Positively Curved 3-Folds & Ricci Flow

Introduction A Ricci flow on $M \times [0, t)$ is a solution to the differential equation:

$$\frac{dg}{dt} = -2\text{Ric} \tag{1}$$

This equation was originally introduced by Hamilton in the 1980's as a tool for studying Riemannian manifolds by providing a means of deforming an arbitrary metric into something more canonical. To understand why we might expect this behavior, you can look at this equation in harmonic coordinates.

$$\frac{dg_{ij}}{dt} = \Delta_g g_{ij} + O(\partial g, g)$$

This form suggests that the time evolution might flow the metric towards something well-understood, just like the heat equation flows a function $u(0, x)$ towards a limiting harmonic function $v(x) = \lim_{t\to\infty} u(t, x)$ satisfying $\Delta v = 0$. This intuition works in some ways and fails in others, for instance due to finite time singularity formation, but we won't go into the details of those phenomena here.

In these lecture notes, we will discuss some basic results in the study of Ricci flow with the goal of proving the following theorem:

Theorem 1 (Hamilton's Theorem, [Ham82]). *Let (M, g) be a compact orientable 3-manifold with Ricci curvature satisfying $Ric > 0$ (i.e Ric_x has signature $(+, +, +)$ as a symmetric bilinear form on T_xM). Then $M \simeq S^3/\Gamma$, where $\Gamma \subset SO(3)$ is a finite subgroup of the isometry group of S^3.*

The proof is short and illustrative of a few important aspects of pre-Perelman Ricci flow theory, so hopefully it will be educational for the geometry-minded reader/lecture attendee. Also, it's interesting to note that the stronger 2-dimensional analog of this theorem is easy to prove.

Theorem 2. *Let (M, g) be a compact, orientable surface with scalar curvature $R > 0$ (or, equivalently, Ricci curvature $Ric > 0$). Then $M \simeq S^2$.*

Proof. By Gauss-Bonnet:

$$\int_M R dv_g = 2\pi\chi(M) > 0 \implies \chi(M) = 2 - 2g > 0 \implies g = 0$$

Since compact surfaces are classified by their genus this implies that $M \simeq S^2$. □

These lecture notes are entirely based on Richard Bamler's course notes [Bam14], heavily digested, with more details in some place and much less in others. We will start by discussing two essential tools in our proof: Hamilton's short-time dispersion estimates and curvature pinching.

Ricci Flow & Hamilton's Estimates Clearly the time evolution of g governs the time evolution of the curvature tensors indirectly. We can derive the corresponding evolution equations using an intricate series of tensor manipulations, which we will not do here in the interest of shortness. We will need the evolution equation of the Ricci tensor later. It can be written as:

$$\text{Index-Free Version: } (\tilde{\nabla}_t \text{Ric})(X, Y) = (\Delta_g \text{Ric})(X, Y) + 2\sum_{i,j} \langle \text{Rm}(X, e_i)e_j, Y\rangle_g \text{Ric}(e_i, e_j) \tag{2}$$

$$\text{Index-Full Version: } \tilde{\nabla}_t \mathrm{Ric}_{ij} = \Delta_g \mathrm{Ric}_{ij} + 2\mathrm{Rm}_{iklj}\mathrm{Ric}^{kl} \tag{3}$$

$$\text{3-Dimensional Version: } \tilde{\nabla}_t \mathrm{Ric}_{ij} = \Delta_g \mathrm{Ric}_{ij} + 2G(\mathrm{Ric})_{iklj}\mathrm{Ric}^{kl} = \Delta_g \mathrm{Ric}_{ij} + Q(\mathrm{Ric})_{ij} \tag{4}$$

Here Rm is the Riemann tensor, G is the expression for Rm_{ijkl} in terms of R_{ij} which we have in 3-dimensions, and $\tilde{\nabla}_t$ is a covariant time derivative associated to the bundle TM over $M \times [0,T)$ and its tensor bundles, with respect to which g_t is parallel. We will utilize this formula later to formulate Hamilton pinching.

As with the curvature tensor evolution equations, the associated scalar quantities typically turn out to be semi-linear heat equations of some persuasion. A particularly important example of this is the following time evolution equation for $|\mathrm{Rm}|^2$.

$$\partial_t |\mathrm{Rm}|^2 = \Delta_g |\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + O(\mathrm{Rm}^3) \leq \Delta_g |\mathrm{Rm}|^2 - 2(\nabla |\mathrm{Rm}|)^2 + C|\mathrm{Rm}|^3$$

We can use this equation to inductively derive heat-type equations for $|\nabla^k \mathrm{Rm}|^2$. The following short-time estimates can be derived from these equations.

Proposition 3 (Hamilton's Estimates, [Ham82])**.** *Let g_t be a Ricci flow on $M \times [0,T)$ for M a closed n-fold. If $|Rm| \leq A$ on $M \times [0,T)$ then for all $k \geq 1$ we have:*

$$|\nabla^k Rm| \leq C_{n,k} A(t^{-k/2} + A^{k/2})$$

This theorem can be proven with by bootstrapping (i.e, induction on k) and using the right version of the maximum principle for scalar parabolic PDE. Likewise, another very important equation is given for the Ricci scalar by:

$$\partial_t R = \Delta R + \frac{2}{n} R^2 + 2|\mathring{\mathrm{Ric}}|^2 \geq \Delta R + \frac{2}{n} R^2 \tag{5}$$

An application of the comparison principle to this PDE yields:

Proposition 4. *Let g_t be a Ricci flow on $M \times [0,T)$, and suppose that $R \geq R_0$ at $t = 0$. Then:*

$$R(\cdot, t) \geq \frac{1}{R_0^{-1} - 2t/n}$$

Both of these estimates will help us later on. In particular, this shows that manifold with positive curvature must degenerate in finite time. We will assume this from the get go in our proof.

The Maximum Principle for Vector Bundles & Pinching The vector bundle maximum principle is a fundamental tool in parabolic PDE over manifolds. It takes a lot of effort to state this theorem, but it is truly just a direct generalization of the maximum principle to a more general geometric setting.

Theorem 5 (Weak Maximum Principle)**.** *Consider the following setup:*

1. *A manifold $M \times [0,T)$ with M a compact manifold with or without boundary.*

2. *A family g_t of metrics on M paramaterized by $[0,T)$.*

3. *A Euclidean vector bundle $E \to M \times [0,T)$ with a metric h and an h-compatible connection ∇. Here we write ∇_t for the lift of the time vector field to E.*

4. *A ∇-parallel sub-bundle $C \subset E$ having fibers $C_{x,t} \subset E_{x,t}$ which are closed and convex.*
5. *A vertical vector-field Φ on E (i.e a section of $ker(\pi^*) \subset TE$ where $\pi^* : TE \to TM$ is the push-forward map induced by $\pi : E \to M$) such that the flow of $\nabla_t - \Phi$ preserves the bundle C.*
6. *A section u of E satisfying the parabolic PDE $\nabla_t u = \Delta u + \Phi(u)$.*

If u takes values in C on the parabolic boundary $M \times \{0\} \cup \partial M \times [0,T)$ then u takes values in C on all of $M \times [0,T)$.

Example 1. If $E = M \times [0,T) \times \mathbb{R}$ (so the trivial bundle), $g_t = g + dt^2$ is the standard metric on

The maximum principle essentially tells us that even though $\nabla_t u = \Delta u + \Phi(u)$ is a PDE, the behavior of u is restricted substantially by the non-linear ODE $\nabla_t F = \Phi(F)$. I like to think of this in terms of a shepherd/sheep metaphor: points $F(t)$ originating in C and flowing by $\nabla_t F = \Phi(F)$ sort of shepherd the various sections u into the C and keep them there.

As with Hamilton's estimates, we're only going to allude to the proof method here. Basically, you examine the "barrier" function:

$$B(t) = \sup_{x \in M}(\inf_{y \in C_{x,t}}(|u(x,t) - y|_h))$$

This is simply the distance between the section u and the convex bundle C restricted to a fixed time t. You can show that B satisfies the ODR $B' \leq -CB$ in a weak sense, and then the fact that u is in C on the boundary means that $B(0) = 0$, so you can integrate (strictly speaking, apply Gronwall's inequality) to show that B stays 0, i.e that u stays inside of C.

The maximum principle is helpful because it lets us formulate estimates our differential equations on vector-bundles by finding these parallel, convex sub-bundles preserved by the $\nabla_t + \Phi$ flow, and then interpreting them in tandem with our other inequalities (like Hamilton's estimates). The example of this phenomenon that we will utilize is called curvature pinching:

Proposition 6. *Let M be a closed 3-manifold, and let $\lambda_3(\sigma) \geq \lambda_2(\sigma) \geq \lambda_1(\sigma)$ be the eigenvalues of a section σ of $Sym^2(TM)$ over $M \times [0,T)$. Then for every $1 > \epsilon > 0$ there exists a $\delta > 0$ such that the following convex sub-bundle $C \subset Sym^2(TM)$ is preserved by $\tilde{\nabla}_t - G$. Here G is as in (4).*

$$C = \{\sigma | \lambda_1(\sigma) \geq \epsilon\lambda_3(\sigma) \geq 0 \text{ and } \lambda_1(\sigma) \geq \lambda_3(\sigma) - \lambda_3(\sigma)^{1-\delta}\} \tag{6}$$

Proof. To check convexity we can use the following lemma.

Lemma 7. *Given a symmetric matrix M, let $\Lambda(M)$ and $\lambda(M)$ be the smallest and largest eigenvalues. If $F'' \geq 0$ and $F(0) = 0$ then the space of matrices:*

$$\{M | M = M^T, \lambda(M) > 0 \text{ and } \lambda(M) \geq F(\Lambda(M))\}$$

is convex.

Proof. Symmetry and positivity are clearly conserved under convex combination. For the other condition, we see that $\lambda(M) \geq F(\Lambda(M))$ if and only if $w^T M w \geq F(v^T M v)$ for all v and w. Then we see that if M and N have this property then for $a, b > 0$ with $a + b = 1$ we have:

$$F(v^T(aM + bN)v) \leq aF(v^T M v) + bF(v^T N v) \leq aw^T M w + bw^T N w = w^T(aM + bN)w$$

Here the second inequality comes from the conditions imposed on F, which imply that $F(x + y) < F(x) + F(y)$ for $a, b > 0$ and that $F(ax) \leq aF(x)$ for $a \leq 1$. □

Lemma 7 immediately implies that $C_{x,t}$ is the intersection of two convex subsets of the set of symmetric matrices with positive signature, which is itself convex. So $C_{x,t}$ is convex.

To see that C is conserved by the flow, consider the following expression for Q in terms of the Riemann tensor, given in (2).

$$Q(\mathrm{Ric})(e_s, e_t) = 2\sum_{i,j} \langle \mathrm{Rm}(e_s, e_i)e_j, e_t\rangle_g \mathrm{Ric}(e_i, e_j) \tag{7}$$

If we choose e_i to be a diagonal basis for Ric this reduces to:

$$Q(\mathrm{Ric})(e_s, e_t) = 2\sum_{i} \langle \mathrm{Rm}(e_s, e_i)e_i, e_t\rangle_g \mathrm{Ric}(e_i, e_i)$$

If $t \neq s$, then the only non-zero Rm term in this sum occurs when $i = r$, $r \neq s$ and $r \neq t$ (by anti-symmetry). But also:

$$\mathrm{Rm}(e_s, e_r)e_r, e_t\rangle_g = \sum_i \mathrm{Rm}(e_s, e_i)e_i, e_t\rangle_g = \mathrm{Ric}(e_s, e_t) = 0$$

So these terms are 0. When $e_s = e_t$, say $s = 1$, we get:

$$Q(\mathrm{Ric})(e_1, e_1) = 2\sum_{i} \langle \mathrm{Rm}(e_1, e_i)e_i, e_1\rangle_g \mathrm{Ric}(e_i, e_i) = 2(s_3\lambda_2(\mathrm{Ric}) + s_2\lambda_3(\mathrm{Ric}))$$

Here s_i are the sectional curvatures. In 3-dimensions we have the formula $\lambda_1(\mathrm{Ric}) = s_2 + s_3$, and analogous formulae for the other sectional curvatures, so:

$$Q(\mathrm{Ric})(e_1, e_1) = 2(\lambda_2^2 + \lambda_3^2 + \lambda_1(\lambda_2 + \lambda_3) - 2\lambda_2\lambda_3)$$

Thus the ODE $\tilde{\nabla}_t \sigma = G(\sigma)$ can be written (at a point, in the eigenbasis of $\sigma \in \mathrm{Sym}^2(TM)$ and in terms of its eigenvalues) as:

$$\dot{\lambda}_1 = 2(\lambda_2^2 + \lambda_3^2 + \lambda_1(\lambda_2 + \lambda_3) - 2\lambda_2\lambda_3)$$

with analogous expressions for other $\dot{\lambda}_i$. This allows us to check whether or not the positivity conditions are preserved. To do so, note that if $\lambda_1 - \epsilon\lambda_3 \geq 0$ and $\lambda_1 - \lambda_3 + \lambda_3^{1-\delta} \geq 0$ everywhere at some time t then:

$$\dot{\lambda}_1 - \epsilon\dot{\lambda}_3 = 2(\lambda_2^2 + \lambda_3^2 + \lambda_1(\lambda_2 + \lambda_3) - 2\lambda_2\lambda_3) - 2\epsilon(\lambda_1^2 + \lambda_2^2 + \lambda_3(\lambda_1 + \lambda_2) - 2\lambda_1\lambda_2)$$

$\square$

The Proof Of Hamilton's Theorem Hamilton's estimates and pinching are the key components in a proof of Theorem 1. With these in hand, we can start the proof. Under the setup, Ric > 0 so by compactness we can choose a small $\epsilon > 0$ so that $\lambda_1(\mathrm{Ric}) \geq \epsilon\lambda_3(\mathrm{Ric})$. by rescaling the solution parabolically ($g_t \mapsto a^{-1}g_{at}$) we can adjust the curvature so that $\lambda_3(\mathrm{Ric}_g) \leq 1$ at $t = 0$. Let $\delta = \delta(\epsilon)$ be as in Proposition 6.

We can now break the proof into 4 steps.

1. There exists a sequence $t_k \to T$ along which $R_{\min} \to \infty$. So curvature blows up uniformly across M, and small curvature pockets don't get left behind.

2. There exists a $T_0 < T$ such that $\mathrm{R} \leq 2R_{\max}$ on $M \times [0, t]$. So we can upgrade the sequence t_k to the entire interval $[T_0, T)$

3. $R_{\min}(t) \leq \frac{1}{(2/3)(T-t)}$ for all $t \in (0,T)$ and $R_{\max}(t) \leq \frac{2}{(2/3)(T-t)}$ for t near T. So we have a good upper bound on how fast the curvature blows up.

4. There exists a normalization g_t^* of g_t which converges to a positively curved Einstein metric on M. Any simply connected Einstein manifold of positive curvature is isometric to S^n with a scaled version of the standard metric.

Proof. (Step 1) Choose a sequence $t_k \to T, x_k \in M$ such that $R(x_k,t_k) \to \infty$ and $R \leq 2R(x_k,t_k)$ on $M \times [0,t_k]$. Since $R_{\max}(t)$ diverges to infinity as $t \to T$, we can make such a choice. If we label $Q_k = R(x_k,t_k)$, then using Theorem 3 we get the following:

$$|\nabla \mathrm{Rm}_{t_k}| \lesssim Q_k(t_k^{-1/2} + Q_k^{1/2}) \lesssim Q_k^{3/2}; |\nabla^2 \mathrm{Rm}_{t_k}| \lesssim Q_k(t_k^{-1} + Q_k) \lesssim Q_k^2;$$

Here we use the fact that we can assume t_k is lower bounded and thus $Q_k \gg t_k^{-1}$. We also use two facts allowing us to turn an estimate on R into an estimate on Rm. First, if the Ricci curvature is positive then $R^2 \simeq |\mathrm{Ric}|^2$, i.e they bound each other up to constant. This is analogous to the statement that $x^2+y^2 \leq (x+y)^2 \leq 2(x^2+y^2)$ when $x,y>0$. Second, in 3-d we have $|\mathrm{Ric}|^2 \simeq |\mathrm{Rm}|^2$ due to the fact that one can be recovered from the other via G from (4).

Now we're going to use our control of the derivatives to show that R isn't too far from Q_k at the other points in M at t_k. We break our problem into two regimes, corresponding to a "small" diameter M and a "large" diameter M, where the scale is set by Q_k and δ.

First assume that $\mathrm{diam}_{t_k} M \leq Q_k^{-1/2-\delta/2}$. Then by interpolation:

$$|R(y,t_k) - Q_k| = |R(y,t_k) - R(x_k,t_k)| \lesssim |\nabla R| \cdot \mathrm{diam}_{t_k} M \lesssim Q_k^{3/2} \cdot Q_k^{-1/2-\delta/2} = Q^{1-\delta/2}$$

Thus when $k \gg 1$ we have:

$$\frac{1}{2}Q_k \leq (1 - CQ_k^{-\delta/2})Q_k \leq R(y,t_k) \leq (1+CQ_k^{-\delta/2})Q_k \leq 2Q_k$$

Thus as Q_k diverges, so does $R_{\min}(t_k)$.

Now assume that $\mathrm{diam}_{t_k} M \geq Q_k^{-1/2-\delta/2}$. This regime is where pinching comes to the rescue. One interpretation of pinching is that it controls the eigenvalues of the traceless Ricci tensor $\mathring{\mathrm{Ric}} = \mathrm{Ric} - \frac{1}{3}Rg_{\mu\nu}$. From this formula we can see that these eigenvalues are $\lambda_i - \frac{1}{3}\sum_j \lambda_j$. Using this, we observe that:

$$|\mathring{\mathrm{Ric}}_{t_k}| \lesssim \sum_i |\lambda_i - \frac{1}{3}\sum_j \lambda_j| \lesssim \sum_{i,j} |\lambda_i - \lambda_j| \lesssim |\lambda_1 - \lambda_3| \leq \lambda_3^{1-\delta} \lesssim CR_{t_k}^{1-\delta}$$

The second to last inequality uses pinching, and the last one uses the fact that $R_{t_k} = \sum_i \lambda_i$. Now we can interpolate, using $r = Q^{-1/2-\delta/2}$:

$$|\nabla \mathring{\mathrm{Ric}}_{t_k}| \lesssim |\mathring{\mathrm{Ric}}_{t_k}|r^{-1} + |\nabla^2 \mathring{\mathrm{Ric}}_{t_k}|r \lesssim Q^{3/2-\delta/2}$$

But since $\mathrm{div}(\mathring{\mathrm{Ric}}_{t_k}) = \frac{1}{6}\nabla R$ by the Bianchi identity, this yields a bound $|\nabla R_{t_k}| \lesssim Q^{3/2-\delta/2}$. Note that we have improved our naive estimate on ∇R_{t_k}. Now we can use this bound to observe that as long as $k \gg 1$ and $B_k = B(x_k, t_k, 10^4 Q_k^{-1/2})$ we have:

$$|R_{t_k}(y) - R_{t_k}(x_k)| \lesssim |\nabla R_{t_k}| \cdot \mathrm{diam}(B_k) \simeq Q_k^{3/2-\delta/2} \cdot Q_k^{-1/2} = Q_k^{1-\delta/2} \leq \frac{1}{2}Q_k$$

$$(1 - CQ_k^{-\delta/2})Q_k \leq R(y,t_k) \leq (1+CQ_k^{-\delta/2})Q_k; \frac{1}{2}Q_k \leq R_{t_k}; \frac{1}{10}Q_k \leq \mathrm{Ric}_{t_k}$$

□

The last inequality for Ric_{t_k} follows from positivity and pinching for $k \gg 1$. Now we have a lower bound on R_{t_k} all over the ball B_k, so we're very close. To finish the argument, we simply observe that by Myer's theorem and this lower bound on the Ricci curvature, the diameter of the ball is at most volume $\sqrt{20}\pi Q_k^{-1/2} < 10^4 Q_k^{-1/2}$, i.e a ball of this radius with this curvature covers the connected component of which it is a part. Therefore, $B_k = M$ and we get the lower curvature bound on R_{t_k} over all of M.

Proof. (Step 2) Suppose otherwise, i.e we can pick two sequences (x_k, t_k') and t_k^* with $t_k' < t_k^*$ and $t_k', t_k^* \to \infty$ such that $R(x_k, t_k') > 2R_{max}(t_k^*)$ and $2R(x_k, t_k') \geq R$ on $M \times [0, t_k']$. In other words, we pick a sequence t_k' satisfying the assumptions in Step 1, then we pick a series of times t_k^* which use the contradiction assumption that we can always find a space-time point ahead of (x_k, t_k') where the curvature stays low as $t \to T$. However, by applying Step 1 we see that for $k \gg 1$:

$$R_{\min}(t_k') \geq \frac{9}{10} R_{\max}(t_k') \geq \frac{18}{10} R_{\max}(t_k^*) \geq \frac{18}{10} R_{\min}(t_k^*) \geq \frac{18}{10} R_{\min}(t_k')$$

The first inequality is Step 1 (pinching), the second inequality is out choice of t_k' and t_k^*, our third is trivial and our fourth uses the monotonicity of $R_{\min}$. This is a contradiction. □

This confirms that any sequence $T_0 < t_k \to T$ satisfying the assumptions of Step 1 would have sufficed. In fact, by taking T_0 close enough to T so that $R_{\max}(t_k)$ satisfies $(1 - CR_{\max}(t_k)^{-\delta/2}) \gg 1/2$ we can ensure that $R_{\min}(t)$ also satisfies $2R_{\min}(t) > R$ on $[0, t]$ for all $t > T_0$. This allows us to upgrade our results from Step 1 to the estimate:

$$(1 - CR^{-\delta/2}(y,t))R(y,t) \leq R(z,t) \leq (1 + CR^{-\delta/2}(y,t))R(y,t) \text{ for all } y, z \in M, t \geq T_0$$

Proof. (Step 3) By applying the comparison principle to $\partial_t R \geq \Delta R + \frac{2}{3}R^2$ we get the estimate:

$$R_{\min}(t) \geq \frac{1}{R_{\min}(t_1)^{-1} + \frac{2}{3}(t_1 - t)}$$

$\phi(t) = \frac{1}{R_{\min}(t_1)^{-1} + \frac{2}{3}(t_1 - t)}$ is a solution to the PDE $\partial_t \phi = \Delta\phi + \frac{2}{3}\phi^2 = \frac{2}{3}\phi^2$ and R is a super-solution. Since, by assumption, the Ricci flow lives through time T, the right hand side above can't blow up before time T. In other words:

$$R_{\min}(t_1)^{-1} + \frac{2}{3}(t_1 - T) \geq 0; R_{\min}(t_1) \leq \frac{1}{\frac{2}{3}(T - t_1)}$$

Since we chose t_1 arbitrarily, this proves the claim for $R_{\min}(t)$. The claim for $R_{\max}(t)$ follows from Step 1 and Step 2. □

Proof. (Step 4) To get our rescaled metric, we set:

$$g_t^* = V(t)^{-2/3} g_t$$

Take note of the time derivative of the volume (a quick computation that you can check).

$$\dot{V}(t) = \int_M R dv_{g_t}$$

Now we ca take the time derivative of g_t^*.

$$\partial_t g_t^* = -2V(t)^{-2/3}\mathrm{Ric}_{g_t} - \frac{2}{3} V(t)^{5/3} \dot{V}(t) g_t =$$

$$-2V(t)^{-3/2}\mathrm{Ric}_{g_t} + \frac{2}{3}V(t)^{-2/3}R(x,t)g_t + \frac{2}{3}V^{-5/3}\left(\int_M (R - R(x,t))dv_{g_t}\right)g_t$$

$$-2V(t)^{-2/3}\overset{\circ}{\mathrm{Ric}}_{g_t} + \frac{2}{3}V(t)^{-5/3}\left(\int_M (R - R(x,t))dv_{g_t}\right)g_t$$

In the above calculation, $R(x,t)$ is a constant within the integral. Now if we take the norm of all sides and insert our estimates, we see that:

$$|\partial_t g_t^*|_{g_t} \lesssim V(t)^{-2/3}\cdot CR^{1-\delta}(x,t) + V(t)^{-5/3}\cdot CR^{1-\delta/2}(x,t)V(t) \leq CR(x,t)^{1-\delta/2}V(t)^{-2/3} \leq C(T-t)^{\delta/2-1}V(t)^{-2/3}$$

If we switch to our rescaled metric g_t^* when taking the norm, we see that:

$$|\partial_t g_t^*|_{g_t^*} \leq C(T-t)^{\delta/2-1}$$

Since $\delta/2 - 1 \in (-1, 0)$, this is an integrable singularity about T, so we can integrate the limit $g_t^* \to T$ exists. There are some details here about boosting to smooth convergence using Hamilton's estimates and the curvatures of the rescaled metric which I won't go into here. Now if we look at $\overset{\circ}{\mathrm{Ric}}^* = \overset{\circ}{\mathrm{Ric}}$, i.e the Ricci tensor of g_t^*, we see that since $R^* = V^{2/3}R$ is bounded, we get:

$$\lim_{t\to T}|\overset{\circ}{\mathrm{Ric}}^*|_{g_t^*} = \lim_{t\to T}V(t)^{2/3}|\overset{\circ}{\mathrm{Ric}}|_{g_t} \leq \lim_{t\to T}V(t)^{2/3}R_t^{1-\delta} = \lim_{t\to T}R_t^*R^{-\delta} \lesssim \lim_{t\to T}R^{-\delta} = 0$$

So the traceless Ricci tensor is pinched to 0 in $t \to T$ limit! This implies that M is Einstein and positive curvature, which implies that it is a space form. □

References

[Bam14] Richard Bamler, *Ricci Flow Lecture Notes* `http://web.stanford.edu/~cmad/Papers/RicciFlowNotes.pdf`

[Ham82] Richard Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry. **17** (1982), no. 2, 255-306.