Math 540: Take Home Midterm

Instructions. Please write solutions to **12 of 15** problems below, completing at least **3 problems in each part**. This assignment is due **in class on 10/14.**

Collaboration. You may collaborate with whoever you want.

Usable Facts. You may use any of the theorems, lemmas, etc discussed in class or in the readings. The following properties of homology will be especially useful.

Homology Properties. Singular homology is a sequence of functors

$$H_n : \mathbf{Top_2} \to \mathbf{Ab} \qquad \text{with} \qquad (X, A) \mapsto H_n(X, A)$$

satisfying the following properties. Here we use $H_n(X)$ to denote $H_n(X, \emptyset)$ as usual.

- **Homotopy.** If $f, g : (X, A) \to (Y, B)$ are homotopic maps of pairs, then the induced maps on homology are equal. That is

$$f_* = g_* : H_n(X, A) \to H_n(Y, B) \qquad \text{for all } n \geq 0$$

- **Dimension.** If X is a one point space, then $H_n(X) = 0$ for all $n > 0$. For singular homology, we also have $H_0(X) = \mathbb{Z}$.
- **Union.** If $X = \sqcup_{i \in I} X_i$ for some index set i, then

$$H_n(X) = \bigoplus_{i \in I} H_n(X_i)$$

- **Exactness.** For any pair (X, A), there is a long exact sequence

$$\cdots \to H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{\jmath_*} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \to \dots$$

  Here $\iota : (A, \emptyset) \to (X, \emptyset)$ and $\jmath : (X, \emptyset) \to (X, A)$ are the obvious inclusions of pairs.
- **Excision.** If $B \subset A \subset X$ with $\bar{B} \subset \mathring{A}$ then the inclusion $(X \setminus B, A \setminus B) \to (X, A)$ induces an isomorphism

$$H_n(X \setminus B, A \setminus B) \simeq H_n(X, A)$$

- **Mayer-Vietoris.** If $U, V \subset X$ are subsets with $\mathring{U} \cup \mathring{V} = X$ then there is a long exact sequence

$$\cdots \to H_n(U \cap V) \xrightarrow{\iota_* \oplus \jmath_*} H_n(U) \oplus H_n(V) \xrightarrow{j_* - k_*} H_n(X) \xrightarrow{\delta} H_{n-1}(U \cap V) \to \cdots$$

- **Kunneth.** Let X and Y be two topological spaces. Then there is a short exact sequence.

$$0 \to \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \to H_n(X \times Y) \to \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y)) \to 0$$

  Moreover, this short exact sequence is natural in the sense that a pair of maps $f : X \to X'$

and $g : Y \to Y'$ induce a commutative diagram

$$\begin{tikzcd}
\bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \arrow[r] \arrow[d, "f_* \otimes g_*"] & H_n(X \times Y) \arrow[r] \arrow[d, "(f\times g)_*"] & \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y)) \arrow[d, "\mathrm{Tor}(f_*, g_*)"] \\
\bigoplus_{i+j=n} H_i(X') \otimes H_j(Y') \arrow[r] & H_n(X \times Y) \arrow[r] & \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y))
\end{tikzcd}$$

- **Universal Coefficients.** Let X be a topological space and let A be an abelian group. Then there is a short exact sequence

$$0 \to H_n(X;\mathbb{Z}) \otimes A \to H_n(X;A) \to \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), A) \to 0$$

Moreover, this is natural in the sense that if $f : X \to X'$ is a continuous map, then

$$\begin{tikzcd}
H_n(X) \otimes A \arrow[r] \arrow[d, "f_* \otimes \mathrm{Id}_A"] & H_n(X;A) \arrow[r] \arrow[d, "f_*"] & \mathrm{Tor}(H_{n-1}(X), A) \arrow[d, "\mathrm{Tor}(f_*, \mathrm{Id}_A)"] \\
H_n(X') \otimes A \arrow[r] & H_n(X';A) \arrow[r] & \mathrm{Tor}(H_{n-1}(X'), A)
\end{tikzcd}$$

Homology Of Familiar Spaces. You are permitted to use the homology of the following spaces with no further justification.

- The sphere S^n in any dimension.
- The torus T^n in any dimension (except in Problem 9).
- The projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$.

Detailed computations of these homology groups can be found in the book.

Notation And Definitions The following notation and definitions were not introduced in class, but will be used in the problems below.

Notation 1. We will use $H_i(X)$ to denote the singular homology of a topological space X with $\mathbb{Z}$ coefficients.

Definition 1. The *wedge sum* $X \vee Y$ of two pointed topological spaces (X, x) and (Y, y) is

$$X \vee Y = (X \sqcup Y)/\sim \qquad \text{where} \qquad x \sim y \text{ and no other relations.}$$

The new basepoint $x \vee y$ of $X \vee Y$ is the point corresponding to x (which is identified with y). Wedge sum is functorial: a pair of pointed maps

$$f : (X, x) \to (X', x') \quad \text{and} \quad f : (Y, y) \to (Y', y')$$

induces an obvious map $f \vee g : X \vee Y \to X' \vee Y'$.

Remark 1. It is common to refer to the wedge without specifying the basepoint, e.g. writing

$$\mathbb{C}P^2 \vee \mathbb{R}P^1$$

to denote the wedge sum with respect to some choice of basepoints.

Definition 2. The *suspension* ΣX of a topological space X is the space

$$[0,1] \times X/\sim \qquad \text{where}(0,x) \sim (0,y) \text{ and } (1,x) \sim (1,y) \text{ for all } x,y \in X$$

That is, every point in $0 \times X$ is crushed to one point, and the same for $1 \times X$. Suspension is functorial: a map $f : X \to Y$ induces a map

$$\Sigma f : \Sigma X \to \Sigma Y$$

given by the quotient of the map $\mathrm{Id} \times f : [0,1] \times X \to [0,1] \times Y$.

Definition 3. A *good pair* (X, A) is a pair where A is closed and there is an open neighborhood U of A that deformation retracts onto A.

Definition 4. A topological space X has *finitely generated* homology if the direct sum

$$\bigoplus_i H_i(X)$$

is finitely generated. Equivalently, $H_i(X) = 0$ for all but finitely many i and $H_i(X)$ is finitely generated for each i.

Definition 5. The *Euler characteristic* $\chi(X)$ of a topological space X with finitely generated homology is

$$\chi(X) = \sum_i (-1)^i \cdot \mathrm{rank}(H_i(X))$$

Part I: Proofs

Instructions. In each problem below, give a rigorous and complete proof of the requested fact.

Problem 1. Show that if (X, x) and (Y, y) are both good pairs, then

$$\tilde{H}_n(X \vee Y) \simeq \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$$

Explain the corresponding statement for ordinary (not reduced) homology.

Problem 2. Let X be a finite CW complex. Show that every point $x \in X$ admits a neighborhood U that deformation retracts onto x. In other words, (X, x) is always good.

Problem 3. Show that if (X, A) is a good pair then

$$H_n(X, A) \simeq \tilde{H}_n(X/A)$$

Here X/A denotes the quotient space identifying every point in A with a single point.

Problem 4. Let X, Y be a pair of topological spaces with finitely generated homology.

(a) Show that for finitely generated abelian groups A and B, we have

$$\operatorname{rank}(A \otimes B) = \operatorname{rank}(A) \cdot \operatorname{rank}(B)$$

(b) Show that the Euler characteristic $\chi(X \times Y)$ of $X \times Y$ is the product $\chi(X) \cdot \chi(Y)$.

Problem 5. Let ΣX be the suspension of a connected topological space X.

(a) Use Mayer-Vietoris to show that

$$\tilde{H}_{n+1}(\Sigma X) = \tilde{H}_n(X) \qquad \text{if } n \geq 0$$

(b) Find a formula for $\chi(\Sigma X)$ in terms of $\chi(X)$.

Part II: Calculations

Instructions. Each of the following problems involves computing homology groups. Where asked, rigorously compute the homology groups using the properties of homology (e.g. excision, LES of a pair, etc). Write clearly and carefully, and always cite the properties that you're using when you use them!

Problem 6. For each of the following pairs of spaces, either show that they are homeomorphic or prove that they are not.

(a) S^{n+1} and ΣS^n.

(b) $\Sigma(\mathbb{R}P^3)$ and $\mathbb{R}P^4$

(c) $S^2 \wedge S^4$ and $\mathbb{C}P^2$

Problem 7. Compute the homology with $\mathbb{Z}$-coefficients and $\mathbb{Z}/3$-coefficients of the space

$$X = (T^2 \times (\mathbb{R}P^3 \vee \mathbb{R}P^2)) \setminus P \qquad \text{where } P \text{ is a point in } T^2 \times (\mathbb{R}P^3 \vee \mathbb{R}P^2)$$

Problem 8. For each n, there is a map $\iota_n : \mathbb{R}P^1 \to \mathbb{R}P^n$ given by $\iota_n[x, y] = [x, y, 0, \ldots, 0] \in \mathbb{R}P^n$ in standard projective coordinate notation. Let

$$X = (\mathbb{R}P^2 \sqcup \mathbb{R}P^3)/\sim$$

where $\sim$ identifies $\iota_2(x)$ with $\iota_3(x)$ for each $x \in \mathbb{R}P^1$. Compute the homology of X.

Problem 9. Let $\iota : T^2 \to T^4$ denote the inclusion of the 2-torus into the 4-torus via the map

$$\iota(s, t) = (s, t, p, p) \in T^4$$

Here we view T^2 as $S^1 \times S^1$ and T^4 as $S^1 \times S^1 \times S^1 \times S^1$. Also, $p \in S^1$ is an arbitrary point in the circle.

(a) Use the Kunneth formula to compute the homology groups of T^n from the homology groups of S^1.

(b) Compute the induced map $\iota_* : H_n(T^2) \to H_n(T^4)$ for each n.

(c) Let $S \subset T^4$ denote the image $\iota(T^2)$. Compute the homology of the quotient T^4/S.

Problem 10. Fix an abstract simplicial complex K with vertex set $V = \{1, \ldots, 5\}$ and simplices

$$\{(1), (2), (3), (4), (5), (15), (24), (12), (23), (13), (34), (45), (35), (123), (345)\}$$

Here $(i_1 i_2 \ldots i_k)$ denotes the face with vertices $i_1, i_2, \ldots, i_k$.

(a) Compute the singular homology of the geometric realization $|K|$ of K.

(b) Draw a picture of a simplicial complex $K' \subset \mathbb{R}^2$ whose corresponding abstract simplicial complex is equivalent to K.

(c) Show that $|K'|$ is homotopy equivalent to a wedge of familiar spaces (circles, projective spaces, etc).

(d) Apply (c) to give an alternative calculation of the singular homology.

Part III: Examples

Instructions. For each of the following problems, give an example of a space, map, etc with the requested properties. You must demonstrate the claimed properties with a rigorous explanation.

Problem 11. Give an example of a topological space that is homotopy equivalent to its own suspension.

Problem 12. Give examples of topological spaces with the following homology groups.

(a) A topological space X with $H_1(X) = \mathbb{Z}/7$.

(b) A topological space Y with

$$H_0(Y) = \mathbb{Z} \qquad H_1(Y) = \mathbb{Z}/7 \oplus \mathbb{Z}/2 \qquad H_2(Y) = \mathbb{Z} \qquad H_4(Y) = \mathbb{Z}$$

Problem 13. Give an example of two topological spaces X and Y and a continuous map $f : X \to Y$ such that

- $H_2(X)$ and $H_2(Y)$ are not zero.
- f is *not* null-homotopic.
- $f_* : H_2(X) \to H_2(Y)$ is the zero map.

Problem 14. Consider the 2-torus T^2. Construct examples of the following.

(a) An continuous map $f : T^2 \to T^2$ where f^n is not homotopic to the identity for any $n \geq 1$.

(b) A continuous map $g : T^2 \to T^2$ where g is not null-homotopic but g^2 is null-homotopic.

Problem 15. Give an example of a topological space X and a map $F : \Sigma X \to \Sigma X$ that is *not* homotopic to the suspension of a map $f : X \to X$.