The Donaldson Flow on Symplectic 4-Folds

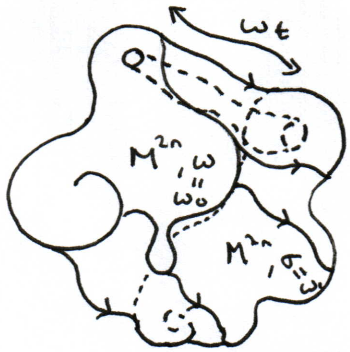
Agenda

- Motivating questions
- Setup + Observations
- Definition of Flow
- Derivation of Flow
- Summary of Results of Donaldson / Krom
- ~~Challenges + Ideas~~ Open Problems

All of talk material is in

1) The Donaldson geometric Flow For Symplectic Four-manifolds [Krom + Salamon] 2015/2016
2) The DF is a locally smooth semiflow [Krom] ← 2015/2016

Motivating Questions



Given symplectic manifold (M^{2n}, ω), when can we isotopy ω to another symplectic σ on M^{2n}?

Necessary: $[\omega] = [\sigma] \in H^2(M^{2n})$. Sufficient?

Conjecture: Yes (?)

Consequences (if true):

- Moser Isotopy $\Rightarrow$ $[\sigma] = [\omega]$ are connected by diff-family: $\exists\ \varphi_t : M \circlearrowleft$ s.t. $\varphi_0 = \mathrm{Id}$ & $\varphi_1^* \sigma = \omega$

Gromov & Taubes results $\Rightarrow$ $\varphi \in \mathrm{Diff}(M)$ isotopic to identity implies $\varphi^* : H^*(M) \circlearrowleft$ is identity. if $M = \mathbb{CP}^n$

Setup + Observations: Find "flow" (natural ω_t) which connects symp forms.

- Let (M, ω, g, J) be symplectic mfld with choice of compatible triple.
- ~~If dim M = 4; then for any ρ symplectic on~~
- For any ρ symplectic, w/ $[\rho] = [\omega]$ say, we can define tensor $R^\rho \in \mathrm{End}(\Lambda^2 M, \Lambda^2 M)$ by:

$$R^\rho \tau := \tau - \frac{2\,\tau \wedge \rho}{\rho \wedge \rho}\rho = \tau - \frac{\tau \wedge \rho}{\mathrm{dvol}_\rho}\rho$$

← on 4-fold

- R^ρ is an involution on $\Lambda^2 M$ acting as 1 on τ w/ $\tau \wedge \rho = 0$ & -1 on ρ.
- Linear Algebra fact: Given a volume form $\mathrm{vol} \in \Lambda^4 \mathbb{R}^4$ and a rank 3 positive bundle $\Lambda^+ \subset \Lambda^2 \mathbb{R}^4$, $\exists$ a unique metric w/

$\tau \wedge \tau = c\,\mathrm{vol}\ \forall \tau \in \Lambda^+$, $c > 0$

(1) $\mathrm{vol}_g = \mathrm{vol}_\rho$ + (2) $\Lambda^+_g \mathbb{R}^4 = \Lambda^+$

Setup + Observations (cont):

- To ρ, we can associate unique Riemannian metric g_ρ s.t.:

$$(1)\ dvol_{g_\rho} = dvol_g = dvol_\omega \quad + \quad (2)\ \Lambda^+_{g_\rho}(M) = R^\rho \Lambda^+_g(M)$$

- Imbues ∞-dim mfld $Sym(M,[\omega]) := \{$space of symplectic ρ w/ $[\rho]=[\omega]\}$ ($\Leftarrow$ use some Banach completion.) w/ ∞-dim Riemannian structure as so:

$$\mathcal{M} = Symp(M,[\omega]) \overset{\text{open}}{\subset} [\omega] + d\Omega^1(M) \leftarrow \text{Affine Banach mfld.}$$

$$\Rightarrow T_\rho Sym(M,[\omega]) = d\Omega^1(M)$$

Metric defn: $\langle \hat\rho, \hat\sigma \rangle_\rho = \int \lambda \wedge *^\rho \xi$ w/ $d\lambda = \hat\rho$, $d\xi = \hat\sigma$, $d*^\rho\lambda = d*^\rho \xi = 0$

Thus, have ∞-dim Riemannian mfld $(\mathcal{M}, \langle,\rangle)$.

Definition of Flow: (Do on side board / save if possible)

- Consider energy functional: $E: \mathcal{M} \to \mathbb{R}$

$$E_\omega(\rho) = \int_M \frac{2|\rho^{+_g}|^2}{|\rho^{+_g}|^2 - |\rho^{-_g}|^2}\, dvol_g = \int_M \frac{\rho^{+_g}\wedge \rho^{+_g}}{\rho\wedge\rho}\, dvol_g$$

- Donaldson Flow is negative Gradient flow of E w.r.t $\langle \cdot,\cdot \rangle_\rho$

$$\boxed{\partial_t \rho = d *^\rho d\Theta^\rho, \quad \Theta^\rho := *\frac{\rho}{\upsilon} - \frac{1}{2}\left|\frac{\rho}{\upsilon}\right|^2 \rho, \quad \upsilon = \frac{dvol_\rho}{dvol_g}}$$

More Derivation

Proof: $d\mu_i(\phi)\hat{\phi} = \frac{\overset{\phi^* \mathcal{L}_{\hat{\phi}}\omega}{\overset{\shortparallel}{d\omega(\hat{\phi}, d\phi\cdot)}} \wedge \sigma}{dvol_\sigma}$

$d\mu_i : T\mathcal{M} \to T\Omega^0 \underset{\text{trivial}}{=} \Omega^0$

$\Omega_i(\rho_\phi(H), \hat{\phi}) = \langle H, \mu_i(\phi)\rangle_{L^2}$?

$\Omega_i(-d\phi \circ X_H^\sigma, \hat{\phi}) = \int_M \omega_i(\hat{\phi}, d\phi \circ X_H)\, dvol_\sigma$

$= \int_M \iota_{X_H} d\omega_i(\hat{\phi}, d\phi\cdot)\, dvol_\sigma$

$= \int_M d\omega_i(\hat{\phi}, d\phi\cdot) \wedge \iota_{X_H}(dvol_\sigma)$

$= \int_M d\omega_i(\hat{\phi}, d\phi\cdot) \wedge dH \wedge \sigma$

$= \int_M H (d\omega_i(\hat{\phi}, d\phi\cdot)\wedge\sigma) \frac{\cdot\, dvol_\sigma}{dvol_\sigma}$

$= \langle H, d\mu_i(\phi)\hat{\phi}\rangle_{L^2}$!

(Prop 2.1) in [S-K]

Yields energy function $\mathcal{E}$ on $\mathcal{M}$, namely:

$\mathcal{E}(\phi) = \frac{1}{2}\sum_i \|\mu_i\|_{L^2}^2$, $\mathcal{E} : \mathcal{M} \to \mathbb{R}$

Gradient flow of $\mathcal{E}$ w.r.t $\langle\, ,\rangle_{L^2}$?

$d\mathcal{E}(\phi)\hat{\phi} = \sum_i \langle d\mu_i \hat{\phi}, \mu_i\rangle_{L^2}$ } defn of moment map

$= -\sum_i \Omega_i(d\phi \circ X_{H_i}, \hat{\phi}) = -\sum_i \langle J_i\, d\phi \circ X_{H_i}, \hat{\phi}\rangle_{L^2}$

$\Omega_i = \langle J, \cdot\rangle$

$\Rightarrow \nabla\mathcal{E} = -\sum_i J_i\, d\phi \circ X_{H_i}$ | $H_i := \frac{\phi^*\omega_i \wedge \sigma}{dvol_\sigma}$

$\Rightarrow$ $\boxed{\partial_t \phi_t = -\nabla\mathcal{E}(\phi_t) = \sum J_i\, d\phi_t \circ X_{H_i}}$

is negative gradient flow!

Use $\rho_t := (\phi_t^{-1})^*\sigma$ to get evoln of ρ_t

$\Rightarrow$ Donaldson flow on ρ_t!

(No more derivations pls).

Current Status / Results of Donaldson Krom

1) [Donaldson] Background ω is unique absolute minimizer of E.

2) [Donaldson] All other critical pts are Index > 0 in hyper Kahler case

3) [Krom] Donaldson Flow exists for short time Ω is ~~smooth~~ regular on $W^{k,p}$ if k is big enough.

For short time?

Open Problems

1) Long-time existence:
   - (a) $vol\, \rho_t \not\to \infty$ or 0
   - (b) $\|\omega\|_{W^{p,k}} \not\to \infty$ in finite time in long-term.

2) Deal w/ saddle pts in general case

Derivation of Flow : How did Donaldson come up with this?

- Consider hyper-Kähler curve :

$$(H, g, \underbrace{\omega_1, \omega_2, \omega_3}_{\bar{\omega}}, \underbrace{J_1, J_2, J_3}_{\bar{J}}) \text{ w/ } J_i \text{ satisfying quaternionic relations} \qquad (g, \omega_i, J_i) \text{ all Kahler.}$$

- Look at mfld (∞-dim) of diffeomorphisms:

$$\{\underset{M \simeq H \text{ diffeomorphic}}{\phi : (M, \sigma) \to (H, g, \bar{\omega}, \bar{J})} \text{ w/ } \phi^*[\omega_1] = [\sigma]\} =: \mathcal{M} \quad \text{new } \mathcal{M} \text{ sorry!}$$

- $\mathcal{M}$ has "hyper-Kahler" structure pulled back from H.
- $T_\phi \mathcal{M} = \Gamma(M, \phi^* TH)$; $\hat{v}, \hat{w} \in T_\phi \mathcal{M}$ then define:

$$\underset{\text{symplectic}}{\Omega_i(\hat{v}, \hat{w})_\phi := \int_M \omega_i(\hat{v}, \hat{w})\, dvol_\sigma}, \quad \underset{\text{complex}}{\mathcal{J}_i \hat{v}_\phi := J_i \hat{v}}, \quad \underset{\text{Riemannian}}{\langle \hat{v}, \hat{w} \rangle_\phi = \int_M \langle \hat{v}, \hat{w} \rangle_g\, dvol_{g\sigma}}$$

- $\mathcal{M}$ has action by $\mathrm{SympMor}(M, \sigma) =: \mathcal{G}$; $\mathcal{M}/\mathcal{G} \simeq \mathrm{Symp}(M, [\omega])$ $\quad \phi \mapsto \phi \circ \psi^{-1}$, ψ symplectomorphism $\quad ([f] \mapsto (f^{-1})^* \omega)$

Derivation continued :

- Action $\mathcal{G} \curvearrowright \mathcal{M}$ is "Hamiltonian" almost:
  - obviously Ω_i is invariant under $\mathcal{G}$ action. So is symplectic action.
  - moreover, $\mathfrak{g} = "T_0 \mathcal{G}"$ = symplectic vectorfields on M
  - can restrict to $\mathfrak{g}_0$ = Hamiltonian vectorfields on M.
  - Thus $\mathfrak{g}_0 \simeq d\Omega^0(M) \simeq \Omega^0(M)/\mathbb{R}$, & we have infinitesimal action of $\mathfrak{g}_0$ on $\mathcal{M}$ realized by map:

$$\rho : \mathfrak{g}_0 \to \mathrm{Vect}(\mathcal{M}) : \underset{\in \Omega^0/\mathbb{R}}{H} \mapsto \{\phi \mapsto -d\phi \circ \underbrace{X_H^\sigma}_{\text{Hamiltonian v-field w.r.t } \sigma} \in \Gamma(M, \phi^* TH) = T_\phi \mathcal{M}\}$$

- Is action of $\mathfrak{g}_0$ given by moment maps $\mu_i : \mathcal{M} \to \Omega^0(M)$? Yes!
- Reminder : Moment map is $\mu : M \to \mathfrak{g}^*$ s.t $\rho(\xi) = X_{d\langle \mu, \xi \rangle}$.
- Map is :

$$\mu_i : \mathcal{M} \to \Omega^0(M) \quad , \quad \mu_i(\phi) := \frac{\phi^* \omega_i \wedge \sigma}{dvol_\sigma}$$