The Obstruction Theory Of Kähler-Einstein Metrics

Julian Chaidez

N.B This brief note is intended to sketch some recent work in Kähler theory. I am not a specialist in this field and I skipped details here and there for the sake of clarity and length. Please read the original papers cited herein if you would like a complete picture.

Introduction The obstruction theory of geometric structures on manifolds and varieties is a fundamental part of modern geometry, in every incarnation. In general, the goal of such theories is to characterize the condition of existence or non-existence of a structure in terms of easily computed or ostensibly unrelated data about the underlying space. Here are three elementary examples:

Example 1. Degree and Genus: The genus of a closed surface Σ is the count of its holes, and the degree of a line bundle L counts the degree of a divisor D corresponding to L. Together this data measure the obstruction to the existence of a global holomorphic section of L on a Riemann surfaces Σ via the Riemann-Roch formula.

Example 2. Stieffel-Whitney Classes: These are distinguished cohomology classes $w_i(\xi) \in H^i(M, \mathbb{Z}/2)$ detecting properties of a real vector bundle ξ on a manifold M. $w_1(TM)$ measures the obstruction to an orientation on M and $w_2(TM)$ measures the obstruction to a spin structure.

Example 3. The Euler Characteristic: $\chi(\Sigma)$ is the alternating sum of the dimensions of the homology groups of Σ. When Σ is a compact oriented surface, Gauss-Bonnet tells us that $\chi(\Sigma)$ is a topological measure of the existence of a positive, zero or negatively curved metric via the formula $\int R dv_g = 2\pi\chi(\Sigma)$.

Note that the first example here can be viewed as a linear geometric PDE problem and that the obstruction is (famously) rooted in algebraic geometry and topology. The second problem falls firmly into the realm of geometric and algebraic topology. A more difficult "non-linear" problem, which is in many ways the proper successor to the last example, is the following:

Question. What is the obstruction to a compact Kähler manifold M admitting a Kähler-Einstein metric?

Note that the Ricci tensor Ric_g corresponding to a Kähler metric g yields a representative of the first Chern class via $c_1(M) = \frac{i}{2\pi}\mathrm{Ric}_g(J\cdot, \cdot) \in H^2(M, \mathbb{Z})$ (with J the complex structure on M). Thus, the positive definiteness of g and the Einstein equations $\mathrm{Ric}_g = \lambda g$ imply that any Kähler M admitting a Kähler-Einstein metric must have first Chern class with a well-defined sign, i.e $c_1(M)$ is either positive, negative or 0 evaluated against any complex curve in M, independent of the curve.

For $c_1(M) < 0$, it was proven independently by Yau and Aubin that any such M admits a unique negatively curved Kähler-Einstein metric, while for $c_1(M) = 0$ the analogous statement was proven by Yau in his celebrated series of papers starting with [Yau78]. The more subtle $c_1(M) > 0$, or Fano case has been a longstanding problem, which was recently solved by Chen, Donaldson and Sun (CDS) in the papers [CDS15i]-[CDS15iii]. It is this series of papers and results that we will focus on here.

K-stability of a Fano The primary reason for the increased difficulty of the positive case is that the simple, obvious criterion for the existence of an Einstein metric, i.e $c_1(M)$ having a definite sign, was quickly found to be insufficient. In the 80's it was conjectured that the proper condition for a $c_1 > 0$ variety was for it to possess no global $c_1(M)$ vector-fields, but even then Gang Tian demonstrated in [Tian97] that sub-variety of $G_4\mathbb{C}^7$ with positive Chern class admitted no Einstein-Kähler metric, even though they met this stronger criterion.

In the same paper, Tian formulated an early version of a necessary algebraic stability condition called K-stability. We will introduce the definition used in [CDS15i] now, and motivate its original introduction in [Tian97] afterwards. In what follows, let X be a Fano manifold of complex dimension n.

Definition 1. A *test configuration* $(\mathcal{X}, i)$ of X is a flat family (fibration) $\pi : \mathcal{X} \to \mathbb{C}$ and a fibration compatible embedding $i : \mathcal{X} \to \mathbb{C}P^N \times \mathbb{C}$ for some N, satisfying the following conditions.

1. $i(\mathcal{X})$ is invariant under a $\mathbb{C}^*$ action on $\mathbb{C}P^N \times \mathbb{C}$ that covers the standard action on $\mathbb{C}$.
2. The fiber of $\mathcal{X}$ at 1 is X and the (central) fiber at 0, X_0, is a normal variety with log terminal singularities. This condition controls how singular X_0 can be. In particular, if $f : Y \to X_0$ is a resolution of Y, then the pullback of the canonical bundle $f^*K_{X_0}$ will agree with K_X up to a sum of divisors $\sum_i \delta_i E_i$ with $\delta_i > -1$. I do not have enough expertise to give a good interpretation of why this particular condition is important.
3. The embedding $X \hookrightarrow \mathcal{X} \to \mathcal{C}P^N$ is given by complete linear system of K_X^{-m} for some $m > 0$. In other words, the embedding $X \to \mathbb{C}P^N$ is the natural one constructed from global sections of some tensor power of the anti-canonical bundle, which is ample because X is Fano. We also need the embedding $X_0 \to \mathbb{C}P^N$ to arise this way.

Because X_0 is invariant under the $\mathbb{C}^*$ action, the dimension $d(k)$ vector space of sections $H^0(X_0, L^k)$ (with $L = K_{X_0}^{-m}$) inherit a $\mathbb{C}^*$ action with some weight $w(k)$. By "general theory" (the phrasing of [CDS15iii]) we know that for large k, $d(k)$ and $w(k)$ are given by polynomials of degree n and $n+1$ respectively. Thus we can expand:

$$\frac{w(k)}{kd(k)} = F_0 + F_1 k^{-1} + O(k^{-2})$$

Definition 2. We define the *Futaki invariant* as $Fut(\mathcal{X}, i) = -F_1$.

Definition 3. A Fano X is called K-stable when $Fut(\mathcal{X}, i) > 0$ for all test configurations $(\mathcal{X}, i)$ such that $X_0 \not\cong X$ and $Fut(\mathcal{X}, i) \geq 0$ if $\mathcal{X}$ is not the trivial family.

In [Tian97], Tian proved that any compact Kähler-Einstein M must satisfy a form of K-stability. He used an equivalent integral formulation of $Fut(\mathcal{X}, i)$ in terms of a total integral over W of an expression in $\mathrm{Ric}(\omega)$ and ω, the Ricci 2-form and the Kähler 2-form. A version of this formula is provided on p. 264-265 of [CDS15iii]. Utilizing this formula, Tian looked at X_t as t varied from 1 to 0 (so from X to the central fiber X_0). He illustrated that if $X_0 \not\cong X_1$ then the C^0 norm of the Kähler potential ϕ_t blows up in the $t \to 0$ limit, where ϕ_t is introduced as the varietion of the KE metric ω_{KE} on X_1 as one moves along the fibers X_t with metrics $\omega_t = \omega_{KE} + \partial\bar{\partial}\phi_t$. Tian was then able to use ϕ_t to estimate the integral expression for $Fut(\mathcal{X})$ from below and demonstrate non-negativity, due to ϕ_t's roll in said expression.

Conceptually, the relationship between the algebro-geometric notion of K-stability and the analytic notion of Kähler-Einstein seems to come from the fact that the existence of such a nice metric on X allows one to control quantities with formulae in terms of the metric on an algebraic family containing X. This paradigm extends even to metric derived quantities which are ultimately metric independent (i.e algebraic or topological invariants), in particular the Futaki invariant.

Later work (see [Ber12]) established that the form of K-stability given above and used in [CDS15i] - [CDS15iii] was necessary.

Theorem 4. *Every Fano Kähler-Einstein manifold is K-stable.*

The converse statement was the focus of CDS. The proof relies on a program, formulated by Donaldson, where a continuity argument is implemented on the existence of singular Kähler-Einstein metrics of cone angle $2\pi\beta$ along a divisor D. We will now outline the steps in this proof. Fix a K-stable Fano X.

1. Fix a $\lambda > 0$ and a smooth divisor D in the linear system $|-\lambda K_X|$. Such a choice is possible for λ large due to Bertini's theorem. Consider Kähler-Einstein metrics on X with cone angle $2\pi\beta$ along D, which satisfy: $\mathrm{Ric}(\omega_\beta) = (1-(1-\beta)\lambda)\omega_\beta + 2\pi(1-\beta)[D]$. Such metrics ω_β are defined to satisfy the Kähler-Einstein equations on $X - D$ and to have Kähler potential in $C^{2,\alpha,\beta}$ for some $\alpha \in (0, \beta^{-1}-1)$.

2. Let I be the set of $\beta \in (0,1]$ such that such an ω_β like those described above exists.

3. I is non-empty. If we choose $\beta = 1/N$ with N large enough, we can show that the existence of such a Kähler-Einstein metric on X is equivalent to the existence of such a metric on an orbifold $\hat{X}$ constructed out of X with an orbifold singularity about D. This orbifold K-E metric is negative curvature, and as in the smooth case the existence theory was solved by Yau and Aubin. Thus I is non-empty.

4. Similarly, the open-ness of I follows from linear elliptic estimates similar to those in the smooth case. In particular, an inverse function theorem on Banach spaces can be used in the context of Kähler-metrics with cone singularities as long as a small additional assumption is met.

5. The closed-ness of I is, as in the Calabi-Yau case, the difficult part and this is where the K-stability comes in. Let g_i be a sequence of K-E metrics on X with cone-singularity β_i converging to β_∞. By older theory of cone-singular Kähler-Einstein metrics, we can assume $\beta_\infty > 1 - \lambda^{-1}$ since otherwise one can show that the curvature of the resulting metric must be negative, and the existence theory of such metrics is well-established.

   In the $\beta_\infty > 1 - \lambda^{-1}$ case, the papers [CDS15i] - [CDS15ii] illustrate that a sequence of singular metrics g_i with cone angle β_i along D smoothly Gromov-Hausdorff converge to a limiting manifold W. CDS show that W is in fact a $\mathbf{Q}$-Fano variety with a Weil divisor Δ that satisfies the assumptions on the central fiber X_0 of a test configuration. They also illustrate that W has a weak cone K-E metric with singularity of angle $2\pi\beta_\infty$ along Δ. Using the theory of Luna slices, CDS thus construct a test configuration $\mathcal{X}$ with central fiber $X_0 = W$ such that Δ is the flat limit of D. They then illustrate that the associated Futaki invariant [1] vanishes, thus implying by the stability assumption that the central fiber (W, Δ) is isomorphic to (X, D). The cone singular metric on W can then be pulled back to a cone metric of angle $2\pi\beta_\infty$. This illustrates closeness.

Step 5 here is the most technically difficult and required a large amount of original analysis by CDS. In particular, [CDS15i] - [CDS15ii] focused primarily on establishing the algebro-geometric properties if the G-H limit of (X, D, g_i).

[1] Technically, a small generalization of the Futaki invariant to families with divisors is used in the end. Of course, the K-stability assumption carries over to this extended case.

Possible Questions As a student and a non-specialist, it is difficult for me to muse productively about the future applications and implications of the techniques developed by CDS. However, there is a point here that I believe deserves some attention. I was originally drawn to this body of work because of the beautiful interplay between the PDE analysis, Riemannian geometry and algebraic geometry. This is a general feature of complex geometry and Kähler theory, but here the authors utilize a less conventional technique that embody this interaction: G-H limits in the algebraic context.

Studying the algebro-geometric properties of smooth Gromov-Hausdorff limits of Kähler manifolds under various analytic assumptions could provide an interesting general set of problems. Here the assumption is extremely strong: CDS study sequences of K-E metrics, where powerful bootstrapping estimates are available. One could conceivably attempt to generalize their results, examining the algebro-geometric properties of geometric limits of Kähler metrics with lower-bounded Ricci curvature over some fixed variety X. Perhaps such limits can often be described and characterized in terms of properties of algebraic families containing X.

References

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