)$ be a distribution on ∂U such that $d\phi_t(\kappa_u) = \kappa_{\phi_t(u)}$ for all $u \in U$, where $\phi : U \times I \to U$ denote the family of smooth maps given by $\phi_t(x, u) := (x, tu)$.*

Finally, suppose that $\tau \in \Omega^{k+1}(\overline{U})$ is a $(k+1)$–form such that:

$$d\tau = 0 \qquad \tau|_X = 0 \qquad (\iota_{\partial X}^*\tau)|_\kappa = 0. \tag{4.2}$$

Then there exists a k–form $\sigma \in \Omega^k(\overline{U})$ with the following properties:

$$d\sigma = \tau \qquad \sigma|_X = 0 \qquad (\iota_{\partial X}^*\sigma)|_\kappa = 0. \tag{4.3}$$

Proof. We use integration over the fiber, as in [14, p. 109]. Note that the maps $\phi_t : U \to \phi_t(U) \subset U$ are diffeomorphisms for each $t > 0$, $\phi_0 = \pi$, $\phi_1 = \mathrm{Id}$ and $\phi_t|_X = \mathrm{Id}$. Therefore we have:

$$\phi_0^*\tau = 0 \qquad \phi_1^*\tau = \tau$$

We may define a vector field Z_t for all $t > 0$ and a k-form σ_t for all $t \geq 0$ as so.

$$Z_t := (\frac{d}{dt}\phi_t) \circ \phi_t^{-1} \text{ for } t > 0 \qquad \sigma_t := \phi_t^*(\iota(Z_t)\tau) \text{ for } t \geq 0$$

Although Z_t is singular at $t = 0$, as in [14] one can verify in local coordinates that σ_t is smooth at $t = 0$. Since $Z_t|_X = 0$, the k–form σ_t satisfies $\sigma_t|_X = 0$. Furthermore, for any vector field $K \in \Gamma(\kappa)$ on ∂X which is parallel to κ, we have $\iota(K)\sigma_t = \phi_t^*(\iota(Z_t)\iota(d\phi_t(K))\tau) = 0$ on the boundary, so that $\iota_{\partial X}^*(\sigma_t)|_\kappa = 0$. Finally, σ_t satisfies the following equation:

$$\tau = \phi_1^*\tau - \phi_0^*\tau = \int_0^1 \frac{d}{dt}(\phi_t^*\tau)dt = \int_0^1 \phi_t^*(\mathcal{L}_{X_t}\tau)dt$$

$$= \int_0^1 d(\phi_t^*(\iota(X_t)\tau))dt = \int_0^1 d\sigma_t dt = d(\int_0^1 \sigma_t dt).$$

Therefore, if we define $\sigma := \int_0^1 \sigma_t dt$, it is simple to verify the desired properties using the corresponding properties for σ_t. □

Lemma 4.15. *Let U be a manifold and $L \subset U$ be a closed submanifold. Let κ_0, κ_1 be rank 1 orientable distributions in TU such that $\kappa_i|_L \cap TL = \{0\}$ and $\kappa_0|_L = \kappa_1|_L$.*

Then there exists a neighborhood $U' \subset U$ of L and a family of smooth embeddings $\psi : U's \times I \to U$ with the following four properties.

$$\psi_t|_{\partial L} = \mathrm{Id} \qquad d(\psi_t)_u = \mathrm{Id} \ \textit{for } u \in L \qquad \psi_0 = \mathrm{Id} \qquad [\psi_1]_*(\kappa_0) = \kappa_1.$$

Furthermore, we can take ψ_t to be t–independent for t near 0 and 1.

Proof. Since κ_0 and κ_1 are orientable, we can pick nonvanishing sections Z_0 and Z_1 We may assume that $Z_0 = Z_1$ along L. We let Z_t denote the family of vector fields $Z_t := (1-t)Z_0 + tZ_1$. Since $Z_0 = Z_1$ along L, we can pick a neighborhood N of L such that Z_t is nowhere vanishing for all t. We also select a sub-manifold $\Sigma \subset N$ with $\dim(\Sigma) = \dim(U) - 1$ and such that:

$$\Sigma \pitchfork Z_t \text{ for all } t \quad \text{and} \quad L \subset \Sigma.$$

We can find such a Σ by, say, picking a metric and using the exponential map on a neighborhood of L in the sub–bundle $\nu L \cap \kappa_0^\perp$ of TL. By shrinking Σ and scaling Z_t to λZ_t, $0 < \lambda < 1$, we can define a smooth family of embeddings:

$$\Psi : (-1,1)_s \times \Sigma \times [0,1]_t \to N \qquad \Psi_t(s,x) = \exp[Z_t]_s(x).$$

Here $\exp[Z_t]$ denotes the flow generated by Z_t. We let $\psi_t = \Psi_t \circ \Psi_0^{-1}$. To see the properties of (4.1), note that $\Psi_t(0,l) = l$ for all $l \in L$ and $d(\Psi_t)_{0,l}(s,u) = sZ_t + u$. This implies the first two properties. The third is trivial, while the fourth is immediate from $[\Phi_t]_*(\partial_t) = Z_t$. We can make ψ_t constant near 0 and 1 by simply reparametrizing with respect to t. □

Lemma 4.16. *Let L be a manifold with boundary and let (T^*L, ω) be the cotangent bundle with the standard symplectic form. Let $\kappa = T(\partial T^*L)^\omega$ denote the characteristic foliation of the boundary ∂T^*L and let $\phi : T^*L \times (0,1] \to T^*L$ denote the family of maps $\phi_t(x,v) = (x,tv)$. Then $[\phi_t]_*(\kappa) = \kappa$.*

Proof. By passing to a chart, we may assume that $L \subset \mathbb{R}^+_{x_1} \times \mathbb{R}^{n-1}_x$ and $T^*L \subset \mathbb{R}^+_{x_1} \times \mathbb{R}^{n-1}_x \times \mathbb{R}^n_p$. Then κ is simply given on $\partial T^*L \subset \{0\} \times \times \mathbb{R}^{n-1}_x \times \mathbb{R}^n_p$ by:

$$\kappa = \mathrm{span}(\partial_{p_1}) = \mathrm{span}(\partial_{x_1})^\omega \subset T(\partial T^*L).$$

Under the scaling map, we have $[\phi_t]_*(\partial_{p_1}) = t{\cdot}\partial_{p_1}$. This implies that $[\phi_t]_*(\kappa) = \kappa$. □

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