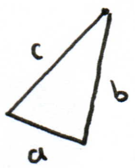
Moduli Spaces (& Gauge Theory):

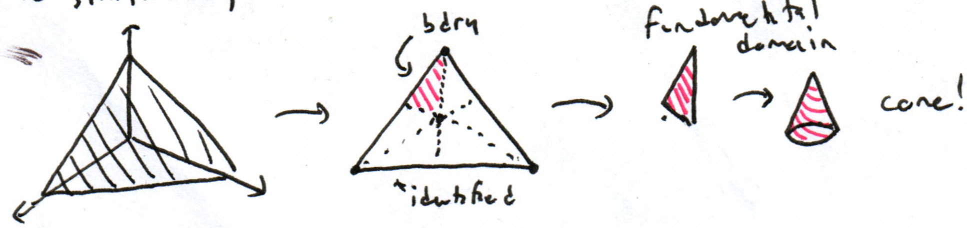
What is a Moduli space? A space of spaces! A meta space if you will.

Examples:

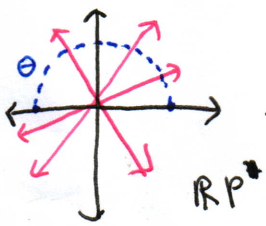
(1) Moduli of triangles up to similarity.



$a<b<c$
$a+b+c=1$



(2) Moduli space of lines ℓ in $\mathbb{R}^n$, $\mathbb{RP}^{n-1}$.



$\longrightarrow$ given by angle, $[0,\pi)$ w/ angle π = angle $0 \Rightarrow \mathbb{RP}^1 = S^1$ circle

Fun Fact: $\mathbb{RP}^2$ is Mobius strip / bdry id :

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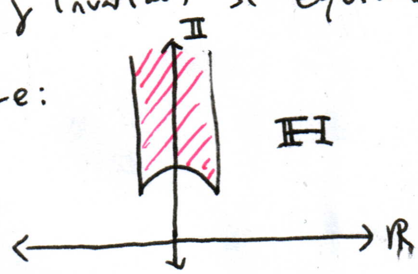
(3) Moduli of Elliptic curves: $w^2 = z^3 + az + b$ loci.
Curves are equivalent if there is change of variables btwn them. (algebraic).
$\Rightarrow$ classified by j invariant $\Leftrightarrow$ equivalent to (conformal) complex structures on T^2 $\leftarrow$ donut.

Moduli looks like:



(4) Gauge moduli! (anti-self dual Yang Mills $\Leftrightarrow$ Seiberg Witten + many more!)

These are much more complicated. $\uparrow$ baby examples (1) - (3)!

Why do geometers care?

- Cool: Moduli spaces are a cool idea. Spaces of spaces are tight as fuck.
- Natural: Moduli spaces "occur naturally" all over the place in geometry.
  Also, sometimes get "functor" $X \mapsto \mathcal{M}(X)$ from category to itself via moduli construction.
- Via "Functor", Invariants:

usual route to invariants   detour to moduli:

$$\begin{tikzcd} X \arrow[rr, "\text{moduli construction}"] \arrow[d, "\text{algebraic topology}"'] & & \mathcal{M}(X) \arrow[d] \\ I(X) & & I(\mathcal{M}(X)) =: \hat{I}(X) \end{tikzcd}$$

invariant → $I(X)$ $I(\mathcal{M}(X)) =: \hat{I}(X)$ ← new invariant!

Examples of $I(X)$:

- Homology / Cohomology (Singular, Simplicial Čech, Sheaf, De Rham etc...)
  De Rham ← actually special.. later!
- Euler characteristic
- Betti #'s
- Intersection form

$\}$ all from ↑

- Chern / Stieffel Whitney #'s / Classes.

Examples of $I(\mathcal{M}(X))$:

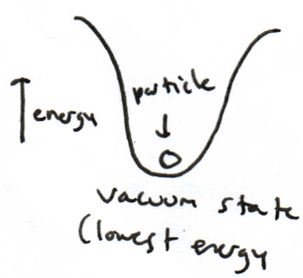
- Donaldson Invariants (I is sort of Chern #'s).
- S-W invariants
- G-W invariants

$\}$ often just $H^0(\mathcal{M}(X))$!

- Floer invariants

Why do physicists care?

- Classical physics: Some moduli are classical vacua of modern quantum theories
- Path Integration / QFT:



Want to evaluate partition function:

$$Z \simeq \int_{\Phi \in \mathcal{F}} e^{i S[\Phi]/\hbar} D\Phi$$

S ← action; $\mathcal{F}$ ← space of fields.

$\mathcal{F}$ is ∞ dimensional! So integration = hard!

BUT if $\mathcal{F}$ has large amount of "gauge" symmetry (along w/ S) then integral should be over $\mathcal{M} := \mathcal{F}/G$ w/ G huuuge group. $\mathcal{M}$ could be f.d moduli space!

So:

$$Z \simeq \int_{[\Phi] \in \mathcal{M}} e^{i S[[\Phi]]/\hbar} D[\Phi]$$

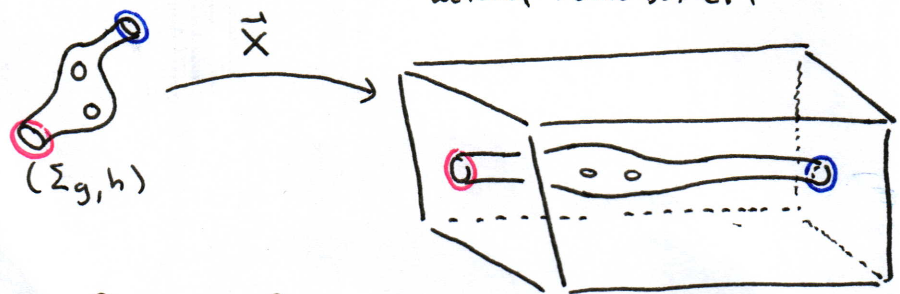
Example of path integral simplification:

Elliptic curves in string theory!
↖ donut T^2's

Theory's action S depends on embedding $\vec{X}$ & conformal structure h on Σ_g genus g surface.
actually metric but CFT

$$\underset{\substack{\uparrow \\ \text{Polyakov amplitude}}}{Z} \simeq \int_{h \in G(\Sigma_g)} dh \int d\vec{X}\, e^{iS(h,\vec{X})/\hbar}$$

$G(\Sigma_g)$ = Space of metrics on Σ_g



Conformal invariance of S w.r.t h $\Rightarrow \int dh \Rightarrow \int d[h]$

Thus:

$$Z \simeq \int_{[h] \in C(\Sigma_g)} d[h] \int d\vec{X}\, e^{iS([h],\vec{X})/\hbar}$$

$C(\Sigma_g)$ = conformal structures

When $g=1$, $\Sigma_g = \Sigma_1 = T^2$

& $C(\Sigma_g) = C(T^2)$ is 2d & :

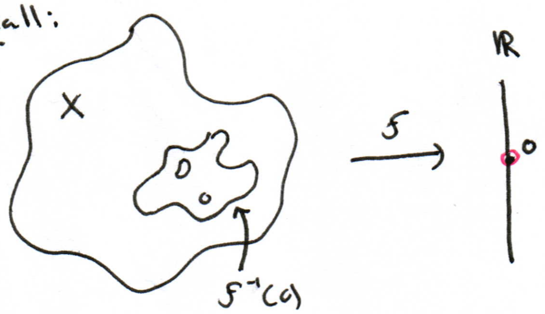
$\mathbb{H}$  our friend from (3)!

How To Make A Moduli Space:

Depends on category: smooth mflds? algebraic varieties? schemes?

I like smooth manifolds $\Rightarrow$ we talk about that.

Recall:

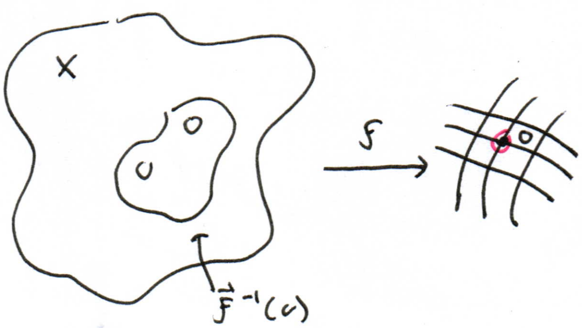


$f^{-1}(c)$ is $n-1$ dim manifold $\Leftarrow \nabla f \neq 0$ in $f^{-1}(c)$.

Ex: if f is polynomial in 2-variables,

$f^{-1}(c)$ is curve: $x^2+y^2-1=0 \Rightarrow f^{-1}(0) =$ 

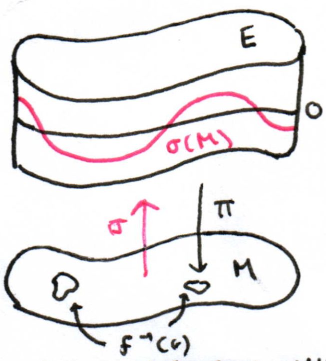
More Generally:



f locally surjective, $d\vec{f}$ non-degenerate (full rank)

$\Rightarrow \vec{f}^{-1}(c)$ is $n-k$ dimensional manifold.

MORE generally



$n+k$ dim w/ Fiber F

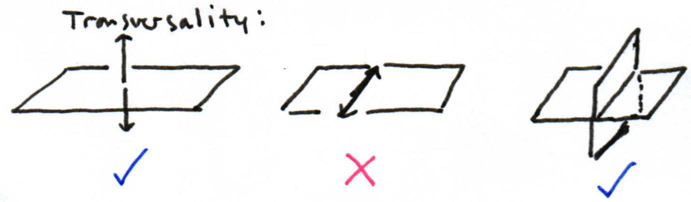
← vector bundle

n-dim

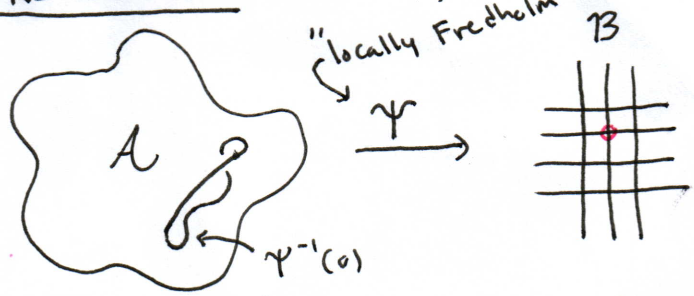
If $\sigma : M \to E$ is section, $d\sigma : TM \to TE \to TF$ surjective ($\sigma(M)$ & 0-section are transverse)

$\Rightarrow \sigma^{-1}(0)$ is $n-k$-dim manifold

Transversality:



MORE GENERALLY!! (sort of)



∞-dim manifold (Banach manifold)

∞-dim Banach space

Under similarly "good" conditions, $\Psi^{-1}(0)$ is smooth manifold of dimension $\underbrace{\text{Index}(D\Psi)}_{\text{not going to define. :(}}$

Examples: PDE from geometry + physics!

(1) De Rham Cohomology / Harmonic k-forms / E & M.

$$\overset{\bullet\text{-forms}}{\Omega^\bullet(M,\mathbb{C})} \xrightarrow{\Delta = d^*d + dd^*} \Omega^\bullet(M,\mathbb{C})$$

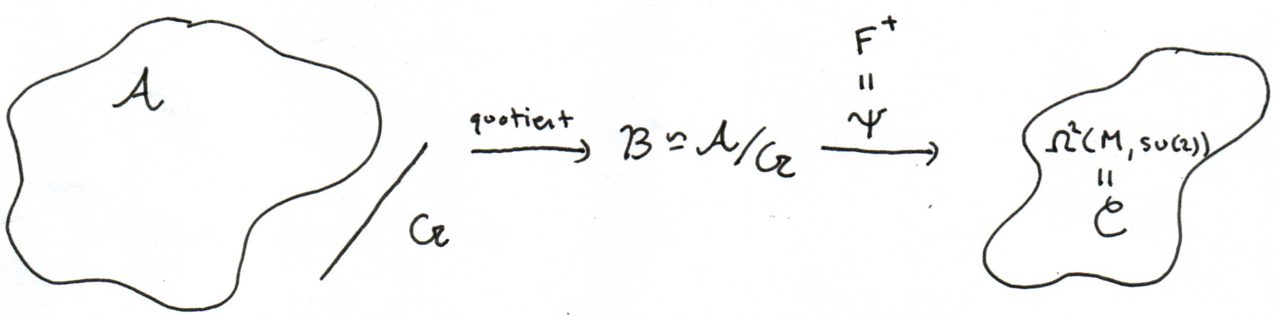
$$\mathcal{A} = \Omega^\bullet(M,\mathbb{C}), \quad \Psi = \Delta, \quad \mathcal{B} = \Omega^\bullet(M,\mathbb{C})$$

$\Psi^{-1}(0) = \overset{\text{harmonic forms}}{\mathcal{H}^\bullet(M,\mathbb{C})} \cong \overset{\text{cohomology}}{H^\bullet(M,\mathbb{C})}$

Moduli space is vector space of harmonic forms.

(2) Gauge theory! Actually, (1) is a U(1) gauge theory...

Donaldson theory: SU(2) / SO(3) gauge theory



$\mathcal{A}$ = SU(2) / SO(3) connections on M

$\mathcal{G}$ = gauge transformations $\{\phi : M \to SU(2)/SO(3)\}$

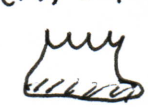
Ψ = ASDYM map $[A] \mapsto F_A^+$

$\mathcal{E} = \Omega^2(M, \overset{\text{Lie algebra}}{su(2)})$, Lie algebra valued 2-forms.

Donaldson ('86):

M simply connected

$\Rightarrow \mathcal{M}(M) = \Psi^{-1}(0)$ is 5 mfld



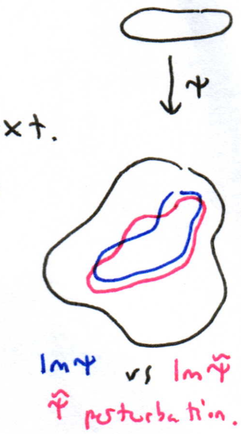
w/ bdry & singularities $\cong$ cone over $\mathbb{CP}^2$.

(3) Moduli of J-holomorphic curves / Strings: Σ_g Rieman surface, (M,ω,J) symplectic.

$\mathcal{A} = \{ \phi : \Sigma_g \to M,\ [\phi \Sigma_g] = a \in H_2(M) \text{ fixed} \}$

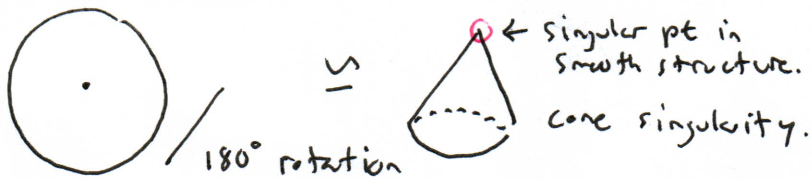
$\mathcal{B} = \Gamma(\Sigma, \mathrm{End}(T\Sigma, \phi^* TM))$, $\Psi(\phi) = J\,d\phi - d\phi\, j$

Challenges



(1) often maps Ψ don't satisfy transversality in ∞ dimensional context.
In finite dimensions, Thom transversality theorem allows you to perturb Ψ slightly to get transverse maps. In infinite dim, life isn't as easy.

(2) Group quotients can cause singular behavior, particularly very large symmetries.
   (a) Gauge group fails to act smoothly.
   (b) bad singularities occur in quotients.



Can be much worse!

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