Hofer's Proof of the Weinstein Conjecture for Overtwisted Contact Structures

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1 Introduction

In this paper, we will be discussing a question about contact manifolds, so let's start by defining them.

Definition 1. A **contact manifold** Y is a $2n+1$-manifold equipped with a hyperplane distribution $\xi \subset TY$, called the **contact structure**, which is locally given as the kernel of a 1-form λ on Y such that $\lambda \wedge (d\lambda)^n$ is a volume form. Such a λ is called a **contact form**. The **Reeb vector-field** R associated to λ is the unique vector-field satisfying $\lambda(R) = 1$ and $i_R d\lambda = 0$. In this paper we will always work with a fixed global contact structure λ on our Y.

Such manifolds naturally live on the boundary of a symplectic manifold X in the sense that $\mathbb{R} \times Y$ can be given the exact symplectic form $\omega_f = d(f\lambda)$ for any $f : \mathbb{R} \to \mathbb{R}^+$ with $f' > 0$. In what follows, we will use ω to denote ω_f with $f(t) = e^t$. Then we can interpret Y as either the boundary of $\mathbb{R}^+ \times Y$ or $\mathbb{R}^- \times Y$.

The Reeb vector-field R emerges naturally in this picture as the restriction to $Y \times 0$ of the Hamiltonian vector field associated to f (as a function on $\mathbb{R} \times Y$) with respect the symplectic form ω_f to any choice of f above. Indeed, by direct computation, we see that R can be characterized as the unique vector-field such that $\lambda(R) = 1$ and $i_R d\lambda = 0$. For then on $Y \times 0$ we have:

$$i_R\omega_f = i_R(f'ds \wedge \lambda + fd\lambda) = f'ds = df$$

It is natural to conjecture about the structure of the orbits of this vector-field, in particular the existence or non-existence of periodic orbits. In particular, in [6] Weinstein made his eponymous conjecture.

Conjecture 1. *[6] (Weinstein) Let Y be a closed contact manifold with Reeb vector field X and $H^1(Y;\mathbb{R}) = 0$. Then the Reeb vector-field R has a periodic orbit.*

Historically, the search for such periodic orbits was actually a principle motivating factor in the definition of contact manifolds. In [6], Weinstein defines a precursor concept called a "hyper-surface of contact type" (which is precisely a contact manifold embedded in a symplectic manifold such that a neighborhood of the embedding is symplectomorphic to the symplectization) in an effort to formulate a sufficient condition for the level set of a Hamiltonian to contain a periodic orbit of the Hamiltonian vector field. He was motivated by a the following result for star-shaped surfaces by Rabinowitz in [4], and then confirmed in the case of contact hyper surfaces by Viterbo by the subsequent (more general) result in [5].

Theorem 2. *[4] A star-shaped constant energy surface in $\mathbb{R}^{2n}$ carries a periodic orbit.*

Theorem 3. *[5] Every contact type hypersurface in $(\mathbb{R}^{2n}, \omega)$ contains a periodic orbit.*

The general case of any closed contact manifold resisted treatment into the 90's. However, in 1993 Hofer managed to prove the Weinstein conjecture on closed 3-manifolds for a class of contact structures called over-twisted, defined as so.

Definition 4. A contact manifold (Y, ξ) is **over-twisted** if if there exists an embedded disc $i : D \hookrightarrow Y$ such that $T(\partial D)_p \subset \xi_p$ for all $p \in \partial D$ and TD_p is tranverse to ξ_p for any $p \in \text{int}(D)$.

Hofer's proof, written in [3], utilized J-holomorphic curve based techniques, which are extremely prominent in modern symplectic topology.

The purpose of this paper is to give a short account of Hofer's J-curve based proofs of the Weinstein conjecture in 3-d for over-twisted contact 3-manifolds. We will discuss some background for the proof in Section 3. In Section 4, we will state the main ingredients to the proof and give the main argument. In Section 5 we will give a few detailed arguments for the main ingredients of the proof, including an example of the bubbling analysis that is the mainstay of Hofer's paper. In Sectionn 6 we very briefly outline an alternate version of the same argument that reveals the existence of a Reeb orbit in a more direct manner.

2 Hofer's Theorem

In [3], Hofer in fact proves a slightly more general version of the Weinstein conjecture, disposing of the assumption that $H^1(Y;\mathbb{R})$ vanishes.

Theorem 5. *([3], Thm 5) Let λ be an over twisted contact form on a closed 3-manifold Y. Then the Reeb vector-field has a periodic orbit.*

The proof of this result is based on the study of J-holomorphic discs with boundary on a distinguished sub-surface. We now briefly review the definitions in J-curve theory and discuss the particular formulation of the J-disc PDE that Hofer uses.

3 Background And Setup

Definition 6. Given a vector bundle $\xi \to X$, a $J \in \mathrm{End}(\xi)$ is called a **complex structure** if $J^2 = -Id$.

Definition 7. An **almost complex structure** $J \in \mathrm{End}(TX)$ on a $2n$-manifold X is a complex structure on the tangent bundle TX. We call J integrable at $p \in X$ if there is a neighborhood U of p and a diffeomorphism $\phi : U \to V \subset \mathbb{C}^n$ so that $J = \phi^* J_0$, where J_0 is the almost complex structure induced by multiplication by i.

On a symplectic manifold, there are special classes of J that are particularly well-behaved. Hofer utilizes one such class, called compatible.

Definition 8. Let (X, ω) be a symplectic manifold. An almost complex structure J is called **compatible** with ω if $\omega(v, Jv) > 0$ for all $v \neq 0$ and $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$.

Definition 9. Let Σ be a Riemann surface with boundary $\partial\Sigma$ and almost complex structure j. A map $u : (\Sigma, \partial\Sigma, j) \to (X, F, J)$ is ***J*-holomorphic** if $J \circ du = du \circ j$. The above notation also means that $u(\partial\Sigma) \subset F$.

In his argument, Hofer examines J-curves on $X = \mathbb{R} \times Y$, utilizing J's of a special form. In particular, consider a complex structure J on the sub-bundle $\pi^*\xi$ of TX where $\pi : X \to Y$ is the projection (we will simply denote this by ξ from now on). We can define an almost complex structure $\tilde{J}$ on TX using J as so. Let $\pi_\xi : TX \to \xi$ denote the fiber-wise projection to ξ along ∂_t and R and let π_R denote the projection to the span of R along ∂_t and ξ. Then:

$$\tilde{J} := R \otimes ds - \partial_s \otimes \lambda + J\pi_\xi$$

Note that $\tilde{J}$ is compatible with ω if and only if $d\lambda(v, Jv) > 0$ for all $v \in \xi - 0$ and $d\lambda(J\cdot, J\cdot) = d\lambda(\cdot, \cdot)$. Furthermore, we can rewrite the J-curve equations for a map $\tilde{u} : \Sigma \to X$ split as $\tilde{u} = (a, u)$ with $a : \Sigma \to \mathbb{R}$ and $u : \Sigma \to Y$. We then see that:

$$R \otimes da - \partial_s \otimes \lambda \circ du + J\pi_\xi du = (R \otimes ds - \partial_s \otimes \lambda + J\pi_\xi)(\partial_s \otimes da + du)$$

$$= \tilde{J} \circ d\tilde{u} = d\tilde{u} \circ j = \partial_s \otimes da \circ j + du \circ j$$

This yields a system of equations for a and u:

$$u^*\lambda \circ j = da \qquad \pi_\xi u + J\pi_\xi u \qquad R \otimes da = \pi_R du \circ j$$

Since $\pi_R = R \otimes \lambda$, the last equation is an easy conseuquence of the first. Thus J-curves on X are precisely the (a, u) satisfying:

$$u^*\lambda \circ j = da \qquad \pi_\xi u + J\pi_\xi u \tag{1}$$

Separate from J-curve theory, we must also discuss some properties of embedded discs in Y.

Definition 10. Let $F \to Y$ be an embedded disc. We call the subset $\Sigma \subset F$ of points p in D such that $\xi_p = TF_p$ the set of **singular points**.

A C^0 generic smooth embedded disc $F \subset Y$ has (by dimensionality arguments and transversality) only finitely many points p where $\xi_p = TF_p$. On such a disc, $\xi \cap TF$ is a 1-plane distribution on $F - \Sigma$, and as such is integrable, giving a foliation of $F - \Sigma$ into curves.

We can characterize the foliations arising in this way using contact Hamiltonian vector-fields.

Definition 11. Let $H : Y \to \mathbb{R}$ be a smooth map. The **contact Hamiltonian vector-field** associated to H is $X_H = HR + \hat{X}_H$ with $\lambda(\hat{X}_H) = 0$ and $i_{\hat{X}_H} d\lambda = (i_X dH)\lambda - dH$.

If $H : Y \to \mathbb{R}$ is a smooth function with 0 a regular value and $F \subset H^{-1}(0)$, then $X_H|_F = \hat{X}_H$ and $\xi \cap TF = \text{span}(X_H)$ on $F - p$, with the singular points Σ being precisely the points where V_H vanishes on F. At these points, the Hessian $(dX_H)_p$ is well-defined as an endomorphism of TF_p, and we can characterize the integral curves of X_H near such p (equivalently, leaves of the foliation induced by ξ) in terms of $(dX_h)_p$.

Definition 12. We say that a singular point $p \in \Sigma$ is **non-degenerate** if $(dX_H)_p$ is an isomorphism, and that it is **nicely elliptic** if $(dX_H)_p$ has positive determinant and real eigenvalues.

One can check in local coordinates that these properties are independent of the choice of H.

4 Main Argument And Important Lemmas

We now provide the main ingredients for the proof of Theorem 5. Let (Y, ξ) be an over-twisted contact 3-manifold with a contact form λ, and let $\inf(\lambda) \in (0, \infty]$ be the minimal period of a Reeb orbit for λ. As above, let X denote Y's symplectization. The first main ingredient is an existence result for embedded discs F of a certain form in Y. In the discussion below, we implicitly use the identification $F \subset Y \subset X$ via $Y \to Y \times 0$ to speak of F as an embedded sub manifold of $X = \mathbb{R} \times Y$.

Lemma 13. *([3], Appendix, Thm 45) Every over-twisted contact structure λ on contact manifold (Y, ξ) has an embedded, over-twisted disc F with exactly 1 elliptic singularity at a point $p \in F$, such that ∂F is a limit cycle of the foliation induced by the line-distribution $\xi \cap TF$.*

From here on, assume that F is a disc with the properties delineated in Lemma 13. The other ingredients concern properties of J-discs with boundary on such an F.

Theorem 14. *[1] (Bishop) If $\tilde{J}$ is integrable near p, then there exists a map $\tilde{u}_t : [0, \epsilon) \times D \to X$ with the properties that $\tilde{u}_0$ is the constant map $D \to p \in F$, $\tilde{u}_t$ is J-holomorphic for all ϵ, $\tilde{u}_t(\partial D) \subset F$ and $\tilde{u}_t : (0, \epsilon) \times D \to X$ is an embedding.*

Lemma 15. *([3], p. 545) We can pick complex structure J on ξ such that the associated almost complex structure on X is integrable near p.*

Proposition 16. *([3], Prop 34) Assume $\tilde{u}_k : (D, \partial D) \to (X, F)$ is a sequence of J-curves such that $\tilde{u}_k(\partial D)$ have winding numbers 1 about p, $dist(\tilde{u}_k(\partial D), p) \geq \epsilon$ and:*

$$\lim_{k \to \infty} \sup \left(\sup_{\psi \in \mathcal{P}} \int_D \tilde{u}_k^* \omega_\phi\right) < \inf(\lambda)$$

then there exists Mobius transformations ϕ_k and a subsequence $\tilde{u}_{k'}$ of $\tilde{u}_k$ such that $\tilde{u}_{k'} \circ \phi_{k'}$ converges to a $\tilde{u}$ with the same properties. Here $\mathcal{P} = \{\phi \in \mathbb{C}^\infty(\mathbb{R}; [1/2, 1]) | \phi' > 0\}$ and $\omega_\phi = d(f\lambda)$ is the associated symplectic form.

Theorem 17. *([3], Thm 17-18) A map $\tilde{u} : (D, \partial D) \to (X, F - p)$ with $Ind(\tilde{u}) = 2$ can be extended to an embedded family $\tilde{u} : (-\epsilon, \epsilon) \times (D, \partial D) \to (X, F - p)$. Moreover, this extension is unique up to reparameterization.*

Lemma 18. *([3], Lem 33) If $u : (D, \partial D) \to (X, F)$ has winding number 1, then $\int_D \tilde{u}^* \omega_\phi \leq vol_\lambda(F)$ for all $\phi \in \mathcal{P}$.*

Lemma 19. *([3], Lem 19) If $\tilde{u} : (D, \partial D) \to (X, F - p)$ is a non-constant J-disc, then it is an immersion transversal to the foliation induced by ξ on F and $\int_{\partial D} u^* \lambda > 0$.*

With the above results, we can now give Hofer's argument.

Proof. (Theorem 5) It suffices to show that λ such that $\inf(\lambda) \leq \mathrm{vol}_\lambda(F)$. Then since $\mathrm{vol}_\lambda(F)$ is finite, this proves that $\inf(\lambda)$ is as well.

Now we argue by contradiction: assume that $\mathrm{vol}_\lambda(F) < \inf(\lambda)$. Take a J integrable near p by Lemma 15. By Bishop's result, Theorem 14, we have a family of holomorphic discs $u_t : [0, \epsilon) \times (D, \partial D) \to (X, F)$. By Lemma 18, these discs obey the bound $\int_D \tilde{u}_t^* \phi \leq \mathrm{vol}_\lambda(F) < \inf(\lambda)$ for all $\phi \in \Sigma$. Thus by Proposition 16, Theorem 17 and an elementary continuity argument, the family can be extended to a family $[0, T] \times (D, \partial D) \to (X, F)$ with $u_T(\partial D) \cap \partial F$ non-empty.

Examining u_T, we see that $u_T(\partial D)$ must be a smooth curve contained within $F \simeq D$ and touching $\partial F \simeq S^1$, which implies that it must be tangent at those points. This contradicts the fact that $u_T(\partial D)$ must be transverse to the foliation induced by ξ (according to Lemma 19) for which ∂F is a limit cycle. $\square$

5 Some Detailed Arguments

In this section, we prove a few of the statements given above, primarily to provide the reader with some idea of the flavor of argument utilized. We begin with the argument for Lemma 15 as a warm up.

Proof. (Lemma 15) The contact version of the Darboux theorem implies that we can take a local contactomorphism from a neighborhood U of p to a neighborhood of a point in S^3 with the standard contact structure induced by $S^3 \subset \mathbb{C}^2$. There, we have an obvious complex structure $\tilde{J}$ compatible with a symplectization, i.e the one induced by the the isomorphism between the symplectization and a neighborhood of S^3 in $\mathbb{C}^4$. On S^3, the Reeb vector-field R_0 is given by $i \cdot \hat{r} = \tilde{J}\hat{r}$ and ξ is given by the orthogonal complex line bundle to $\hat{r}$, which is closed under $\tilde{J}$-mulitplication.

Thus if we let $J_0 := \tilde{J}_0|_{\xi_0}$, and use the fact that the construction of going from a complex structure on $\ker\lambda = \xi$ to an almost complex structure on ξ is compatible with diffeomorphism we see that the pullback J of J_0 precisely yields the integrable, compatible split $\tilde{J}$ in a neighborhood of $p \in Y \subset X$. We can then extend this to the rest of X by taking the associated metric g_J on ξ in a neighborhood U of p, using a partition of unity which is identically 1 near p to combine this with a compatible metric $g_{J'}$ on a V with $U \cup V = Y$, and then recovering a global compatible J from the interpolated compatible metric. □

Next we provide a proof of Lemma 18.

Proof. (Lemma 18) First observe that, for any $\phi \in \mathcal{P}$, we have by Stokes theorem:

$$\int_D u^*\omega_\phi = \int_{\partial D} u^*(\phi\lambda) \leq \int_{\partial D} u^*\lambda$$

Then, since $\int_F d\lambda = \int_{\partial F} \lambda = 0$ (since $T(\partial F) \subset \xi = \ker\lambda$) and $u(\partial D)$ splits F into two parts F' and F'', we have by Stokes theorem:

$$\int_{\partial D} u^*\lambda \leq \min(|\int_F d\lambda|, |\int_{F'} d\lambda|) \leq \frac{1}{2}\int_F |d\lambda| \leq \frac{1}{2}\mathrm{Vol}_\lambda(F)$$

□

Last, we prove Lemma 19.

Proof. (Lemma 19) Let $\tilde{u} = (a, u)$ be a non-constant J-disc with boundary on $F - p$. Then we see that the pullback of $d\lambda$ by u has the form (in local coordinates x, y on D):

$$u^*d\lambda = \frac{1}{2}(|\pi_\xi u_x|_J^2 + |\pi_\xi u_y|^2)dx \wedge dy$$

Now using the fact that (a, u) satisfies Equation 1, we can calculate that:

$$(-\Delta a)dx \wedge dy = d(da \circ j) = d(-u^*\lambda) = -u^*d\lambda = \frac{1}{2}(|\pi_\xi u_x|_J^2 + |\pi_\xi u_y|^2)dx \wedge dy$$

Also, $a|_{\partial D} = 0$ because $\tilde{u}(\partial D) \in 0 \times F \subset X$ by assumption.

Now, to see that $\int_{\partial D} u^*\lambda > 0$, assume otherwise. Then by Stokes we have $\int_D u^*d\lambda = 0$ and thus by the calculation above that $\Delta a = 0$. Thus by the maximum and minimum principle, a must be 0 and so both a and du are 0, and $\tilde{u}$ must be constant, a contradiction.

Using the above results, we can reason that u is transverse to the foliation induced by ξ on F. Now we know that $\int_D u^* d\lambda > 0$ and thus that $-\Delta a \leq 0$ but is not identically 0, and that it has boundary values 0. By the strong maximum principle for sub-harmonic functions, we have $a < 0$ on $\mathrm{int}(D)$ and thus that $\partial_n a(z) > 0$ for $z \in \partial D$ and ∂_n the outward normal derivative (i.e it is increasing towards the boundary). Now we observe using Equation 1 that:

$$\lambda(u_*(j\partial_n))(u(z)) = u^*\lambda(j\partial_n)(z) = da(\partial_n)(z) = \partial_n a(z) > 0 \qquad z \in \partial D$$

In particular, since $j\partial_n$ spans the tangent space of ∂D this implies that u_* is non-zero on that tangent space at all $z \in \partial D$ and that the image is transverse to the foliation generated by ξ, since λ is by definition 0 there. This concludes the proof. □

For our last, and most illustrative proof, we will give the argument for Proposition 16. Here we will use the most heavily utilized technique in [3], namely bubble-off analysis. That is, we will examine the structure of the limiting curves which result from a sequence of J-discs satisfying L^2 energy bounds. We will require the following Lemma, which is proven by very similar techniques.

Lemma 20. *([3], Lemma 35) Let $\tilde{u}$ be a non-constant J disc with boundary in $F - p$ such that $\int_{\phi \in \mathcal{P}} \tilde{u}^*\omega_\phi < C$ and list$(u(\partial D), p) > \epsilon$. Then there exists some energy quantum $\hbar = \hbar(\epsilon, C) > 0$ independent of u such that:*

$$\int_D u^* d\lambda \geq \hbar$$

Lemma's like this for $u^*d\lambda$ replaced with $\tilde{u}^*\omega$ are considered standard in J-cure theory (see, for instance, Proposition 4.14 in [2]). With Lemma 20 in hand, we can begin our proof.

Proof. Let $\tilde{u}_k$ be a sequence of J-discs obeying the assumptions of the proposition. Let $e(\tilde{u}, \phi) = \sup_{z \in D} |\nabla \tilde{u} \circ \phi|_{\tilde{J}}$. If $e(\tilde{u}_k, \phi_k) < C$ uniformly for some sequence of ϕ_k, then by standard elliptic bootstrapping arguments we can find a uniformly convergent subsequence of $u_k \circ \phi_k$ on compact subsets to a smooth J-curve $\tilde{u}$, and using the derivative bounds near the boundary and the fact that $\tilde{u}_k$ is bounded away from p, we can insure that $\tilde{u}(\partial D) \subset F - p$ is a smooth curve. The winding number is stable in the limit of smooth convergence of curves which are bounded away from the puncture at p.

Thus we can assume that $\sup_{z \in D} |\nabla \tilde{u}_k \circ \phi_k|_{\tilde{J}}$ has a divergent subsequence for any choice of ϕ_k. Fix ϕ_k so that:

$$\sup_{z \in D} |\nabla \tilde{u}_k \circ \phi_k|_{\tilde{J} = \inf_{\phi \in \mathrm{Mob}(D)} \sup_{z \in D} |\nabla \tilde{u}_k \circ \phi|_{\tilde{J}}}$$

We assume from now on that $\tilde{u}_k$ is reparameterized by this ϕ_k already. Then by our divergence assumption, we have many sequences $z_k, \epsilon_k, R_k := |\nabla \tilde{u}_k(z_k)|_{\tilde{J}}$ with $z_k \to z$ such that:

$$\epsilon_k R_k \to \infty, R_k \to \infty, \epsilon_k \to 0$$

$$|\nabla \tilde{u}(z)| \leq 2|\nabla \tilde{u}(z_k)| \qquad |z - z_k| \leq \epsilon_k$$

There is a small analytical Lemma that we need to use here to show that we can make such choices (Lemma 26 in [3]), but we will not state it explicitly.

Now by conformally rescaling, $\tilde{v}_k(z) := (a(z_k + z/R_k) - a(z_k), u(z_k + z/R_k))$, we acquire a sequence $\tilde{v}_k$ of J-discs satisfying uniform derivative gradient bounds (by 2) and with energy $\int_{B_{\epsilon_k R_k}} \tilde{v}_k^* \omega_\phi \leq \int_D \tilde{u}_k^* \omega_\phi <$

$\inf(\lambda)$ for all $\phi \in \mathcal{P}$. First, note that this gives convergence to a J-plane $\tilde{v}$ with the same energy estimates. Such a rescaled limit is commonly called a **bubble**. Second, note that Lemma [?] implies that:

$$\hbar(\epsilon, \inf(\lambda) \leq \lim_{k\to\infty} \int_{B_{\epsilon_k R_k}} \tilde{v}_k^* \omega_\phi$$

But this implies that the number of limit points z must be less than or equal to:

$$N := 2\lfloor \frac{\sup_{\phi\in\mathcal{P}} \int_D \tilde{u}^*\omega_\phi}{\hbar(\epsilon, \inf(\lambda)} \rfloor$$

, which is finite because the numerator is bounded by $\mathrm{vol}_\lambda(F)$ (by Lemma 18). Indeed, were this not the case, we could choose $2N$ points $z^{(i)}$ and a large K such that for each of the $2N$ $z^{(i)}$ and an accompanying choice of $z_k^{(i)}, \epsilon_k^{(i)}, R_k^{(i)}, \tilde{u}_k^{(i)}$ and $v_k^{(i)}$ we have:

$$\sup_{\phi\in\mathcal{P}} \int_{B^{(i)}_{\epsilon_K^{(i)}}(z_K^{(i)})} (\tilde{u}_k^{(i)})^*\omega_\phi = \sup_{\phi\in\mathcal{P}} \int_{B^{(i)}_{\epsilon_K^{(i)} R_K^{(i)}}} (\tilde{v}_k^{(i)})^*\omega_\phi > \frac{\tau}{2}$$

We also choose K so that each $B^{(i)}_{\epsilon_K^{(i)}}(z_K^{(i)}$ is disjoint. This choice would thus imply that $\tilde{u}_K$ violated the L^2 energy bound. Thus we must have a "finiteness of bubbling points" in the sense that there can only be finitely many z arising as limits of these sequences z_k.

Now, if a bubbling point z is in the interior of D, then we can find a limiting J-plane satisfying:

$$|\nabla\tilde{u}(0)| = 1, |\nabla\tilde{u}| \leq 1, \sup_{\phi\in\mathcal{P}} \int \tilde{u}^*\omega_\phi < \inf(\lambda)$$

However, it can be shown (Theorem 31 in [3]) that there exists a sequence $\rho_k \to \infty$ such that the circles about 0 $S(\rho_k) \subset \mathbb{C}$ have image $\tilde{u}(S(\rho_k))$ limiting to a Reeb orbit. Thus Stokes theorem gives us a bound $\inf(\lambda) \leq \sup_{\phi\in\mathcal{P}} \int \tilde{u}^*\omega_\phi$, which is a contradiction.

Thus any bubbling must occur on the boundary. The same argument as above can be used to show that the number of such points is finite (thus time using shrinking half-discs about the boundary instead of balls). The limiting map resulting from conformal rescaling about the boundary blow up points is instead a map of the upper-half plane into X (which can be extended to a map of the closed disc using the energy bounds). Thus we have a limit $\tilde{v}$ of the maps $\tilde{u}_k$ (possibly after reparameterizing) converging in C^∞ away from the bubble points and some number $\tilde{v}_i$ of bubbled off J-discs with boundary on $F - p$.

Now, a non-constant $\tilde{v}_i$ must have winding number ≥ 1 since otherwise it would have negative action (in the < 0 case) or it would be constant because it would be somewhere tangent to the foliation (in the $= 0$ case) Lemma 19). Since the curves $\tilde{u}_k(\partial D)$ limit to the union of $\tilde{v}(\partial D)$ and $\tilde{v}_i(\partial D)$, the sum of the winding numbers of $\tilde{v}$ and the finite $\tilde{v}_i$ are equal to the winding number of $\tilde{u}_k$ for sufficiently large k (which is 1 by assumption).

This implies that either $\tilde{u}_k$ limit to $\tilde{v}$ with no bubbles or to a single $\tilde{v}_1$ with $\tilde{v}$ constant. Either way, the limit is a J-disc with the correct winding number and the other prescribed properties.

□

6 A Brief Discussion Of An Alternate Argument

In this last short section, we briefly mention an alternative structuring for the proof of Theorem 5, using the same basic ingredients, which shows more directly where the Reeb orbit is coming from. Using the same special J as in Theorem 5 and the dimension result in Theorem 17, we know that the moduli space of discs with boundary on $F-p$ is 1-dimensional, with a compact end consisting of the constant map to p.

Now, we cannot have any uniform energy bounds, since then a proof by contradiction similar to the one executed in the proof of Theorem 5 would result in a contradiction. Thus some bubbling must occur at the other end of the moduli space. The bubbling cannot result in a disc bubble since this disc bubble would touch the boundary, leading to a contradiction like before. So the bubble must be a plane bubble, with boundary necessarily limiting to a Reed orbit. Thus a Reeb orbit must exist.

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