Computing The EHZ Capacity Of Polytopes

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Agenda

Goal Of Talk: Describe an algorithm to calculate the Ekeland-Hofer-Zehnder capacity of a generic polytope in $\mathbb{R}^4$, and provide the theory necessary to prove that it works. Demonstrate results from an implementation of the algorithm.

Agenda:

1. Background
2. Dynamics On Polytopes
3. Combinatorial Conley-Zehnder Index
4. Outline Of Algorithm
5. Applications, Computations, Demonstration

Background

We're interested in calculating the EHZ capacity for polytopes, so let's recall the definition.

Definition Let $X \subset \mathbb{R}^{2n}$ be a convex domain such that $Y = \partial X$ is a C^∞ hypersurface and such that $0 \in \text{interior}(X)$. Then we define the **Ekeland-Hofer-Zehnder Capacity** c_{EHZ} of X as:

$$c_{EHZ}(X) := \min\{\mathcal{A}(\gamma) | \gamma \text{ is a closed Reeb trajectory on } Y\}$$

Here Y is given the standard contact form λ induced by the standard radial Liouville vector-field $X(p) = \frac{1}{2}p$ on $\mathbb{R}^{2n}$. $\square$

Background

- c_{EHZ} makes sense for polytopes because <u>any</u> capacity c on C^∞ convex domains extends to C^0 ones in a unique way.
- **Issues:** This extension is, a-priori, not computation friendly. Furthermore, there is no geometric interpretation in general.
- However, for the EHZ capacity we have the following result.

Theorem (Artstein-Avidan,Ostrover, 2014)

If $X \subset \mathbb{R}^{2n}$ is a convex domain with C^0 boundary and $0 \in interior(X)$, then:

$$c_{EHZ}(X) = min\{\mathcal{A}(\gamma) | \gamma \text{ a } \underline{\text{generalized}} \text{ Reeb characteristic on } \partial X\} \quad \square$$

Background

By generalized Reeb trajectory, we mean:

Definition A **generalized Reeb trajectory** on Y is a C^0 curve $\gamma : I \to Y$ which is:

1. C^∞ except at a measure 0 set.
2. Has right and left derivatives $\dot{\gamma}_\pm$ everywhere.
3. Satisfies, for every $t \in I$,:

$$\dot{\gamma}_\pm(t) \in J_0 N(\gamma(t))$$

where $N(\gamma(t))$ is the out-pointing normal cone of X at $\gamma(t)$. $\square$

Remark 1: The outward pointing normal cone is defined by

$$N(p) := \{v | \langle v, p - q\rangle \geq 0 \forall q \in X\}$$

Remark 2: On a smooth parts of ∂X, this definition coincides with the usual definition of a Reeb trajectory (up to reparameterization).

Dynamics On Polytopes

Now let's specialize to a convex polytope $X \subset \mathbb{R}^4$ containing 0 and consider orbits there.

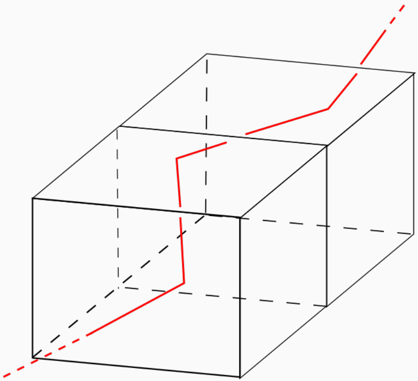
Definition $Y = \partial X$ breaks into a disjoint union of k-dimensional strata, each of which is an affinely embedded polytope in $\mathbb{R}^4$ (points, line segments, polygons, etc). We call a connected k-stratum a k-**face**. $\square$

- For a point p on a 3-face F (smooth stratum) of Y, $JN(p)$ is just equal to all non-negative multiples of $\hat{n}$ where $\hat{n}$ is the normal vector to F.
- In other words, the generalized trajectories on any 3-face are just line segments parallel to the vector $J\hat{n}$.
- The nicest possible thing that a curve could do away from the 3-faces is just move between the 3-faces at discrete times.

Point: We would like to imagine nice generalized trajectories as piece-wise linear curves which break along $0, 1$ and 2-faces.

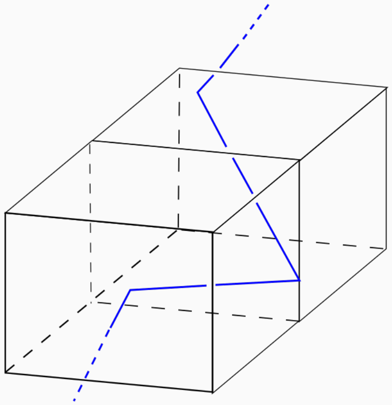
Dynamics On Polytopes

Definition A generalized Reeb trajectory $\gamma : [0, L] \to Y$ is **Type 1** if $\gamma^{-1}(Y - \{\text{3-faces}\})$ is finite and $\gamma^{-1}(\{\text{1-faces}\}) = \gamma^{-1}(\{\text{0-faces}\}) = \emptyset$.



This is essentially the nicest kind of generalized Reeb trajectory.

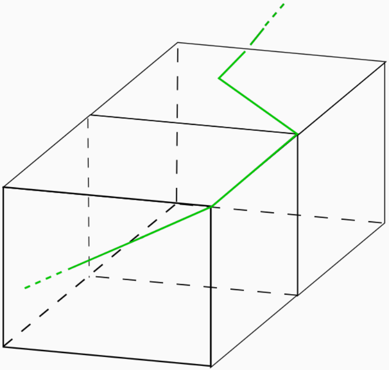
Definition A generalized Reeb trajectory $\gamma : [0, L] \to Y$ is **Type 2** if $\gamma^{-1}(Y - \{\text{3-faces}\})$ is finite, but $\gamma^{-1}(\{\text{0-faces}\})$ and/or $\gamma^{-1}(\{\text{1-faces}\})$ are non-empty.



Notice how this trajectory hits a 1-face, but still in an isolated way.

Dynamics On Polytopes

Definition A generalized Reeb trajectory $\gamma : [0, L] \to Y$ is **Type 3** if it is given by a path composition of Type 2 trajectories and whole parameterized 1-faces.



This trajectory hits the 2-faces, 1-faces and 0-faces at isolated times, except potentially for intervals where the path hits an whole 1-face.

Our first result says that, for a generic set of polytopes, only Type 1-3 trajectories are needed to calculate $c_{EHZ}(X)$

Theorem (Thm 1, C-H, 2017)

Let $X \subset \mathbb{R}^4$ be a convex polytope containing 0. Assume that:

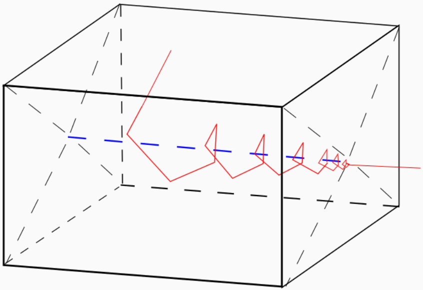
1. *Every 2-face E of Y is symplectic, i.e the tangent 2-plane TE is a symplectic subspace of $\mathbb{R}^4$.*
2. *The 3-faces are in general position (3 3-faces intersect at a line segment, 4 3-faces intersect at a point, etc.)*

Then Y possesses a minimizing Reeb trajectory of Type 1,2 or 3.

Remark 1: We hope to get rid of Assumption (2) soon. **Remark 2:** We have an effective version of this result that (roughly speaking) bounds the number of $0, 1$ and 2-face crossings. More later...

Dynamics On Polytopes

Why is this a non-trivial result? In general, bad things can happen around a 1-face E where $J_0 N(p) \cap TE \neq \emptyset$. We call these **vorticial 1-faces**, due to the picture below. These are the 1-faces that generalized Reeb trajectories are allowed to travel along, and they already appeared in the definition of Type 3 trajectories.



A good example of such an issue is this "infinite spiral" trajectory.

Idea Of Proof Of Theorem

Here is a very rough description of the strategy for proving the Theorem.

1. Define "combinatorial" analogs of the rotation number ρ and Conley-Zehnder index CZ from smooth contact geometry for the polytope setting.
2. Prove analogs of the following 2 facts from smooth convex contact geometry:
    2.1 Every convex hypersurface in $\mathbb{R}^4$ has a minimizing orbit γ with $CZ(\gamma) \leq 4$.
    2.2 The linearized Reeb flow on a convex hypersurface is "positively rotating", i.e the Conley-Zehnder is increasing along any trajectory.
3. Show that, due to 2.2, any generalized trajectory which is not Type 1, 2 or 3 has "infinite" combinatorial Conley-Zehnder index.
4. Conclude by 2.1 that at least one minimizing trajectory must be of Type 1, 2 or 3.

Brief Review Of CZ

To define the combinatorial version of ρ and CZ, let's recall one way of defining them in the smooth world.

Definition The **mod 2 rotation number** $\bar{\rho} : Sp(2,\mathbb{R}) \simeq SL(2,\mathbb{R}) \to S^1$ is defined as:

$$\bar{\rho}(A) := \begin{cases} \pm 1 & \text{if } A \text{ is } \pm \text{ hyperbolic or } \pm \text{ shear} \\ e^{i\theta} & \text{if } A \text{ is a positive rotation with eigenvalues } e^{\pm i\theta}. \\ e^{-i\theta} & \text{if } A \text{ is a negative rotation with eigenvalues } e^{\pm i\theta}. \end{cases}$$

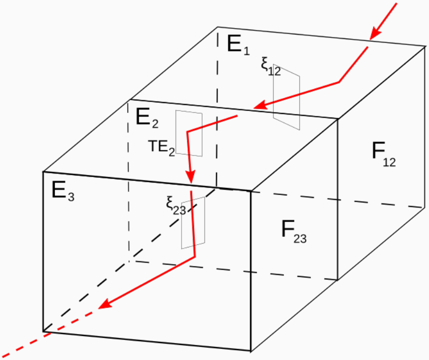
Definition The **rotation number** $\rho : \widetilde{Sp}(2,\mathbb{R}) \to \mathbb{R}$ is the lift of ρ to the universal cover of $\widetilde{Sp}(2,\mathbb{R})$. The rotation number of a path $\psi : [0,L] \to Sp(2,\mathbb{R})$ with $\psi(0) = 1$ is $\rho(\tilde{\psi}(L))$.

Definition The **Conley-Zehnder index** $CZ(\psi)$ of a path is defined as:

$$CZ(\psi) = \lfloor \rho(\tilde{\psi}(L)) \rfloor + \lceil \rho(\tilde{\psi}(L)) \rceil$$

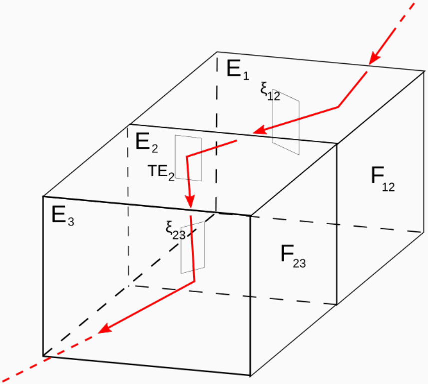
A Combinatorial Conley-Zehnder Index

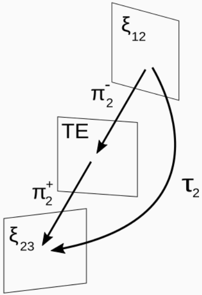
To explain the combinatorial CZ, it helps to first think about Type 1 orbits. Consider a closed Type 1 trajectory $\gamma : \mathbb{R}/L\mathbb{Z} \to Y$. The orbit hits 2-faces at a finite set of times $\{t_i\}_1^k$. Let E_i denote the 2-face hit by γ at time t_i, and F_{ii+1} denote the 3-face between E_i and E_{i+1}. Let TE_i denote the tangent plane to E_i and ξ_{ii+1} denote a fixed fiber of the contact distribution over the 3-face F_{ii+1}.



A Combinatorial Conley-Zehnder Index

For each 2-face E_i, we can define a projection $\pi_i^- : \xi_{i-1i} \to TE_i$ projecting from the contact distribution to the tangent plane TE_i along the Reeb vector on F_{i-1i} (which is constant). Likewise we can define a projection $\pi_i^+ : TE_i \to \xi_{ii+1}$ on the other side. The composition, $\tau_i = \pi_i^+ \pi_i^-$, is a sort of "transition map" between ξ on the two faces.





A Combinatorial Conley-Zender Index

The maps $\tau_i : \xi_{i-1i} \to \xi_{ii+1}$ are symplectomorphisms with respect to the restricted symplectic form $\omega_0|_{\xi}$. Furthermore, each τ_i is conjugate to a rotation. Thus by picking trivializations of ξ_{i-1i} over each 3-face (there are in fact canonical ones that we should use), we can define CZ:

Definition We define the **rotation number** of a Type 1 generalized Reeb orbit γ on Y as:

$$\rho(\gamma) = \rho(\tilde{\tau}_k \tilde{\tau}_{k-1} \cdots \tilde{\tau}_1)$$

and similarly for the **Conley-Zehnder index**. Here $\tilde{\tau}_i$ is the unique lift of τ_i with $\rho(\tilde{\tau}_i) \in (-1, 1)$. □

Remark 1 Note that this definition depends only on the data of the sequence of 2-faces $\{E_i\}_1^k$ and not on the actual path γ.
Remark 2 Rotation numbers and CZ indices can also be assigned to Type 2 and Type 3 but the definition is less straight forward.

Effective Refinements Of Theorem

Having described the rotation number and CZ index above, we can actually state a more effective refinement of our first Theorem.

Theorem (C-H, 2017)

Let $X \subset \mathbb{R}^4$ satisfy the assumptions of Theorem 1. One of the following is true:

1. *Y has an action minimizing Type 1 or 2 orbit γ with $CZ(\gamma) \leq 4$.*
2. *Y has an action minimizing Type 3 orbit.*

Now that we've covered all of the setup above, we can at last describe the algorithm...

The Algorithm

Algorithm: Let $X \subset \mathbb{R}^4$ satisfy the assumptions of Theorem 1. To compute c_{EHZ}:

1. First, check for Type 1 or 2 orbits:
2. Construct the following graph $G(X)$. The nodes are 2-faces E of X. There is an edge $E \to F$ when E and F bound a common 3-face G and there is a Reeb flow line from E to F in G.
3. Each edge comes with a affine flow map $f(EF)$ representing the Reeb flow from E to F and an affine action map $a(EF)$ computing the action needed to flow from a point $e \in E$ to $f(EF)(e) \in F$ (when this flow is defined).

The Algorithm

5. Find all cycles $p = E_1 \dots E_k$ in this graph with CZ index $CZ(p) \leq 4$.
6. Identify which of the above cycles p support an orbit. This is essentially reduced to the linear algebra problem of finding a fixed point of the affine map $f(E_{k-1}E_k) \circ \cdots \circ f(E_1E_2)$ subject to some linear constraints.
7. Minimize the action over all found cycles from Step 4 (if any were found). This takes care of potential Type 1 or 2 orbits.
8. Checking for Type 3 orbits is similar (actually simpler). We can essentially just enumerate the vorticial 1-faces and then follow the Reeb flow emanating from each one, checking whether or not it closes up.

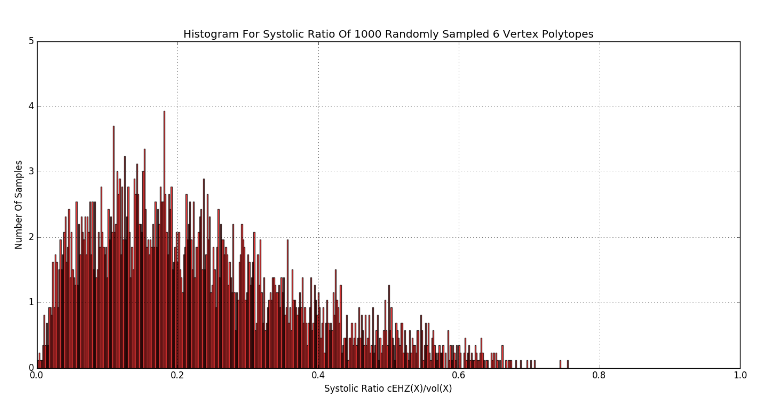
Applications And Computations

The last part of this talk is about calculations that we have performed and will perform with this algorithm. We implemented a version of this algorithm as a Python package. Applications include:

1. Gathering data on the EHZ capacity of random polytopes.
2. Testing conjectures for c_{EHZ}. For instance, automated searching for counter-examples to Viterbo's conjecture.
3. Examining structure of spectrum of closed orbits in detail.
4. Looking for data to make conjectures...

Data Gathering

This algorithm is fast enough to facilitate large scale data gathering for polytopes with small numbers of vertices, and testing of individual polytopes with much larger numbers of vertices.

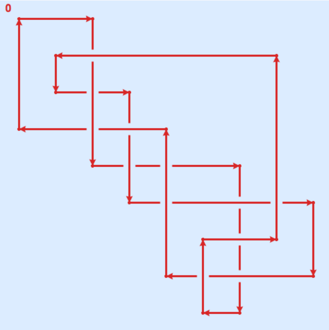


Testing Conjectures

- The original goal of this project was to design a program that could perform a large scale automated test of differnet conjectures, in particular the Viterbo conjecture.
- For many polytope counter-examples, the example could be smoothed to a C^∞ counter-example.
- We can test Viterbo via a biased random walk. Take some starting polytope and perturb the vertices of the polytope randomly, checking to see if the systolic ratio is improved. If it is, keep the new polytope and repeat. Otherwise, keep the old polytope and repeat.
- This method has yielded interesting examples of apparently local maxima of the systolic ratio function: for example, a 6 vertex polytopes with systolic ratio of .97, or an 8 vertex polytope with systolic ratio of .95.
- There seem to be interesting geometric obstructions keeping the ratio from being improved by this method.

Testing Conjectures

Another conjecture that we will test (in progress) is the conjectural unknotedness of action minimizing Reeb orbits. Our Python module includes an export function for Reeb orbits to SnapPy, so we can understand the knot topology of these orbits.



This is an complicated programmatically generated diagram for an unknotted minimizing orbit on a 24-cell.

Open Problems

- Nice analytic formulae for c_{EHZ} of polytopes or for a zeta function?
- Higher dimensional algorithms? Face structure becomes more complicated, need analogue of our CZ index and positive rotation property.
- More efficient algorithm? Would be helpful for polytopes in higher dimensions and polytopes with large number of vertices.
- More development of PL contact geometry ideas in this project...