SHOCK BOUNDARIES IN 2D QUASILINEAR WAVE EQUATIONS

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In this paper we study shock formation in 2+1 quasilinear waves solving $g^{\mu\nu}(\partial\Phi)\partial_\mu\partial_\nu\Phi = 0$, under the assumption of radial symmetry and small initial conditions. We apply the methods recently developed by Christodolou to understand the 3+1 Euler equations to acquire a more detailed understanding of the behavior of Φ near the light-cone. We derive estimates for the C^2 decay of our solution and the asymptotic lifetime of solutions. We also discuss the character of singularity at the shock boundary and the geometry of the boundary itself. We conclude the paper by constructing stable families of data which exhibit the various behavior that we discuss.

1. Introduction

In this paper we will study shock development in radially symmetric solutions to quasilinear perturbations of the linear wave equation in 2 spatial dimensions. That is, we will study the finite time blow up of derivatives of solutions to equations of the following form[1].

$$[\eta^{\mu\nu} + h^{\mu\nu}(\partial\Phi)]\partial_\mu\partial_\nu\Phi = 0 \tag{1}$$

Here $\eta^{\mu\nu}$ is the flat Minkwoski metric in $2+1$ dimensions; Φ is a solution with initial data $\mathring{\Phi} := \Phi|_{t=0}$ and $\dot{\Phi} := \partial_t\Phi|_{t=0}$, chosen to be radially symmetric functions in $C_0^\infty(\mathbb{R}^2)$ with $\mathring{\epsilon} := \|\mathring{\Phi}\|_{C^2} + \|\dot{\Phi}\|_{C^1}$; $\partial\Phi$ denotes the gradient of Φ; and $h^{\mu\nu} : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ is any smooth symmetric (2,0)-tensor field in a neighborhood of 0 where our perturbation $h^{\mu\nu}$ satisfies $h^{\mu\nu}(0) = 0$.

Our ultimate goal in this treatise is to characterize the following properties of solutions Φ, for arbitrary compactly supported, radially symmetric initial data $(\Phi|_{t=0}, \partial_t\Phi|_{t=0})$ and arbitrary perturbations $h^{\mu\nu}$: the asymptotic size of the classical lifespan T_L of Φ in terms of $\mathring{\epsilon}$, i.e how long Φ remains smooth; the type of singularity that forms in Φ, i.e which derivative reaches infinity at finite time; and the shape of the shock boundary which forms. Our first main result is encapsulated in the following theorem.

Theorem 1.1. *For any non-trivial initial data $(\mathring{\Phi}, \dot{\Phi})$ (compactly supported in the unit ball with $\mathring{\epsilon} := \|\mathring{\Phi}\|_{C^2} + \|\dot{\Phi}\|_{C^1}$) and non-linearity $h^{\mu\nu}$, the corresponding solution Φ to (1) falls into one of the following categories <u>determined entirely by $h^{\mu\nu}$</u>.*

(1) $h^{\mu\nu}$ does not satisfy the 1st null condition, so Φ has a classical lifespan of $\Theta(\mathring{\epsilon}^{-2})$ time ending in the blow up of $\|\partial^2\Phi\|_{C^0}$ within a strip about the light-cone.
(2) $h^{\mu\nu}$ satisfies the 1st null condition but not the 2nd null condition, so Φ has a classical lifespan of $\Theta(\exp(\mathring{\epsilon}^{-2}))$ time ending in the blow up of $\|\partial^2\Phi\|_{C^0}$ within a strip about the light-cone.
(3) $h^{\mu\nu}$ satisfies both the 1st and 2nd null condition, and for small enough data has a unique global smooth solution within a strip about the light-cone of size dependent on the size of the data.

Our exposition will proceed as so. Sections 1 & 2 will contain setup along with a small amount of background for non-linear waves and geometric optics. After this, we will jump

[1]We will use Einstein notation freely and regularly, as we did in our problem definition. When we use Greek indices, we contract with respect to the Minkowski metric. When we use Roman indices, we contract with respect to the Euclidean metric (i.e the dot product). We will also raise and lower indices by the same convention.

directly into proving the estimates necessary to realize our main result. These estimates will be of 3 flavors, corresponding to 3 time periods in which we will observe and estimate our solution Ψ; a short-term period, medium-term period and long-term period. The estimates in the first two periods will be derivable via a repackaging of the estimates covering the last period. Thus, in Section 3 we will use a continuity/bootstrap arguments to derive dispersive estimates for $\partial_i\Phi$ and its derivatives within a strip of constant width covering the long-term phase. Some of these estimates will be degenerate, and this degeneracy will control the development of singularities. In turn, this degeneracy will be controlled by an important space-time function μ, which will be the real focus of our analysis.

In Section 4, we will demonstrate the short and medium term estimates by citing classic local well-posedness results and using the estimates from the previous section. In Section 5, we will use the results of Sections 4 and 5 to analyze μ, by deriving various nice formulae [2] describing its behavior up to slowly growing or decaying terms. In Section 6, we will analysis these formulae to demonstrate shock formation for a wide variety of compactly supported data at small scales, and demonstrate global existence for all others. We will also give a more detailed discussion of our results than the terse Theorem 1.1.

In Section 7, we will apply the results of the previous sections to describe the geometry of the shock boundary in detail. In the last part, Section 8, we will build various open, stable families of radially symmetric data which produce the various types of boundary behavior described in Section 7.

2. A Geometric Formulation

The first step towards a better geometric formulation of this problem is the translation our equation into $1+1$ dimensions using the assumed radial symmetry. Changing to radial coordinates $(t, r, \theta_1, \theta_2)$, imposing our assumption that $\partial_{\theta_i}\Phi = 0$ and performing some algebraic manipulations on (1), we can demonstrate that it suffices to analyze solutions to the following equation in $1+1$ dimensions (i.e with coordinates t, r).

$$g^{\mu\nu}\partial_\mu\partial_\nu\Phi = [\eta^{\mu\nu} + h^{\mu\nu}(\partial\Phi)]\partial_\mu\partial_\nu\Phi = \frac{1+a(\partial\Phi)}{r}\partial_r\Phi \tag{2}$$

In (1), Φ is a solution to our new $1+1$ dimension problem with initial data $\mathring{\Phi} := \Phi|_{t=0}$ and $\dot{\Phi} := \partial_t\Phi|_{t=0}$ are even functions in $C_0^\infty(\mathbb{R})$. $\eta^{\mu\nu}$ is the $1+1$ dimensional Minkowski metric. $h^{\mu\nu} : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ is a smooth symmetric (2,0)-tensor field in a neighborhood of 0 where $h^{\mu\nu}(0) = 0$, $h^{tt} = 0$ identically, h^{rr} is an even function in r and $h^{tr} = h^{rt}$ is an odd function of r. $a : \mathbb{R}^2 \to \mathbb{R}$ is smooth, even function on a neighborhood of 0 with $a(0) = 0$. Note that we will write $g^{\mu\nu}$, $h^{\mu\nu}$ and a without noting their dependence on $\partial\Phi$, as this will be understood throughout the paper. All of the aforementioned properties of $h^{\mu\nu}$ and a follow either from the radial symmetry of the original problem (i.e the evenness and oddness assumptions) or from some degree of freedom afforded to us by simple algebraic manipulation (like the assumption that $h^{tt} = 0$).

Now that we have a $1+1$ dimensional formulation of our equation, we can transform it one step further into a coupled system of PDE. Previous work such as [7] suggests that we will see blow up in the second derivatives of our solution Φ. Thus, instead of studying Φ and equation (2) directly, we will examine $\Psi_\sigma := [\partial_t + \partial_r]\Phi$ and $\Psi_\rho := [\partial_t - \partial_r]\Phi$ via a system of equations, equivalent to (2). This system will essentially be derived by differentiating (2) by $\partial_t + \partial_r$ and $\partial_t - \partial_r$, and coupling the result with the equation $\partial_\rho\Phi_\sigma = \partial_\sigma\Phi_\rho$. Note that we will denote the covector $(\Psi_i)_{i\in\{\sigma,\rho\}}$ simply as Ψ.

[2]See Proposition 5.1, specifically formulae (38) and (40) for a preview.

Our choice of differentiation by $\partial_t + \partial_r$ and $\partial_t - \partial_r$ as opposed to Ψ_t and Ψ_r is an important one; Ψ_σ and Ψ_ρ obey the following important identity as a consequence of (2).

$$\frac{1}{2}r^{-1}(\Psi_\sigma+\Psi_\rho)+r^{-1}O(\Psi_i^2) = L\Psi_\rho+O(\Psi_i L\Psi_j)+O(\Psi_i \underline{L}\Psi_j) = \underline{L}\Psi_\sigma+O(\sqrt{g}-g^{tr}-1)\cdot O(\underline{L}\Psi_i) \tag{3}$$

By $O(\Psi_i)$ we mean that all of the terms collected into the coefficient term $O(\Psi_i)$ are of order 1 in either Ψ_i or Ψ_j, or higher order. The other $O(\cdot)$ terms are analogous. Note that we provided an extra bit of detail for $\underline{L}\Psi_\sigma + O(\sqrt{g} - g^{tr} - 1) \cdot O(\underline{L}\Psi_i)$, giving a more explicit formula for the $O(\Psi_i)$ coefficients of the $\underline{L}\Psi_i$ remainder. This will be particularly important as we utilize this identity in the algebra steps of our proofs below. Keep it in mind.

Instead of expressing the system for Ψ_σ and Ψ_ρ in terms of ∂_t and ∂_r, our analysis will require that we put the equations in a more tractable, geometric form. To do so we must introduce the following vector-fields, L^i and $\underline{L}^i$. These are light-like with respect to the 1+1-dimensional Lorentzian metric $g_{\mu\nu}$, which we define as satisfying $g_{\kappa\mu}g^{\mu\nu} = \delta^\nu_\kappa$. In other words, L^i and $\underline{L}^i$ are analogous the flat light-like vector fields L^i_{flat} defined by $L^t_{\text{flat}} = L^r_{\text{flat}} = 1$ and $\underline{L}^i_{\text{flat}}$ defined by $\underline{L}^t_{\text{flat}} = -\underline{L}^r_{\text{flat}} = 1$ in Minkowski space in the geometry of $g_{\mu\nu}$. $g_{\mu\nu}$ is a natural part of the equation (2), and its geometry will play a critical roll in the forthcoming analysis. We can write L^i and $\underline{L}^i$ explicitly as so.

$$L^t = \underline{L}^t = 1; L^r = \sqrt{1 + h^{rr} + (h^{tr})^2} - h^{tr} = \sqrt{g} - g^{tr} \tag{4}$$

$$\underline{L}^r = -\sqrt{1 + h^{rr} + (h^{tr})^2} - h^{tr} = -\sqrt{g} - g^{tr} \tag{5}$$

Above, we use the quantity $g := -\det g^{\mu\nu}$ for succinctness. We will denote the derivations $L^i\partial_i$ and $\underline{L}^i\partial_i$ simply as L and $\underline{L}$, respectively (and likewise for L^i_{flat} and $\underline{L}^i_{\text{flat}}$). Note that $\Psi_\sigma = L_{\text{flat}}\Phi$ and $\Psi_\rho = \underline{L}_{\text{flat}}\Phi$; this is no accident! The critical identity (3) is a direct consequence of this.

The analogy between the non-linear light-like vector fields and their flat equivalents will manifest itself in many important ways, via formulae which we now discuss. First, there is a factorization identity, analogous to the factorization of the 1+1 wave equation as $\eta^{\mu\nu}\partial_\mu\partial_\nu = L^t_{\text{flat}}\underline{L}^i_{\text{flat}} = \underline{L}_{\text{flat}}L_{\text{flat}}$.

$$L\underline{L} + \frac{L[\sqrt{g}+g^{tr}]}{2\sqrt{g}}[L-\underline{L}] = \underline{L}L + \frac{\underline{L}[\sqrt{g}-g^{tr}]}{2\sqrt{g}}[\underline{L}-L] = -g^{\mu\nu}\partial_\mu\partial_\nu \tag{6}$$

$$[L,\underline{L}] = \frac{\underline{L}[\sqrt{g}-g^{tr}] + L[\sqrt{g}+g^{tr}]}{2\sqrt{g}}[\underline{L}-L] \tag{7}$$

We also have the following change of basis identities, analogous to the identities $\partial_t = \frac{1}{2}(L_{\text{flat}} + \underline{L}_{\text{flat}})$ and $\partial_t = \frac{1}{2}(L_{\text{flat}} - \underline{L}_{\text{flat}})$ which we can use to express ∂_t and ∂_r in terms of L and $\underline{L}$.

$$\partial_t = \frac{\sqrt{g}+g^{rt}}{2\sqrt{g}}L + \frac{\sqrt{g}-g^{rt}}{2\sqrt{g}}\underline{L}; \partial_r = \frac{1}{2\sqrt{g}}L - \frac{1}{2\sqrt{g}}\underline{L} \tag{8}$$

$$\partial_t+\partial_r = \frac{1+\sqrt{g}+g^{rt}}{2\sqrt{g}}L + \frac{-1+\sqrt{g}-g^{rt}}{2\sqrt{g}}\underline{L}; \partial_t-\partial_r = \frac{-1+\sqrt{g}+g^{rt}}{2\sqrt{g}}L + \frac{1+\sqrt{g}-g^{rt}}{2\sqrt{g}}\underline{L} \tag{9}$$

Our geometric reformulation will also require us to switch to geometric coordinates. That is, we will trade r for u, defined by the eikonal equation $Lu = 0$ with initial values $u|_{t=0} = 1 - r$. The level sets of u are thus the integral curves of L, in physical terms the "beams of light" emitted from position $u = 1 - r$ in the positive r direction at time 0 in the geometry of $g_{\mu\nu}$. The linear analog of u in the function $u_{\text{flat}} = 1 - t + r$, which satisfies $L_{\text{flat}}u_{\text{flat}} = 0$ for the same initial values.

As in the 3-dimensional case studied in [5], the key quantity characterizing shock development for solutions to (2) with small[3] initial data is the density, in space-time, of the level curves of u. In the case of radial symmetry described by our problem, this can be measured using the quantity μ defined as:

$$\mu^{-1} := -g^{\mu\nu}\partial_\mu t\partial_\nu u = \frac{1}{2}\underline{L}u \tag{10}$$

This quantity is typically called the *inverse foliation density*. Our analysis will show that, for all types of non-linearities $h^{\mu\nu}(\Psi)$ and small initial data (fitting the constraints above) which yield a solution with a finite classical lifespan, the first shock formed will be characterized by $\mu \to 0$.

The inverse foliation will play two important roles in the analysis below. First, as in [5], μ will allow us to rescale the inward light-like direction $\underline{L}$ such that, after the rescaling, our reformulated PDE will contain only negligible quadratic non-linear terms. This will prevent our efforts at deriving analogues of the peeling estimates in the linear 2D wave equation from being obstructed by non-linear behavior, when the initial data is sufficiently small. The effect of the rescaling is captured through the following commutator relation:

$$[L,\mu\underline{L}] = \mu[L,\underline{L}] + L\mu\cdot\underline{L} = -\frac{1}{2\sqrt{g}}[\mu\underline{L}[\sqrt{g}-g^{tr}] + \mu L[\sqrt{g}+g^{tr}]]L \tag{11}$$

Second, the foliation density obeys two non-linear transport equations which will allow us to describe its behavior in great detail. This equation is given below.

$$L\mu = -\frac{\mu^2}{2}[L,\underline{L}]u = -\frac{1}{2\sqrt{g}}\left[\mu\underline{L}[\sqrt{g}-g^{tr}] + \mu L[\sqrt{g}+g^{tr}]\right] \tag{12}$$

$$\mu\underline{L}\mu = -\mu^3\underline{L}^2u = -\mu(\mu\underline{L}(\mu\underline{L}u) + \mu[\underline{L},\mu]\underline{L}u \tag{13}$$

As we shall see, given estimates on $\mu L\Psi_i$ and $\mu\underline{L}\Psi_i$ this equation empowers us with a rough description of the longterm (large t) behavior of μ.

Having introduced the necessary language, we can finally give the appropriate formulation of our system for Ψ_σ and Ψ_ρ. By differentiating (2) with L_{flat} and $\underline{L}_{\text{flat}}$, grouping terms and utilizing the identity [ID], we get the following system of equations for Ψ_σ and Ψ_ρ entirely in terms of $L, \underline{L}$ and μ.

(14)
$$L(r^{1/2}\mu\underline{L}\Psi_\rho) = r^{1/2}[\dot{P}^{ij}\mu L + \dot{\underline{P}}^{ij}\mu\underline{L}]\Psi_i\cdot L\Psi_j + r^{-1/2}[P^{ij}\mu L + \underline{P}^{ij}\mu\underline{L} + \hat{P}^{ij}]\Psi_i\cdot\Psi_j + \boxed{\frac{3}{2}r^{-1/2}\mu L\Psi_\rho}$$

(15)
$$L(r^{3/2}\mu\underline{L}\Psi_\sigma) = r^{3/2}[\dot{Q}^{ij}\mu L + \dot{\underline{Q}}^{ij}\mu\underline{L}]\Psi_i\cdot L\Psi_j + r^{1/2}[Q^{ij}\mu L + \underline{Q}^{ij}\mu\underline{L} + \mu\hat{Q}^{ij}]\Psi_i\cdot\Psi_j + \boxed{\frac{1}{2}r^{1/2}\mu L\Psi_\sigma}$$

(16)
$$\mu\underline{L}(r^{3/2}L\Psi_\rho) = r^{3/2}[\dot{R}^{ij}\mu L + \dot{\underline{R}}^{ij}\mu\underline{L}]\Psi_i\cdot L\Psi_j + r^{1/2}[R^{ij}\mu L + \underline{R}^{ij}\mu\underline{L} + \mu\hat{R}^{ij}]\Psi_i\cdot\Psi_j + \boxed{\frac{-1}{2}r^{1/2}\mu\underline{L}\Psi_\rho}$$

(17)
$$\mu\underline{L}(r^{1/2}L\Psi_\sigma) = r^{1/2}[\dot{S}^{ij}\mu L + \dot{\underline{S}}^{ij}\mu\underline{L}]\Psi_i\cdot L\Psi_j + r^{-1/2}[S^{ij}\mu L + \underline{S}^{ij}\mu\underline{L} + \mu\hat{S}^{ij}]\Psi_i\cdot\Psi_j + \boxed{\frac{-3}{2}r^{-1/2}\mu\underline{L}\Psi_\sigma}$$

[3]"Small data" in this paper will mean compactly supported of C^2 size $\mathring{\epsilon} > 0$ at $t = 0$ in the annulus $r \in (0, 1)$.

We also have the following combined equations for Ψ_σ and Ψ_ρ.

$$\begin{aligned}
&\mu\underline{L}L(r^{1/2}(\Psi_\sigma + \Psi_\rho)) = r^{1/2}[\dot{A}^{ij}\mu L + \dot{\underline{A}}^{ij}\mu\underline{L}]\Psi_i \cdot L\Psi_j + \\
&r^{-1/2}[A^{ij}\mu L + \underline{A}^{ij}\mu\underline{L} + \mu\hat{A}^{ij}]\Psi_i \cdot \Psi_j + \boxed{\frac{-1}{4}\mu r^{-3/2}(\Psi_\sigma + \Psi_\rho)}
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\mu\underline{L}L(r^{1/2}(\Psi_\sigma - \Psi_\rho)) = r^{1/2}[\dot{B}^{ij}\mu L + \dot{\underline{B}}^{ij}\mu\underline{L}]\Psi_i \cdot L\Psi_j + \\
&r^{-1/2}[B^{ij}\mu L + \underline{B}^{ij}\mu\underline{L} + \mu\hat{B}^{ij}]\Psi_i \cdot \Psi_j + \boxed{\frac{-3}{4}\mu r^{-3/2}(\Psi_\sigma - \Psi_\rho)}
\end{aligned} \tag{19}$$

Here the terms $P^{ij}, \dot{P}^{ij}, \hat{P}^{ij}$ etc. in (14), along with the analogous Q, R, S, A and B terms in the other 5 equations, are simply smooth $(2,0)$ tensors which are functions of Ψ only. Thus, it is important to note that when Ψ is small (less than 1) these terms can be interpreted as being bounded by a constant factor. That is, $P^{ij}, \dot{Q}^{ij}$ etc. would be $O(1)$ in this situation. Also notice the boxed pieces of the equations above; these are the important pieces, which will effectively control our ability to make estimates. The other terms are quadratic in Ψ and devoid of a $\underline{L}\Psi_i \cdot \underline{L}\Psi_j$ term thanks to the rescaling of the $\underline{L}$ derivative by μ. As we shall see, this will make the unboxed terms negligible during the analysis.

3. Estimates Of Rescaled Derivatives Near The Light Cone

Equipped with our reformulation, we can prove the key estimates in our analysis that describe 2D shocks.

Our estimates in this section will not be global in nature; instead, we will focus on a space-time strip of some finite spatial width. Let $\epsilon \in (0,1)$ and $u_0, w \in \mathbb{R}^+$ with $0 < u_0 < w$. Choose the definition of the coordinate u given by the equation $Lu = 0, u|_{t=0} = w - r$. We will denote the region where $t \in [0,T)$ and $u \in [0,u_0]$ as $M(T,u_0,w)$. Moreover, we will denote the time slice $M(T,u_0,w) \cap \{(t,u)|t = s\}$ as Σ_s and the space slice $M(T,u_0,w) \cap \{(t,u)|u = v\}$ as C_v. $M(T,u_0,w)$ is, pictorially, the strip of space-time composed of the light-rays emitted from the annulus Σ_0 in the radially outward direction.

For all of the Ψ_i which we study below, we will denote the initial data as $\mathring{\Psi}_i := \Psi_i|_{t=0}$ and $\dot{\Psi}_i := \partial_t\Psi_i|_{t=0}$. We will also denote the size of the initial data as $\mathring{\epsilon} = \|\mathring{\Psi}_i\|_{C^2} + \|\dot{\Psi}_i\|_{C^1}$. We will assume that the initial data is compactly supported within Σ_0 and that $\mathring{\epsilon} \le \epsilon$. The maximum classical lifespan of Ψ_i, for some u_0, ϵ and initial data $\mathring{\Psi}_i$, $\dot{\Psi}_i$ as above, is the first time t at which a derivative of Ψ_i blows up at some point in Σ_t. We will denote it as $T_L = \sup\{t > 0 | \Psi_i \in C^\infty(M(T,u_0,w))\}$. When the supremum does not exist, we say that $T_L = \infty$ and that we have smooth existence within the strips $M(T,u_0,w)$ for all T.

Finally, in the proofs below we will use $A \lesssim B$ or $A \simeq B$ for two quantities A and B to denote that $A \le C \cdot B$ or $D \cdot B \le A \le C \cdot B$ respectively, where C and D are constants dependent only on our choice of w, u_0 and the non-linearity $h^{\mu\nu}$.

With this setup completed, we are ready to proceed to the proof of our estimates.

Proposition 3.1. *For any u_0, we can pick an ϵ as above such that within $M(T,u_0,w)$ we have the following estimates for $t \le \min(C\exp(\mathring{\epsilon}^{-1/2}), T_L)$.*

$$|\Psi_\sigma| \le C\mathring{\epsilon}(1+t)^{-1/2}; |\mu\underline{L}\Psi_\sigma| \le C\mathring{\epsilon}(1+t)^{-1}; |L\Psi_\sigma| \le C\mathring{\epsilon}(1+t)^{-3/2} \tag{20}$$

$$|\Psi_\rho| \le C\mathring{\epsilon}(1+t)^{-1/2}; |\mu\underline{L}\Psi_\rho| \le C\mathring{\epsilon}(1+t)^{-1/2}; |L\Psi_\rho| \le C\mathring{\epsilon}(1+t)^{-3/2} \tag{21}$$

$$|1-\mu| \le C\mathring{\epsilon}(1+t)^{1/2}; |1-r+t-u| \le C\mathring{\epsilon}(1+t)^{1/2} \tag{22}$$

Here C is a constant dependent only on u_0, w and $h^{\mu\nu}$.

Proof. Our proof consists of a continuity argument. Let $B \subset [0, T_L)$ be the set of times such that (20), (21) and (22) hold with $C\mathring{\epsilon}$ replaced by $\mathring{\epsilon}^{1/2}$. By local existence results we can pick ϵ small enough so that this set is non-empty, connected and relatively closed. We now proceed to improve these bootstrap assumptions to show that the set is relatively open as well.

Note that under our bootstrap assumptions we know that $|h^{\mu\nu}(\Psi)| \lesssim \mathring{\epsilon}^{1/2}(1+t)^{-1/2}$, $|g^{\mu\nu}| \simeq 1$, $|P^{ij}| \lesssim 1$ and $r \simeq t - u + 1 \simeq 1 + t$. Particularly, the last similarity allows us to replace any of the $1+t$ terms in the inequalities (20), (21) and (22) with r. Also remember that r is bounded below by our assumptions on the initial range of u, so $(1+t)^{-1} \simeq r^{-1} \lesssim 1$.

We'll start by improving our estimates on Ψ_σ and Ψ_ρ. Plugging our bootstrap assumptions into (2) and (2) gets us the following.

$$\mu\underline{L}L(r^{1/2}(\Psi_\sigma + \Psi_\rho)) = O(\mathring{\epsilon}r^{-3/2}) + \frac{-1}{4}\mu r^{-3/2}(\Psi_\sigma + \Psi_\rho)$$

$$\mu\underline{L}L(r^{1/2}(\Psi_\sigma - \Psi_\rho)) = O(\mathring{\epsilon}r^{-3/2}) + \frac{-3}{4}\mu r^{-3/2}(\Psi_\sigma - \Psi_\rho)$$

We can integrate these expressions with respect to u on $[0, u_0]$, noting that $\mu\underline{L}u = 2$. $|\partial_u t| = |\partial_t u|^{-1} = |\mu| \lesssim (1+t)^{-1/2}$ by our bootstrap assumptions. Since we are integrating u over a compact domain, we can bound the result by some constant dependent on U_0 multiplied by a bound on the supremum of the integrand and a bound on $\partial_u t$. Note that we integrate to $u = 0$ where the solution vanishes by assumption. Thus, integration gives us the following.

$$|L(r^{1/2}(\Psi_\sigma + \Psi_\rho))| \lesssim \mathring{\epsilon}(1 + \mathring{\epsilon}^{1/2}r^{1/2})r^{-3/2} + (1 + \mathring{\epsilon}^{1/2}r^{1/2})^2 r^{-2}\|r^{1/2}(\Psi_\sigma + \Psi_\rho)\|_{C^0(\Sigma_t)}$$

$$|L(r^{1/2}(\Psi_\sigma - \Psi_\rho))| \lesssim \mathring{\epsilon}(1 + \mathring{\epsilon}^{1/2}r^{1/2})r^{-3/2} + (1 + \mathring{\epsilon}^{1/2}r^{1/2})^2 r^{-2}\|r^{1/2}(\Psi_\sigma - \Psi_\rho)\|_{C^0(\Sigma_t)}$$

Now we integrate the inequalities above along the integral curves C_u of L, noting that by our bootstrap assumptions we have $L \simeq \partial_t + \partial_r$. Thus integrating with respect to L should give us (up to constant) the above bound integrated with respect to r or,equivalently, with respect to t since $\frac{dr}{dt} \simeq 1$ along these integral curves. Thus, coupled with the smallness of the initial data, integration gives us the following inequality.

$$\|r^{1/2}(\Psi_\sigma + \Psi_\rho)\|_{C^0}(\Sigma_t) \lesssim \mathring{\epsilon}(1 + \mathring{\epsilon}^{1/2}\lg(r)) + \int_{u_0}^{r}(1 + \mathring{\epsilon}^{1/2}s^{1/2}))^2 s^{-2}\|s^{1/2}(\Psi_\sigma + \Psi_\rho)\|_{\Sigma_s}ds \tag{23}$$

$$\|r^{1/2}(\Psi_\sigma - \Psi_\rho)\|_{C^0}(\Sigma_t) \lesssim \mathring{\epsilon}(1 + \mathring{\epsilon}^{1/2}\lg(r)) + \int_{u_0}^{r}(1 + \mathring{\epsilon}^{1/2}s^{1/2}))^2 s^{-2}\|s^{1/2}(\Psi_\sigma - \Psi_\rho)\|_{\Sigma_s}ds \tag{24}$$

Applying Gronwall's lemma, the equivalence $r \simeq t - u + 1 \simeq 1 + t$ and the assumption that $t \lesssim \exp(\mathring{\epsilon}^{-1/2})$ then yields bounds on $\Psi_\sigma + \Psi_\rho$ and $\Psi_\sigma - \Psi_\rho$.

$$\|(1+t)^{1/2}(\Psi_\sigma + \Psi_\rho)\|_{C^0}(\Sigma_t) \lesssim \mathring{\epsilon}(1 + \mathring{\epsilon}^{1/2}\lg(1+t))\exp\left(\int_0^t(1 + \mathring{\epsilon}^{1/2}(1+s)^{1/2}))^2(1+s)^{-2}ds\right) \lesssim \mathring{\epsilon}$$

$$\|(1+t)^{1/2}(\Psi_\sigma - \Psi_\rho)\|_{C^0}(\Sigma_t) \lesssim \mathring{\epsilon}(1 + \mathring{\epsilon}^{1/2}\lg(1+t))\exp\left(\int_0^t(1 + \mathring{\epsilon}^{1/2}(1+s)^{1/2}))^2(1+s)^{-2}ds\right) \lesssim \mathring{\epsilon}$$

These give us obvious estimates for Ψ_σ and Ψ_ρ themselves, i.e $|\Psi_\sigma|, |\Psi_\rho| \lesssim \mathring{\epsilon}(1+t)^{-1/2}$.

Thus we have our desired estimate for Ψ_ρ and Ψ_σ. We can apply these bounds to improve our estimates on $L\Psi_\sigma$ and $L\Psi_\rho$, by plugging them into (16) and (17). Along with our bootstrap assumptions, this gives us the following.

$$\mu\underline{L}(r^{3/2}L\Psi_\rho) = O(\mathring{\epsilon}r^{-1/2}) + O(r^{-1/2}\mu\Psi_\rho) - \frac{1}{2}\mu\underline{L}(r^{1/2}\Psi_\rho)$$

$$\mu\underline{L}(r^{1/2}L\Psi_\sigma) = O(\mathring{\epsilon}r^{-3/2}) + O(r^{-1/2}\mu\Psi_\sigma) - \frac{3}{2}\mu\underline{L}(r^{1/2}\Psi_\sigma)$$

Integrating both of these with respect to u and using the same approximations as before, we then get the following estimates within the range of t to which we are restricted.

$$\|(1+t)^{3/2}L\Psi_\rho\|_{C^0(\Sigma_t)} = O(\mathring{\epsilon}) + ((1+t)^{1/2} + \mu)\|\Psi_\rho\|_{C^0(\Sigma_t)} \lesssim \mathring{\epsilon}$$

$$\|(1+t)^{1/2}L\Psi_\sigma\|_{C^0(\Sigma_t)} = O(\mathring{\epsilon}(1+t)^{-1}) + O((1+t)^{-3/2}\mu + (1+t)^{-1/2})\|\Psi_\sigma\|_{C^0(\Sigma_t)} \lesssim \mathring{\epsilon}(1+t)^{-1}$$

This yields our desired estimates for $L\Psi_\sigma$ and $L\Psi_\rho$. These in hand, we at last get to the $\mu\underline{L}\Psi_\sigma$ and $\mu\underline{L}\Psi_\rho$ bounds. Plugging the results above into (14) and (3) gives us the following

$$|L(r^{1/2}\mu\underline{L}\Psi_\rho)| = O(\mathring{\epsilon}r^{-3/2}); |\mu\underline{L}\Psi_\rho| \lesssim \mathring{\epsilon}(1+t)^{-1/2}$$

$$|\mu\underline{L}\Psi_\sigma| \lesssim \mathring{\epsilon}r^{-3/2} + O(\sqrt{g} - g^{tr} - 1) \cdot O(\mu\underline{L}\Psi_\rho) \lesssim \mathring{\epsilon}(1+t)^{-1}$$

This finishes our improvements on (20) and (21). Given these estimates, we can make quick work of (22). First, we see that (12) yields the inequality $|L\mu| \leq C\mathring{\epsilon}(1+t)^{-1/2}$ immediately. Integrating this and using the fact that $\mu = 1$ when $t = 0$, we get the first inequality of (22). Finally, we see that $|L(1-r+t-u)| = O(\Psi)$ and thus $|L(1-r+t-u)| \leq C\mathring{\epsilon}(1+t)^{-1/2}$. Integrating this inequality from $t = 0$ onward and noting that $1 - r = u$ at $t = 0$, we acquire the final inequality (22), closing our bootstrap argument. □

There are several important points which follow closely from the above proof. First, it is important to note that with a better estimate on μ, we could have achieved global estimates which beat (20),(21) and (22). We can make this rigorous as so.

Corollary 3.1. *If $|\sqrt{g} - g^{rt} - 1| = O(\Psi_\rho^2) + O(\Psi_\sigma)$,*[4] *then we can strengthen estimates (20),(21) and (22) such that they hold on all $M(T, u_0, w)$ with $t \leq T_L$. Additionally, we can get the estimates $|1-\mu| \leq C\mathring{\epsilon}\ln(e+t)$, $|\mu\underline{L}\Psi_\sigma| \leq C\mathring{\epsilon}\lg(e+t)(1+t)^{-3/2}$ and $|\Psi_\sigma| \leq C\mathring{\epsilon}\lg(e+t)(1+t)^{-1}$.*

Proof. If we assume that $|\sqrt{g} - g^{rt} - 1| = O(\Psi_\rho^2) + O(\Psi_\sigma)$ and use the improved estimates from above in our bootstrap assumptions, then we can follow the argument given in the proposition above. When we do so, we see that (23) instead has the following form.

$$\|(1+t)^{1/2}\Psi_\sigma\|_{C^0}(\Sigma_t) \lesssim \mathring{\epsilon} + \int_0^t (1 + \mathring{\epsilon}^{1/2}\lg(e+s))^2(1+s)^{-2}\|(1+s)^{1/2}\Psi_\sigma\|_{C^0(\Sigma_s)}ds$$

Using Gronwall's lemma on this inequality leads to a global estimate $|\Psi_\sigma| \leq C(1+t)^{-1/2}$. The same improvement occurs in the Gronwall argument bounding $|\Psi_\rho|$. This then gives us our other estimates from (20) and (21), which did not rely on our time bound. Using the formula (**??**) and the fact that $|\sqrt{g} - g^{rt} - 1| = O(\Psi_\rho^2) + O(\Psi_\sigma)$, we can then close the bootstrap argument for $\mu\underline{L}\Psi_\sigma$, integrate to get the new estimate for $|\Psi_\sigma|$ and then plug these in to close the estimate for $|1-\mu| \leq C\mathring{\epsilon}\log(e+t)$. □

Aside from this improvement, we will also require the following auxilary estimates, which follow immediately from our main estimates. These will help us control error terms when we examine μ in detail in a forthcoming section.

Corollary 3.2. *Choose u_0, w and other conditions in keeping with Proposition 3.1, and let $\mathring{\epsilon}^{-3/2} \leq 1 + s < 1 + t \leq \min(T_L, \exp(\mathring{\epsilon}^{-2}))$. Then, at any point $(t, u) \in \Sigma_t$ we have the following estimates.*

(25) $$|[r^{1/2}\mu\underline{L}\Psi_\rho](u,t) - [r^{1/2}\mu\underline{L}\Psi_\rho](u,s)| \leq C\mathring{\epsilon}^{3/2}$$

[4]By this we mean that the highest order Ψ_ρ term is order 2 or greater in the function $\sqrt{g} - g^{rt}$ of Ψ, and likewise the highest order Ψ_σ term is order 1 or greater

Here C depends only on the constants in Proposition 3.1.

Proof. Here we just note that, since $r^{-1} \simeq (1+t)^{-1} \leq \mathring{\epsilon}^{3/2}$, we can plug the estimates on $|L\Psi_\rho|$ from Proposition 3.1 into the (14), integrate along the characteristics C_u and use our upper-bound on $t \leq \min(T_L, \exp(\mathring{\epsilon}^{-2}))$ to get the desired bound. $\square$

Corollary 3.3. *Choosing T, u_0, w and other conditions in keeping with Corollary 3.1, and let $\mathring{\epsilon}^{-3/2} \leq 1+s \leq 1+t$. Then at any point $(t,u) \in M(T,u_0,w)$ we have the following estimates.*

$$|[r^{1/2}\mu\underline{L}\Psi_\rho](u,t) - [r^{1/2}\mu\underline{L}\Psi_\rho](u,s)| \leq C\mathring{\epsilon}^{3/2} \tag{26}$$

$$|[r^{1/2}\Psi_\rho](u,t) - [r^{1/2}\Psi_\rho](u,s)| \leq C\mathring{\epsilon}^{3/2} \tag{27}$$

$$|[r\Psi_\rho\mu\underline{L}\Psi_\rho](u,t) - [r\Psi_\rho\mu\underline{L}\Psi_\rho](u,s)| \leq C\mathring{\epsilon}^{5/2} \tag{28}$$

Here C is a constant dependent only on u_0, w and $h^{\mu\nu}$.

Proof. To prove (26) we simply repeat the argument from Corollary 3.2 using the improved estimates from Corollary 3.1. For (27), we can plug our previously derived estimates into a variant of the main equations (14 - 2) to note the following.

$$|\mu\underline{L}L(r^{1/2}\Psi_\rho)| \lesssim \mathring{\epsilon}^2(1+t)^{-3/2} + \mathring{\epsilon}(1+t)^{-2} \lesssim \mathring{\epsilon}^{3/2}(1+t)^{-3/2}$$

Integrating this inequality twice, first in terms of u and then along the characteristics C_u from $\mathring{\epsilon}^{-1}$ to t, we confirm the estimate (27). At last, we note the following.

$$|L(r\Psi_\rho\mu\underline{L}\Psi_\rho)| \lesssim |L(r^{1/2}\Psi_\rho)r^{1/2}\mu\underline{L}\Psi_\rho| + |r^{1/2}\Psi_\rho L(r^{1/2}\mu\underline{L}\Psi_\rho)| \lesssim \mathring{\epsilon}|L(r^{1/2}\Psi_\rho)| + \mathring{\epsilon}|L(r^{1/2}\mu\underline{L}\Psi_\rho)|$$

Now we just apply the previous estimates on $|L(r^{1/2}\Psi_\rho)|$ and $|L(r^{1/2}\mu\underline{L}\Psi_\rho)|$ used to derive (26) and (27) and integrate to get (28). $\square$

4. Approximation Of Φ By A Radiation Field In The Medium Term

A critical aspect of our results is the connection that they provides between the asymptotic behavior of μ and the initial data of the corresponding solution. A key quantity governing this relationship, which will be the subject of this section, is the classic *Friedlander radiation field* $F_0[\mathring{\Phi}, \dot{\Phi}]$, which is the first component of the Taylor expansion at null infinity of the solution to the linear equation $\Box\tilde{\Phi} = 0$ with initial data $\mathring{\Phi}, \dot{\Phi}$. That is to say, F_0 described the long-term behavior of solutions to the linear wave equation in the limit of $r+t \to \infty$. The formula for F_0 is given by the following convolution.

$$F_0[\mathring{\Phi}, \dot{\Phi}](r-t) := \frac{1}{\sqrt{8\pi}}\chi_-^{-1/2} * (R[\dot{\Phi}] - R[\mathring{\Phi}]) \tag{29}$$

Here R is the 2-dimensional radon transform, $*$ denotes the convolution and $\chi_-^{-1/2}$ is defined as so.

$$\chi_-^{-1/2}(s) := \begin{cases} (-s)^{-1/2} & s < 0 \\ 0 & \text{else} \end{cases}$$

Our goal in this section will be to show that the radiation field provides a good approximation of the solution to Φ, the solution to our non-linear equation, at time $\mathring{\epsilon}^{-3/2}$ and in a strip of some arbitrary fixed width u_0 around the light-cone, i.e in $\Sigma_{\mathring{\epsilon}^{-3/2}}$ for some choice of u_0 and w. Establishing this type of estimate will rely on a short-time estimate dependent on well-posedness, which we begin with here.

We know from general local well-posedness results (see, for instance, [8]) that for any data $\mathring{\Phi}, \dot{\Phi}$ we have the following estimate.

Proposition 4.1. *Let Φ be a solution to (1) for smooth data with $\|\mathring{\Phi}\|_{C^3}, \|\overset{\bullet}{\Phi}\|_{C^2} \lesssim \mathring{\epsilon}$. Then for any $T_0 > 0$ there exists an ϵ such that $\mathring{\epsilon} \leq \epsilon$ implies $\|\Phi\|_{C^3} \leq C\mathring{\epsilon}$ on $[0, T_0] \times \mathbb{R}$ where C is a constant independent of the data.*

This allows us to show that Φ, the solution to our non-linear equation, stays close to Φ', the solution to the linear wave equation for short time and small initial data, as so.

Proposition 4.2. *Let $T_0, \epsilon, \Phi, \mathring{\epsilon}$ be as in Proposition 4.1, and let Φ' be the solution to $\Box\Phi = 0$ with the same initial data as Φ. Then within the time range $t \in [0, T_0]$, we have that $\|\Phi - \Phi'\|_{C^2(\Sigma_t)}, \|\Phi - \Phi'\|_{C^2(\Sigma_t)} \leq C\mathring{\epsilon}^2$ and supp$([\Phi - \Phi']|_{\Sigma_t}) \subset [0, t+1]$. Here C is independent of the initial data but dependent on T_0.*

Proof. The statement about the support of $\Phi - \Phi'$ follows from the fact that Φ and Φ' are both supported in those intervals.

Now we acquire the estimate given above. We know by finite propagation speed that, within this time frame, the functions Φ, Φ' and $\Phi - \Phi'$ will be compactly supported (spatially) within $\{r \leq T_0 + 1\}$. Now we study the function $\Phi - \Phi'$, noting that it solves the in inhomogenous wave equation $\Box(\Phi - \Phi') = F(\partial\Phi, \partial^2\Phi)$ with trivial initial data and F quadratic order in $\partial\Phi$ and $\partial^2\Phi$. The following estimate, which applied standard Sobolev inequalities and energy estimates, gives us our result.

$$\|\Phi - \Phi'\|_{C^2(\Sigma_t)} \lesssim \|\Phi - \Phi'\|_{H^3(\Sigma_t)} \lesssim \|\Phi - \Phi'\|_{L^2(\Sigma_t)} + (E_3(\Phi - \Phi'))^{1/2}$$

$$\lesssim \|\Phi - \Phi'\|_{L^2(\Sigma_0)} + \int_0^{T_0} (E_0[\Phi - \Phi'](s))^{1/2} ds + (E_3[\Phi - \Phi'](0))^{1/2} + \int_0^{T_0} \|F\|_{H^3} ds$$

$$= (T_0 + 1) \int_0^{T_0} \|F\|_{H^3} \lesssim \int_0^{T_0} T_0 \|F\|_{C^2(\Sigma_t)} \leq C\mathring{\epsilon}^2$$

Here $E_3[\Phi - \Phi'] = \sum_{|\alpha| \leq 3} E_0[\partial^\alpha(\Phi - \Phi')]$ and $E_0[\Phi - \Phi'](t)$ is simply the standard energy at time t. Above we apply the Sobolev inequality in dimension 2, then a standard coercive estimate for bounding Sobolev norms in terms of higher energies and the L^2 norm, then an integral estimate for the energy of $\Phi - \Phi'$ in terms of the initial higher energies and an integral of the forcing term F. We then apply the fact that the initial data for $\Phi - \Phi'$ is trivial, the fact that $|F| = O((\partial\Phi)^2) + O((\partial^2\Phi)^2) + O(|\partial\Phi \cdot \partial^2\Phi|)$ and our estimates on $\partial\Phi, \partial^2\Phi$ and the fact that the C^2 norm bounds the H^2 norm on a compact domain (where $[0, t]$ for $t \leq T_0$) to get our final result. $\Box$

Proposition 4.3. *Let Φ be a solution to (1) with initial data $\mathring{\Phi}$ and $\overset{\bullet}{\Phi}$ of size $\mathring{\epsilon}$. Let $F_0 := F_0[\mathring{\Phi}, \overset{\bullet}{\Phi}]$ be the radiation field correspond to this initial data. Choose a w, u_0 as above, and consider the time slices Σ_t of $M(T, u_0, w)$. Let $\mathring{\epsilon}$ be chosen smaller than ϵ such that we have the estimates in Proposition 4.2 hold for $T_0 = 2w$. Then when $t = \mathring{\epsilon}^{-3/2}$, $|\underline{L}_{flat}^2 F_0 - r^{1/2}\underline{L}_{flat}^2 \Phi| \leq C\mathring{\epsilon}^{5/4}$ in Σ_t.*

Proof. Let $\tilde{\Phi}$ be the solution to the linear wave equation with initial data $(\mathring{\Phi}, \overset{\bullet}{\Phi})$. We will divide the proof of our statement into three comparison steps. First, we demonstrate that $|[r^{1/2}\underline{L}_{\text{flat}}^2(\Phi - \tilde{\Phi})](\mathring{\epsilon}^{-3/2}, r)| \lesssim \mathring{\epsilon}^2$, then that $|[r^{1/2}\underline{L}_{\text{flat}}^2(\tilde{\Phi} - r^{-1/2}F_0)](\mathring{\epsilon}^{-3/2}, r)| \lesssim \mathring{\epsilon}^{3/2}$ and then that $|[\underline{L}_{\text{flat}}^2 F_0 - r^{1/2}\underline{L}_{\text{flat}}^2(r^{-1/2}F_0)](\mathring{\epsilon}^{-1}, r)| \lesssim \mathring{\epsilon}^{3/2}$.

For our first estimate, we first must consider our solution at time $T_0 = 2w$. By Proposition 4.2 and standard estimates on the wave equation we know that the Φ restricted to the time slice $\{t = T\}$ is bounded in size by $\mathring{\epsilon}$. Thus, if we define the eikonal strip $M(T, u_0, w) \cap \{t \geq T_0\}$ beginning at time $T_0 = 2w$ with u being defined as $w - r$ at $t = T_0$, we get the decay estimates afforded by Proposition 3.1. Note that we also have these decay estimates for the linear wave equation $\tilde{\Phi}$ in the strip $M(T,u_0,w)\cap\{t\geq T_0\}$ through standard dispersive results.

With this in mind, we proceed in proving $|[r^{1/2}\underline{L}^2_{\text{flat}}(\Phi-\tilde{\Phi})](\mathring{\epsilon}^{-3/2},r)|\lesssim\mathring{\epsilon}^2$ by setting up a system of equations of the following form for the functions $\tilde{\Psi}_t:=\partial_t(\Phi-\tilde{\Phi})$ and $\tilde{\Psi}_t:=\partial_r(\Phi-\tilde{\Phi})$. We will examine this equation *only in the strip* $M(T,u_0,w)\cap\{t\geq T_0\}$.

$$L(r^{1/2}\underline{L}\tilde{\Psi}_t)=\frac{1}{2}r^{-1/2}L\tilde{\Psi}_t+O(\mathring{\epsilon}^2r^{-1/2});\underline{L}(r^{1/2}L\tilde{\Psi}_t)=\frac{1}{2}r^{-1/2}\underline{L}\tilde{\Psi}_t+O(\mathring{\epsilon}^2r^{-1/2})$$

$$L(r^{1/2}\underline{L}\tilde{\Psi}_r)=\frac{1}{2}r^{-1/2}L\tilde{\Psi}_r-\frac{1}{2}r^{-3/2}\tilde{\Psi}_r+O(\mathring{\epsilon}^2r^{-1/2});\underline{L}(r^{1/2}L\tilde{\Psi}_r)=r^{-1/2}\underline{L}\tilde{\Psi}_r-\frac{1}{2}r^{-3/2}\tilde{\Psi}_r+O(\mathring{\epsilon}^2r^{-1/2})$$

Here the $O(\mathring{\epsilon}^2r^{-1/2})$ terms come from quadratic pieces in terms of Ψ, the solution to the non-linear system for which we already have estimates within $M(T,u_0,w)\cap\{t\geq T_0\}$, since $\mu\simeq 1$ within the strip $M(T,u_0,w)$ in the time range $[0,\mathring{\epsilon}^{-3/2}]$. Now we can mimic the proof of Proposition 3.1 to get similar estimates for $\tilde{\Psi}$ within the same space-time region. First, we integrate along the $\underline{L}$ (i.e with respect to u, to the outside of the light-cone where $\tilde{\Psi}$ vanishes), we get the following estimates.

$$\|L\tilde{\Psi}_t\|_{C^0(\Sigma_t)}\lesssim\mathring{\epsilon}^2r^{-1}+r^{-1}\|\underline{L}\tilde{\Psi}_t\|_{C^0(\Sigma_t)};\|L\tilde{\Psi}_r\|_{C^0(\Sigma_t)}\lesssim\mathring{\epsilon}^2r^{-1}+r^{-1}\|\underline{L}\tilde{\Psi}_r\|_{C^0(\Sigma_t)};$$

$$\|L(r^{1/2}\underline{L}\tilde{\Psi}_t)\|_{C^0(\Sigma_t)}\lesssim\mathring{\epsilon}^2r^{-1/2}+r^{-3/2}\|\underline{L}\tilde{\Psi}_t\|_{C^0(\Sigma_t)};\|L(r^{1/2}\underline{L}\tilde{\Psi}_r)\|_{C^0(\Sigma_t)}\lesssim\mathring{\epsilon}^2r^{-1/2}+r^{-3/2}\|\underline{L}\tilde{\Psi}_r\|_{C^0(\Sigma_t)}$$

Integrating the last line of inequalities along the characteristics C_u from $t=T_0$ to $t=\mathring{\epsilon}^{-3/2}$, using our bound on t and applying the estimate from Proposition 4.2 which bounds the size of our initial data at $t=T_0$, we get the following inequalities.

$$\|r^{1/2}\underline{L}\tilde{\Psi}_t\|_{C^0(\Sigma_t)}\lesssim\mathring{\epsilon}^{5/4}+\int_0^t(1+s)^{-2}\|r^{1/2}\underline{L}\tilde{\Psi}_t\|_{C^0(\Sigma_s)}\,ds$$

$$\|r^{1/2}\underline{L}\tilde{\Psi}_r\|_{C^0(\Sigma_t)}\lesssim\mathring{\epsilon}^{5/4}+\int_0^t(1+s)^{-2}\|r^{1/2}\underline{L}\tilde{\Psi}_r\|_{C^0(\Sigma_s)}\,ds$$

This integral exploits the fact that the initial data of $\tilde{\Psi}$ is small, by assumption. Applying Gronwall's lemma to this inequality then gets us the estimates $|L\Psi_t|\lesssim\mathring{\epsilon}^{5/4}(1+t)^{-1/2}$ and $|L\Psi_r|\lesssim\mathring{\epsilon}^{5/4}(1+t)^{-1/2}$. Plugging this estimate back into the estimates on $L\tilde{\Psi}_t$ and $L\tilde{\Psi}_r$ gets us the estimates $|L\Psi_t|\lesssim\mathring{\epsilon}^{5/4}(1+t)^{-1}$ and $|L\Psi_r|\lesssim\mathring{\epsilon}^{5/4}(1+t)^{-1}$. Collectively these estimates imply the estimate on $|r^{1/2}\underline{L}^2_{\text{flat}}(\Phi-\tilde{\Phi})|(\mathring{\epsilon}^{-3/2},r)|$ that we wanted.

The other estimates are either straight forward or citable. Our second estimate is proven as Theorem 6.2.1 in [4], which states that, in our strip of interest, we have the estimate $|\partial^\alpha Z(\tilde{\Psi}'-F_0/r^{1/2})|\lesssim C(1+t)^{-3/2}$. Our third estimate is obvious given that $r\simeq\mathring{\epsilon}^{-3/2}$ and $\underline{L}_{\text{flat}}(r^{-1/2})=-r^{-3/2}$. □

Proposition 4.4. *Let* Φ *be a solution to (1) with initial data* $\mathring{\Phi}$ *and* $\dot{\mathring{\Phi}}$ *of size* $\mathring{\epsilon}$. *Let* $F_0:=F_0[\mathring{\Phi},\dot{\mathring{\Phi}}]$ *be the radiation field correspond to this initial data. Choose a* w,u_0 *as above, and consider the time slices* Σ_t *of* $M(T,u_0,w)$. *Let* $\mathring{\epsilon}$ *be chosen smaller than* ϵ *such that we have the estimates in Proposition 4.2 hold for* $T_0=2w$. *Then when* $t=\mathring{\epsilon}^{-3/2}$, *in* Σ_t *we have:*

$$|\underline{L}_{flat}F_0\cdot\underline{L}^2_{flat}F_0-r\underline{L}_{flat}\Phi\underline{L}^2_{flat}\Phi|\leq C\mathring{\epsilon}^{5/2}$$

Proof. I'm going to write this in later. It should definitely be true that $|\underline{L}_{\text{flat}}F_0\cdot\underline{L}^2_{\text{flat}}F_0-r\underline{L}_{\text{flat}}\Phi\underline{L}^2_{\text{flat}}\Phi|\leq C\mathring{\epsilon}^{2+}$ by a proof similar to that of the previous proposition, and the value of 2+ doesn't actually matter in the proofs where we use this. □

5. A Detailed Examination of μ

Having derived the necessary estimates above, we are now free to analyze the behavior of μ in depth. Before we begin, however, let's discuss the parts of the non-linear term $h^{\mu\nu}(\Psi)$ and the initial data $\mathring{\Psi}, \dot{\Psi}$ that will play an important part in the emergence of shocks at small scales.

We can isolate the shock forming effects of $h^{\mu\nu}(\Psi)$ via the Taylor expansion of the quantity $\sqrt{g} - g^{rt}$ derived from the perturbation $h^{\mu\nu}$. As it turns out, the taylor coefficients $c_1 := \partial_\rho[\sqrt{g} - g^{rt}](\Psi = 0)$ and $c_2 := \partial_\rho^2[\sqrt{g} - g^{rt}](\Psi = 0)$ are the only important quantities that the non-linearity contributes to shock development in all cases. Due to the estimates derived above, the other terms be suppressed by sufficiently fast decay, at least in the small data regime.

In our radially symmetric context, it is important to note that the condition that $c_1 = 0$ and the condition that $c_1, c_2 = 0$ are equivalent to the *1st null condition* and *2nd null condition* being satisfied respectively. The 1st null-condition, historically called the "classic" null condition due to its roll in the 3-dimensional theory, theis simply the condition that $\{x^\mu | \eta^{\mu\nu} x^\mu x^\nu = 0\} \subset \{x^\mu | h^{\mu\nu\alpha} x^\mu x^\nu x^\alpha\}$ where here $\eta^{\mu\nu}$ is the 1+1 Minkowski metric, i.e the the light-cone is a subset of the kernel of $h^{\mu\nu\alpha}$ viewed as a multi-linear form on $\mathbb{R}^{1+1}$. Likewise, the 2nd null condition is satisfied when $\{x^\mu | \eta^{\mu\nu} x^\mu x^\nu = 0\} \subset \{x^\mu | h^{\mu\nu\alpha} x^\mu x^\nu x^\alpha\}$ and $\{x^\mu | \eta^{\mu\nu} x^\mu x^\nu = 0\} \subset \{x^\mu | h^{\mu\nu\alpha\beta} x^\mu x^\nu x^\alpha x^\beta\}$. An alternate (and possibly more enlightening point of view) is that the 1st null condition states that we can factorize the Taylor expansion as so $g^{\mu\nu}$ as $(1 + a^\alpha \Psi_\alpha)\eta^{\mu\nu} + O(\Psi^2)$, and with the 2nd null condition $g^{\mu\nu}$ as $(1 + a^\alpha \Psi_\alpha + a^{\alpha\beta}\Psi^\alpha\Psi^\beta)\eta^{\mu\nu} + O(\Psi^3)$. Thus, in some sense, the null conditions measures to what order (in Ψ) $g^{\mu\nu}$ is conformal to $\eta^{\mu\nu}$.

The important aspects of the initial data at small scales will be captured appropriately by the combinations of the functions $\mathring{\Psi}_\rho(r)/\mathring{\epsilon}$ and $\underline{L}_{\text{flat}}\Psi_\rho(r)/\mathring{\epsilon}$. Note that the rescaling by $\mathring{\epsilon}$ will later allow us to get a description of μ that is invariant with scaling, and which is thus easier to discuss in a context where we can and will flippantly rescale our initial data.

The effects of the non-linearities and the data will work in concert via the following two key, scale-invariant functions of u.

$$S_1[h, \mathring{\Psi}, \dot{\Psi}](u) = c_1 \cdot \mathring{\epsilon}^{-1} r^{1/2} [\underline{L}_{\text{flat}} \Psi_\rho](T, w - u) = c_1 \cdot \mathring{\epsilon}^{-1} r^{1/2} [\underline{L}_{\text{flat}}^2 \Phi](T, w - u) \tag{30}$$

$$S_2[h, \mathring{\Psi}, \dot{\Psi}](u) = c_2 \cdot \mathring{\epsilon}^{-2} r \Psi_\rho \cdot \underline{L}_{\text{flat}}[\Psi_\rho](T, w - u) = c_2 \cdot \mathring{\epsilon}^{-2} r \underline{L}_{\text{flat}}\Phi \cdot \underline{L}_{\text{flat}}^2 \Phi(T, w - u) \tag{31}$$

We will abbreviate these as $S_1(u)$ and $S_2(u)$. Having defined these key quantities, we are now prepared to give a nice description of μ's behavior.

Proposition 5.1. *Let $u_0, w, \epsilon, h^{\mu\nu}, \Psi, \mathring{\epsilon}$ and ϵ be as above. Furthermore, let F_0 be defined as the radiation field corresponding to data close to the initial data of Ψ, in the same sense as in Proposition 4.3. We then have one of the following formulae for $\mu(t, u)$ in the following 3 cases.*

1. *$h^{\mu\nu}$ does not satisfy either null condition. Thus we have formula (38) for $\mathring{\epsilon}^{-3/2} \leq t \leq \min(\exp(\mathring{\epsilon}^{-1/2}), T_L)$.*
2. *$h^{\mu\nu}$ satisfies the 1st null condition, but not the 2nd. Thus we have formula* (40) *for $\mathring{\epsilon}^{-3/2} \leq t \leq T_L$.*
3. *$h^{\mu\nu}$ satisfies the 2nd null condition, $\mu(t, u) = 1 + O(\mathring{\epsilon})$ for $\mathring{\epsilon}^{-3/2} \leq t \leq T_L$.*

Proof. The key to this description is the formula (12). This expression, along with our estimates from the previous section, yields the following identity.

$$L\mu = \mu \underline{L}\Psi_\rho \cdot (c_1 + c_2 \Psi_\rho) + O(\Psi_\rho^2 \cdot \mu \underline{L}\Psi_\rho) + O(\Psi_\sigma \cdot \mu \underline{L}\Psi_\rho) + O(\mu \underline{L}\Psi_\sigma) + O(\mu L \Psi_i) \tag{32}$$

Above, we divide the formula for $L\mu$ into pieces that decay at different rates according to our estimates in Proposition 3.1. We will ignore terms that decay so fast that they will remain negligible when we integrate the formula above to get a formula for μ. The only terms that do not fall into this category are the terms with coefficients c_1 and c_2 given above.

To maintain control of $c_1 \cdot \mu\underline{L}\Psi_\rho$ and $c_2 \cdot \Psi_\rho\mu\underline{L}\Psi_\rho$ in our time range of interest, we'll recall Corollaries 3.2 and 3.3 in the following form.

$$[\mu\underline{L}\Psi_\rho](s,u) = \frac{r(t,u)^{1/2}[\mu\underline{L}\Psi_\rho](t,u)}{\sqrt{r(s,u)}} + O((1+s)^{-1/2}\mathring{\epsilon}^{3/2}); \tag{33}$$

$$[\Psi_\rho\mu\underline{L}\Psi_\rho](s,u) = \frac{r(t,u)[\Psi_\rho\mu\underline{L}\Psi_\rho](t,u)}{r(s,u)} + O((1+s)^{-1}\mathring{\epsilon}^{5/2}); \tag{34}$$

We also have the following important approximations telling us that, due the fact that $\underline{L} = \underline{L}_m + O(\mathring{\epsilon})$ and that $\mu \simeq 1$ at time $\mathring{\epsilon}^{-3/2}$, we know the following.

$$[r^{1/2}\mu\underline{L}\Psi_\rho](\mathring{\epsilon}^{-3/2},u) = [r^{1/2}\underline{L}_{\text{flat}}\Psi_\rho](\mathring{\epsilon}^{-3/2},u) + O(\mathring{\epsilon}^2) = [r^{1/2}\underline{L}^2_{\text{flat}}\Phi](\mathring{\epsilon}^{-3/2},u) + O(\mathring{\epsilon}^2) \tag{35}$$

$$[r\Psi_\rho\mu\underline{L}\Psi_\rho](\mathring{\epsilon}^{-3/2},u) = [r^{1/2}\underline{L}_{\text{flat}}\Psi_\rho](\mathring{\epsilon}^{-3/2},u) + O(\mathring{\epsilon}^3) = [r^{1/2}\underline{L}^2_{\text{flat}}\Phi](\mathring{\epsilon}^{-3/2},u) + O(\mathring{\epsilon}^3) \tag{36}$$

With these formulae in mind, we can begin to analyze the three cases given in the proposition statement. Rephrased in terms of the coefficients c_i defined above, these are $c_1 \neq 0$; $c_1 = 0$ and $c_2 \neq 0$; and $c_1 = c_2 = 0$. In the first case we are in the regime of Proposition 3.1 and Corollary 3.2, so (32) can be given as the following formula.

$$L\mu(s,u) = (1+O(\mathring{\epsilon}))(c_1\frac{r(t,u)^{1/2}[\mu\underline{L}\Psi_\rho](t,u)}{\sqrt{r(s,u)}} + \mathring{\epsilon}^{3/2}(1+s)^{-1/2}) + O(\mathring{\epsilon}(1+s)^{-1})$$

Integrating this formula from $s = \mathring{\epsilon}^{-3/2}$ to $s = t$, and using the fact that $\mu(\mathring{\epsilon}^{-3/2},u) = 1 + O(\mathring{\epsilon})$, gets us the following formula for μ when t is greater than $\mathring{\epsilon}^{-3/2}$.

$$\mu(t,u) = 1 + \frac{1}{2}c_1(r(t,u)^{1/2} - r(\mathring{\epsilon}^{3/2},u)^{1/2}) \cdot ([r^{1/2}\mu\underline{L}\Psi_\rho](t,u) + O(\mathring{\epsilon}^{3/2})) + O(\mathring{\epsilon}\ln(e+t)) \tag{37}$$

Applying (35) and then Proposition 4.3, we can then replace the above formula for μ with the formula below.

$$\mu(t,u) = 1 + \frac{1}{2}c_1(r(t,u)^{1/2} - r(\mathring{\epsilon}^{-3/2},u)^{1/2})([\underline{L}^2_{\text{flat}}F_0](u) + O(\mathring{\epsilon}^{5/4})) + O(\mathring{\epsilon}\ln(e+t)) \tag{38}$$

That is our first formula for μ. Now we move on to the second case, i.e $c_1 = 0$ and $c_2 \neq 0$. In this case, we are in the regime of Corollaries 3.1 and 3.3, and (32) can be written as so.

$$L\mu(s,u) = (1+O(\mathring{\epsilon}))(c_2\frac{r(t,u)[\Psi_\rho\mu\underline{L}\Psi_\rho](t,u)}{r(s,u)} + O((1+s)^{-1}\mathring{\epsilon}^{5/2})) + O(\mathring{\epsilon}(1+s)^{-5/4})$$

Integrating this formula from $s = \mathring{\epsilon}^{-3/2}$ to $s = t$, and using the fact that $\mu(\mathring{\epsilon}^{-3/2},u) = 1 + O(\mathring{\epsilon})$, gets us the following formula for μ when t is greater than $\mathring{\epsilon}^{-3/2}$.

$$\mu = 1 + c_2(\lg(r(t,u)) - \lg(r(\mathring{\epsilon}^{3/2},u)))([r\Psi_\rho\mu\underline{L}\Psi_\rho](t,u) + O(\mathring{\epsilon}^{5/2})) + O(\mathring{\epsilon}) \tag{39}$$

Then, applying (36) and Proposition 4.3, we have the following.

$$\mu = 1 + c_2(\lg(r(t,u)) - \lg(r(\mathring{\epsilon}^{3/2},u)))([\underline{L}_{\text{flat}}F_0](u) \cdot [\underline{L}^2_{\text{flat}}F_0](u) + O(\mathring{\epsilon}^{5/2})) + O(\mathring{\epsilon}) \tag{40}$$

This is our second formula for μ. Our last formula simply follows from the observation that in (12) all of the terms are $O(\mathring{\epsilon}(1+t)^{-5/4})$ and thus $\mu = 1 + O(\mathring{\epsilon})$ by integration along the characteristics C_u. ∎

6. A Description of Shock Development In All Cases

The formulae for μ derived in the last section will allow us to analyze shock formation by reducing our work to a study of $\underline{L}_{\text{flat}}^2 F_0$ and $\underline{L}_{\text{flat}} F_0 \cdot \underline{L}_{\text{flat}}^2 F_0$, which are give by simple integral formulae from our initial data.

Proposition 6.1. *Consider $F_0[\mathring{\Phi}, \overset{\bullet}{\Phi}]$ for initial data $\mathring{\Phi}, \overset{\bullet}{\Phi}$. $[\underline{L}_{flat}^2 F_0](-\rho)$ and $[\underline{L}_{flat} F_0 \cdot \underline{L}_{flat}^2 F_0](-\rho)$ have a negative minimal value and a positive maximal value if and only if $\mathring{\Phi}, \overset{\bullet}{\Phi}$ is non-trivial.*

Proof. Instead of studying $[\underline{L}_{\text{flat}}^2 F_0](-\rho)$ and $[\underline{L}_{\text{flat}} F_0 \cdot \underline{L}_{\text{flat}}^2 F_0](-\rho)$, we can look at $[\partial_\rho^2 F_0](\rho)$ and $[\partial_\rho((\partial_\rho F_0)^2)](\rho)$. Showing that these quantities achieve negative minima and positive maxima is evidently equivalent to showing the same statement for our original quantities, and the latter are simpler to work with. Also, it will be important to that we note some non-trivial facts about the Radon transform, notably that it is invertible and linear as a map between smooth functions on $\mathbb{R}^2$ supported in the ball of radius 1. The linearity and support are obvious from the definition. The regularity and bijectivity is proven, for example, in [3]

Now, the fact that trivial data correspond to $F_0 \equiv 0$ and thus $[\underline{L}_{\text{flat}}^2 F_0](-\rho) \equiv [\underline{L}_{\text{flat}} F_0 \cdot \underline{L}_{\text{flat}}^2 F_0](-\rho) \equiv 0$ identically is obvious from the formula (29) for F_0. We prove the implication in the opposite direction as so. Suppose that either $\partial_\rho^2 F_0$ or $\frac{1}{2}\partial_\rho((\partial_\rho F_0)^2)$ has either a non-negative minimum or a non-positive maximum, or in other words that one of these functions is either everywhere non-negative or everywhere non-positive. Then note that $\lim_{\rho\to\pm\infty}(\partial_\rho^n F_0)^p = 0$. This can be seen immediately by examining (29), then noting that $\lim_{\rho\to\pm\infty} \chi_+^{-1/2}(\rho) = 0$. This implies that the total integral $\int_{-\infty}^{\infty} \partial_\rho((\partial_\rho^n F_0)^p) = 0$ and thus that $\partial_\rho(\partial_\rho^n F_0)^p \geq 0$ or $\partial_\rho(\partial_\rho^n F_0)^p \leq 0$ everywhere implies that $\partial_\rho^n F_0 = 0$ identically. Applying this to $\partial_\rho^2 F_0$ and $\frac{1}{2}\partial_\rho((\partial_\rho F_0)^2)$, this implies that $F_0 \equiv 0$ in any of the cases that we're considering.

Thus, suppose that $F_0 \equiv 0$ identically. Now we show that this implies that $R[\overset{\bullet}{\Phi}] - R'[\overset{\bullet}{\Phi}] \equiv 0$. Let $\mathcal{F}$ be the Fourier transform with $\mathcal{F}^{-1}$ its inverse. Then observe that $F_0 \simeq \chi_-^{-1/2} * (R[\overset{\bullet}{\Phi}] - R'[\overset{\bullet}{\Phi}]) \equiv 0$ if and only if $\mathcal{F}(\chi_-^{-1/2}) \cdot \mathcal{F}(R[\overset{\bullet}{\Phi}] - R'[\overset{\bullet}{\Phi}]) \equiv 0$. Now note that since $\chi_-^{-1/2}$ is a real valued homogeneous distribution with homogeneity $-\frac{1}{2}$, we have that $\mathcal{F}(\chi_-^{-1/2})$ is a homogeneous distribution of homogeneity $-1 + \frac{1}{2} = -\frac{1}{2}$ where $\mathcal{F}(\chi_-^{-1/2})(\xi) = [\mathcal{F}(\chi_-^{-1/2})(-\xi)]^*$ (where $*$ here denotes the complex conjugate). Moreover, this distribution is given by a function defined outside of $\{0\}$ since the Fourier transform converges for non-zero frequency ξ by an integration by parts argument. All of these facts about $\mathcal{F}(\chi_-^{-1/2})$ imply that it is non-zero everywhere but 0 and thus $\mathcal{F}(\chi_-^{-1/2}) \cdot \mathcal{F}(R[\overset{\bullet}{\Phi}] - R'[\overset{\bullet}{\Phi}]) \equiv 0$ implies that $\mathcal{F}(R[\overset{\bullet}{\Phi}] - R'[\overset{\bullet}{\Phi}]) \equiv 0$ since $\mathcal{F}(R[\overset{\bullet}{\Phi}] - R'[\overset{\bullet}{\Phi}])$ is the Fourier transform of a smooth bump function, and is thus Schwatrtz. Therefore, $R[\overset{\bullet}{\Phi}] - R'[\overset{\bullet}{\Phi}] \equiv 0$.

Now we have reduced our problem to the assumption that $R[\overset{\bullet}{\Phi}] - R[\mathring{\Phi}]' \equiv 0$. Now, observe that $R[\overset{\bullet}{\Phi}]$ is even by our observations about the Radon transform, while $R[\mathring{\Phi}]'$ is the derivative of an even function and thus odd. Thus, $R[\overset{\bullet}{\Phi}] - R[\mathring{\Phi}]' \equiv 0$ if and only if $R[\overset{\bullet}{\Phi}] \equiv 0$ and $R[\mathring{\Phi}]' \equiv 0$. The latter statement is equivalent to $R[\mathring{\Phi}] \equiv 0$ by integrating. Finally, the linearity of the Radon transform means that $\mathring{\Phi} \equiv 0$ and $\overset{\bullet}{\Phi} \equiv 0$, finishing our proof. □

Corollary 6.1. *$c_1[\underline{L}_{flat}^2 F_0](u)$ and $c_2[\underline{L}_{flat}^2 F_0](u)$ both achieve a negative minimum at some values of $\rho = t - r$, ρ_1 and ρ_2 respectively. Let $P_1 := \min(2, \rho_1)$ and $P_2 := \min(2, \rho_2)$.*

This corollary, coupled with the result from Proposition 5.1, guarantees the formation of shocks within a strip $M(T, u_0, w)$ near the light cone for data size $\mathring{\epsilon}$ small enough and appropriate choice of w, u_0. We demonstrate this fact with the following proof of Theorem 1.1.

Proof. We divide this into three cases, as in Proposition 5.1: $c_1 \neq 0$ and $c_2 \neq 0$; $c_1 = 0$ but $c_2 \neq 0$; and $c_1 = c_2 = 0$. In the first case, let P_1 be as in Corollary 6.1. If we let $T_0 = P_1$ in Proposition 4.3, choose ϵ accordingly and choose the size of the data $\mathring{\epsilon} \leq \epsilon$ then we have the formulae (37) and (38) for μ within the strip $M(T_0, 3P_1/2, 2P_1)$ for time $t \geq T_0$ (and smooth existence for times before) by Proposition 5.1.

Now, when we choose $T_0 = P_1$, our u coordinate as defined in Proposition 4.3 ranges in value from 0 to $3P_1/2$ and thus the value of ρ ranges from -1 to $3P_1/2$. Thus, by our definition of P_1 we know that $c_1[\underline{L}^2_{\text{flat}} F_0](u)$ obtains its (negative) minimal value at $u = \rho_1 + 1$ which is in the range of u within $M(T, u_0, w)$.

To show that this implies blow up, we first note that when $\mu \leq \frac{1}{4}$ and we are in the strip $M(T, u_0, w)$ with $\mathring{\epsilon}^{-3/2} < t < \mathring{\epsilon}^{-5/2}$, formula (37) gives us the following explicit lower bound for $|\mu \underline{L} \Psi_\rho|$.

$$|\mu \underline{L} \Psi_\rho| \geq r(t,u)^{-1/2} \left(\frac{1}{c_1 (r(t,u)^{1/2} - r(\mathring{\epsilon}, u)^{1/2})} - D \mathring{\epsilon}^{3/2} \right)$$

Here D is a constant not depending on the size of the data. Note that for $\mathring{\epsilon}$ small enough the above quantity is in fact a non-zero lower bound in the time span specified, since the first term is at least $\Omega(\mathring{\epsilon}^{5/4})$ while the latter term is $O(\mathring{\epsilon}^{3/2})$ which is smaller. However, the formula (38) tells us that $\mu = 0$ in $O(\mathring{\epsilon}^{-2})$ time. Thus, for small enough $\mathring{\epsilon}$, this confirms our result for the first case ($c_1 \neq 0$) of our theorem.

Now we move on to our second case, where $c_2 \neq 0$ and $c_1 = 0$. Here, let P_2 be as in Corollary 6.1. Now we choose $T_0 = P_2$ in Proposition 4.3, choose ϵ accordingly and choose the size of the data $\mathring{\epsilon} \leq \epsilon$ so that we have the formulae (39) and (40) for μ within the strip $M(T_0, 3P_2/2, 2P_2)$ for time $t \geq T_0$ (and smooth existence for times before) by Proposition 5.1.

As with the first case, when we choose $T_0 = P_2$, our u coordinate as defined in Proposition 4.3 ranges in value from 0 to $3P_2/2$ and thus the value of ρ ranges from -1 to $3P_2/2$. Thus, by our definition of P_1 we know that $c_1[\underline{L}^2_{\text{flat}} F_0](u)$ obtains its (negative) minimal value at $u = \rho_1 + 1$ which is in the range of u within $M(T, u_0, w)$.

The parallels with case one continue as we show that this implies blow up. Again, first note that when $\mu \leq \frac{1}{4}$ and we are in the strip $M(T, u_0, w)$ with $\mathring{\epsilon}^{-3/2} < t < \exp(\mathring{\epsilon}^{-9/4})$, formula (39) gives us the following explicit lower bound for $|\Psi_\rho \cdot \mu \underline{L} \Psi_\rho|$.

$$|\Psi_\rho \cdot \mu \underline{L} \Psi_\rho| \geq r(t,u)^{-1} \left(\frac{1}{2c_2 (\lg(r(t,u)) - \lg(r(\mathring{\epsilon}^{-3/2}, u)))} - D \mathring{\epsilon}^{5/2} \right)$$

Here D is a constant not depending on the size of the data. Note that for $\mathring{\epsilon}$ small enough the above quantity is in fact a non-zero lower bound in the time span specified, since the first term is at least $\Omega(\mathring{\epsilon}^{9/4})$ while the latter term is $O(\mathring{\epsilon}^{5/2})$ which is smaller. However, note that the formula (40) tells us that $\mu = 0$ in $\Theta(\exp(\mathring{\epsilon}^{-2}))$ time along the characteristic C_u corresponding to the negative minimum value of $c_2[\underline{L}_{\text{flat}}((\underline{L}_{\text{flat}} F_0)^2)](u)$. Thus, for small enough $\mathring{\epsilon}$, $\mu \underline{L} \Psi_\rho$ blows up in $\Theta(\exp(\mathring{\epsilon}^{-2}))$. Note that the quantity $|\Psi_\rho \cdot \mu \underline{L} \Psi_\rho|$ cannot blow up by $|\Psi_\rho| \to 0$ because of our bounds on $|\mu \underline{L} \Psi_\rho|$. This confirms our result for the second case ($c_1 = 0$ and $c_2 \neq 0$) of our theorem.

Finally, the fact that $\mu = 1 + O(\mathring{\epsilon})$ in case three means that we can use the estimates from Proposition 3.1 and Proposition 4.3 to get, for small enough $\mathring{\epsilon}$, global existence in a strip of any width about the forward pointing light cone. □

This proof encapsulates our first significant results. In particular, it indicates that *initial shock formation is determined by structural aspects of the non-linearity, not by the initial data.* This is an extremely strong result, which demonstrates shock formation in every situations as opposed to simply "generic" situations as in [1] or [7]. These results also allow us to continue our analysis at the boundary itself, as in the next section.

7. The Geometry Of The Shock Boundary

8. Open Families Of Data Displaying Various Boundary Behaviors

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