The Vector Bundle Maximum Principle : Consider

- (X, g) a closed Riemannian mfld
- $(E \to X, h)$ a Hermitian $\mathbb{C}$-vector bundle on X
- $K \subset E$ a fiberwise convex, parallel sub-bundle w/ C^∞ boundary ∂K
- $\psi : X \to K$ a section satisfying:

$$-\Delta\psi + a^i \nabla_i \psi + \Phi(\psi) = 0$$

- $\Phi : E \to \mathrm{Vert}(TE) \subset TE$ s.t. $\forall\, e \in \partial K$ we have:

$$\langle \nu_K(e), \Phi(e) \rangle \leq 0 \quad , \nu_K : \partial K \to \mathrm{Vert}(TE) \text{ is } \underline{\text{inward}} \text{ normal to } \partial K_{\pi(e)}$$

Then $\psi(p) \in \partial K$ for $\underline{\text{some } p}$ $\Rightarrow$ $\psi(p) \in \partial K$ for $\underline{\text{all } p}$.

Proof: Let $\bar{d} : E \to \mathbb{R}$ be defined by $\bar{d}(e) = \mathrm{dist}(e, \partial K_{\pi(e)})$, & define $f : X \to \mathbb{R}$ by $f(x) := \bar{d}(\psi(x))$.

$$-\nabla^i \partial_i f + a^i \partial_i f = (-\nabla^i + a^i)(\partial_j \bar{d}\, \nabla_i \psi^j) = \partial_j \bar{d}\,(-\Delta\psi^j + a^i \nabla_i \psi) - g^{ie} \nabla_k \partial_j \bar{d}\, \nabla_i \psi^j \nabla_e \psi^k$$

$$= \partial_j \bar{d}\, \Phi^j(\psi) - I(\psi, \nabla\psi) = \langle \mathrm{grad}\, \bar{d}, \Phi \rangle - I(\psi, \nabla\psi)$$

$$-I = -\cancel{\mathrm{Hess}_{ij}(\bar{d})}\, \mathrm{tr}(\mathrm{Hess}_{\bar{d}}(\nabla\psi, \nabla\psi))?$$

Let $c : B_\varepsilon(\partial K) \to \partial K$ be the map $c(e) := \{\text{closest elt of } \partial K \text{ to } e\}$

Then $-\partial_j \bar{d}(c(\psi)) = -\nu_K(c(\psi))$ so:

$$-\Delta f + a^i \partial_j f \geq -(\partial_j \bar{d}(\psi) - \partial_j \bar{d}(c(\psi)))\Phi^j - I(\psi, \nabla\psi)$$

$\geq \langle \phi, \psi - c(\psi) \rangle - I(\psi, \nabla\psi)$ ← Use Φ Lipschitz,

$\geq \langle \phi, \nu_K(c(\psi)) \rangle f - I(\psi, \nabla\psi)$ ← Use $\psi = c(\psi) + \bar{d}(\psi) \cdot \nu_K(c(\psi))$ near bdry.

↓ need $I \leq 0$.

$\geq \langle \phi, \nu_K(c(\psi)) \rangle f$ Thus:

↑ need $\langle \phi, \nu_K(c(\psi)) \rangle$ ← can use negative, $\underline{\text{bdd}}$.

$$-\Delta f + a^i \partial_i f + \langle \phi, \nu_K(c(\psi)) \rangle f \geq 0$$

$\Rightarrow$ classical max ppl $\Rightarrow$ $f = 0$ on interior $\Rightarrow f \equiv 0$ everywhere.

~~(Evans)~~
(Evans, p. 350)
thm 4, (ii)