Math 535A: Take Home Final

Instructions. Complete as many of the following problems as you can. You may use the following results.

- Results stated in class. Please reference the precise statement.
- Results stated in Ch. 1-18 or the appendix of Lee. Please reference the page number.
- Results from the problems in this assignment. However, you must avoid any circular dependence between problems.

The maximum score is 40 points, so you do not need to do all of the problems. This assignment is due on **in class on Friday 4/26.**

Background Definitions

The following definitions and review will be needed for multiple questions.

Definition 1. Let X and Y be smooth manifolds. We say that W is a *connect sum* of X and Y if there is an embedded, codimension one sphere

$$S \subset W \quad \text{such that} \quad W \setminus S \simeq (X \simeq A) \sqcup (Y \setminus B)$$

where $A \subset X$ and $B \subset Y$ are embedded closed balls in the interiors X and Y, respectively. In this case, we write

$$W \simeq X \# Y$$

Definition 2. The *Euler characteristic* $\chi(C)$ of a (co)chain complex (C, d) with finite dimensional cohomology is the alternating sum

$$\chi(C, d) := \sum_i (-1)^i \cdot \dim(H^i(C, d))$$

Review 1. Fix a compact Lie group G with Lie algebra $\mathfrak{g}$. Recall (from class) that the de Rham cohomology of a connected, compact Lie group can be computed as the cohomology of the complex

$$\Lambda^\bullet(\mathfrak{g}^*) \quad \text{with differential} \quad d_{\mathfrak{g}}$$

Here the differential is the unique map satisfying the following two properties. First, it satisfies a Leibniz rule.

$$d_{\mathfrak{g}}(\theta \wedge \phi) = d_{\mathfrak{g}}(\theta) \wedge \phi + (-1)^k \theta \wedge d_{\mathfrak{g}}(\phi) \qquad \text{if } \theta \in \Lambda^k(\mathfrak{g}^*)$$

Second, consider the differential restricted to $\Lambda^1(\mathfrak{g}^*)$, which is a map

$$d_{\mathfrak{g}} : \mathfrak{g}^* \simeq \Lambda^1(\mathfrak{g}^*) \to \Lambda^2(\mathfrak{g}^*) \simeq (\Lambda^2(\mathfrak{g}))^*$$

Then this map is dual to the map

$$\Lambda^2(\mathfrak{g}) \to \mathfrak{g} \qquad \text{given by} \qquad u \wedge v \mapsto [v, u]$$

Problems

Problem 1. (4 points) Let W be an connected, oriented closed manifold of dimension $n \geq 2$ that is a connect sum of X and Y. Show that

$$H^k_{\mathrm{dR}}(W) \simeq \begin{cases} H^k_{\mathrm{dR}}(X) \oplus H^k_{\mathrm{dR}}(Y) & \text{if } 0 < k < n \\ \mathbb{R} & \text{else} \end{cases}$$

Problem 2. (4 points) Show that the 4-manifold $(S^1 \times S^3)\#T^4$ is not diffeomorphic to a connect sum $M\#M$ of a closed manifold M with itself.

Problem 3. (4 points) Suppose that θ is a closed 2-form on the torus T^4 with

$$\int_\Sigma \theta = 0$$

for every closed, oriented, embedded surface Σ in T^4. Show that $\theta = d\mu$ for a 1-form μ.

Problem 4. A flow $\Phi : \mathbb{R} \times X \to X$ on a smooth manifold is *periodic* if

$$\Phi(T, x) = x \text{ for some } T > 0 \text{ and every } x$$

An *isotopy of periodic flows* from a flow Φ to a flow Ψ is a smooth map

$$H : [0,1]_s \times \mathbb{R}_t \times X \to X$$

such that $H_s : \mathbb{R} \times X \to X$ is a periodic flow for each $s \in [0,1]$ and $H_0 = \Phi$ and $H_1 = \Psi$.

(a) (1 point) Show that the periodic flows

$$\Phi : \mathbb{R} \times T^2 \to T^2 \qquad \text{given by} \qquad \Phi(t,x,y) = (x+t, y+t)$$

$$\Psi : \mathbb{R} \times T^2 \to T^2 \qquad \text{given by} \qquad \Phi(t,x,y) = (x-t, y+t)$$

are periodic.

(b) (4 points) Show that they are not isotopic through periodic flows.

Problem 5. Let $SO(n)$ be the special orthogonal group in dimension n, defined by

$$SO(n) = \{A \in GL(n;\mathbb{R}) \;:\; A^T A = \mathrm{Id} \text{ and } \det(A) > 0\} \subset GL(n;\mathbb{R})$$

(a) (4 points) Compute the Lie algebra $\mathfrak{so}(n)$ as a Lie sub-algebra of $\mathfrak{gl}(n;\mathbb{R})$.

(b) (4 points) Find a basis of $\mathfrak{so}(3)$ and write the Lie bracket in that basis.

(c) (6 points) Compute the differential $d_{\mathfrak{g}}$ in the basis of $\Lambda^\bullet(\mathfrak{g}^*)$ induced (dually) by the basis from (b).

(d) (4 points) Compute the de Rham cohomology of $SO(3)$. (You may use the fact that $SO(3)$ is connected).

Problem 6. (4 points) Show that if (C, d) is a finite dimensional complex, then

$$\chi(C, d) := \sum_i (-1)^i \cdot \dim(C^i)$$

where C^i is the dimension of the ith graded piece.

Problem 7. (4 points) Show that every compact Lie group has Euler characteristic zero.

Problem 8. (4 points) Let G and H be two even-dimensional compact connected Lie groups. Show that there is no Lie group structure on any connect sum $G\#H$.

Section 5:

Problem 9. A *symplectic manifold* (X, ω) is a smooth $2n$-manifold with a 2-form ω such that

$$d\omega = 0 \qquad \text{and} \qquad \omega^n \qquad \text{is a volume form (i.e. vanishes nowhere)}$$

Here $\omega^n \in \Omega^{2n}(X)$ is the nth wedge power of ω.

(a) (4 points) Show that the cohomology class $[\omega]$ is non-zero if X is closed.

(b) (4 points) Give an example of a closed 4-manifold that does not admit a symplectic form.

Problem 10. A *contact manifold* (Y, α) is a $(2n+1)$-dimensional manifold such that

$$\alpha \wedge d\alpha^n \quad \text{is a volume form (i.e. vanishes nowhere)}$$

Here again $d\alpha^n$ denotes the nth wedge power of $d\alpha$.

(a) (4 points) Consider the kernel $\ker(d\alpha)$, i.e. the kernel of the bundle map

$$TY \to T^*Y \quad \text{given by} \quad v \mapsto \iota_v(d\alpha)$$

Show that this is a rank 1 sub-bundle of TY. This is the *characteristic sub-bundle* $K \subset TY$ of (Y, α).

(b) (2 points) A *transverse sub-manifold* $\Sigma \subset Y$ of (Y, α) is a sub-manifold such that

$$T\Sigma_p \text{ is transverse to } K_p \text{ within } T_pY$$

at every point $p \in Y$. Show that if Σ is transverse and codimension one, then Σ is closed.

Problem 11. Let $E \to X$ be a rank 2 vector bundle and suppose that $\Phi : E \to E$ is a bundle automorphism with $\Phi^2 + \Phi + \mathrm{Id} = 0$.

(a) (4 points) Show that E is orientable.

(b) (4 points) Show that if E admits a non-vanishing section, then E is trivial.

Problem 12. Let $E \to X$ be a smooth vector bundle over a closed manifold.

(a) (4 points) Show that E has a nowhere vanishing section if $\mathrm{rank}(E) > \dim(X)$.

(b) (2 points) Show that (a) if false if $\mathrm{rank}(E) = \dim(X)$.