Essential tori in spaces of symplectic embeddings

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December 2, 2018

Abstract

Given two n–dimensional symplectic ellipsoids whose symplectic sizes satisfy certain inequalities, we show that a map from the n–torus to the space of symplectic embeddings from one ellipsoid to the other induces an injective map at the level of singular homology with mod 2 coefficients. The proof uses parametrized moduli space of J–holomorphic cylinders in completed symplectic cobordisms, provided by bordism maps of full contact homology. This is a pre-arxiv draft, please do not distribute.

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1 Introduction

The study of symplectic embeddings is a major area of focus in symplectic geometry. Remarkably, the space of such embeddings can have a rich and complex structure, even when the domain and range manifolds are relatively simple.

Symplectic embeddings between ellipsoids are a well–studied instance of this phenomenon. For a nondecreasing sequence of positive real numbers $a = (a_1, a_2, \ldots, a_n)$ define the *symplectic ellipsoid* $E(a)$ as:

$$E(a) = E(a_1, a_2, \ldots, a_n) = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \frac{\pi|z_1|^2}{a_1} + \ldots \frac{\pi|z_n|^2}{a_n} \leq 1 \right\}. \quad (1.1)$$

Imbued with the restriction of the standard Liouville form $\lambda = \frac{1}{2}\sum_{i=1}^{n} x_i dy_i - y_i dx_i$ on $\mathbb{C}^n$, $E(a)$ is a compact, exact symplectic manifold with boundary. A special case is the *symplectic ball* $B^{2n}(r)$, which is simply $E(a)$ for $a = (r, \ldots, r)$.

The types of results that one can prove about symplectic embeddings, together with the tools used to do so, are surveyed at length by Schlenk in [18]. Most research has thus far sought to address the existence problem. Let us recall some of the more striking progress in this direction. The first nontrivial result was Gromov's eponymous *non–squeezing theorem*, proven in the seminal paper [7].

Theorem 1.1 ([7])**.** *There exists a symplectic embedding $\varphi : B^{2n}(r) \to B^2(R) \times \mathbb{C}^{2n-2}$ if and only if $r \leq R$.*

Theorem 1.1, in particular, demonstrated that there are obstructions to symplectic embeddings beyond the volume and initiated the study of quantitative symplectic geometry. It can be thought of as a result about ellipsoid embeddings, since $B^2(R) \times \mathbb{C}^{2n-2}$ can be viewed as the degenerate ellipsoid $E(R, \infty, \ldots, \infty)$.

In dimension 4, the question of when the ellipsoid $E(a, b)$ symplectically embeds into the ellipsoid $E(a', b')$ was answered by McDuff in [10]. Let $\{N_k(a, b)\}_{k \geq 0}$ denote the sequence of nonnegative integer linear combinations of a and b, ordered nondecreasingly with repetitions.

Theorem 1.2 ([10])**.** *There exists a symplectic embedding $\mathrm{int}(E(a, b)) \to E(a', b')$ if and only if $N_k(a, b) \leq N_k(a', b')$, for every nonnegative integer k.*

Building on techniques introduced in [10], the authors of [11] study the function

$$c_0(a) = \inf \left\{ \lambda \;\middle|\; E(1, a) \text{ symplectically embeds into } B^4(\lambda) \right\}$$

and show that, in particular, for $a \in [1, (\frac{1+\sqrt{5}}{2})^4]$, c_0 is given by a piecewise linear function involving the Fibonacci numbers, which they call the "Fibonacci staircase".

In the more recent work [5] and [6], the authors studied the stabilized ellipsoid embedding problem in higher dimensions, and proved that the functions:

$$c_n(a) = \inf \left\{ \lambda \;\middle|\; E(1,a) \times \mathbb{C}^n \text{ symplectically embeds into } B^4(\lambda) \times \mathbb{C}^n \right\}$$

exhibit a similar behavior involving staircases and Fibonacci numbers.

Beyond problems of existence, one can ask about the algebraic topology of the space of symplectic embeddings $\mathrm{SympEmb}(U, V)$ between two symplectic $2n$–dimensional manifolds U and V. Very little is known about such questions in dimensions higher than 4. For instance, in [10] McDuff demonstrated that the space of embeddings between 4–dimensional symplectic ellipsoids is connected whenever it is nonempty. The equivalent question in higher dimensions is open. For other results in dimension 4, see [1] and [8].

More recently, in [13], the second author developed methods to show that the contractibility of certain loops of symplectic embeddings of ellipsoids depends on the relative sizes of the two ellipsoids.

1.1 Main result

In this paper, we build upon the methods developed in [13] to tackle the question of describing the higher homology groups of spaces of symplectic embeddings between ellipsoids in any dimension. In particular, we produce certain torus families of embeddings between ellipsoids whose contractibility depends on the relative symplectic sizes of the ellipsoids involved.

More precisely, we will be studying families of symplectic embeddings that are restrictions of the following unitary maps. For $\theta = (\theta_1, \ldots, \theta_n) \in T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, let U_θ denote the unitary transformation:

$$\mathrm{U}_\theta(z_1, \ldots, z_n) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n). \tag{1.2}$$

For any symplectic ellipsoid $E(a)$ and $E(b)$ such that $a_i < b_i$, we may define a family of ellipsoid embeddings by restricting the domain.

$$\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b)) \qquad \Phi(\theta) := U_\theta|_{E(a)} \tag{1.3}$$

The following theorem about the family Φ is the main result of this paper.

Theorem 1.3 (Main theorem)**.** *Let $E(a)$ and $E(b)$ be ellipsoids in $\mathbb{C}^n$ such that $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ satisfy:*

$$a_i < b_i < a_{i+1} \quad \textit{and} \quad b_n < 2a_1.$$

Let $\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b))$ be given by $\Phi(\theta) = \mathrm{U}_\theta\,|_{E(a)}$ for U_θ defined as in (1.2). Then the induced map on homology with $\mathbb{Z}_2$–coefficients is injective.

$$\Phi_* : H_*(T^n; \mathbb{Z}_2) \hookrightarrow H_*(\mathrm{SympEmb}(E(a), E(b)); \mathbb{Z}_2).$$

To appreciate the nontriviality of Theorem 1.3, we note that the torus map $U : T^n \to U(n)$ is very non–injective on $\mathbb{Z}_2$–homology.

Lemma 1.4. *Consider the map $U : T^n \to U(n)$ given by $\theta \mapsto U_\theta$. Then the induced map $U_* : H_*(T^n; \mathbb{Z}_2) \to H_*(U(n); \mathbb{Z}_2)$ on homology with $\mathbb{Z}_2$–coefficients is:*

(a) surjective if $* = 0$ or $* = 1$.

(b) identically 0 if $* \geq 2$.

Proof. To show (a), we simply note that T^n and $U(n)$ are connected so $U_0 = \mathrm{Id}$. Furtheremore, if we consider the loop $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to T^n$ given by $\theta \mapsto (\theta, 0, \ldots, 0)$, we see that the composition:

$$\det_{\mathbb{C}} \circ U \circ \gamma : \mathbb{R}/2\pi\mathbb{Z} \to U(1) \simeq \mathbb{R}/2\pi\mathbb{Z}$$

is the identity. Since $\det_{\mathbb{C}} : U(n) \to U(1)$ induces an isomorphism on H_1, the induced map of U must be surjective on H_1.

To show (b) we proceed as so. For each subset $L \subset \{1, \ldots, n\}$ with size $|L| = k$, define the k–torus by:

$$T_L = \{(\theta_1, \ldots, \theta_n) \in T^n \mid \theta_j = 0, \ \forall j \notin L\}.$$

The homology group $H_k(T^n; \mathbb{Z}_2)$ is generated by the classes $[T_L]$, where L runs over all subsets of size k. It suffices to show that $U_*[T_L] = 0$ for all L with $|L| \geq 2$. We can factorize $T_L = T_J \times T_K$ for $J \sqcup K = L$ and $|J| = 2$. Then we have a factorization:

$$T_L = T_J \times T_K \xrightarrow{\iota_J \times \iota_K} U(2) \times U(n-2) \xrightarrow{j} U(n)$$

Here j is the inclusions of a product of unitary subgroups, and ι_J and ι_K are inclusions of the tori into these unitary subgroups. It suffices to show that $(\iota_J \times \iota_K)_*[T_L] = [\iota_J]_*[T_J] \otimes [\iota_K]_*[T_K] = 0$, or simply that $[\iota_J]_*[T_J] = 0 \in H_2(U(2); \mathbb{Z}_2)$.

Now we simply note that $\dim(U(2)) = 4$ and $H^*(U(2); \mathbb{Z}_2) \simeq \mathbb{Z}_2[c_1, c_3]$ where c_i is a generator of index i. In particular, $H_2(U(2); \mathbb{Z}_2) \simeq H^2(U(2); \mathbb{Z}_2) = 0$. □

Lemma 1.4 can be used to prove that the torus map $\Phi : T^n \to \mathrm{Symp}(E(a), E(b))$ similarly fails to be injective when $E(a)$ is very small relative to $E(b)$.

Proposition 1.5. *Let $E(a)$ and $E(b)$ be ellipsoids in $\mathbb{C}^n$ such that $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ satisfy*

$$a_i < a_{i+1}, \quad b_i < b_{i+1}, \quad a_n < b_1.$$

Let $\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b))$ be given by (1.3). Then the induced map Φ_* on homology with $\mathbb{Z}_2$–coefficients has $\mathrm{Im}(\Phi_*)$ rank 1 in degree $* = 0, 1$, and rank 0 in degree $* \geq 2$.

Proof. Let $D : \mathrm{SympEmb}(E(a), E(b)) \to U(n)$ denote the map $\varphi \mapsto r(d\varphi|_0)$, given by taking derivatives $d\varphi|_0 \in \mathrm{Sp}(2n)$ at the origin and composing with a retration $r : \mathrm{Sp}(2n) \to U(n)$. Note that, for a and b meeting our assumptions, we can factor the identity $\mathrm{Id} : U(n) \to U(n)$ and $\Phi : T^n \to \mathrm{SympEmb}(E(a), E(b))$ as:

$$\mathrm{Id} : U(n) \xrightarrow{\mathrm{res}} \mathrm{Symp}(E(a), E(b)) \xrightarrow{D} U(n)$$

$$\Phi : T^n \xrightarrow{U} U(n) \xrightarrow{\mathrm{res}} \mathrm{SympEmb}(E(a), E(b))$$

Here $\mathrm{res}(\varphi) := \varphi|_{E(a)}$ denotes restriction of domain. In particular, $\mathrm{res} : U(n) \to \mathrm{Symp}(E(a), E(b))$ is injective on homology and $\mathrm{Im}(\Phi_*) \simeq \mathrm{Im}(U_*)$ as graded $\mathbb{Z}_2$-vectorspaces. The result thus follows from Lemma 1.4. $\square$

Remark 1.6. One can recover the result of Theorem 1.4 in [13] by looking at the first order homology $H_1(T^n; \mathbb{Z}_2)$ in the conclusion of Theorem 1.3 for $n = 2$.

Organization. The rest of the paper is organized as so. In Section §2, we give the proof of Theorem 1.3. The final two section are dedicated to demonstrating some technical results needed to deduce the steps of the proof. In §3, we recall the definition of the contact dg–algebra (constructed in full generality by Pardon in [14]) together with the releant computations for symplectic ellipsoids. In §4, we prove some technical results about the topology of all symplectic embeddings spaces.

Acknowledgements. We would like to thank our advisor, Michael Hutchings, for all the helpful discussions. JC was supported by the National Science Foundation Graduate Research Fellowship under Grant No. 1752814.

2 Proof of the main result

In this section, we prove Theorem 1.3, modulo some technical results discussed in §3-4. The strategy is, roughly, the following.

We assume for contradiction that the map Φ_* induced by the family Φ of (1.3) is not injective in degree k. Using results in §4, we use use this assumption to find a certain family of embeddings, parameterized by a union of k-tori and built from Φ, is null-bordant via a family of embeddings Ψ over some compact space P. We use Ψ to construct a moduli space of holomorphic curves $\mathcal{M}_I(\mathfrak{J})$ in cobordisms and some associated evaluation maps ev_I to a k–torus T^k. We then show that the degree of restricted to the boundary is 1 mod 2, which is a contradiction since the evaluation map extends to the bounding manifold P.

2.1 Moduli spaces in cobordisms

We now introduce the spaces of holomorphic curves that are relevant to our proof. Throughout §2.1-2.3, we will assume that $E(a)$ and $E(b)$ are ellipsoids with $a_i/a_j \notin \mathbb{Q}$ and $b_i/b_j \notin \mathbb{Q}$ for any $i \neq j$. In particular, this implies that the natural contact forms on the boundaries $\partial E(a)$ and $\partial E(b)$ are non-degenerate (see Example 3.8).

Recall that, for $(Y_\pm, \lambda_\pm)$ closed contact manifolds of the same odd dimension, a *compact symplectic cobordism* from (Y_+, λ_+) to (Y_-, λ_-) is a compact symplectic manifold (W, ω) with boundary $\partial W = -Y_- \sqcup Y_+$ such that $\omega|_{Y_\pm} = d\lambda_\pm$. Given a compact symplectic cobordism (W, ω), one can find neighborhoods N_- of Y_- and N_+ of Y_+ in W, and symplectomorphisms

$$(N_-, \omega) \simeq ([0, \epsilon) \times Y_-, d(e^s\lambda_-)) \qquad (N_+, \omega) \simeq ((-\epsilon, 0] \times Y_+, d(e^s\lambda_+)),$$

where s denotes the coordinate on $[0, \epsilon)$ and $(-\epsilon, 0]$. Using these identifications, we can complete the compact symplectic cobordism (W, ω) by adding cylindrical ends $(-\infty, 0] \times Y_-$ and $[0, \infty) \times Y_+$ to obtain the *completed symplectic cobordism*

$$\widehat{W} = [0, \infty) \times Y_+ \cup_{Y_+} W \cup_{Y_-} (-\infty, 0] \times Y_-.$$

An almost complex structure J on a completed symplectic cobordism $\widehat{W}$ as above is called *compatible* if:

- On $[0, \infty) \times Y_+$ and $(-\infty, 0] \times Y_-$, the almost complex structure J is $\mathbb{R}$–invariant, maps ∂_s (the $\mathbb{R}$ direction) to $R_{\lambda_\pm}$, and maps $\xi_\pm$ to itself compatibly with $d\lambda_\pm$.
- On the compact symplectic cobordism W, the almost complex structure J is tamed by ω.

Call $\mathcal{J}(\widehat{W})$ the set of all such compatible almost complex structures on $\widehat{W}$.

Notation 2.1. For the remainder of this section, we adopt the following notation.

Fix a subset $I \subset \{1, \ldots, n\}$. For each $i \in \{1, \ldots, n\}$, let Σ_i denote a copy of the twice punctured Riemann sphere $\mathbb{R} \times S^1 \simeq \mathbb{CP}^1 \setminus \{0, \infty\}$ with the usual complex structure. Label the positive and negative punctures of Σ_i by p_i^+ and p_i^- respectively. Denote $\Sigma_I = \sqcup_{i \in I} \Sigma_i$.

Moreover, for a symplectic embedding $\varphi : E(a) \to E(b)$, let $\widehat{W}_\varphi$ denote the completed symplectic cobordism obtained by completing the compact symplectic cobordism $W_\varphi = E(b) \setminus \mathrm{int}(\varphi(E(a)))$. Finally, let $\gamma_i^+ = \partial E(b) \cap \{z_i = 0\}$ and let $\gamma_i^- = \partial E(a) \cap \{z_i = 0\}$.

Definition 2.2 (Unparameterized moduli space)**.** For any symplectic embedding $\varphi \in \mathrm{Symp}(E(a), E(b))$ and any admissible almost–complex structure $J_\varphi \in \mathcal{J}(\widehat{W}_\varphi)$, we denote by $\mathcal{M}_I(\widehat{W}_\varphi; J_\varphi)$ the moduli space:

$$\mathcal{M}_I(\widehat{W}_\varphi; J_\varphi) = \left\{ u : \Sigma_I \to \widehat{W}_\varphi \;\middle|\; (du)^{0,1}_{J_\varphi} = 0, u \to \gamma_i^+ \text{ at } p_i^+ \text{ and } u \to \varphi(\gamma_i^-) \text{ at } p_i^- \right\} / \mathbb{C}^\times$$

That is, $u : \Sigma_I \to \widehat{W}_\varphi$ is a J_φ–holomorphic curve such that u asymptotes to the trivial cylinder over γ_i^+ in $\mathbb{R}^+ \times \partial_+ W_{\Phi_p} \simeq \mathbb{R}^+ \times \partial E(b)$ at the puncture p_i^+ and u asymptotes to the trivial cylinder over $\varphi(\gamma_i^-)$ in $\mathbb{R}^- \times \partial_- W_\varphi \simeq \mathbb{R}^- \times \varphi(E(a))$ at the puncture p_i^-, for each $i \in I$. We quotient the space of such maps by the group of domain reparameterizations, which is $\mathbb{C}^\times$.

Definition 2.3 (Parameterized moduli space over P)**.** Given a compact manifold with boundary P and a P–parameterized family of symplectic embeddings $\Psi : P \times E(a) \to E(b)$, let $\mathcal{J}(\Psi)$ denote the set of P–parameterized families $\mathfrak{J} = \{J_p\}_{p \in P}$ of almost complex structures $J_p \in \mathcal{J}(\widehat{W}_{\Psi_p})$ for each Ψ_p. Let $\mathcal{M}_I(\mathfrak{J})$ denote the parameterized moduli space of pairs:

$$\mathcal{M}_I(\mathfrak{J}) = \left\{ (p, u) \;\middle|\; p \in P,\ u \in \mathcal{M}_I(\widehat{W}_{\Psi_p}; J_p) \right\}.$$

Given a family of symplectic embeddings $\Psi : P \times E(a) \to E(b)$ where $\mathrm{Im}(\Psi_p)$ is independent of p for p near ∂P, we let $W_{\partial P} = E(b) \setminus \Psi_p(E(a))$ for $p \in \partial P$.

2.2 Transversality and compactness

This section is devoted to a proof of generic transversality (Lemma 2.4) and compactness (Lemma 2.5) for the moduli space $\mathcal{M}_I(\mathfrak{J})$.

Lemma 2.4 (Transversality)**.** *There exists a $J_{\partial P} \in \mathcal{J}(\widehat{W}_{\partial P})$ and a family of almost complex structures $\mathfrak{J} \in \mathcal{J}(\Psi)$ with $\mathfrak{J}|_{\partial P} \equiv J_{\partial P}$ such that the moduli space $\mathcal{M}_I(\widehat{W}_{\partial P}, J_{\partial P})$ is a 0–dimensional manifold and the moduli space $\mathcal{M}_I(\mathfrak{J})$ is a $(k+1)$–dimensional manifold with boundary $\partial\mathcal{M}_I(\mathfrak{J}) \simeq \partial P \times \mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$.*

Proof. This essentially follows from the results of [19] and [20, §7], which we now discuss in some detail.

Observe that all of the J_p–holomorphic curves u appearing in points $(p, u) \in \mathcal{M}_I(\mathfrak{J})$, for any choice of $\mathfrak{J}$, are somewhere injective. By [20, Theorems 7.1–7.2], there exists a comeager subset $\mathcal{J}^{\mathrm{reg}}(\widehat{W}_{\partial P}) \subset \mathcal{J}(\widehat{W}_{\partial P})$ such that, for any $J_{\partial P} \in \mathcal{J}^{\mathrm{reg}}(\widehat{W}_{\partial P})$, the moduli space $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$, which consists entirely of somewhere–injective curves, is a manifold with dimension equal to the virtual dimension:

$$\mathrm{vdim}(\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})) = \sum_{i \in I} (\dim(\widehat{W}_{\partial P}) - 3)\chi(u) + 2c_1^\tau(u_*[\Sigma_I]) + \mu_{CZ}(\gamma_i^+) - \mu_{CZ}(\gamma_i^-).$$

Since $c_1(\widehat{W}_{\partial P}) = 0$, $\chi(u) = 0$, and each component of u has two punctures and $\mu_{CZ}(\gamma_i^+) = \mu_{CZ}(\gamma_i^-)$, the virtual dimension is 0.

By the parametric version of transversality (see [20, Remark 7.4] and [19, §4.5]), there exists a family $\mathfrak{J} \in \mathcal{J}(\Psi)$, such that $\mathfrak{J}|_{\partial P} \equiv J_{\partial P}$ and $\mathcal{M}_I(\mathfrak{J})$ is a manifold with boundary $\partial\mathcal{M}_I(\mathfrak{J}) = \partial P \times \mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ of dimension:

$$\dim(\mathcal{M}_I(\mathfrak{J})) = \dim(P) + \mathrm{vdim}(\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})) = k + 1.$$

□

Lemma 2.5 (Compactness)**.** *Let* $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < 2a_1$ *and choose* $J_{\partial P} \in \mathcal{J}(\widehat{W}_{\partial P})$ *and* $\mathfrak{J} \in \mathcal{J}(\Psi)$ *as in Lemma 2.4. Then the moduli spaces* $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ *and* $\mathcal{M}_I(\mathfrak{J})$ *are compact.*

Proof. This is a simple application of SFT compactness, of which we give a simplified discussion below (see Review 2.6).

Thus, let u^i be a sequence in $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ or $\mathcal{M}_I(\mathfrak{J})$. We need to show that the limit building v (under the notation of Review 2.6) has no symplectization levels. By considering components of Σ_I and $\#_i\Sigma_i$ individually, we can assume that $|I| = 1$, i.e. that Σ_I has one component and each u^i is positively asymptotic to a single γ_l^+ for some $l \in \{1, \ldots, n\}$.

By the action monotonicity, the ends of u_j^+, u_j^- and u^W must aymptote to a set of orbits with total action between a_l and b_l. Due to our assumptions on a_l and b_l, there are no such orbits besides γ_l^- and γ_l^+ themselves. Since one symplectization component must exist in the compactification, u^W must be a cylinder from γ_l^+ to γ_l^- and there can be no other levels by action considerations. □

Review 2.6 (SFT Compactness)**.** Here we give a very simplified review of a version of SFT compactness as proven in [2, §10].

Let W and P be connected, closed manifolds with boundary, and let $\{(\lambda_p, J_p)\}_{p\in P}$ be a P–parameterized family of Liouville form/compatible almost–complex structure pairs. Fix a closed surface with punctures $\dot{\Sigma}$, and consider a sequence $p_i \in P$ and $u^i : \dot{\Sigma} \to (\widehat{W}, J_{p_i})$ of J_{p_i}–holomorphic curves satisfying a uniform energy bound.

The SFT compactness theorem states that, after passing to a subsequence, $p_i \to p \in P$ and u^i converges to a J_p*–holomorphic building*:

$$v = (u_1^+, \ldots, u_\alpha^+, u^W, u_1^-, \ldots, u_\beta^-)$$

Here $\alpha, \beta \in \mathbb{Z}^{\geq 0}$ are integers and the elements of the tuple (called *levels*):

$$u_j^* : \Sigma_{(j)}^* \to \widehat{Y}_* \text{ for } * \in \{+,-\} \quad \text{and} \quad u^W : \Sigma^W \to \widehat{W}_{\partial P}$$

denote J_p–holomorphic curves (modulo translation, in the symplectization case). The asymptotics of the u_i^* and u^W are supposed to be compatible, in the sense that the negative ends of u_i^* and the positive ends of u_{i+1}^* must agree (and likewise for u_a^+ and u^W, etc.). Note that each symplectization level u_i^* can be disconnected, but it must have at least one component that is not a trivial cylinder $\mathbb{R} \times \gamma$.

The surfaces $\Sigma_{(i)}$ can be glued together along the boundary punctures asymptotic to matching Reeb orbits, and this glued surface $\#_i \Sigma_{(i)}$ is homeomorphic to $\dot{\Sigma}$. There is some of additional data, beyond the holomorphic curves themselves, associated to a holomorphic building. However, we suppress this data since it will play no role in our argument.

2.3 Counting the curves

This section is devoted to proving the following lemma.

Lemma 2.7 (Odd signed count)**.** *The 0–dimensional moduli space* $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ *has an odd number of points.*

Proof. First note that we can write the moduli space $\mathcal{M}(\widehat{W}_{\partial P}, J_{\partial P})$ in terms of the moduli spaces $\mathcal{M}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-, [0])$ from Construction 3.4, as so:

$$\mathcal{M}_I(W_{\partial P}; J_{\partial P}) \simeq \prod_{i \in I} \mathcal{M}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-, [0]).$$

Here $[0]$ denotes the unique homotopy class of cylinder in $\pi_2(W_{\partial P}; \gamma_i^+ \sqcup \gamma_i^-)$ (by abuse of notation, we suppress the dependence of this class on i). The argument in Lemma 2.5 establishes that $\mathcal{M}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-, [0])$ is compact, and in fact that there can be no genus 0 holomorphic buildings from γ_i^+ to γ_i^- other than those of $\mathcal{M}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-, [0])$. Thus we have:

$$\overline{\mathcal{M}}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-, [0]) = \mathcal{M}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-, [0]).$$

The compactified moduli space is therefore cut out transversely. It suffices to show that $\overline{\mathcal{M}}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-, [0])$ has an odd number of points.

To prove this, consider the cobordism $W_{\partial P}$ from $E(b)$ to $E(a)$ and the cobordism $V_{\partial P} := E(c \cdot a) \setminus E(b)$ from $E(c \cdot a)$ to $E(b)$. Here $c \gg 0$ is some constant such that $ca_i > b_i$ for all i. We denote the cobordism maps by:

$$CH[V_{\partial P}] : CH(\partial E(c{\cdot}a)) \to CH(\partial E(b)) \qquad CH[W_{\partial P}] : CH(\partial E(b)) \to CH(\partial E(a)).$$

By Lemma 3.9, the map $CH[V_{\partial P}]$ and $CH[W_{\partial P}]$ are isomorphisms. Furthermore, the composition cobordism map $CH[W_{\partial P}] \circ CH[V_{\partial P}]$ agrees with the obvious map:

$$CH(\partial E(c \cdot a)) \simeq \Lambda[\mathcal{P}_{\mathrm{g}}(\partial E(c \cdot a))] \simeq \Lambda[\mathcal{P}_{\mathrm{g}}(\partial E(a))] \simeq \Lambda[\mathcal{P}_{\mathrm{g}}(\partial E(a))].$$

This map is induced by the bijection between orbits $\mathcal{P}_{\mathrm{g}}(\partial E(c\cdot a)) \simeq \mathcal{P}_{\mathrm{g}}(\partial E(a))$ (see [14, Lemma 1.2] and the proof for a discussion of this fact).

The computation in Example 3.8 implies that on $\partial E(c\cdot a)$, $\partial E(a)$, and $\partial E(b)$ any non–simple orbit η satisfies $|\eta| \geq 4n-2$ and any set of 2 or more orbits Γ satisfies $|\Gamma| \geq 4n-2$. In particular, we have the following isomorphisms for $1 \leq i \leq n$

$$CH_{2(n+i-2)}(\partial E(c \cdot a)) \simeq \mathbb{Q}\langle \gamma_i^{ca} \rangle \qquad CH_{2(n+i-2)}(\partial E(a)) \simeq \mathbb{Q}\langle \gamma_i^{a} \rangle.$$

Here γ_i^{ca} and γ_i^a are two simple orbits of $\partial E(c \cdot a)$ and $\partial E(a)$ identified under the bijection $\mathcal{P}_{\mathrm{g}}(\partial E(c\cdot a)) \simeq \mathcal{P}_{\mathrm{g}}(\partial E(a))$. By the discussion above about the composition cobordism map, we can pick generators $[\gamma_i^{ca}]$ and $[\gamma_i^a]$ of $CH_{2(n+i-2)}(\partial E(c\cdot a))$ and $CH_{2(n+i-2)}(\partial E(a))$ respectively such that:

$$CH[W_{\partial P}] \circ CH[V_{\partial P}]([\gamma_i^{ca}]) = [\gamma_i^a].$$

The definition of the cobordism maps on homology classes on the chain level, given in Construction 3.4, then implies that the following contact homology class must be 0.

$$\left[\frac{\#\overline{\mathcal{M}}(\widehat{W}_{\partial P}; \gamma_i^b, \gamma_i^a, [0])}{|\operatorname{Aut}(\gamma_i^b, \gamma_i^a, ; [0])|} \cdot \frac{\#_\theta \overline{\mathcal{M}}(\widehat{V}_{\partial P}; \gamma_i^{ca}, \gamma_i^b, [0])}{|\operatorname{Aut}(\gamma_i^{ca}, \gamma_i^b, ; [0])|} \cdot \gamma_i^a - \gamma_i^a \right] = 0 \in CH(\partial E(a)).$$

The automorphism groups $\operatorname{Aut}(\gamma_i^{ca}, \gamma_i^b, [0])$ and $\operatorname{Aut}(\gamma_i^b, \gamma_i^a, [0])$ are trivial, since γ_i^a, γ_i^b, and γ_i^{ca} are all simple, and the above expression can be null–homologous if and only if it yields a zero multiple of γ_i^a. Thus we must have:

$$\#\overline{\mathcal{M}}(\widehat{W}_{\partial P}; \gamma_i^b, \gamma_i^a, [0]) \cdot \#_\theta \overline{\mathcal{M}}(\widehat{V}_{\partial P}; \gamma_i^{ca}, \gamma_i^b, [0]) - \mathrm{Id} = 0.$$

The above identity is as maps of orientation lines, but identifying the orientation lines with $\mathbb{Z}$ yields the analogous identity for a pair of integer–valued point counts. This implies that $\#\overline{\mathcal{M}}(\widehat{W}_{\partial P}; \gamma_i^b, \gamma_i^a, [0])$ must be ± 1 as an integer–valued point count, which is the desired result. □

Remark 2.8 (Moduli space differences)**.** A confusing point, worth mentioning, is that the moduli spaces $\mathcal{M}_I(\widehat{W}_{\partial P}; J_{\partial P})$ and $\mathcal{M}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-, [0])$ consist of slightly different objects. Namely, the moduli space $\mathcal{M}(\widehat{W}_{\partial P}; \gamma_i^+, \gamma_i^-, [0])$ consists of reparameterization equivalence classes $[u, \Sigma, m]$ of maps u from *any* genus 0, twice punctured Riemann surface Σ (so the domain complex structure can vary), and includes the data of an asymptotic marker m_p which must asymptote to fixed base points b_p of the limiting orbits γ_p, for every puncture p.

In the case of a genus 0 surface with two punctures, however, the associated Teichmuller space is trivial (see [19]), and a careful reading of the equivalence relations between trees described in [14, §2.1] shows that the asymptotic markers at input and output vertices are allowed to vary.

2.4 Proof of main theorem

Given the moduli-space constructions of §2.1-2.3, we can now proceed to the proof of Theorem 1.3.

Proof. As in Lemma 1.4, for each subset $L \subset \{1, \dots, n\}$ with size $|L| = k$ we define

$$T_L = \{(\theta_1, \dots, \theta_n) \in T^n \mid \theta_j = 0, \ \forall j \notin L\}\,. \tag{2.1}$$

We again note that the homology group $H_k(T^n; \mathbb{Z}_2)$ is generated by the classes $[T_L]$, where L runs over all subsets of size k.

We will now argue by contradiction to prove Theorem 1.3. To that end, fix an integer $k \in \{1, \dots, n\}$ and assume there is a nonzero homology class $[A] = \sum_L c_L[T_L]$ in $H_k(T^n; \mathbb{Z}_2)$ such that $\Phi_*[A] = 0$ in $H_k(\mathrm{SympEmb}(E(a), E(b)); \mathbb{Z}_2)$. Fix a subset $I = \{i_1, i_2, \dots, i_k\}$ for which $c_I = 1$ in the expansion of the class $[A]$.

Since $\Phi_*([A])$, using Proposition 4.8, we can assume there exists a smooth $k+1$–dimensional manifold P with boundary $\partial P = \sqcup_{c_L \neq 0} T_L$ and a smooth map

$$\Psi : P \to \mathrm{SympEmb}(E(a), E(b)),$$

such that $\Psi|_{T_L} = \Phi|_{T_L}$ for each component T_L of ∂P.

We may, by passing to sub-ellipsoids, assume that $E(a)$ and $E(b)$ with sub-ellipsoids satisfying $a_i/a_j \notin \mathbb{Q}$ and $b_i/b_j \notin \mathbb{Q}$ for all $i \neq j$. Lemmas 2.4 and 2.5 then imply that there exist $J_{\partial P} \in \mathcal{J}(\widehat{W}_{\partial P})$ and $\mathfrak{J} \in \mathcal{J}(\Psi)$ such that the parametrized moduli space $\mathcal{M}_I(\mathfrak{J})$ is a compact $(k+1)$–dimensional manifold with boundary $P \times \mathcal{M}(\widehat{W}_{\partial P}; J_{\partial P})$. We define the evaluation map

$$\mathrm{ev}_I : \mathcal{M}_I(\mathfrak{J}) \to \gamma^-_{i_1} \times \gamma^-_{i_2} \times \cdots \times \gamma^-_{i_k} \simeq T^k,$$

as follows. For each $i_l \in I$, fix a point $q^+_{i_l} \in \gamma^+_{i_l}$. Each $(p, u) \in \mathcal{M}^I(\mathfrak{J})$ has k cylinder components $u_1, \dots, u_k$ such that u_l has a positive end at $\gamma^+_{i_l}$ and a negative end at $\gamma^-_{i_l}$. Let $t_l \in S^1$ be such that $\lim_{s\to\infty} u_l(s, t_l) = q^+_{i_l}$ and let $q^-_{i_l} = \lim_{s\to-\infty} u_l(s, t_l)$. Define $\mathrm{ev}_I(p, u) = (q^-_{i_1}, q^-_{i_2}, \dots, q^-_{i_k})$.

Let $\pi : \mathcal{M}_I(\mathfrak{J}) \to P$, $(p, u) \mapsto p$, denote the projection to the parameter space. Lemma 2.7 implies that for each T_L in the decomposition of A, $\pi^{-1}(T_L)$ is a disjoint union of an odd number of k–tori.

Notice that for each $L \neq I$, the restriction of ev_I to each component of $\pi^{-1}(T_L)$ is not surjective and thus it has degree 0. By contrast, the restriction of ev_I to each component of $\pi^{-1}(T_I)$ is bijective and has degree 1. Hence, we can conclude that

$$\sum_C \deg((\mathrm{ev}_I)|_C) \equiv 1 \bmod 2,$$

where the sum is taken over the connected components C of the boundary $\partial\mathcal{M}_I(\mathfrak{J})$. Since ev_I is defined on all of $\mathcal{M}_I(\mathfrak{J})$, the above sum of the degrees must be 0 mod 2, hence providing the contradiction. $\square$

3 Contact homology algebra

In this section, we review the main Floer–theoretic tool in this paper, the full contact homology $CH(Y, \xi)$ of a closed contact manifold (Y, ξ).

3.1 Contact dg–algebra

In this section, we describe the construction of the contact dg–algebra of a contact manifold and the cobordism dg–algebra maps induced by exact symplectic cobordisms. This is largely a review of Pardon's construction of contact homology in [14].

We begin by fixing notation for the choices of Floer data that are needed to define the relevant chain groups and cobordism maps.

Setup 3.1 (Contact manifold)**.** Fix the following setup and notation.

(a) (Y, ξ) is a closed contact $(2n+1)$–manifold with nondegenerate contact form α and associated Reeb vector-field R.

(b) J is a $d\alpha$–compatible complex structure on ξ and $\widehat{J}$ is the associated $\mathbb{R}$–invariant almost complex structure on the symplectization $\widehat{Y} := \mathbb{R} \times Y$.

(c) $\theta \in \Theta(Y, \alpha, J)$ is a choice of virtual perturbation data (in the sense of [14] §1.1) for the compactified moduli spaces of holomorphic curves $\overline{\mathcal{M}}_{\mathrm{I}}(\widehat{Y}; \gamma^+, \Gamma^-, \beta)/\mathbb{R}$ for any γ^+, Γ^- and β (see [14] for elaboration).

We use $\mathsf{Data}(Y, \xi)$ to denote the set of choices of data associated to a fixed contact manifold (Y, ξ).

Setup 3.2 (Exact symplectic cobordism)**.** Fix the following setup and notation.

(a) For each $* \in \{+, -\}$, let Y_*, ξ_* and α_* be contact data, J_* and $\widehat{J}_*$ be complex data, and θ_* be virtual perturbation data, all as in Setup 3.1.

Furthermore, fix the following setup and notation for corresponding cobordism data.

(b) (W, λ) is an exact symplectic $(2n+2)$–cobordism with $\partial_+ W = Y_+$ and $\partial_- W = Y_-$ with completion $(\widehat{W}, \widehat{\lambda})$ such that $\widehat{W} = (-\infty, 0] \times Y_- \cup W \cup [0, \infty) \times \widehat{Y}_+$ and $\widehat{\lambda}$ agrees with $d(e^t \alpha_*)$ and λ on the appropriate pieces.

(c) J is a $d\lambda$ compatible complex structure on W agreeing with J_* on symplectic collar neighborhoods of Y_*, and $\widehat{J}$ is the associated complex structure on the Liouville completion $\widehat{W}$.

(d) $\theta \in \Theta(W, \lambda, \widehat{J})$ is a choice of virtual perturbation data (in the sense of [14] §1.3) for the compactified moduli spaces of holomorphic curves $\overline{\mathcal{M}}(\widehat{W}; \gamma^+, \Gamma^-, \beta)$ for all γ^+, Γ^- and β.

We use $\mathsf{Data}[W, \lambda]$ to denote the set of choices of data as above for a fixed deformation class $[W, \lambda]$ of exact symplectic cobordism. For each $* \in \{+, -\}$, there is a projection map of Floer data:

$$\pi_* : \mathsf{Data}[W, \lambda] \to \mathsf{Data}(Y_*, \xi_*) \qquad (\lambda, J, \theta) \mapsto (\alpha_*, J_*, \pi_*\theta).$$

Here we use the projection map of VFC data $\pi_* : \Theta(W, \lambda, J) \to \Theta(Y_*, \alpha_*, J_*)$ described in [14], §1.3. Note that the product map

$$\pi_+ \times \pi_- : \Theta(W, \lambda, J) \to \Theta(Y_+, \alpha_+, J_+) \times \Theta(Y_-, \alpha_-, J_-)$$

is surjective.

Given the setup discussed above, we give an overview of the construction of the contact dg–algebra and the cobordism maps.

Construction 3.3. (Contact DG–Algebra) Consider a contact manifold (Y, ξ) along with a choice of data $\mathcal{D} \in \mathsf{Data}(Y, \xi; Z, [P])$, all as in Setup 3.1, and let $m(\xi)$ denote the integer:

$$m(\xi) := 2 \cdot \gcd \{\langle c_1(\xi), [\Sigma] \rangle \mid [\Sigma] \in H_2(Y)\}.$$

We now give the construction of the $(\mathbb{Z}_2, H_1(Y; \mathbb{Z}))$–bigraded differential algebra $CC(Y, \xi)_{\mathcal{D}}$ with differential $\partial_{\mathcal{D}}$) of bi–degree $(-1, 0)$, called the *contact dg–algebra* or the *full contact homology algebra*. The $\mathbb{Z}_2$ part of the grading is called the *homological grading* and the $H_1(Y; \mathbb{Z})$ part of the grading is called the *loop grading*. The subalgebra $CC_\circ(Y, \xi)_{\mathcal{D}}$ of loop grading 0 elements will have a $\mathbb{Z}_{m(\xi)}$ lift of the homological grading. We denote the homologies of these dg–algebras by:

$$CH(Y, \xi)_{\mathcal{D}} := H(CC(Y, \xi)_{\mathcal{D}}) \qquad CH_\circ(Y, \xi)_{\mathcal{D}} := H(CC_\circ(Y, \xi)_{\mathcal{D}}).$$

(Algebra) To define the algebra of the total complex $CC(Y, \xi)_{\mathcal{D}}$, consider the set $\mathcal{P}(Y, \alpha)$ of unparameterized Reeb orbits of (Y, α). We can divide $\mathcal{P}(Y, \alpha)$ into good orbits $\mathcal{P}_g(Y, \alpha)$ and bad orbits $\mathcal{P}_b(Y, \alpha)$. An orbit is *bad* if it is an even cover of an orbit η with $CZ_\tau(\eta) \equiv 1 \bmod 2$ for some (any) trivialization of $\xi|_\eta$. An orbit is *good* if it is not bad.

To each good orbit we can associate an $(\mathbb{Z}_2, H_1(Y; \mathbb{Z}))$–graded orientation line $\mathfrak{o}_\gamma$ supported in bigrading $(|\gamma|, [\gamma])$ where $|\gamma| = CZ(\gamma, [\Sigma]) + n - 3 \bmod 2$ (see for instance [3]) and $[\gamma] \in H_1(Y; \mathbb{Z})$ is simply the homology class of the orbit. For a finite set of orbits $\Gamma \subset \mathcal{P}_g(Y, \alpha)$ of good orbits, we define $\mathfrak{o}_\Gamma := \otimes_{\gamma \in \Gamma} \mathfrak{o}_\gamma$, $|\Gamma| := \sum_{\gamma \in \Gamma} |\gamma|$

and $[\Gamma] = \sum_{\gamma\in\Gamma}[\gamma]$. Notationally, we will often $\mathfrak{o}_\gamma$ or $\mathfrak{o}_\Gamma$ to also mean $\mathfrak{o}_\Gamma \otimes \mathbb{Q}$, suppressing the tensor up to $\mathbb{Q}$. We then define the contact dg–algebra as:

$$CC(Y,\xi)_{\mathcal{D}} := \bigwedge\Big(\bigoplus_{\gamma\in\mathcal{P}_g(Y,\alpha)} \mathfrak{o}_\gamma\Big),$$

that is, $CC(Y,\xi)_{\mathcal{D}}$ is the free graded commutative unital $\mathbb{Z}_2$-graded $\mathbb{Q}$-algebra generated by $\mathfrak{o}_\gamma$ for $\gamma \in \mathcal{P}_g(Y,\alpha)$.

If we let $\mathcal{P}_{g,\circ}(Y,\alpha)$ denote the space of 0 loop–graded orbits, then we can define the subalgebra $CC_\circ(Y,\xi)_{\mathcal{D}}$ by:

$$CC(Y,\xi)_{\mathcal{D}} := \bigwedge\Big(\bigoplus_{\gamma\in\mathcal{P}_{g,\circ}(Y,\alpha)} \mathfrak{o}_\gamma\Big).$$

The $\mathbb{Z}_{m(\xi)}$–grading on $CC_\circ(Y,\xi)_{\mathcal{D}}$ is defined as $CZ(\gamma,[\Sigma]) + n - 3 \bmod m(\xi)$.

(Differential) The differential $\partial_{\mathcal{D}} : CC(Y,\xi)_{\mathcal{D}} \to CC(Y,\xi)_{\mathcal{D}}$ is defined as so. For any pure element $x \in \mathfrak{o}_{\gamma_+}$ in the algebra:

$$\partial_{\mathcal{D}} x := \sum_{\mu(\gamma^+,\Gamma^-,\beta)=1} \frac{\#_\theta\overline{\mathcal{M}}_{\mathrm{I}}(\widehat{Y};\gamma^+,\Gamma^-,\beta)}{|\mathrm{Aut}(\gamma^+,\Gamma^-,\beta)|}\cdot x.$$

On a product xy of elements $x, y \in CC(Y,\alpha)$, $\partial_{\mathcal{D}}$ satisfies the graded Leibniz rule $\partial_{\mathcal{D}}(xy) = \partial_{\mathcal{D}}(x)y + (-1)^{|x|}x\partial_{\mathcal{D}}(y)$ for any $x, y \in CC(Y,\xi)_{\mathcal{D}}$.

Here $\overline{\mathcal{M}}_{\mathrm{I}}(\widehat{Y};\gamma^+,\Gamma^-,\beta)$ is the compactified moduli space of genus 0 curves in the symplectization $\widehat{Y}$ with one positive puncture asymptotic to γ^+ and k negative punctures asymptotic to Γ^- (along with appropriate asymptotic marker behavior at the punctures). The symbol $\#_\theta$ denotes taking a virtual moduli count in ${}_{\Gamma^-} \otimes {}^{\vee}_{\gamma^+}$ with respect to the VFC data θ comin with the data $\mathcal{D}$. See [14, §1.2, 2.3] for a full discussion.

Construction 3.4 (Cobordism Maps)**.** Consider an exact symplectic cobordism (W,λ) between contact manifolds (Y_+,ξ_+) and (Y_-,ξ_-), along with a choice of data $\mathcal{D} \in \mathsf{Data}(W,V,\lambda)$ as in Setup 3.2, where $\mathcal{A} = \pi_+\mathcal{D}$ and $\mathcal{B} = \pi_-\mathcal{D}$. Then we can construct a morphism of differential $\mathbb{Z}_2$–graded algebras:

$$CC(W,\lambda)_{\mathcal{D}} : CC(Y_+,\xi_+)_{\mathcal{A}} \to CC(Y_-,\xi_-)_{\mathcal{B}}.$$

To define this cobordism map, we declare its value on generators and extend it to an algebra map. On an $x \in \mathfrak{o}_{\gamma^+}$ for $\gamma^+ \in \mathcal{P}(Y_+,\alpha_+)$, it is given by the sum:

$$CC(W,\lambda)_{\mathcal{D}}(x) = \sum_{(\Gamma_-,\beta)|\mu(\gamma_+,\Gamma_-,\beta)=0} \frac{\#_\theta\overline{\mathcal{M}}_{\mathrm{II}}(\widehat{W};\gamma_+,\Gamma_-,\beta)}{|\,\mathrm{Aut}(\gamma_+,\Gamma_-,\beta)|}\cdot x.$$

Here $\overline{\mathcal{M}}_{\mathrm{II}}(\widehat{W};\gamma_+,\Gamma_-,\beta)$ is the compactified moduli space of genus 0 in the completion $\widehat{W}$ of W with one positive puncture asymptotic to γ^+ and k negative punctures asymptotic to Γ^- (along with appropriate asymptotic marker behavior at the punctures). The symbol $\#_\theta$ denotes taking a virtual moduli count in ${}_{\Gamma^-}\otimes^\vee_{\gamma^+}$ with respect to the VFC data θ, as with the differential. See [14, §1.3, 2.3] for a full discussion.

If $W \simeq Y \times [0,1]$ is cylindrical (as a smooth manifold) then $CC(W,\lambda)_{\mathcal{D}}$ respects the $H_1(Y;\mathbb{Z})$ grading and provides a map of differential $\mathbb{Z}_{m(\xi)}$–graded algebras:

$$CC_\circ(W,\lambda)_{\mathcal{D}} : CC_\circ(Y_+,\xi_+)_{\mathcal{A}} \to CC_\circ(Y_-,\xi_-)_{\mathcal{B}}.$$

The salient features of the construction in [14] can be summarized in the following Theorem, various parts of which are covered in [14, §1.3 − 1.8].

Theorem 3.5 ([14] Contact homology)**.** *The dg–algebra and maps described in Constructions 3.3 and 3.4 have the following properties.*

(a) (Homology) The map $\partial_{\mathcal{D}}$ of Construction 3.3 is a differential, i.e. $\partial_{\mathcal{D}}^2 = 0$. The homology algebra $CH(Y,\xi) \simeq CH(Y,\xi)_{\mathcal{D}}$ is independent of the choice of data $\mathcal{D} \in \mathsf{Data}(Y,\xi)$ up to canonical isomorphism.

(b) (Cobordism) The cobordism map $CC(W,\lambda)_{\mathcal{D}}$ of Construction 3.4 is a map of dg-algebras, i.e. $CC(W,\lambda)_{\mathcal{D}} \circ \partial_{\pi_+\mathcal{D}} - \partial_{\pi_-\mathcal{D}} \circ CC(W,\lambda)_{\mathcal{D}} = 0$. The maps on homology $CH[W,\lambda]_{\mathcal{D}}$, given a choice of data $\mathcal{D} \in \mathsf{Data}(W,\lambda)$, induces a map $CH[W,\lambda] : CH(Y_+,\xi_+) \to CH(Y_-,\xi_-)$ depending only on the deformation class of the Liouville cobordism $[W,\lambda]$.

(c) (Nullhomologous subalgebra) The homology algebra $CH_\circ(Y,\xi) \simeq CH_\circ(Y,\xi)_{\mathcal{D}}$ is independent of $\mathcal{D}$ up to canonical isomorphism, and if $W \simeq Y \times [0,1]$ then we have a map $CH_\circ[W,\lambda] : CH_\circ(Y_+,\xi_+) \to CH_\circ(Y_-,\xi_-)$ depending only on the deformation class $[W,\lambda]$.

(d) (Contactomorphism) An isotopy class of contactomorphism $[\Phi] : (Y_0,\xi_0) \to (Y_1,\xi_1)$ induces isomorphisms $CH[\Phi] : CH(Y_0,\xi_0) \simeq CH(Y_1,\xi_1)$ and $CH_\circ[\Phi] : CH_\circ(Y_0,\xi_0) \simeq CH_\circ(Y_1,\xi_1)$.

(e) (Transversality) If the moduli spaces of either Construction 3.3 or 3.4:

$$\overline{\mathcal{M}}_{\mathrm{I}}(\widehat{Y};\gamma^+,\Gamma^-,\beta) \qquad \overline{\mathcal{M}}_{\mathrm{II}}(\widehat{W};\gamma^+,\Gamma^-,\beta)$$

are regular (i.e. the linearized $\overline{\partial}_J$ operators on each component (including broken components) is surjective, see [14, §2.11]) then:

$$\#_\theta\overline{\mathcal{M}}_{\mathrm{I}}(\widehat{Y};\gamma^+,\Gamma^-,\beta) = \#\overline{\mathcal{M}}_{\mathrm{I}}(\widehat{Y};\gamma^+,\Gamma^-,\beta)$$

$$\#_\theta \overline{\mathcal{M}}_{\mathrm{II}}(\widehat{W};\gamma^+,\Gamma^-,\beta) = \#\overline{\mathcal{M}}_{\mathrm{II}}(\widehat{W};\gamma^+,\Gamma^-,\beta)$$

Here $\#S$ for a oriented zero–dimensional compact manifold S is equivalent to an oriented point count. In particular, if either moduli space is empty then the virtual count is 0.

3.2 Computations for ellipsoids

We now compute the examples of contact homology and cobordism maps that are relevant to our construction.

Definition 3.6. A contact form α on a closed contact manifold (Y,ξ) is *lacunary* if, for every orbit $\gamma \in \mathcal{P}_{\mathrm{g}}(Y,\xi)$, the grading $|\gamma| \in \mathbb{Z}_2$ is 0.

The contact homology dg–algebra of a contact manifold with lacunary contact form has vanishing differential for grading reasons. In particular, we have:

Lemma 3.7. *Let (Y,ξ) be a contact manifold with lacunary contact form α. Then:*

$$CH(Y,\xi) \simeq \bigwedge\Big(\mathbb{Q}[\mathcal{P}_{\mathrm{g}}(Y,\alpha)]\Big) \qquad CH_\circ(Y,\xi) \simeq \bigwedge\Big(\mathbb{Q}[\mathcal{P}_{\mathrm{g},\circ}(Y,\alpha)]\Big).$$

Example 3.8 (Ellipsoids)**.** Let $(a_1,\dots,a_n) \in (0,\infty)^n$ satisfy $a_i/a_j \notin \mathbb{Q}$ for each $i \neq j$. Recall that the ellipsoid $E(a_1,\dots,a_n) \subset \mathbb{C}^n$ is given by (1.1).

Consider the contact hypersurface $Y = \partial E(a_1,\dots,a_n) \subset \mathbb{C}^n$ with contact form $\alpha = \lambda|_Y$, where λ is the Liouville form $\lambda = \frac{1}{2}\sum_i x_i dy_i - y_i dx_i$. We now calculate the Reeb vector field, orbits, orbit actions and orbit Conley–Zehnder indices of (Y,α). In particular, we show that the contact form α is lacunary, thus computing the contact homology by Lemma 3.7.

The Reeb vector field R_α is given by $R_\alpha = 2\pi \sum_i a_i^{-1} \frac{\partial}{\partial\theta_i}$. Here θ_i is the angular coordinate in the ith $\mathbb{C}$ factor $\mathbb{C}_i$ of $\mathbb{C}^n$. The flow $\Phi_\alpha : Y \times \mathbb{R} \to Y$ by R_α is given by:

$$z = (z_1,\dots,z_n) \mapsto \Phi^t_\alpha(z) = (e^{2\pi t/a_1} z_1, \dots, e^{2\pi t/a_n}). \tag{3.1}$$

Due to our assumption that $a_i/a_j \notin \mathbb{Q}$ for each $i \neq j$, there are precisely n simple orbits γ_i for $1 \leq i \leq n$. Each γ_i is a parameterization of the curve of points in Y with $z_j = 0$ for all $j \neq i$. The action of γ_i^m is given by $\mathcal{A}_\alpha(\gamma_i^m) = ma_i$ by (3.1), since the action $\mathcal{A}_\alpha(\gamma) = \int_{S^1} \gamma^*\alpha$ of a Reeb orbit coincides with the period.

The Conley–Zehnder index of γ_i^m can be computed as so. Given any contact manifold (Y,α) with $c_1(\xi) = 0$, a non-degenerate Reeb orbit γ and a homotopy class of symplectic trivialization τ of $\gamma^*\xi$, the Conley-Zehnder index $\mathrm{CZ}(\gamma)$ can be written as:

$$\mathrm{CZ}(\gamma) = \mathrm{CZ}(\gamma,\tau) + 2c_1(\gamma,\tau)$$

Here $\mathrm{CZ}(\gamma, \tau)$ is the Conley-Zehnder index of the linearized Reeb flow with respect to the trivialization τ and $c_1(\gamma, \tau)$ is the relative first Chern number with respect to τ of the pullback $u^*\xi$ of ξ to a capping disk u of γ.

Returning to the case where (Y, ξ) is the boundary $\partial E(a_1, \dots, a_n)$ as above we note that, along γ_i^m, the fiber $\xi_{\gamma(t)}$ agrees at each t with the orthogonal complex subspace to the ith component $\mathbb{C}_i^\perp = \oplus_{j \neq i} \mathbb{C}_j \subset \mathbb{C}^n \simeq T\mathbb{C}^n_{\gamma(t)}$. By 3.1, linearized flow in this trivialization is a direct sum of loops $t \mapsto e^{2\pi i m a_i t/a_j}$ for $j \neq i$ for $t \in [0,1]$. Thus we have

$$\mathrm{CZ}(\gamma_i^m, \tau) = \sum_{j \neq i} \Big(2 \lfloor \frac{m a_i}{a_j} \rfloor + 1 \Big).$$

The relative Chern number $c_1(\gamma_i^m, \tau)$ is given by:

$$c_1(\gamma_i^m, \tau) = m$$

Thus we have the following formula for the Conley-Zehnder index of γ_i^m:

$$\mathrm{CZ}(\gamma_i^m) = n - 1 + 2 \left| \{ L \in \mathrm{Spec}(Y, \alpha) \mid L \leq m a_i \} \right|.$$

Finally, note that since $H_1(Y, \mathbb{Z}) = 0$ and $c_1(\xi) = 0$ for (Y, ξ), $CH(Y, \xi) = CH_\circ(Y, \xi)$ and the algebra is $\mathbb{Z}$–graded. By the above calculation, we have

$$|\gamma_i^m| = 2n - 4 + 2 |\{ L \in \mathrm{Spec}(Y, \alpha) | L \leq m a_i \}| \in \mathbb{Z}.$$

Therefore the grading is even and the contact form is lacunary. The contact homology is thus computed by the formula in Lemma 3.7.

Lemma 3.9. *Let (Y, ξ) be a closed contact manifold. Let λ be a Liouville form with Liouville vector field Z positively transverse to the fibers of $\pi : Y \times I \to I$. Then the deformation class $[Y \times I, \lambda]$ is independent of λ and induces the identity map $CH[Y \times I, \lambda] = \mathrm{Id}$ on $CH(Y, \xi)$.*

Proof. First we show that such cobordisms determine a single deformation class. Let Y_+ and Y_- denote the positive and negative boundary copies of Y, respectively. Let $(Y \times \mathbb{R}^+, \widehat{\lambda})$ denote the partial completion of $Y \times [0,1]$ by attaching $Y \times [1, \infty)$ with Liouville form $e^t \lambda|_{Y_+}$. We denote the extended Liouville vector field on $Y \times \mathbb{R}^+$ by Z. Let $T > 0$ be a time such that $\Phi_T^Z(y, 0) \in Y \times (1, \infty)$ for all $y \in Y$, let $\Sigma = \Phi_T^Z(Y \times 0)$, and let $\tau : Y \to \mathbb{R}^+$ be the smooth function of time such that $\Phi^Z_{\tau(y)}(y, 1) \in \Sigma$. Then the family λ_t of Liouville forms:

$$\lambda_s := \Psi_s^* \widehat{\lambda} \qquad \Psi_s : Y \to Y \text{ defined as } \Psi_s(y, r) = \Phi^Z_{(sr) \cdot \tau(y)}(y, r)$$

is a deformation of Liouville structures on $Y \times I$ with $\lambda_0 = \lambda$ and λ_1 having the property that $\Phi_T^{Z_1}(y, 0) = (y, 1)$.

Thus we may assume these properties without altering the deformation class. Pulling back by the diffeomorphism $Y \times I \to Y \times I$ given by $(y,r) \mapsto \Phi^Z_{Tr}(y,0)$ yields a form with Liouville vector-field $Z = \partial t$. Thus the Liouville form can be assumed to be $\lambda = e^t \lambda|_{Y_-}$. Any two such such forms are deformation equivalent, since the space of contact forms is contractible.

Second, we note that $CH[Y \times I, \lambda] = \mathrm{Id}$. In particular, we can choose $\lambda = e^t \alpha$ for some contact form α, and then this is [14, Lemma 1.2]. $\square$

4 The space of symplectic embeddings

In this section, we discuss some basic results about the Fréchet manifold of symplectic embeddings $\mathrm{SympEmb}(U,V)$ between symplectic manifolds with boundary. In §4.1, we construct the Fréchet manifold structure on $\mathrm{SympEmb}(U,V)$. In §4.2, we discuss the relationship between the bordism groups and homology groups of a Fréchet manifold. Last, we prove a version of the Weinstein neighborhood with boundary as Proposotion 4.13 in §4.3.

4.1 Fréchet manifold structure

Let (U,ω_U) and (V,ω_V) be n–dimensional compact symplectic manifolds with nonempty contact boundaries. We now give a proof of the folklore result that the space of symplectic embeddings from U to V is a Fréchet manifold.

Proposition 4.1. *The space* $\mathrm{SympEmb}(U,V)$ *of symplectic embeddings* $\varphi : U \to \mathrm{int}(V)$ *with the* C^∞ *compact open topology is a metrizable Fréchet manifold.*

Proof. Let $(U \times V, \omega_{U\times V})$, with $\omega_{U\times V} = \pi_U^*\omega_U - \pi_V^*\omega_V$, denote the product symplectic manifold with corners. Given a symplectic embedding $\varphi : U \to \mathrm{int}(V)$, we may associate the graph $\Gamma(\varphi) \subset U \times V$ given by:

$$\Gamma(\varphi) := \{(u, \varphi(u)) \in U \times V\}.$$

The graph is a Lagrangian submanifold with boundary transverse to the characteristic foliation $T(\partial U)^\omega$ on the contact hypersurface $\partial U \times \mathrm{int}(V)$. By the Weinstein neighborhood theorem with boundary, Proposition 4.13, there is a neighborhood A of U, a neighborhood B of $\Gamma(\varphi)$ and a symplectomorphism $\psi : A \simeq B$ with $\psi|_U : U \to \Gamma(\varphi)$ given by $u \mapsto (u, \varphi(u))$ and $\psi^*\omega_{U\times V} = \omega_{\mathrm{std}}$.

Let $\mathcal{A}(\varphi,\psi) \subset \ker(d : \Omega^1(L) \to \Omega^2(L))$ and $\mathcal{B}(\varphi,\psi) \subset \mathrm{SympEmb}(U,V)$ denote the open subsets given by:

$$\mathcal{A}(\varphi,\psi) := \{\alpha \in \Omega^1(L) | d\alpha = 0 \text{ and } \mathrm{Im}(\alpha) \subset A\}$$

$$\mathcal{B}(\varphi, \psi) := \{\phi \in \mathrm{SympEmb}(U, V) | \, \mathrm{Im}(\varphi) \subset B\}.$$

Then we have maps $\Phi : \mathcal{A}(\varphi, \psi) \to \mathcal{B}(\varphi, \psi)$ and $\Psi : \mathcal{A}(\varphi, \psi) \to \mathcal{B}(\varphi, \psi)$ given by:

$$\alpha \mapsto \Phi[\alpha] := (\pi_V \circ \psi \circ \alpha) \circ (\pi_U \circ \psi \circ \alpha)^{-1}$$

$$\phi \mapsto \Psi[\phi] := (\psi^{-1} \circ (\mathrm{Id} \times \phi)) \circ (\pi_L \circ \psi^{-1} \circ (\mathrm{Id} \times \phi))^{-1}.$$

It is a tedious but straight forward calculation to check that $\Phi \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Phi = \mathrm{Id}$. The fact that Φ and Ψ are continuous in the C^∞ compact open topologies on the domain and images follows from the fact that function composition defines a continuous map $C^\infty(M, N) \times C^\infty(N, O) \to C^\infty(M, O)$ for any compact manifolds M, N and O (in fact, smooth; see [16, Theorem 42.13]).

Since $C^\infty(U, V)$ is metrizable under the compact open C^∞–topology (see [16, Corollary 41.12]), the subspace $\mathrm{SympEmb}(U, V)$ is also metrizable. □

Lemma 4.2. *Let L be a compact manifold with boundary and let $\sigma : L \to T^*L$ be a section. Then $\sigma(L)$ is Lagrangian if and only if σ is closed.*

Proof. The same as the closed case, see [12, Proposition 3.4.2]. □

4.2 Bordism groups of Fréchet manifolds

We now discuss (unoriented) bordism groups and their structure in the case of Fréchet maifolds. We begin by defining the relevant notions of (continuous and smooth) bordism.

Definition 4.3 (Bordisms)**.** Let X be a topological space and $f : Z \to X$ be a map from a closed manifold. We say that the pair (Z, f) is *null–bordant* if there exists a pair (Y, g) of a compact manifold with boundary Y and a continuous map $g : Y \to X$ such that $\partial Y = Z$ and $g|_{\partial Y} = f$. Given a pair of manifold/map pairs (Z_i, f_i) for $i \in \{0, 1\}$, we say that (Z_0, f_0) and (Z_1, f_1) are *bordant* if $(Z_0 \sqcup Z_1, f_0 \sqcup f_1)$ is null–bordant.

Definition 4.4 (Smooth bordism)**.** Let X be a Fréchet manifold and $f : Z \to X$ be a smooth map from a smooth closed manifold. Then (Z, f) is *smoothly null–bordant* if it is null–bordant via a pair (Y, g) where $g : Y \to X$ be a smooth map of Banach manifolds with boundary. Similarly, a pair (Z_i, f_i) for $i\{0, 1\}$ is *smoothly bordant* if $(Z_0 \sqcup Z_1, f_0 \sqcup Z_1)$ is smoothly null-bordant.

The above notions come with accompanying versions of the bordism group.

Definition 4.5 (Bordism Group Of X)**.** The *n–th bordism group* $\Omega_n(X;\mathbb{Z}_2)$ of a topological space X is group generated by equivalence classes $[Z,f]$ of pairs (Z,f), where Z is a closed n–dimensional manifold and $f : Z \to X$ is a continuous map, modulo the relation that $(Z_0, f_0) \sim (Z_1, f_1)$ if the pair is bordant. Addition is defined by disjoint union:

$$[Z_0, f_0] + [Z_1, f_1] := [Z_0 \sqcup Z_1, f_0 \sqcup f_1].$$

Definition 4.6 (Smooth bordism group of X)**.** The *n–th smooth bordism group* $\Omega_n^\infty(X;\mathbb{Z}_2)$ of a Fréchet manifold X is group generated by equivalence classes $[Z,f]$ of pairs (Z,f), where Z is a closed n–dimensional manifold and $f : Z \to X$ is a smooth map, modulo the relation that $(Z_0, f_0) \sim (Z_1, f_1)$ if the pair is smoothly bordant. Addition in the group $\Omega_*^\infty(X;\mathbb{Z}_2)$ is defined by disjoint union as before.

Lemma 4.7. *The natural map $\Omega_*^\infty(X;\mathbb{Z}_2) \to \Omega_*(X;\mathbb{Z}_2)$ is an isomorphism.*

Proof. The argument uses smooth approximation and is identical to the case where X is a finite dimensional smooth manifold, which can be found in [4, Section I.9]. □

Given the above terminology, we can now prove the main result of this subsection, Proposition 4.8. It provides a class of submanifolds for which being null–bordant and being null–homologous are equivalent.

Proposition 4.8. *Let X be a metrizable Fréchet manifold, and let $f : Z \to X$ be a smooth map from a closed manifold Z with Stieffel–Whitney class $w(Z) = 1 \in H^*(Z;\mathbb{Z}_2)$. Then $f_*[Z] = 0 \in H_*(X;\mathbb{Z}_2)$ if and only $[Z,f] = 0 \in \Omega_*^\infty(X;\mathbb{Z}_2)$.*

Proof. Proposition 4.8 will follow immediately from the following results. First, by Lemma 4.7, it suffices to show $f_*[Z] = 0 \in H_*(X;\mathbb{Z}_2)$ if and only $[Z,f] = 0 \in \Omega_*(X;\mathbb{Z}_2)$. By Proposition 4.9, we can replace X with a CW complex. Lemma 4.10 proves the result in this context. □

Proposition 4.9 ([15, Theorem 14])**.** *A metrizable Fréchet manifold is homotopy equivalent to a CW complex.*

Lemma 4.10. *Let X homotopy equivalent to a CW complex, and let $f : Z \to X$ be a continuous map from a closed manifold Z with Stieffel–Whitney class $w(Z) = 1 \in H^*(Z;\mathbb{Z}_2)$. Then $f_*[Z] = 0 \in H_*(X;\mathbb{Z}_2)$ if and only $[Z,f] = 0 \in \Omega_*(X;\mathbb{Z}_2)$.*

Remark 4.11. Crucially, we make no finiteness assumptions on the CW structure.

Proof. ($\Rightarrow$) Suppose that $f_*[Z] = 0 \in H_2(Z;\mathbb{Z}_2)$. Pick a homotopy equivalence $\varphi : X \simeq X'$ with a CW complex X'. Such an equivalence induces an isomorphism of unoriented bordism groups $\Omega_*(X;\mathbb{Z}_2) \simeq \Omega_*(X';\mathbb{Z}_2)$, so it suffices to show that the pair $(Z, \varphi \circ f)$ is null–bordant, or equivalently to assume that X is a CW complex to begin with.

So assume that X is a CW complex. By Lemma 4.12, we can find a finite sub–complex $A \subset X$ such that $f(Z) \subset A$ and $f_*[Z] = 0 \in H_*(A; \mathbb{Z}_2)$. By Theorem 17.2 of [4], $[Z, f] = 0 \in \Omega_*(A; \mathbb{Z}_2)$ if and only if the Stieffel–Whitney numbers $\mathrm{sw}_{\alpha,I}[Z, f]$ are identically 0. Recall that the Stieffel–Whitney number $\mathrm{sw}_{\alpha,I}[Z, f]$ associated to $[Z, f]$, a cohomology class $\alpha \in H_k(A; \mathbb{Z}_2)$ and a partition $I = (i_1, \ldots, i_k)$ of $\dim(Z) - k$ is defined to be:

$$\mathrm{sw}_{\alpha,I}[Z, f] = \langle w_{i_1}(Z) w_{i_2}(Z) \ldots w_{i_k}(Z) f^* \alpha, [Z] \rangle \in \mathbb{Z}_2.$$

Here $w_j(Z) \in H^j(Z; \mathbb{Z}_2)$ denotes the j–th Stieffel–Whitney class of Z. By assumption, $w(Z) = 1$ and so $w_j(Z) = 0$ for all $j \neq 0$. In particular, the only possible nonzero Stieffel–Whitney numbers have $I = (0)$. But we see that:

$$\mathrm{sw}_{\alpha,(0)}[Z, f] = \langle f^* \alpha, [Z] \rangle = \langle \alpha, f_*[Z] \rangle = 0.$$

Therefore, $\mathrm{sw}_{\alpha,I}[Z, f] \equiv 0$ and $[Z, f]$ must be null–bordant.

($\Leftarrow$) This direction is completely obvious, since the map $\Omega_*(X) \to H_*(X; \mathbb{Z}_2)$ given by $[Z, f] \mapsto f_*[Z]$ is well defined. □

Lemma 4.12. *Let X be a CW complex, and let $f : Z \to X$ be a map from a closed manifold Z with $f_*[Z] = 0 \in H_*(X; \mathbb{Z}_2)$. Then there exists a finite sub–complex $A \subset X$ with $f(Z) \subset A$ and $f_*[Z] = 0 \in H_*(A; \mathbb{Z}_2)$.*

Proof. A very convenient tool for this is the stratifold homology theory of [9], which we now review briefly.

Given a space M, the n–th stratifold group $sH_n(M; \mathbb{Z}_2)$ with $\mathbb{Z}_2$–coefficients (see Proposition 4.4 in [9]) is generated by equivalence classes of pairs (S, g) of a compact, regular stratifold S and a continuous map $g : S \to M$. Two pairs (S_i, g_i) for $i \in \{0, 1\}$ are equivalent if they are bordant by a c–stratifold, i.e. if there is a pair (T, h) of a compact, regular c–stratifold and a continuous map $g : T \to M$ such that $(\partial T, h|_{\partial T}) = (S_0 \sqcup S_1, g_0 \sqcup g_1)$ (see Chapter 3 and Section 4.4 of [9]). Given a map $\varphi : M \to N$ of spaces, the pushforward map $\varphi_* : sH(M; \mathbb{Z}_2) \to sH(M; \mathbb{Z}_2)$ on stratifold homology is given (on generators) by $[S, g] \mapsto [S, \varphi \circ g] = \varphi_*[\Sigma, g]$.

Stratifold homology satisfies the Eilenberg–Steenrod axioms (see Chapter 20 of [9]), and thus if M is a CW complex then there is a natural isomorphism $sH_*(M; \mathbb{Z}_2) \simeq H_*(M; \mathbb{Z}_2)$. If M is a manifold of dimension n, the fundamental class $[M] \in sH_n(M; \mathbb{Z}_2)$ is given by the tautological equivalence class $[M] = [M, \mathrm{Id}]$.

The proof of the lemma is simple with the above machinery in place. Since $f_*[Z] = 0$, the pair (Z, f) must be null–bordant via some compact c–stratifold (Y, g). Since Y and its image $g(Y)$ are both compact, we can choose a sub–complex

$A \subset X$ such that $g(T) \subset A \subset X$. Then the pair (Z, f) are null–bordant by (Y, g) in A as well, so that $[Z, f] = 0 \in sH_*(A; \mathbb{Z}_2)$ and thus $f_*[Z] = 0 \in H_*(A; \mathbb{Z}_2)$ via the isomorphism $sH_*(A; \mathbb{Z}_2) \simeq H_*(A; \mathbb{Z}_2)$. □

4.3 Weinstein neighborhood theorem with boundary

In this section, we prove the analogue of the Weinstein neighborhood theorem for a Lagrangian L with boundary, within a symplectic manifold X with boundary. We could find no reference for this fact in the literature.

Proposition 4.13 (Weinstein neighborhood theorem with boundary)**.** *Let (X, ω) be a symplectic manifold with boundary ∂X and let $L \subset X$ be a properly embedded, Lagrangain submanifold with boundary $\partial L \subset \partial X$ transverse to $T(\partial X)^\omega$.*

Then there exists a neighborhood $U \subset T^*L$ of L (as the zero section), a neighborhood $V \subset X$ of L and a diffeomorphism $f : U \simeq V$ such that $\varphi^*(\omega|_V) = \omega_{\text{std}}|_U$.

Proof. The proof has two steps. First, we construct neighborhoods $U \subset T^*L$ and $V \subset X$ of L, and a diffeomorphism $\varphi : U \simeq V$ such that:

$$\varphi|_L = \text{Id} \qquad \varphi^*(\omega|_V)|_L = \omega_{\text{std}}|_L \qquad T(\partial U)^{\omega_{\text{std}}} = T(\partial U)^{\varphi^*\omega} \tag{4.1}$$

Here $T(\partial U)^{\omega_{\text{std}}} \subset T(\partial U)$ is the symplectic perpendicular to $T(\partial U)$ with respect to ω_{std} (and similarly for $T(\partial U)^{\varphi^*\omega}$. Second, we apply Lemma 4.14 and a Moser type argument to conclude the result.

(Step 1) Let J be a compatible almost complex structure on X and g be the induced metric on L. Recall that the normal bundle $\nu_g L$ with respect to g is a bundle over L with Lagrangian fiber, and that $J : TL \to \nu_g L$ gives a natural isomorphism. Let $\Phi^g : T^*L \to TL$ denote the bundle isomorphism induced by the metric g and let $\exp^g$ denote the exponential map with respect to g.

Since L is compact, we can choose a tubular neighborhood U' of νL such that $\exp^g : U \to X$ is a diffeomorphism onto its image V. We then let $U := [J \circ \Phi^g]^{-1}(U') \subset T^*L$ and let:

$$\phi^g : U \simeq V \qquad (x, v) \mapsto \exp^g_x(J \circ \Phi_g(v)).$$

Note that $\phi^g|_L = \text{Id}$ and $[\phi^g]^*\omega|_L = \omega_{\text{std}}|_L$ by the same calculations as in [12, Theorem 3.4.13]. We now must modify U, V and ϕ^g to satisfy the last condition of (4.1).

To this end, we apply Lemma 4.15. Taking $\kappa_0 = T(\partial U)^{\omega_{\text{std}}}$ and $\kappa_1 = T(\partial U)^{[\phi^g]^*\omega}$, we acquire a neighborhood $N \subset \partial(T^*L)$ of ∂L and a family of embeddings $\psi : N \times I \to \partial(T^*L)$ with the following four properties.

$$\psi_t|_{\partial L} = \text{Id} \qquad d(\psi_t)_u = \text{Id} \text{ for } u \in \partial L \qquad \psi_0 = \text{Id}$$

$$[\psi_1]_*(T(\partial U)^{\omega_{\text{std}}}) = T(\partial U)^{[\phi^g]^*\omega}.$$

Note here that we are using the fact that $T(\partial U)^{\omega_{\text{std}}}|_L = T(\partial U)^{[\phi^g]^*\omega}|_L$ already by the construction of ϕ^g. By shrinking N and U, we can simply assume that $N = \partial U$. Let $\text{tc} : [0,1) \times \partial U \simeq T \subset U$ be tubular neighborhood coordinates near boundary. By choosing the tubular neighborhood coordinates $\text{tc} : [0,1) \times \partial U \simeq T$ appropriately, we can also assume that $\text{tc}([0,1) \times \partial L) = L \cap T$. We define a map $\Phi : U \to T^*L$ by:

$$\Phi(u) = \begin{cases} (s, \psi_{1-s}(v)) & \text{if } u = (s,v) \in [0,1) \times \partial U \text{ via tc} \\ u & \text{otherwise.} \end{cases}$$

The map Φ has the following properties which are analogous to those of ψ_s.

$$\Phi|_L = \text{Id} \quad d(\Phi)_u = \text{Id} \ \text{ for } u \in L \qquad \Phi_*(T(\partial L)^{\omega_{\text{std}}}) = T(\partial L)^{[\phi^g]^*\omega}$$

Also note that Φ is smooth since ψ_t is constant for t near 0 and 1. We thus define f as the composition $\varphi = \phi^g \circ \Phi$. It is immediate that f has the properties in (4.1).

(Step 2) We closely follows the Moser type argument of [12, Lemma 3.2.1]. By shrinking U, we may assume that it is an open disk bundle. Let $\omega_t = (1-t)\omega_{\text{std}} + tf^*\omega$ and $\tau = \frac{d}{dt}(\omega_t) = f^*\omega - \omega_{\text{std}}$. Let $\kappa = T(\partial U)^{\omega_t}$ (by the previous work, it does not depend on t). Note that τ satisfies all of the assumptions of Lemma 4.14(4.3). We prove that κ is invariant under the scaling map $\phi_t(x,u) = (x,tu)$ in Lemma 4.16. We can thus find a σ satisfying the properties listed in (4.2).

Let Z_t be the unique family of vector fields satisfying $\sigma = \iota(Z_t)\omega_t$. Due to the properties of σ, Z_t satisfies the following properties for each t.

$$Z_t|_L = 0 \qquad Z_t|_{\partial U} \in T(\partial U) \text{ for all } t$$

The first property is immediate, while the latter is a consequence of the following.

$$\omega_t(Z_t, \cdot)|_\kappa = \sigma|_\kappa = 0 \implies Z_t \in (\kappa)^{\omega_t} = T(\partial U)$$

These two properties imply that Z_t generates a map $\Psi : U' \times [0,1] \to U$ for some smaller tubular neighborhood $U' \subset U$ with the property that $\Psi_t|_L = \text{Id}$ and $\Psi_t^*\omega_t = \omega_0$ (see [12, §3.2], as the reasoning is identical to the closed case). In particular, we get a map $\Psi_1 : U' \to U$ with $\Psi_1|_L = \text{Id}$ and $\Psi_1^* f^*\omega$. By shrinking U, taking $\varphi = f \circ \Psi_1$ and taking $V = \varphi(U)$, we at last acquire the desired result. □

The remainder of this section is devoted to proving the various lemmas that we used in the proof above.

Lemma 4.14 (Fiber integration with boundary)**.** *Let X be a compact manifold with boundary, $\pi : E \to X$ be a rank k vector bundle with metric and $\pi : U \to X$ be the* *(open) disk bundle of E with closure $\overline{U}$. Let $\kappa \subset T(\partial U)$ be a distribution on ∂U such that $d\phi_t(\kappa_u) = \kappa_{\phi_t(u)}$ for all $u \in U$, where $\phi : U \times I \to U$ denote the family of smooth maps given by $\phi_t(x, u) := (x, tu)$.*

Finally, suppose that $\tau \in \Omega^{k+1}(\overline{U})$ is a $(k+1)$–form such that:

$$d\tau = 0 \qquad \tau|_X = 0 \qquad (\iota_{\partial X}^* \tau)|_\kappa = 0. \tag{4.2}$$

Then there exists a k–form $\sigma \in \Omega^k(\overline{U})$ with the following properties:

$$d\sigma = \tau \qquad \sigma|_X = 0 \qquad (\iota_{\partial X}^* \sigma)|_\kappa = 0. \tag{4.3}$$

Proof. We use integration over the fiber, as in [12, p. 109]. Note that the maps $\phi_t : U \to \phi_t(U) \subset U$ are diffeomorphisms for each $t > 0$, $\phi_0 = \pi$, $\phi_1 = \mathrm{Id}$ and $\phi_t|_X = \mathrm{Id}$. Therefore we have:

$$\phi_0^* \tau = 0 \qquad \phi_1^* \tau = \tau$$

We may define a vector field Z_t for all $t > 0$ and a k-form σ_t for all $t \geq 0$ as so.

$$Z_t := (\frac{d}{dt}\phi_t) \circ \phi_t^{-1} \text{ for } t > 0 \qquad \sigma_t := \phi_t^*(\iota(Z_t)\tau) \text{ for } t \geq 0$$

Although Z_t is singular at $t = 0$, as in [12] one can verify in local coordinates that σ_t is smooth at $t = 0$. Since $Z_t|_X = 0$, the k–form σ_t satisfies $\sigma_t|_X = 0$. Furthermore, for any vector field $K \in \Gamma(\kappa)$ on ∂X which is parallel to κ, we have $\iota(K)\sigma_t = \phi_t^*(\iota(Z_t)\iota(d\phi_t(K))\tau) = 0$ on the boundary, so that $\iota_{\partial X}^*(\sigma_t)|_\kappa = 0$. Finally, σ_t satisfies the following equation:

$$\tau = \phi_1^*\tau - \phi_0^*\tau = \int_0^1 \frac{d}{dt}(\phi_t^*\tau)dt = \int_0^1 \phi_t^*(\mathcal{L}_{X_t}\tau)dt$$
$$= \int_0^1 d(\phi_t^*(\iota(X_t)\tau))dt = \int_0^1 d\sigma_t dt = d(\int_0^1 \sigma_t dt).$$

Therefore, if we define $\sigma := \int_0^1 \sigma_t dt$, it is simple to verify the desired properties using the corresponding properties for σ_t. □

Lemma 4.15. *Let U be a manifold and $L \subset U$ be a closed submanifold. Let κ_0, κ_1 be rank 1 orientable distributions in TU such that $\kappa_i|_L \cap TL = \{0\}$ and $\kappa_0|_L = \kappa_1|_L$.*

Then there exists a neighborhood $U' \subset U$ of L and a family of smooth embeddings $\psi : U's \times I \to U$ with the following four properties.

$$\psi_t|_{\partial L} = \mathrm{Id} \qquad d(\psi_t)_u = \mathrm{Id} \textit{ for } u \in L \qquad \psi_0 = \mathrm{Id} \qquad [\psi_1]_*(\kappa_0) = \kappa_1.$$

Furthermore, we can take ψ_t to be t–independent for t near 0 and 1.

Proof. Since κ_0 and κ_1 are orientable, we can pick nonvanishing sections Z_0 and Z_1 We may assume that $Z_0 = Z_1$ along L. We let Z_t denote the family of vector fields $Z_t := (1-t)Z_0 + tZ_1$. Since $Z_0 = Z_1$ along L, we can pick a neighborhood N of L such that Z_t is nowhere vanishing for all t. We also select a sub-manifold $\Sigma \subset N$ with $\dim(\Sigma) = \dim(U) - 1$ and such that:

$$\Sigma \pitchfork Z_t \text{ for all } t \quad \text{and} \quad L \subset \Sigma.$$

We can find such a Σ by, say, picking a metric and using the exponential map on a neighborhood of L in the sub–bundle $\nu L \cap \kappa_0^\perp$ of TL. By shrinking Σ and scaling Z_t to λZ_t, $0 < \lambda < 1$, we can define a smooth family of embeddings:

$$\Psi : (-1,1)_s \times \Sigma \times [0,1]_t \to N \qquad \Psi_t(s,x) = \exp[Z_t]_s(x).$$

Here $\exp[Z_t]$ denotes the flow generated by Z_t. We let $\psi_t = \Psi_t \circ \Psi_0^{-1}$. To see the properties of (4.1), note that $\Psi_t(0,l) = l$ for all $l \in L$ and $d(\Psi_t)_{0,l}(s,u) = sZ_t + u$. This implies the first two properties. The third is trivial, while the fourth is immediate from $[\Phi_t]_*(\partial_t) = Z_t$. We can make ψ_t constant near 0 and 1 by simply reparameterizing with respect to t. □

Lemma 4.16. *Let L be a manifold with boundary and let (T^*L, ω) be the cotangent bundle with the standard symplectic form. Let $\kappa = T(\partial T^*L)^\omega$ denote the characteristic foliation of the boundary ∂T^*L and let $\phi : T^*L \times (0,1] \to T^*L$ denote the family of maps $\phi_t(x,v) = (x,tv)$. Then $[\phi_t]_*(\kappa) = \kappa$.*

Proof. By passing to a chart, we may assume that $L \subset \mathbb{R}^+_{x_1} \times \mathbb{R}^{n-1}_x$ and $T^*L \subset \mathbb{R}^+_{x_1} \times \mathbb{R}^{n-1}_x \times \mathbb{R}^n_p$. Then κ is simply given on $\partial T^*L \subset \{0\} \times \times \mathbb{R}^{n-1}_x \times \mathbb{R}^n_p$ by:

$$\kappa = \operatorname{span}(\partial_{p_1}) = \operatorname{span}(\partial_{x_1})^\omega \subset T(\partial T^*L).$$

Under the scaling map, we have $[\phi_t]_*(\partial_{p_1}) = t{\cdot}\partial_{p_1}$. This implies that $[\phi_t]_*(\kappa) = \kappa$. □

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