- M9227 - Week 2 Homework -

1 Problems

Problem 1 (Schrodinger Equation) The one of the most famous conservation laws is called the **Schrodinger equation**. This equation is of central importance in quantum mechanics. We will study the following version:

$$w[t+1] = Uw[t]$$

Here $w[t]$ is a time dependent n-dimensional vector (let's call it the wave function) and U is a complex $n \times n$ matrix satisfying the following equation:

$$U^\dagger U = I$$

I is, of course, the identity matrix mentioned in class. $U^\dagger$ is the *conjugate transpose* of U. This means that you reflect the matrix across the middle diagonal and take the complex conjugate of all of the entries; we call such a matrix *unitary*. For example:

$$M = \begin{pmatrix} 1 & i \\ 0 & 1+i \end{pmatrix}; M^\dagger = \begin{pmatrix} 1 & 0 \\ -i & 1-i \end{pmatrix}; N = \begin{pmatrix} 2+i & 3-i \\ 2+i & 5 \end{pmatrix}; N^\dagger = \begin{pmatrix} 2-i & 2-i \\ 3+i & 1+i \end{pmatrix};$$

Do a few more examples if you'd like. The equation $U^\dagger U = I$ means that $U^\dagger$ is the inverse of U (basically by definition).

From here on, let M be any complex matrix, and let u and v be any complex vectors. Also let U be a unitary matrix, $w[t]$ be the wave function, as defined above and $w[0]$ be the wave function at time 0.

Problem 1.A Prove that $(M^\dagger)^\dagger = M$. In other words, show that conjugating and transposing twice gives you back your original matrix.

Problem 1.B Show that $u \cdot (Mv) = (M^\dagger u) \cdot v$.

Problem 1.C Use (B) to show that $(Uu) \cdot (Uv) = u \cdot v$.

Problem 1.D Use (C) to show that $w[1] \cdot w[1] = w[0] \cdot w[0]$, using the equation $w[t+1] = Uw[t]$. Generalize this to show that $w[t] \cdot w[t] = w[0] \cdot w[0]$. This means that the Schrodinger Equation preserves the dot product.

In quantum mechanics, the wave function represents the "state" of a particular system, and in some contexts it can be interpreted as a "wave of probability". What does this mean? Well, suppose that we were back on our circle world, a world of n points. Then we could interpret the number $p_i[0] = |w_i[0]|^2$ as the probability of finding a quantum particle at the point i in space at time 0.

Problem 1.E When we interpret $p_i[0]$ in this way, why does it become important that the quantity $\sum_{i=1}^n p_i[0]$ be equal to 1? Convince yourself that $\sum_1^n p_i[0] = w[0] \cdot w[0]$.

Problem 1.F Now explain how the fact that $w[t] \cdot w[t]$ conserves the dot product implies that the total probability $\sum_i p_i[t]$ is always equal to 1.

Problem 1.G (Hard) Suppose that U is diagonalizable (i.e, that $U = M\Lambda M^{-1}$ for some invertible M and diagonal matrix Λ). Prove that the eigenvalues of U must be roots of unity.

Problem 2 (Fibonacci Eigenvalues) The Fibonacci sequence is the sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ where each number is the sum of the two numbers before it. It's quite famous. Did you know that there's a closed form for the Fibonacci sequence? In other words, if f_n is the nth FIbonacci number I can write $f_n = a^n - b^n$ for some a and b. In this problem you will find a and b.

Problem 2.A Define the 2d vector v_n as $v_n = (f_n, f_{n-1})$. So $v_1 = (1, 0), v_2 = (1, 1)$ et cetera. Find the 2×2 matrix F such that:

$$v_{n+1} = F v_n$$

Problem 2.B Find the eigenvalues λ_1 and λ_2 of F. (Hint: Use the characteristic polynomial from the notes).

Problem 2.C Find the eigenvectors e_1 and e_2 corresponding to those eigenvalues.

Problem 2.D Express $v_1 = (1, 0)$ as a linear combination of e_1 and e_2. That is, $v_1 = c_1 e_1 + c_2 e_2$. So find c_1 and c_2.

Problem 2.E Now convince yourself that $v_n = F^{n-1} v_1 = \lambda_1^{n-1} c_1 e_1 + \lambda_2^{n-1} c_2 e_2$. Use this formula to find the first component of v_n in terms of $\lambda_1, \lambda_2, c_1, c_2$ and the first components of e_1 and e_2. This should give you the answer to the problem.

Problem 2.F Can you use this to find the limit of f_n / f_{n-1} as n goes to infinity (the Golden Ratio)?