



WHAT IS . . .

# a Legendrian Knot?

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Legendrian knots lie at the intersection of knot theory and contact topology. This can be construed to mean that Legendrian knots arise when contact topology imposes extra structure on knot theory or that they mediate the injection of knot-theoretic ideas into the study of contact topology and its applications. We will take the first view to explain what Legendrian knots are and the second to motivate their study.

A (smooth) *knot* is a smooth embedding of the circle into  $\mathbb{R}^3$  (or into any 3-manifold, but we will mostly stay in  $\mathbb{R}^3$ ). A *contact structure* is a special type of *plane field*—just as a vector field assigns a vector to each point in space, a plane field assigns an entire plane of directions to each point. Figure 1 presents the *standard contact structure*  $\xi_0$  on  $\mathbb{R}^3$ , which will be described in more detail below. Though we will not give the general definition of a contact structure here, the idea is that the planes in a contact structure twist so much that there is no surface whose tangent planes are all part of the contact structure. On the other hand, there are curves whose tangent vectors do lie in the contact structure; such curves are called *Legendrian*, and a knot that is also a Legendrian curve is a *Legendrian knot*.

Until the end of the article, we will restrict our attention to Legendrian knots in the standard contact  $(\mathbb{R}^3, \xi_0)$ . To get a feel for the standard contact structure, consider the track that the wheel of a unicycle makes in a parking lot, with the caveat that the wheel never points north-south. The state of the unicycle may be described by the coordinates  $(q, p, z)$  in  $\mathbb{R}^3$ , where  $(q, z)$  is the

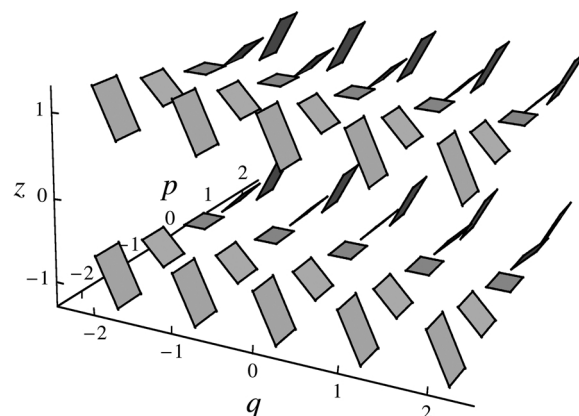


Figure 1. The standard contact structure on  $\mathbb{R}^3$  (figure created using Mathematica).

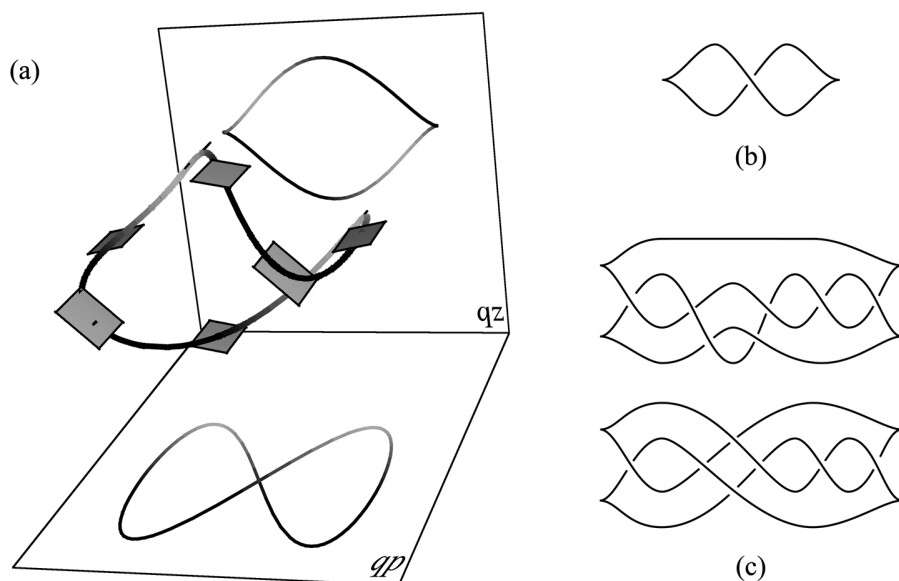
position of the unicycle in the parking lot and  $p$  is the slope of the wheel with respect to the  $q$ - and  $z$ -axes when viewed from above. At any given time, the unicycle can swivel in place, move forward or backward in the direction its wheel points, or use some linear combination of these motions. Thus, the path  $\gamma(t) = (q(t), p(t), z(t))$  that describes the motion of the unicycle must be tangent to the plane field spanned by  $\partial_p$  and  $\partial_q + p\partial_z$ , and hence it must satisfy

$$(1) \quad z'(t) - q'(t)p(t) = 0.$$

This plane field is the aforementioned standard contact structure, and the curve  $\gamma(t)$  is Legendrian.

The constraints imposed on a knot by equation (1) are best visualized using projections to the  $qz$  and  $qp$  planes, as shown in Figure 2(a). Equation (1) implies that the  $p$  coordinate of a Legendrian curve is determined by the slope of its  $qz$  or *front* projection. In terms of the unicycle, the

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**Figure 2. (a) A Legendrian unknot, with its front projection in back and its Lagrangian projection below (figure created using Mathematica); (b) Front projections of another Legendrian unknot; (c) The Chekanov-Eliashberg examples.**

front projection is the track left on the parking lot. Several front projections of Legendrian knots are shown in Figure 2. Notice that the crossing data is always the same: the strand of more negative slope crosses in front. Any closed curve in the  $qz$  plane, immersed except at finitely many cusps and having no vertical tangents, is the front projection of some Legendrian knot.

Equation (1) also implies that the  $z$  coordinate of a Legendrian curve can be recovered (up to a constant) by integrating the quantity  $q'(t)p(t)$  along its  $qp$ , or *Lagrangian*, projection. Since the  $z$  coordinate of a knot must return to its starting value as we go around the knot, Green's theorem shows that the Lagrangian projection of a Legendrian knot must bound the zero signed area. In particular, the round circle cannot be the Lagrangian projection of a Legendrian knot.

Two fundamental goals of knot theory are to understand “geography”—how to distinguish or even classify knots—and to investigate the geometry of a knot's position in 3-space. These two goals apply to Legendrian knot theory as well. Here, we declare two smooth knots to be equivalent if one can be smoothly deformed into the other through smooth knots; similarly, two Legendrian knots are equivalent if one can be smoothly deformed into the other through Legendrian knots.

Two equivalent Legendrian knots are equivalent as smooth knots, but the converse is false. To see why, we introduce two “classical” invariants. The *Thurston-Bennequin number* measures the linking between a Legendrian knot and a second copy of the knot pushed off in a direction (such as  $\partial_z$ ) that

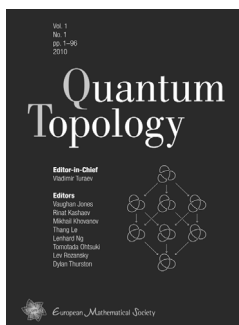
is always transverse to  $\xi_0$ . The knot in Figure 2(a) has  $tb = -1$ , while the knot in Figure 2(b) (also an unknot!) has  $tb = -2$ . The *rotation number* of an oriented Legendrian knot measures the winding number of the tangent vector of the Lagrangian projection of the knot. This can also distinguish the knots in Figure 2(a,b).

The classical invariants refine the “geography” question for Legendrian knots in a fixed smooth knot class: which pairs  $(tb, r)$  of classical invariants can be realized by a Legendrian knot, and how many Legendrian knot classes have the same pair? For the unknot, Eliashberg and Fraser proved that any  $(tb, r)$  pair whose components have opposite parity and that satisfies the *Bennequin bound*  $tb + |r| \leq -1$  is realized by exactly one Legendrian knot class, and only those pairs are realized. In fact, every Legendrian unknot can be constructed by adding a sequence of zig-zags (called *stabilizations*) to the front diagram of the knot in Figure 2(a).

Torus knots, the figure-8 knot, and certain torus links have similar classifications, but in general, the story is more complicated: both Chekanov and Eliashberg discovered that the two Legendrian knots in Figure 2(c) are in the same knot class, have  $tb = 1$  and  $r = 0$ , but are not equivalent. There are several types of “nonclassical” invariants that can distinguish these, and other, knots: Legendrian contact homology, a special case of Eliashberg, Givental, and Hofer's Symplectic Field Theory; Chekanov and Pushkar and Traynor's theory of generating families; and Ozsváth, Szabó, and Thurston's Legendrian invariant in Knot Floer Homology. These recently developed invariants



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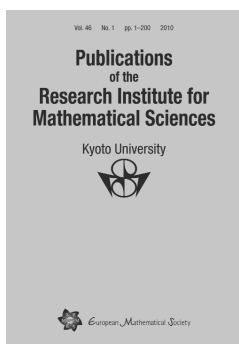
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detect subtle contact-topological features of a Legendrian knot and its complement, and a picture of what these features are is just beginning to emerge.

Legendrian knots are not only interesting in their own right, but also provide tools for the investigation of contact topology and knot theory. In contact topology, Legendrian knots are frequently employed as probes of the ambient contact structure. The first example of this was Bennequin's discovery of an "exotic" contact structure  $\xi'$  on  $\mathbb{R}^3$ : he found a Legendrian unknot in  $(\mathbb{R}^3, \xi')$  that violated the Bennequin bound. Legendrian knots are also used as surgery loci for the construction of new contact manifolds. Legendrian knots influence smooth knot theory through their close ties to known geometric (slice genus) and quantum (HOMFLY, Kauffman, Khovanov, ...) knot invariants and the construction of new invariants like Ng's Knot Contact Homology, which uses higher-dimensional Legendrian knots. Further, surgery on Legendrian knots (and/or its analog for knots transverse to the contact structure) plays a significant role in Kronheimer and Mrowka's proof that every non-trivial knot has Property P. Finally, Legendrian knots have recently been brought back to their historical roots as "wave fronts" in geometric optics with Chekanov and Pushkar's solution of Arnol'd's 4-cusp conjecture. Despite this progress, however, our understanding of the geography, geometry, and applications of Legendrian knots is still in its opening chapters.

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