Heegard Diagrams

Agenda:

- Handle Decompositions / Terminology
- Simplifications of Decomposition Data
- Specialization to 3-d / Heegard Diagrams
- Handle Moves:
  - Creation / Cancellation of Pairs
  - Slides
- Classifying Lens spaces.

Ref: Gompf & Stipsicz [GS]


$\simeq L(2,1) \simeq \mathbb{RP}^3$

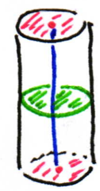
Handle Decompositions

Thm 1: Every compact manifold $(X, \partial X)$ admits decomposition into handles $H_1, H_2, \ldots, H_N$.

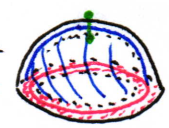
Def: A k-handle in n-dimensions is $H \simeq \underbrace{D^k}_{k \text{ dim disk}} \times D^{n-k}$ where we care about factors.

Terminology:

3d 1-handle $D^1 \times D^2 \simeq$



3d 2-handle $D^2 \times D^1 \simeq$



- attaching sphere $(\partial D^k \times \{0\})$ = A(H)
- core $(D^k \times \{0\})$
- belt sphere $(\{0\} \times \partial D^{n-k})$
- cocore $(\{0\} \times D^{n-k})$
- not usually named $(\partial D^k \times D^{n-k})$ (I will call $S(H)$)

N.B - belt / cocore are dual to attaching sphere / core.

Thm 1 (Elaborated): Data of decomposition consists of:

- Handles H_i
- Embeddings $\varphi_i : S(H_i) \hookrightarrow \partial(H_1 \sqcup_{\varphi_2} H_2 \sqcup \ldots \sqcup_{\varphi_{i-1}} H_{i-1})$
- Denote $X_{i-1} = H_1 \sqcup_{\varphi_2} \sqcup \ldots \sqcup_{\varphi_{i-1}} H_{i-1}$

Simplifications of Data

- If $\varphi, \varphi' : S(H) \hookrightarrow \partial X$ are two embeddings that are isotopic then
$$X \sqcup_\varphi H \underset{\text{diff}}{\simeq} X \sqcup_{\varphi'} H$$
- $\left\{ \begin{array}{l} \text{Iso classes of} \\ i : S(H) \hookrightarrow \partial X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Iso classes of pairs of} \\ \text{iso class of } A(H) \hookrightarrow \partial X \ + \\ \text{iso class of trivialization of } \nu(\mathrm{Im}(H)) \end{array} \right\}$

i.e isotopy classes of framed embeddings

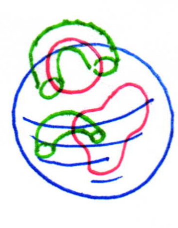
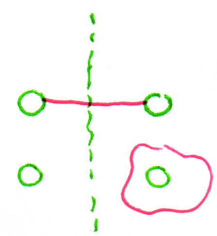
- If X is connected; handle decomposition can have 1 0-handle
- ~~If X is closed, can have 1 n-handle~~.

- If X is closed, $\tilde{X} = \bigsqcup_{k \le n-1}$ k handles / gluing has boundary $\sqcup S^{n-1} = \partial \tilde{X}$, & any attachment of remaining handles produces same mfld.

Specializing to 3-d: For 3-mfld, closed, Y:

need to specify:

- 1 0-handle $\ominus$ D^3 $\Rightarrow$
- Attachments of $A(H_i) = S^0$ to $S^2 \simeq \partial D^3 = \delta(0\text{-handle})$ for 1-handles H_i.
- Framed embeddings of $S(H_i) \simeq S^1$ for 2-handles H_i. $\infty \cup \mathbb{R}^2$


$\Rightarrow$

$(*)$

Def: A Heegard Diagram is the specification of above data as diagram like $(*)$.

Handle Moves

- Cerf theory $\Rightarrow$ Any two Heegaard diagrams / handle decompositions are related by sequence of following moves (+ isotopy).

    1) Handle pair creation

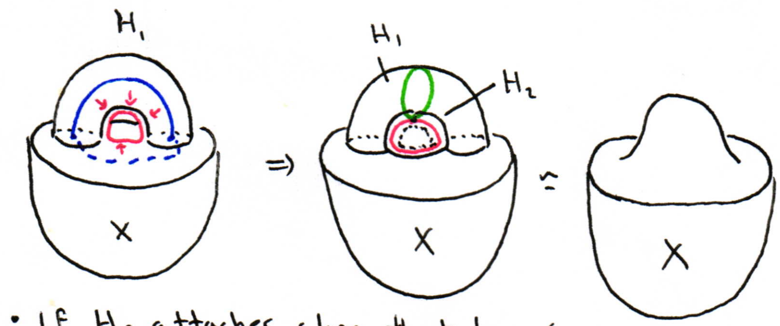
    2) Handle pair cancellation

    3) Handle slides

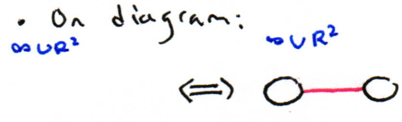
In 3-d, suffices to do all of these moves with 1st, 2 handles.

Creations / Cancellations

- Consider $X \sqcup H_1 \sqcup H_2$ w/ H_k a k-handle
- Can take core of H_1 & close it to a loop, then isotope to the boundary of $X \sqcup H_1$



- If H_2 attaches along that loop (or one isotopic to it) then $X \cong X \sqcup H_1 \sqcup H_2$
- Call H_1 & H_2 a <u>cancelling pair</u>.
- On diagram:



$\Rightarrow$ = <u>creation</u>

$\Leftarrow$ = <u>cancellation</u>

Handle Slides

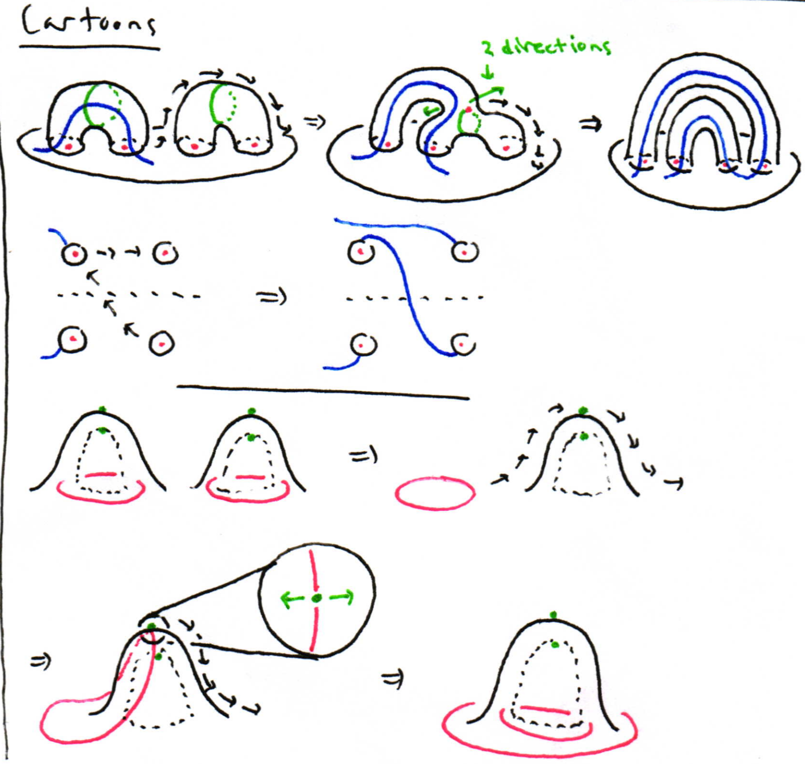
Def: Given 2 k-handles H_1 & H_2 attached to ∂X, a <u>handle slide</u> is given by following:

- Isotope the attaching sphere $A(H_1)$ across the boundary component of $X \sqcup H_2$ corresponding to ∂H_2.
- At some point, attaching sphere will intersect belt sphere B of H_2, $B(H_2)$, Perturb so intersection is transverse.
- At intersection (which is at <u>1 pt</u>), p,
  $\dim T_p A(H_1) + \dim T_p B(H_2) = \dim \partial X - 1$

$\Rightarrow$ 2 normal directions to push $A(H_1)$ off of ∂H_2 component.

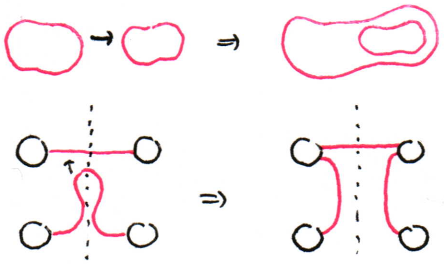
- 1 direction $\Rightarrow$ undoes isotopy
- other direction $\Rightarrow$ <u>handle slide result</u>

Cartoons



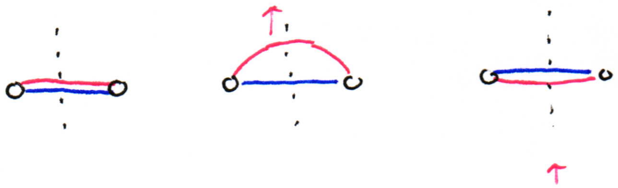
Cartoons Continued

$\infty \cup \mathbb{R}^2$ $\qquad\qquad$ $\infty \cup \mathbb{R}^2$



Last comment:

- Allowed to slide features of diagram "over infinite", i.e. use isotopies of S^2 not available in $\mathbb{R}^2$.



Lens Spaces: Many good definitions.

Def 1: A Lens space is a closed 3-mfld with a genus 1 Heegard splitting

Def 2: A Lens space is any 3-mfld given by quotienting $S^3 = \{v \in \mathbb{C}^2 \mid |v| = 1\}$ by $\mathbb{Z}_p$ action generated by:

$$\left.(z_1, z_2) \mapsto (e^{2\pi i/p} z_1 \,,\, e^{2\pi i q/p} z_2)\right\} \text{ called } L(p,q)$$

Proof of $2 \Rightarrow 1$:

- Split $S^3 \subset \mathbb{C}^2$ into:

  $\Sigma = \{(z_1, z_2) \mid |z_1| = |z_2| = \frac{1}{\sqrt{2}}\}$

  $H_1 = \{(z_1, z_2) \mid |z_1| > |z_2|,\ |z_1|^2 + |z_2|^2 = 1\}$

  $H_2 = \{(z_1, z_2) \mid |z_1| < |z_2|,\ \text{''} \quad \text{''}\}$
- H_1 is disk bundle over $S^1 = U(1) \times \{0\}$

$\Rightarrow H_1 \simeq D^2 \times S^1$. Ditto for H_2

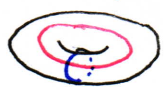
Continued

- Easy to see $\mathrm{Im}(H_1)$ under $q: S^3 \to L(p,q)$ is $D^2 \times S^1$ also. (also disk bundle over torus).
- $\mathrm{Im}(\Sigma)$ is similarly torus (oriented circle bundle over S^1).

$\Rightarrow H_1, H_2, \Sigma$ descends to genus 1 Heegard splitting of $L(p,q)$.

Heegard diagram:

- For $L(p,s)$, 2-handle attachment image is in H^1 class $p[\ell] + q[m]$



Ex: $L(5,\cancel{?})$: $\qquad\qquad$ $L(3,2)$

We're going to prove:

1) $\pi_1(L(p,q)) \simeq \mathbb{Z}/p$

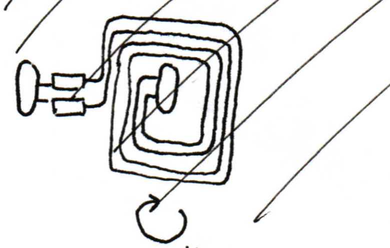
   Cor: $L(p,q) \not\simeq L(p',q')$ if $p \neq q$.

2) $L(p,q) \simeq L(p,q+p)$

3) $L(p,q) \simeq L(p, mp-r)$ $\left(\text{here } q = mp + r,\ r = q \bmod p\right)$

~~General drawing of $L(p,q)$ diagram:~~

~~Let $q = mp + r$.~~



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