- M9227 - Week 2: Discrete Physics -

1 Introduction & Story Time

The point of this lesson will be to demonstrate how we can use linear algebra and very simple analysis to understand the way that a discrete, 1-dimensional world would work physically. To motivate this world, we can tell a story which we (the teachers) find pretty funny.

So suppose that we lived in a circle universe. Yes, not a 3-d space or a globe, or even a plane. A circle, which up close looks like a line (just like the surface of the earth looks like a plane up close). And imagine that the entirety of 21st century civilization had developed on this circle. Things are going pretty good, but then the singularity happens and the computers take over. Now, Wachowski style, the machines set up a virtual reality world (the Circle World Matrix) to farm energy from people (which is somehow an energetically favorable process).

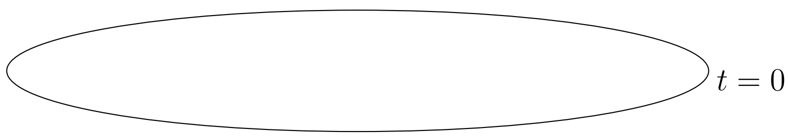

Figure 1: Circle World

Figure 2: Circle World Up Close. The line keeps going, to the left and to the right.

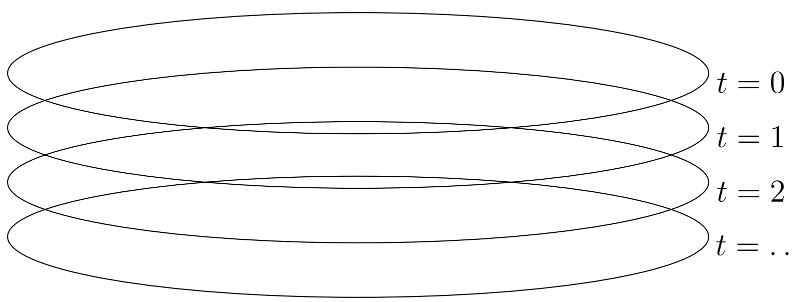

Figure 3: The Circle World Matrix. Here we show some time steps of the world too.

Figure 4: The Circle World Matrix Up Close: It's finite, just a bunch of points!

This is all fine and dandy for our evil 1-dimensional machine overlords, but (as every laptop owner knows) computers have a finite amount of memory. So they need to make a finite world approximation; each point in *space* gets a slot of memory and there are finitely many points. Moreover, the machines need to make *time* discrete. This way, they can use numerical methods (computer overlords are good at that) to model what the real circle world should look and feel like. But in order to do this, they need a working model of physics in this world. Physics which reproduces the behaviors of the real circle world at small scales. To solve this problem for them, we will need to guid some more background with linear algebra.

2 Eigenvectors

If I have a matrix, one important way that we can characterize it is in terms of its **eigenvectors**. These are vectors that the matrix scales, but does not rotate. Algebraically, we can define these vectors as so.

Definition 2.1. An **eigenvector** of a matrix M is a vector v satisfying the equation $Mv = \lambda v$, where λ is some *complex* number. λ is called the **eigenvalue** of v.

So what is the picture that we should see for an eigenvector? Well suppose that we have the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 2.5 \end{pmatrix}$, $v = (1, 2)$. Then $Mv = (3, 6) = 3 \cdot (1, 2) = 3v$. Likewise, if $w = (-2, 1)$, then $Mv = (-1, .5) = .5w$. So we have two eigenvectors for M, v and w. When we apply M to v, we see the following picture.

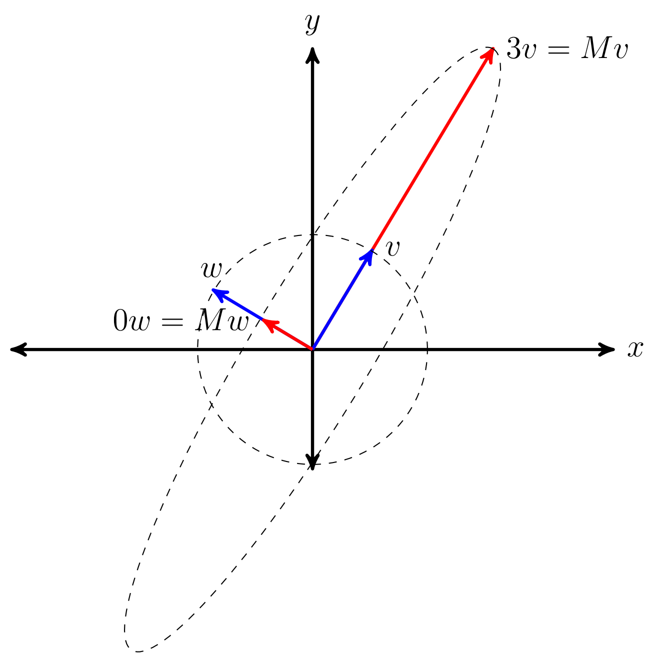

Figure 5: The circle with radii v and w gets deformed into an ellipse.

Notice how the eigenvectors here are along the semi-major (longest) and semi-minor (shortest) axes of the ellipse here. So sometimes we can think of eigenvectors as the axes along which deformation occurs. We can also see eigenvectors as vectors in the lines that are *fixed* by a matrix. Notice that is $Mv = \lambda v$, then $M(cv) = \lambda cv$ for any number c. So the matrix doesn't just scale the vector v, it scales all of its multiples. So the line $u(x) = xv$ for varying x is fixed by M.

These pictures help when eigenvalues are real, but things aren't always this simple. Let's look at some counter-examples to reasonable statements about eigenvalues that we could make.

Example 2.1. *Not all matrices with real entries have real eigenvectors. A good example of this is a* 90*-degree rotation matrix:*

$$R_{90} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Proof. It's geometrically obvious that any line going through the origin is not fixed by a rotation about the origin. Algebraically, we can see that if $M(a,b) = (-b,a) = \lambda(a,b)$ for non-zero (a,b), then this implies that $\lambda a + b = 0$ and $a - \lambda b = 0$. But this implies that $b = \lambda^{-1}a$ so $\lambda a + \lambda^{-1}a = a(\lambda + \lambda^{-1})$. If a isn't 0, this implies that $\lambda + \lambda^{-1} = 0$ and $\lambda^2 + 1 = 0$, which isn't possible for a real number. If a is 0, then $\lambda b = 0$ and since (a,b) isn't 0 (by assumption) λ must be 0. But then this implies that $\lambda a + b = b = 0$. So this is also not possible. $\square$

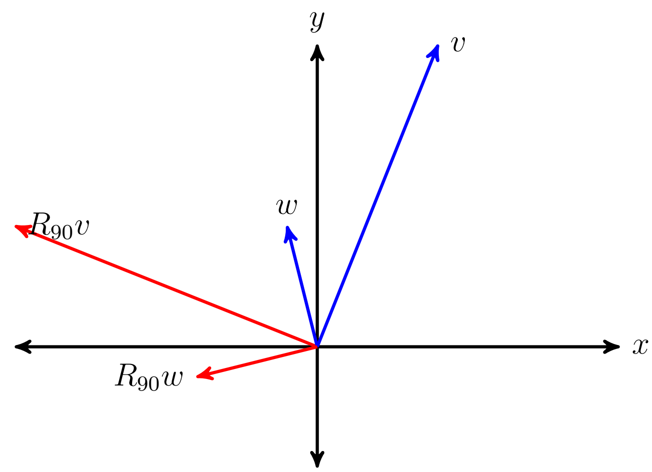

Figure 6: None of these vectors are fixed when they're rotated!

One thing you might be thinking right now is: "Wow that was a little complicated. Is there any way that we can figure that out more easily?" The answer is yes, we can do so using a little tool called the characteristic polynomial of a matrix.

Definition 2.2. The **characteristic polynomial** $p_M(\lambda)$ is defined as the polynomial $\det(\lambda Id - M)$.

Proposition 2.1. *Any eigenvalue λ of M satisfies $p_M(\lambda) = 0$.*

Proof. Remember when we showed that a matrix is invertible if and only if its determinant is non-zero? Well, clearly $M - \lambda Id$ is not invertible when λ is an eigenvalue. Why? Well, if λ is an eigenvalue then it has a non-zero eigenvector v satisfying $Mv = \lambda v$, i.e $(M - \lambda Id)v = 0$. But then $(M - \lambda Id)^{-1}0 = v$, which can't happen for non-zero v. So $M - \lambda Id$ is not invertible and $p_M(\lambda) = \det(M - \lambda Id) = 0$. $\square$

3 A Dot/Inner Product

One really important structure that we can use to study vector spaces, which has particular relevance to the study of eigenvalues, is the dot product.

Definition 3.1. The dot product is a map that takes a pair of complex vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ and produces a number $v \cdot w = \sum_i v_i w_i^*$. This can also be written as $w^\dagger v$ where $w^\dagger$ is the conjugate transpose of w (i.e the vector turned on its side with every entry complex conjugated).

Notice that the dot product has the following three properties:

1. It is conjugate symmetric: $v \cdot w = (w \cdot v)^*$.
2. It is linear in the first variable (and anti-linear in the second variable): $(au + bv) \cdot w = a(u \cdot w) + b(v \cdot w)$.
3. And it is positive definite: $v \cdot v = 0$ when $v = 0$, but otherwise $v \cdot v > 0$.

Notice that the last expression $v \cdot v$ is just another way of writing the magnitude of v, $\sum_i |v_i|^2$. Anything satisfying these properties is called a **Hermetian inner product**. These are very important objects, since they play a big role in geometry and analysis.

Why do they play such a big role? Well, one reason is because they imbue a space with a sense of *length* and *distance*. Consider the following. If v and w are real, then $|v| = \sqrt{v \cdot v}$ is just the length of v (using the regular length formula everyone learns in high school). Likewise, $|v - w|$ is the distance between two vectors, the length of their difference. Another reason, equally important, is that the dot product allows us to formulate *angles*. In particular, we have the following statement.

Proposition 3.1. *Let v and w be non-zero real vectors which meet at an angle θ at the origin. Then $\frac{v \cdot w}{|v||w|} = \cos(\theta)$.*

Proof. We can just think about the 2-d case by choosing our coordinates properly, and we can furthermore assume that one vector is on the x-axis and the other vector is in the upper half plane. In that case, we see that given a vector $v = (a, 0)$ and $w = (c, d)$ we have:

$$\frac{v \cdot w}{|v||w|} = \frac{ac + 0d}{\sqrt{a^2 + 0^2}c^2 + d^2} = \frac{c}{\sqrt{c^2 + d^2}}$$

Now note that the vector w makes a right triangle with the x-axis (drawn in the figure below) and it has hypotenuse $|w| = \sqrt{c^2 + d^2}$ and side lengths a, c. The angle at the origin, which is θ, thus satisfies $\cos(\theta) = \frac{|\text{opposite}|}{|\text{hypotenuse}|}$ (SohCahToa).

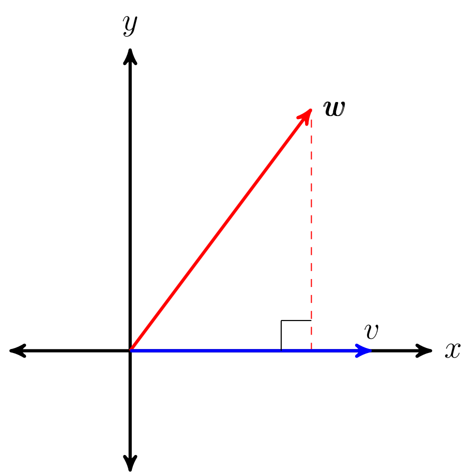

Figure 7: How do we get that angle?

$\square$

So, in other words, the dot product is nice because it lets us talk about geometry using linear algebra. There are two ways that we're going to use the dot product. First, we'll need the following statement.

Proposition 3.2. *Suppose we have a collection $e_1, \dots, e_k$ of vectors such that $\langle e_i, e_j \rangle = 0$ if $i \neq j$ (so they're all perpendicular). Then they are all linearly independent.*

Proof. Suppose for the sake of contradiction that $e_i = \sum_{j \neq i} c_j e_j$ (so e_i is *not* linearly independent from the others). Then $e_i \cdot e_i e = \sum_{i \neq j} c_j e_j \cdot e_i = \sum_{i \neq j} c_j 0 = 0$. Here we use the bilinearity of the dot product. □

In particular, this implies that a set of n perpendicular vectors $e_1, \dots, e_n$ in n-dimensional space forms a basis (i.e any vector can be expressed as a linear combination of them, see last weeks homework). This is very useful when coupled with the following statement.

Second, we'll need the following geometric statement about eigenvectors.

Proposition 3.3. *Suppose that M is matrix such that $M = M^\dagger$. Consider eigenvectors v and w such that $Mv = \alpha v$ and $Mw = \beta w$ but $\alpha \neq w$. Then $v \cdot w = 0$.*

Proof. First, it is important to note that $v \cdot Mw = M^\dagger w \cdot v$. You should check this yourself, it's very easy. Second, we see that M must have real eigenvalues. Indeed, given any complex matrix A we see that $p_{A^\dagger}(\lambda) = \det(A^\dagger - \lambda Id) = (p_{A^\dagger}(\lambda))^*$ (that is, $p_A(\lambda)$ with every coefficient conjugated) so the conjugates of the eigenvalues of A are the eigenvalues of $A^\dagger$. But if $A = A^\dagger$ (as with M) then the eigenvalues must be their own conjugates. This means they must be real.

Now we can prove the statement as so:

$$\alpha(v \cdot w) = (\alpha v) \cdot w = (Mv) \cdot w = v \cdot (M^\dagger w) = v \cdot (Mw) = v \cdot (\beta v) = \beta(v \cdot w)$$

But since $\beta \neq \alpha$, this is a contradiction. □

In other words, the eigenvectors of such a matrix are all perpendicular (except perhaps when they have the same eigenvalue).

4 The Heat Equation

Now we're in a position to start discussing some physical math. We're going to derive an equation modeling the way heat works in our world. Here's how.

4.1 The Model

Our world is a circle made up of n points (with n possible very very large). We can number the points $1, \dots, n$. The temperature at each point is just a number, h_i, and we can collect these into a length n vector h. Now the temperature is of course, a function of time t. In the matrix, though, time is an integers ($t = 0, 1, 2, 3, \dots$). We'll write the time dependent version of h as $h[t]$. Thus, $h_i[t]$ represents the temperature at point i at time t. Now we want to answer the following question,

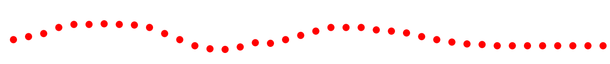

Figure 8: Points vs. Temperature. The dots represents the temperature at the point below.

which is the following: given the vector $h[t]$ (at a specific time), how do we find the vector $h[t+1]$? In other words, how does heat evolve? Well we propose the following simple model.

Model 4.1.1. *At every time t, a point in this world will pass half of its heat to its two neighbors. Since the two neighbors are the same (the world is symmetric), it naturally must pass a quarter of the heat left and a quarter of the heat right. It keeps the other half of the heat for itself.*

This seems like a reasonable model. After all, heat moves in the direction of lower temperatures and hot objects don't just lose their heat instantly (they keep some of it). You should think about how this model captures these effects. Algebraically, we can write this model as:

$$D_t h_i[t+1] = h_i[t+1] - h_i[t] = \frac{1}{4}(h_{i-1}[t] + h_{i+1}[t] - 2h_i[t]) = D_i^2 h[t]$$

Note that this isn't *precisely* correct. When $i = 1$, the index $i - 1$ needs to be replaced with n. Likewise, if $i = n$ we need to replace $i + 1$ with 1. This is because our world is circular, so points 1 and n are adjacent. But we can fix that by defining D_i^2 is the following matrix:

$$D_i^2 = \frac{1}{4}\begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & -2 & \dots & 0 & 0 \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & -2 & 1 \\ 1 & 0 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}$$

4.2 Analysis Of Model

So now we have a model, and we want to know that it works. Particularly, we want to know that it reproduces the behavior that we want to see in heat in real life. The most fundamental characterization (to us) seems to be the following.

Hypothesis 4.2.1. *Heat should spread out over time and eventually settle down into* **equilibrium**, *i.e a steady state where everything is the same temperature.*

We're going to prove this with eigen-analysis; by identifying the eigenvectors and eigenvalues of D_i^2 and using them to derive the long-term behavior of the heat vector $h[t]$. To start out, we'll find the eigenvectors and eigenvalues of D_i.

Proposition 4.2.1. *The eigenvectors of D_i^2 are $f_k = (1, e^{2\pi ik/n}, e^{2\cdot 2\pi ik/n}, e^{3\cdot 2\pi ik/n}, \dots, e^{(n-1)\cdot 2\pi ik/n})$ with eigenvalues $\lambda_k = \frac{1}{2}(\cos(\frac{2\pi k}{n}) - 1)$.*

Proof. This proof will be a little hard, but we can get through it! It starts with two trig identities called the sum to product formulae. You can find these with Google:

$$\frac{1}{4}(\cos(x) + \cos(y)) = \frac{1}{2}\cos(\frac{x-y}{2})\cos(\frac{x+y}{2}); \frac{1}{4}(\sin(x) + \sin(y)) = \frac{1}{2}\cos(\frac{x-y}{2})\sin(\frac{x+y}{2})$$

In particular, suppose that $x = \frac{(j+1)\cdot 2\pi k}{n}$ and $y = \frac{(j-1)\cdot 2\pi k}{n}$ (looks pretty familiar). Then:

$$\frac{1}{4}(\cos(\frac{(j+1)\cdot 2\pi k}{n}) + \cos(\frac{(j-1)\cdot 2\pi k}{n}) = \frac{1}{2}\cos(\frac{2\pi k}{n})\cos(\frac{j\cdot 2\pi k}{n})$$

$$\frac{i}{4}(\sin(\frac{(j+1)\cdot 2\pi k}{n}) + \sin(\frac{(j-1)\cdot 2\pi k}{n}) = \frac{1}{2}i\cos(\frac{2\pi k}{n})\sin(\frac{j\cdot 2\pi k}{n})$$

Notice how I introduced a factor of $i = \sqrt{-1}$. We will use this! Now consider f_k, particularly its entries $e^{(j-1)\cdot 2\pi ik/n}$, $e^{j\cdot 2\pi ik/n}$ and $e^{(j+1)\cdot 2\pi ik/n}$. If we add the two formulae above together and use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$[1], we get:

$$\frac{1}{4}(e^{(j+1)\cdot 2\pi ik/n} + e^{(j-1)\cdot 2\pi ik/n} - 2e^{j\cdot 2\pi ik/n}) = \frac{1}{2}\cos(\frac{2\pi k}{n})(\cos(\frac{j\cdot 2\pi k}{n}) + i\sin(\frac{j\cdot 2\pi k}{n})) - \frac{1}{2}e^{j\cdot 2\pi ik/n} =$$

$$\frac{1}{2}(\cos(\frac{2\pi k}{n}) - 1)e^{j\cdot 2\pi k/n}$$

But this is just the jth entry of $D_i^2 f_k$. So $D_i^2 f_k = \frac{1}{2}(\cos(\frac{2\pi k}{n}) - 1)f_k$. □

This is great! Even better, these eigenvectors are all perpendicular! We can prove this directly as so:

Proposition 4.2.2. *$f_k \cdot f_j = 0$ if $k \neq j$, and $f_k \cdot f_k = n$.*

Proof. First, observe that the numbers $e^{(k-j)\cdot 2\pi i/n}$ are *nth root of unity*. An nth root of unity is a complex number whose nth power is 1, so $x^n = 1$ or $x^n - 1 = 0$. The polynomial $x^n - 1$ factors into $(x-1)(\sum_{\alpha=0}^{n-1} x^\alpha)$, so for a specific nth root of unity we know that when $x - 1 \neq 0$ it must be true that $\sum_{\alpha=0}^{n-1} x^\alpha = 0$. In particular, if $k \neq j$ then $x = e^{(k-j)\cdot 2\pi i/n} \neq 1$ and $(e^{(k-j)\cdot 2\pi i/n})^n = 1$, so $\sum_{\alpha=0}^{n-1} e^{\alpha(k-j)\cdot 2\pi i/n} = 0$. But this is just $f_k \cdot f_j$ when $k \neq j$ (check!). When $k = j$, then this is just the square length of f_k, which is n because every entry of f_k is a root of unity and is thus length squared 1, and there are n entries. □

By Proposition 3.2, this means that the vectors f_k are all linearly independent. If you did the homework from Week 1, you know that a set of n linearly independent vectors in an n-dimensional vector space form a *basis*: any vector v can be expressed as a combination $v = \sum_k c_k f_k$ where c_k are constants. In our case, the basis f_k is called the *Fourier* basis. Every f_k is an oscillator of frequency k/n, and we can express any function in terms of a sum of oscillators. This fact will let us prove our hypothesis, as below.

Proof. Consider any initial heat vector $h[0]$. Since the set of f_k form a basis, we can express this as a sum $h[0] = \sum_k c_k f_k$ for constants c_k. Now consider the matrix $1 + D_i^2$. According to our model, we have that:

$$h[t+1] = (1 + D_i^2)h[t] = (1 + D_i^2)^t h[0]$$

Now note that we can break this up using $h[0] = \sum_k c_k f_k$:

$$(1 + D_i^2)^t h[0] = \sum_k c_k (1 + D_i^2)^t f_k = \sum_k c_k (1 + \lambda_k)^t f_k$$

We can replace $1 + D_i^2$ with λ_k because that's how $1 + D_i^2$ acts on f_k. Now observe that $1 + \lambda_k = \frac{1}{2} + \frac{1}{2}\cos(\frac{2\pi k}{n})$. This number is always smaller than 1, except when $k = 0$, when $1 + \lambda_k = 1$. In that case, $f_k = (1, 1, 1, \ldots, 1)$. It's the constant heat vector. This implies that when t is very very big, $(1 + \lambda_k)^t$ is almost 0 for $k \neq 0$. When $k = 0$, $(1 + \lambda_k)^t$ is just 1. This, in turn, implies that for very large t we know that:

$$h[t] = \sum_k c_k (1 + \lambda_k)^t f_k \simeq c_0 f_0$$

That's because every other term disappears. But this just says that at very large times, the heat $h[t]$ becomes constant (i.e achieves equilibrium). This is what we wanted to prove! □

[1]This is a famous formula, you should know it.