A Polygonal Proof to the Isoperimetric Inequality

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Abstract

The word "isoperimetric" comes from the greek words "is" and "perimetros" which means equal perimeter. The II gives a relationship between shapes of equal perimeter and their areas. Vice versa, you can think about fixing the area of a shape and look at the range of possible perimeters. In both situations, we find out that the circle minimizes perimeter if area is fixed, and maximizes area if perimeter is fixed. We will define the isoperimetric inequality in $\mathbf{R^2}$ below.

The Isoperimetric Inequality in $\mathbf{R^2}$

Let γ be a simple and closed curve. Let A be the area enclosed by γ, and L be the length of γ. Then it follows that

$$L^2 \geq 4\pi A$$

Author's Motive

Even though the isoperimetric inequality seems intuitive and has been around since the Second Century BC, many current proofs rely on other complex inequalities. The idea of this paper is to find a basic proof to this inequality. Although this is an academic paper, we also intend it to be a workbook. We encourage the reader to follow through with the exercises for maximal understanding.

Introduction

0.0.1 Intuition

The center of the polygonal proof comes from the idea that any simple smooth curve can be approximated by a n-sided polygon as $n \to \infty$. So if we can show that all polygons obey the isoperimetric inequality and that the polygonal form of the isoperimetric inequality tends to the isoperimetric inequality for all curves, we are done.

Notice, the II can be rewritten as $\frac{L^2}{A} \geq 4\pi$. Hence, to show that the

> **Polygonal form of the isoperimetric inequality tends to the Curvature form of the isoperimetric inequality (i.e.** $\lim_{n\to\infty} \min \frac{L^2(P_n)}{A(P_n)} = 4\pi$**)**

we simply need to show $\forall n \in \mathbb{N}$, $\forall P_n \in$ set of all n-sided polygons with the same area or perimeter, $\min \frac{L(P_n)^2}{A(P_n)} \geq 4\pi$.

Finding such a polygon comes in two flavors: fix area, minimize perimeter, or fix perimeter and maximize area. You will see the use of both approaches throughout this paper. We will denote $\frac{L^2}{A}$ as the **Isoperimetric Ratio (IR)**, throughout this paper. **IR** will denote **Isoperimetric Ratio**, and **II** will denote **isoperimetric inequality.**

0.0.2 Sufficiency To Consider Convex Bodies

Here, we show quickly that it suffices to prove the II for convex bodies in order to prove the II for all bodies. Recall the II states

$$L^2 \geq 4\pi A$$

Suppose $L(S)$ is the perimeter of some nonconvex shape S, and $A(S)$ likewise for its area. Then, we can take the convex hull of S, call it $\overline{S}$, where $L(S) \geq L(\overline{S})$, and $A(\overline{S}) \geq A(S)$.

From the II on convex bodies, it follows

$$L^2(S) \geq L^2(\overline{S}) \geq 4\pi A(\overline{S}) \geq 4\pi A(S) \to L^2(S) \geq A(S)$$

for all nonconvex bodies, S as desired. Hence, it suffices to also look strictly at convex polygons in our proof.

0.0.3 First Milestone

To understand how the II looks like for polygons, we first try to understand how the inequality applies to the most basic polygon: triangles. For all triangles, is $\min \frac{L^2}{A} \geq 4\pi$? We will show later in this paper, that proving simple characteristics of triangles can be extremely important in decomposing and understanding any n-sided polygon.

The II in Triangles

Before jumping straight to talking about any triangle. It is worth exploring how the II looks like for right triangles to gain some intuition for the general triangle case and build up a set of examples to study.

0.1 Right Triangles

Consider any right triangle T_{right}, we can label T_{right} as the following below, where $l_1, l_2, l_3 \in \mathbf{R}$ are the corresponding side lengths, and θ is the corresponding angle between l_2 and l_3. Hence, the perimeter of T_{right}, $L(T_{right}) = l_1 + l_2 + l_3$, and the area of T_{right}, $A(T_{right}) = \frac{l_1 l_2}{2}$. For sake of simplicity, assume A(T_{right}) = 1. Note: $l_3 = \sqrt{l_1^2 + l_2^2}$ and $l_2 = \frac{2}{l_1}$. Refer to Figure 1 for reference.

We want to show: $\forall T_{right}$, $L^2(T_{right}) \geq xA(T_{right})$, where $x \geq 4\pi$ to prove the II holds for all right triangles. More precisely, we seek to find min $\frac{L(T_{right})^2}{A(T_{right})}$.
$\frac{L(T_{right})^2}{A(T_{right})} = \frac{(l_1+l_2+l_3)^2}{1} = l_1^2 + l_2^2 + l_3^2 + 2(l_1 l_2) + 2(l_2 l_3) + 2(l_1 l_3)$

$$= 2l_1^2 + 2l_2^2 + 2l_1 l_2 + 2l_1\sqrt{l_1^2 + l_2^2} + 2l_2\sqrt{l_1^2 + l_2^2}$$

$$= 2l_1^2 + 2l_2^2 + 4 + 2(l_1 + \frac{2}{l_1})\sqrt{l_1^2 + \frac{4}{l_1^2}}$$

$$= 2l_1^2 + \frac{8}{l_1^2} + 4 + 2(l_1 + \frac{2}{l_1})\sqrt{l_1^2 + \frac{4}{l_1^2}}$$

To find the min$\frac{L^2(T_{right})}{A(T_{right})}$, we treat it as any classic minimization problem. We take its derivative, set it to zero, and find the polygon that corresponds to it. In this case, we've transformed $\frac{L^2(T_{right})}{A(T_{right})}$ into a function, f, that depends only on l_1. **We encourage the reader to compute the l_1 that minimizes the IR before reading the solution below.**

As shown earlier, $\frac{L(T_{right})^2}{A(T_{right})} = f(l_1)$ then $f'(l_1) = \frac{4(l_1^6 - 4\sqrt{\frac{l_1^4+4}{l_1^2}}l_1 + \sqrt{\frac{l_1^4+4}{l_1^2}}l_1^5 - 8)}{l_1^4\sqrt{\frac{l_1^4+4}{l_1^2}}}$.
$f'(l_1) = 0$ when $l_1 = \sqrt{2}$. This implies that $l_2 = \frac{2}{\sqrt{2}} = \sqrt{2}$. Since our triangle is a right triangle, and $l_1, l_2 = \sqrt{2}$ we have shown that the **45-45-90 degree triangle (i.e. an isosceles triangle)** achieves the smallest IR among all right triangles with the same area. Just for a sanity check, let's explicitly compute the IR for the 45-45-90 degree triangle.

$$\min \frac{L^2(T_{right})}{A(T_{right})} = (2 + 2\sqrt{2})^2 \approx 23.3 \geq 4\pi$$

0.2 General Triangles

In the previous section, we showed that for any right triangle, the isosceles right triangle achieves the minimal IR. Now, we wonder what the case is for the general triangle. This journey will take two steps.

0.2.1 Step 1: General to Isosceles (Figure. 2)

As we saw earlier in right triangles, algebraic manipulation is feasible, but it is also messy. In the general triangle case, such calculations can get messier as we can't make any assumptions on the angles of the triangle. In the general case, we look towards geometric intuition for help.

Recall, that the area of any triangle $\triangle ABC$ is $\frac{1}{2}bh$ as shown in Figure. 2. Hence, if we fix base b, and slide vertex C along $\bar{l}$, we preserve h, and hence we preserve the area of $\triangle ABC$. Now, some of us may think that the isosceles version of $\triangle ABC$ achieves a smaller IR (i.e $\triangle ABM$). You are correct, however a rigorous proof is needed. As a word of caution, the field of isoperimetric inequalities has many intuitive truths that are also surprisingly difficult to prove.

We now show that for any general triangle $T_{general}$, we can reduce its IR by considering its isosceles form $T_{isosceles}$ ($T_{general}$ is represented by $\triangle ABC$ and $T_{isosceles}$ is represented by $\triangle ABM$ as shown in Figure. 2).

Consider the isosceles form of $\triangle ABC$, $T_{isosceles}$, where the point C sits directly above the midpoint of AB. Denote, the position of C in $T_{isosceles}$ as M. Let $z =$ the length of the legs in $T_{isosceles}$ and denote θ' as the base angle. Since $T_{isosceles}$ is symmetric, we simply need to show that if we slide point C to the right from M however far we want (you can choose left if you prefer), we never achieve a smaller perimeter than the one in $T_{isosceles}$. Since, the perimeter of $T_{isosceles}$, $L(T_{isosceles}) = b + 2z$, and the perimeter of $T_{general}$, $L(T_{general}) = a + b + c$.

To show, $L(T_{isosceles})^2 \leq L(T_{general})^2$, it suffices to show $2z \leq a + c$.

Proof. Without loss of generality, assume $c \geq a$ and for the sake of simplicity, assume $h = 1$. Since $c \geq a \Rightarrow \theta \geq \phi$. Furthermore, $c \geq a \Rightarrow c \geq z$ and $a \leq z$. Hence, $\theta' \leq \theta$ and $\theta' \geq \phi$. We also know that $h = a\sin(\theta) = b\sin(\phi) = z\sin(\theta')$.
1) $L(T_{general}) = a + b = (\frac{1}{\sin\theta} + \frac{1}{\sin\phi})$ 2) $L(T_{isosceles}) = 2z = \frac{2}{\sin\theta'}$

$\theta' \leq \theta \Rightarrow \sin\theta' \leq \sin\theta \Rightarrow 2z = \frac{2}{\sin\theta'} \geq \frac{2}{\sin\theta} \geq \frac{1}{\sin\theta} + \frac{1}{\sin\phi} = a + b$ □

Lemma 1. *As shown above, given any triangle $\triangle ABC$, if we fix one side of the triangle say $\overline{AB}$, and $L(\triangle ABC)$, then the isosceles triangle $\triangle ABM$ achieves* $\min \frac{L^2(\triangle ABC)}{A(\triangle ABC)}$.

0.2.2 Step 2: Isosceles to Equilateral

Now that we have achieved a smaller IR by going from any general triangle to its isosceles form, we wonder if we can do better than the isosceles form. Now, your intuition might say an equilateral triangle, but once again we will prove this rigorously.

We want to show: Given any isosceles triangle, $T_{isosceles}$, with perimeter $L(T_{isosceles})$ and area $A(T_{isosceles})$, we can minimize $\frac{L(T_{isosceles})^2}{A(T_{isosceles})}$ by considering the equilateral triangle $\triangle A'B'C$ formed with the same area (refer to Figure. 3).

Proof. For the sake of simplicity, assume $A(T_{isosceles}) = 1$. Consider one half of $T_{isosceles}$, by the Pythagorean Theorem, we know that

$$z^2 = (\frac{b}{2})^2 + h^2$$

$$A(T_{isosceles}) = 1 = \frac{1}{2}bh \Rightarrow h = \frac{2}{b}$$

$$L(T_{isosceles}) = b + 2z = b + 2\sqrt{(\frac{b}{2})^2 + (\frac{2}{b})^2}$$

To minimize $L(T_{isosceles})$, we simply find its derivative, $L'(T_{isosceles})$, and see when $L'(T_{isosceles}) = 0$

$$L'(T_{isosceles}) = 1 + \frac{(\frac{b}{2} - \frac{8}{b^3})}{\sqrt{(\frac{b}{2})^2 + (\frac{2}{b})^2}}$$

$$= 1 + 2(\frac{b^4 - 16}{4b^3\sqrt{(\frac{b}{2})^2 + (\frac{2}{b})^2}})$$

$$L'(T_{isosceles}) = 0 \Rightarrow b = \frac{2i}{\sqrt[4]{3}}, \frac{2}{\sqrt[4]{3}}$$

Reject the imaginary root. Again by the Pythagorean Theorem, $(\frac{1}{\sqrt[4]{3}})^2 + h^2 = (\frac{2}{\sqrt[4]{3}})^2 \Rightarrow h = \sqrt[4]{3}$. Now, we quickly conduct a sanity check that $A(T_{isosceles}) = 1$.

$$A(T_{isosceles}) = \frac{1}{2}bh = \frac{1}{2}(\frac{2}{\sqrt[4]{3}})(\sqrt[4]{3}) = 1$$

Notice, an isosceles triangle with $b = \frac{2}{\sqrt[4]{3}}$, and area 1 is an equilateral triangle. Here is a quick recap of what we have shown so far:

1. Among all isosceles triangles with fixed perimeter/area, the equilateral triangle achieves the smallest IR.

2. Among all general triangles with fixed perimeter/area, the isosceles triangle achieves smaller IR.

(1)+(2) $\Rightarrow$ Among all general triangles with fixed perimeter/area, the equilateral triangle achieves the smallest IR. □

Quadrilaterals

Now, what kind of quadrilateral do you think minimizes the IR? It turns out that our previous work with triangles wasn't exactly all wasted. Notice, that any quadrilateral can be divided into two triangles by drawing a straight line between any two opposite vertices.

Consider the quadrilateral, Q_{ABCD} with vertices A,B,C,D as shown in Figure. 4. Let $A(Q_{ABCD})$ denote the area of Q_{ABCD}, $L(Q_{ABCD})$ likewise for its perimeter. We adopt the same strategy here as in the triangle case: fix area and minimize perimeter.

(1) Our strategy here is to consider one triangle first. Fix $\triangle BCD$ and consider $\triangle ABD$. Recall, that in 0.2.1, we showed that we can reduce the IR by converting any triangle into its isosceles form (while preserving its area), by sliding point A along the line parallel to line segment $\overline{\text{BD}}$ that also intersects point A, towards the left if $\overline{\text{AB}} > \overline{\text{AD}}$ and vice versa for the right.

(2) Denote the new position of point A after sliding, A'. Likewise, we can apply the same strategy to $\triangle BCD$: fix $\triangle BA'D$, and slide point C along the line parallel to line segment $\overline{\text{BD}}$ that intersects C, towards the left if $\overline{\text{BC}} > \overline{\text{CD}}$ and vice versa for the right.

Now, we have transformed quadrilateral ABCD into a rhombus A'BC'D with the same area as quadrilateral ABCD but a smaller perimeter. Now, we apply (1) and (2) again but to triangles $\triangle A'BC'$ and $\triangle A'DC'$. This time, because these two triangles are the same, steps (1) and (2) leave us with two identical triangles $\triangle A'B'C'$, $\triangle A'D'C'$ as shown in Step 3 of Figure. 4. This implies that all the side lengths of our new quadrilateral A'B'C'D' are the same which implies that it **must be the square with area** $A(Q_{ABCD})$.

General n-sided Polygon

So far, we have seen what the II says about some basic n-sided polygons. For triangles ($n = 3$), we saw that the equilateral triangle achieves the minimal IR. For quadrilaterals ($n = 4$), the square achieves the minimal IR. So far, the shape that optimizes the IR seems to always be the n-sided polygon where all the side lengths are equal. In fact, such a polygon is so special that mathematicians call it the **normal n-sided polygon**.

Some of you might be inclined to guess that the normal n -sided polygon is the polygon that optimizes the IR for any n-sided polygon. But how do we prove it?

For $n \geq 5$, things get a bit tricker. Unlike the quadrilateral case, there's no way to force two halves of of the polygon to be the same shape. We kind of hit jackpot for $n = 4$, since both triangles shared the same base, but for $n = 6$, applying the isosceles strategy to any triangle has no reason to force all the other triangles to be the same. The same problem but with greater severity occurs for odd n, starting from $n = 5$.

To avoid dealing with polygons directly, some authors [1] in the past have used an indirect proof. That is, they show the existence of an n-sided polygon that minimizes the IR for all n-sided polygons, and conclude from existence, that it cannot be a shape other than the normal polygon. In this paper, we will give two proofs: the indirect proof mentioned above, and a direct proof (showing the existence and forcing the normal n polygon to be the solution simultaneously).

0.3 An Indirect Proof to the II for n-sided Polygons

Setup: Choose some $n \in \mathbb{N}$ (number of sides), $M \in \mathbb{R}$ (perimeter of the n-sided polygons). Let S_n denote the set of polygons with perimeter M, with at most n vertices, and where one of the vertices is fixed at the origin. $\forall P \in S_n$, let $L(P)$ denote its perimeter, and let $A(P)$ denote its area. As mentioned above, the indirect proof of the II for n-sided polygons requires three steps:

1. Show the existence of some polygon $\overline{P} \in S_n$ s.t.
$$\forall P \in S_n,\ A(\overline{P}) \geq A(P)$$
2. Among all polygons with n sides in S_n, the normal n-agon maximizes area.
3. Among all polygons with at most n sides in S_n, the normal n-agon maximizes area.

Intuition: Recall, the IR is $\frac{L(P)^2}{A(P)}$. The entire motive here since the beginning was to find the n-agon, $\overline{P}$, that achieves the smallest IR, then using the geometry of $\overline{P}$, explicitly compute the IR. Finally show that this ratio converges to 4π as $n \to \infty$ (decreasing monotone sequence/shown in Section 0.5). If we go back to our old trusty analysis, we should recall that any continuous function over a compact set achieves a minimum and maximum.

0.3.1 Step 1: Show the existence of some polygon $\overline{P} \in S_n$ s.t. $\forall P \in S_n,\ A(\overline{P}) \geq A(P)$

a) $A(P)$ is a continuous function

Proof. $A(P)$ is continuous for all $P \in S_n$, since any micro-movement of any single vertex of P, changes its area by a small amount, so changing all vertices at the same time, changes the polygon's area by a determinable amount. An epsilon-delta proof is not given here for the time being, but may be included in the future. □

b) S_n is a compact set.

Proving compactness here requires us to define what it means for two polygons in S_n to be close. Notice, S_n contains polygons with different number of sides, so we cannot simply take a sum of the euclidean norm between pairs of vertices as our metric. Instead, we use the Hausdorff distance.

Definition 0.1 (Hausdorff Distance)**.** Consider any two polygons $P_1, P_2 \in S_n$, let $V(P)$ denote the set of vertices of polygon P, then the **Hausdorff Distance** between them is defined as

$$d_H(P_1, P_2) = \inf\{\epsilon \geq 0 | \forall v_i \in V(P_1), \forall z_i \in V(P_2), |v_i - z_j| \leq \epsilon\}$$

(i.e. the Hausforff distance is simply the greatest of all distances from a point in one set to the closest point in the other set [2]).

Proof. We will first show that S_n is bounded.

$\forall P \in S_n$, one of the vertices is fixed at the origin $\rightarrow$ the largest Hausdorff distance, $d_H(P_1, P_2)$, between any two polygons $P_1, P_2 \in S_n$ is when the closest point from any vertex in one polygon to the other polygon is the origin. (You can imagine these two polygons to almost be reflections of each other). However, since any vertex is bounded by distance M from the origin, $d_H(P_1, P_2)$ is also bounded by M.

The proof for closure is not included in this paper. $\square$

0.3.2 Step 2: Among all polygons with n sides in S_n, the normal n-agon maximizes area.

Proof. Assume $\exists \overline{P_n} \in S_n$ s.t. $\min \frac{L^2(P_n)}{A(P_n)} = \frac{L^2(\overline{P_n})}{A(\overline{P_n})}$. Let $v_1, v_2, ..., v_n$ denote the vertices of $\overline{P_n}$. Then we can consider any set of three consecutive vertices v_i, v_{i+1}, v_{i+2} for some $i \in \mathbb{N}_{[1,n-2]}$. Consider the triangle, T, formed by these three vertices, and fix the rest of $\overline{P_n}$, call it T^c. Notice, $A(\overline{P_n}) = A(T) + A(T^c)$. Likewise, $L(\overline{P_n}) = L(T) + L(T^c) - 2\overline{v_1v_3}$. Since, we already assumed that $\overline{P_n}$ optimizes the IR, that implies $A(T)$ must be maximal, and $L(T)$ must be minimal, since $A(T^c), L(T^c), \overline{v_1v_3}$ are all fixed quantities. By Lemma 1, if we fix $\overline{v_1v_3}$ (i.e. the base), and the perimeter of T, we can achieve the greatest area, if we make it isosceles (i.e. $\overline{v_1v_2} = \overline{v_2v_3}$). Hence, T is already isosceles. Since this is true for any set of three consecutive vertices, that implies $\overline{v_1v_2} = \overline{v_2v_3} = ... = \overline{v_nv_1}$ which in turn implies $\overline{P_n}$ is the normal n-sided polygon. $\square$

0.3.3 Step 3: Among all polygons with at most n sides in S_n, the normal n-agon maximizes area.

Proof. Recall that S_n contains polygons with fixed perimeter and at most n edges. Hence, to claim Step 3, we need to rule out polygons with less than n edges. By Step 2, we know that among all $P_m \in S_n$ with m sides, $m \leq n$, $\exists \overline{P_m}$ that maximizes area. Let C_m denote the set of polygons with m edges in S_n. Next, we partition S_n into groups of polygons with the same number of edges.

$$S_n = \{C_3, C_4, C_5, ..., C_n\}$$

$$\forall C_i \in S_n, \exists \overline{P_i} \in C_i, s.t. \forall P \in S_n, A(\overline{P_i}) \geq A(P)$$

Hence, S_n reduces to just the set $\{\overline{P_3}, \overline{P_4}, .., \overline{P_n}\}$ when looking for the polygon that achieves maximal area. Now, it suffices to show $\forall m \leq n, A(\overline{P_n}) \geq A(\overline{P_m})$ to eliminate all other polygons with less than n sides. For any normal polygon with n sides and perimeter M, its area is $\frac{M^2}{4n\tan(\frac{\pi}{n})}$. In section 0.5, we showed that $4n\tan(\frac{\pi}{n})$ is a monotonic decreasing function. Hence, $\forall m \leq n, A(\overline{P_n}) \geq A(\overline{P_m})$. □

Direct Proofs to the II for n-sided Polygons

0.3.4 A Spark for Inspiration: Steiner Symmetrization

Steiner Symmetrization is a good building block to start from when one would like to transform a simple, closed curve, X, into another simple, closed curve, X' where $A(X) = A(X'), P(X) \leq P(X')$. Here, we illustrate the process only on polygons.

Exercise: Show that among all trapezoids with the same area, the trapezoid with the smallest perimeter is the one where the non-parallel sides have the same length. (Will be used later in the algorithm)

0.3.5 The Steiner Symmetrization Algorithm

Let X_0 be some polygon with n sides and perimeter M. Denote $\{v_1, v_2, .., v_n\}$ to be the set of vertices for X. Choose any two consecutive vertices, $v_i, v_{(i+1)\%n}$, let l denote the edge between them. Draw a line E, s.t. $E \perp l$ and E intersects the midpoint of l. Next, for $\forall v_h \in X, h \notin \{i, (i+1)\%n\}$, draw a line Z , s.t. $Z \parallel l$ and goes through v_h. Please refer to Figure 5. for an illustration.

Now, X has been split into triangles and trapezoids. Let $S = \{S_1, S_2, .., S_k\}$ denote the partitions of X, labeled from top to bottom (bottom partition has edge l). For triangles, we know if we fix the base of the triangle, the isosceles form of the triangle minimizes the IR. For trapezoids, the area is $\frac{1}{2}(b_1 + b_2)h$, where h is the height, b_1, b_2 are the respective lengths of the ceiling and floor of the trapezoid, so simply shifting b_1 or b_2 left or right does not alter h, and thus area is fixed.

Now that we know how to optimize triangles and trapezoids for the smallest IR, we can turn back to our partitions, S. Let's look at each slice of the polygon from top to bottom.

$$\forall S_i \in S$$

- If S_i is a triangle, then we apply the isosceles transformation in (0.2.1).
- If S_i is a trapezoid, then we make the non-parallel sides have equal length (isosceles trapezoid).

Once, we have gone through all the partitions we have finished one iteration of Steiner Symmetrization. Let X_j denote the resulting polygon after j-iterations of Steiner Symmetrization.

Notice, in every iteration we are able to minimize the perimeter of the entire polygon since we only adjust the perimeter of the individual partitions S_i, while fixing the remaining partitions. For instance, with triangles the base is never adjusted, and the same is true for trapezoids.

Now the power of Steiner Symmetrization comes from repeating this iteration. After we have symmetrized according to one edge of X, we can apply the same iteration on a new edge. Remember, after every iteration we preserve area but we achieve a new perimeter that is bounded above by the perimeter of the polygon we started off with.

So are we done?

Unfortunately not. As you may have noticed in Figure 5, there may be polygons where the line of slicing (indicated by the red dashed lines) does not intersect two vertices. The problem about this, is that Steiner Symmetrization now adds additional vertices to our original polygon. So in one sense, we made the shape more symmetric but on the other we also introduced more vertices.

0.3.6 Second Attempt: Triangle/Trapezoid Fitting

Well, let's not give up yet! We've already done a huge amount of work: reducing a isoperimetrical optimization problem for an entire polygon into one for smaller polygons. Recall, in the previous section the problem of introducing new vertices arises when the lines of slicing under Steiner Symmetrization did not intersect two vertices. Well instead of slicing according to Steiner, why don't we simply break X_1 into triangles and quadrilaterals that share bases with one another?

To be more precise: Let $v_1, v_2, .., v_n$ denote the vertices of X_1, where the vertices are oriented clockwise. If n is odd, $\forall j \in \mathbb{N}_{[2,\frac{n}{2}-1]}$ draw a dashed line between v_j, v_{n-j} until $v_{j+1} = v_{n-j}$ i.e.(v_j, v_{n-j} are two adjacent vertices). If n is even, $\forall j \in \mathbb{N}_{[0,\frac{n}{2}]}$ draw a dashed line between v_{j+3}, v_n until $v_{j+1} = v_{n-j}$ i.e.(v_j, v_{n-j} are two adjacent vertices). **Note: these are only a few ways of slicing a polygon into neighboring triangles/quadrilaterals**. There can be more research done about the rate of convergence with respect to slicing rules. However, my thought is that slicing rules that generate more slices per iteration of the Steiner Symmetrization have a faster convergence rate.

Please refer to Figure. 6 for the rest of this section. So suppose, we have finished slicing our polygon according to this new method. Now, remember we want to avoid the problem of introducing new vertices, and these new vertices are introduced when we try to symmetrize trapezoids. So now, we need to find some way to symmetrize quadrilaterals, while preserving area and reducing perimeter.

An idea that has not been proven yet, is to do two things. First, want to prove that given any quadrilateral $ABCD$, as shown in Figure. 7, if we fix $\overline{AB}, \overline{CD}$, and preserve perimeter, then the area, $A(ABCD)$ is maximal when it is a trapezoid with bases $\overline{AB}, \overline{CD}$ (i.e. when $\overline{BC} = \overline{AD}$).

Secondly, we show that our polygon under an infinite repetition of Steiner symmetrization iterations converges to the normal n-agon. To show this, we we would want to show that the length of any edge converges to $\frac{M}{n}$, where M is the perimeter of the polygon we started off with. We encourage the reader to look into the codebase, and Figure. 8 for further proof.

0.3.7 Equal Edges + Triangle/Trapezoid Fixing

This last strategy is an algorithm that runs basically two steps:

1. Given any polygon P with m sides. Label its vertices counterclock-wise. $\forall i \in \mathbb{N}_{[1,n-2]}$, consider the triangle formed by vertices, $i, i+1, i+2$ and apply the isosceles triangle transformation (mentioned earlier in the general triangles section). We will show with code below that such a process converges any polygon P to another polygon P' with the same perimeter but with equal side lengths.

2. Recall, what we really want is the normal n-agon which also has equal angles. To get equal angles, we split our polygon P' (from Step 1) into adjacent triangles and quadrilaterals. We convert every quadrilateral into a trapezoid by simply redistributing the side lengths of one pair of opposite sides equally. By fixing the other two sides, we guarantee that the perimeter of the P' is fixed under the quadrilateral$\rightarrow$ trapezoid process. This process as shown below will send $P' \rightarrow \overline{P}$, the normal n-agon.

One might be inclined to ask what difference does equal edging imply on the convergence rate for the IR of any polygon. As shown in Figure. 9, starting off with an equal edged polygon does indeed reduce the number of iterations needed for IR convergence. A potential field of research in the future can look into the total computational complexity of 0.3.6 vs. 0.3.7. `https://github.com/jchaitheguy/isoperimetric_inequalities/`

Conclusion

0.4 IR for Polygons to Curves

Now that we have reached the conclusion that the normal n-sided polygon, $\overline{P_n}$ minimizes $\frac{L^2(P_n)}{A(P_n)}, \forall P_n \in S_n$, we now need to show that

$$\lim_{n\to\infty} \min \frac{L^2(P_n)}{A(P_n)} = 4\pi$$

Proof. From 0.3.4, we know

$$\min \frac{L^2(P_n)}{A(P_n)} = \frac{L^2(\overline{P_n})}{A(\overline{P_n})}$$

Fix, $A(\overline{P_n}) = 1$. For any normal n-agon, $\overline{P_n}$, we know that all of its interior angles are the same, so the area of $\overline{P_n}$ is simply split equally among n identical slices. Each slice is an isosceles triangle with base $\frac{L(\overline{P_n})}{n}$, and height, h (currently an unknown). Denote each slice as I_n.

$$\tan(\frac{\pi}{n}) = \frac{\frac{L(\overline{P_n})}{2n}}{h} \to h = \frac{L}{2n\tan(\frac{\pi}{n})}$$

(From analyzing Figure. 6)

$$A(I_n) = \frac{A(\overline{P_n})}{n} = \frac{1}{n} = \frac{1}{2}(base * height) = \frac{h * L(\overline{P_n})}{2n} = \frac{\frac{L}{2n\tan(\frac{\pi}{n})} * L(\overline{P_n})}{2n}$$

$$\Rightarrow \boxed{\frac{L(\overline{P_n})^2}{A(\overline{P_n})} = 4n\tan(\frac{\pi}{n})}$$

Recall $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

$$\lim_{n\to\infty} \frac{L(\overline{P_n})^2}{A(\overline{P_n})} = 4\lim_{n\to\infty} \frac{n\sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})} = 4\lim_{n\to\infty} \frac{\frac{n\sin(\frac{\pi}{n})}{\frac{\pi}{n}}}{\cos(\frac{\pi}{n})} = 4\lim_{n\to\infty} \frac{\pi}{\cos(\frac{\pi}{n})} = 4\frac{\pi}{1} = 4\pi$$

. $\square$

0.5 Determining the direction of convergence for the IR of polygons

We show below $f(n) = 4n\tan(\frac{\pi}{n})$ is a monotonic decreasing function. Suffices to show $\forall n \geq 3,\ f'(n) \leq 0$, since polygons only make sense from triangles and higher.

Proof. By Product Rule,

$$f'(n) = (4n)(\frac{d}{dn}\tan(\frac{\pi}{n})) + (\tan(\frac{\pi}{n}))(\frac{d}{dn}4n) = 4n\sec^2(\frac{\pi}{n})(-\frac{\pi}{n^2}) + 4\tan(\frac{\pi}{n})$$
$$= \frac{-4\pi\sec^2(\frac{\pi}{n})}{n} + 4\tan(\frac{\pi}{n})$$

Need to show $|\frac{-4\pi\sec^2(\frac{\pi}{n})}{n}| \geq |4\tan(\frac{\pi}{n})|$

$$|\frac{-4\pi\sec^2(\frac{\pi}{n})}{n}| = \frac{4\pi}{n\cos^2(\frac{\pi}{n})}$$
$$|4\tan(\frac{\pi}{n})| = \frac{4\sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})} = \frac{4n\cos(\frac{\pi}{n})\sin(\frac{\pi}{n})}{n\cos^2(\frac{\pi}{n})}$$

Recall, for small x, $\sin(x) \approx x$

$$\lim_{n\to\infty} 4n\cos(\frac{\pi}{n})\sin(\frac{\pi}{n}) = 2n\sin(\frac{2\pi}{n}) \approx 2n(\frac{2\pi}{n}) = 4\pi$$

Hence,

$$\forall n \geq 3, 4n\cos(\frac{\pi}{n})\sin(\frac{\pi}{n}) \leq 4\pi \Rightarrow |\frac{-4\pi\sec^2(\frac{\pi}{n})}{n}| \geq |4\tan(\frac{\pi}{n})| \Rightarrow f'(n) \leq 0$$

□

0.6 Limitations and Future Work

1. S_n compactness: As of now, we have not shown that S_n is closed using the Hausdorff measure as it requires more complex technicalities, such as defining the topology we are in.

2. $A(P)$ is continuous: Similar problem to the one above.

3. We have shown with the codebase that $\forall$ quadrilaterals $ABCD$ with perimeter M, ceiling length a, and floor length b, if we were to fix there a, b, M the isosceles trapezoid with ceiling length a, floor length b, leg length $\frac{M-a-b}{2}$ achieves greater area. However, we have not shown that the isosceles trapezoid does indeed achieve the greatest area under these restrictions.

4. We have finished the second half of this paper with algorithmic strategies. However, for the theoretical mathematician we encourage them to show the convergence of 0.3.6 and 0.3.7 by hand. Do be warned, the computations can get pretty complicated.

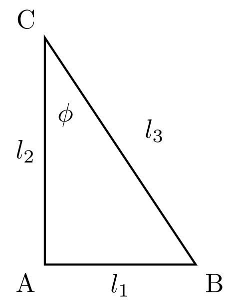


Figure 1: Right Triangle

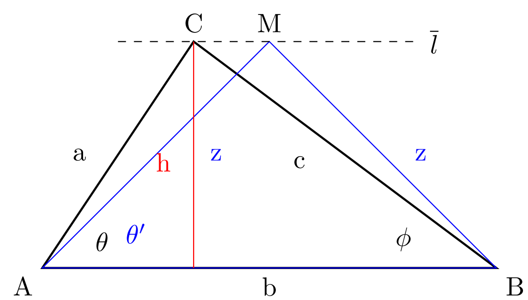


Figure 2: General Triangle to Isosceles

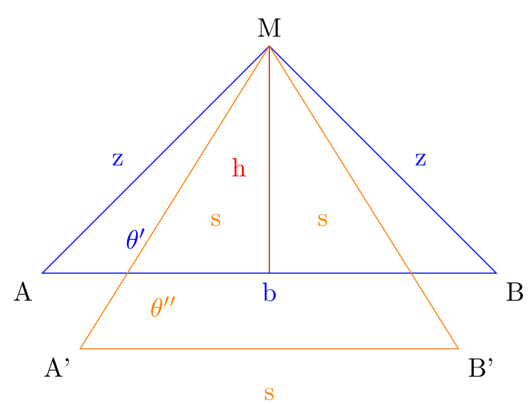
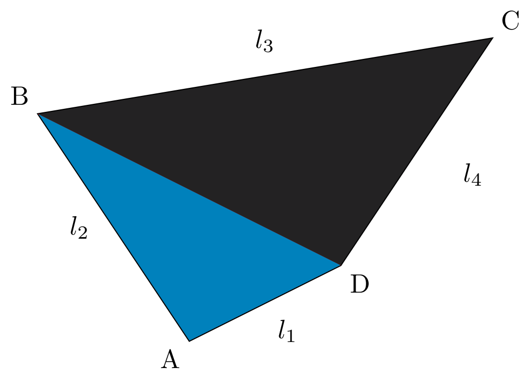
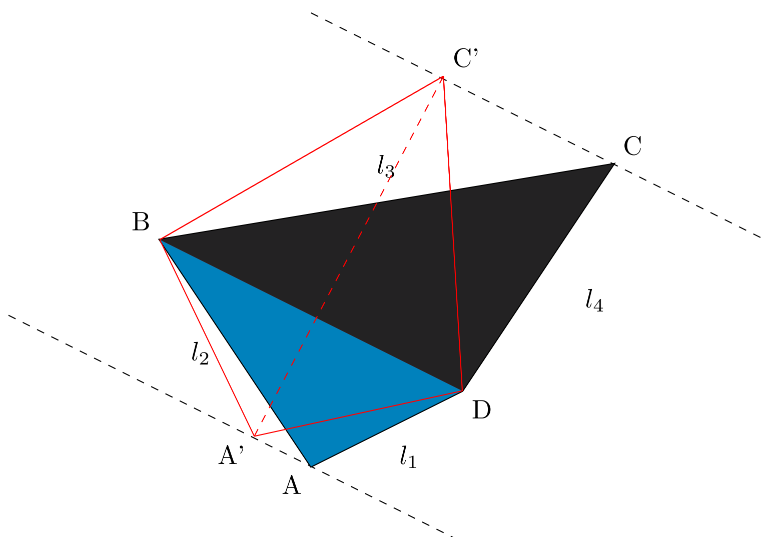


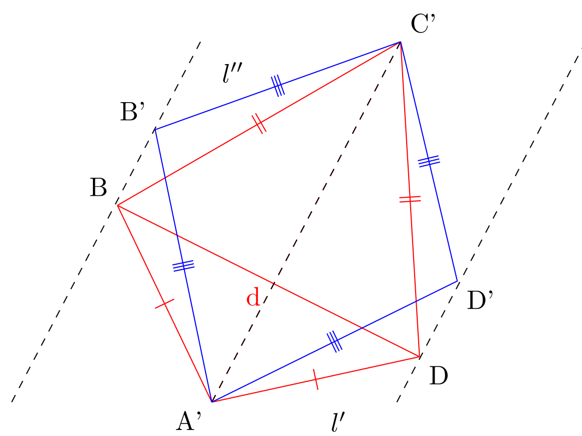
Figure 3: Isosceles Triangle to Equilateral



(a) General Quadrilateral

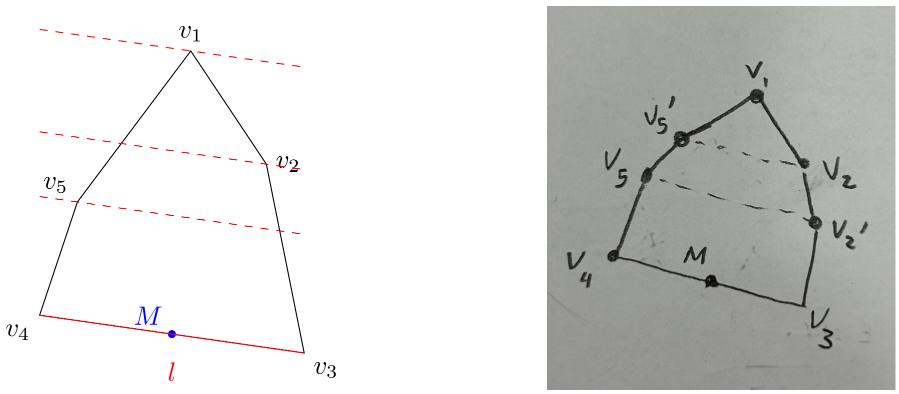


(b) Quadrilateral to Rhombus



(c) Rhombus to Square

Figure 4: Quadrilaterals to Squares



(a) Slicing Step

(b) Symmetrizing Step

Figure 5: Steiner Symmetrization

Citations

1. From the Triangle Inequality to the II by S. Kesavan (The Institute of Mathematical Sciences, CIT Campus, Taramani) `https://www.imsc.res.in/~kesh/tri.pdf`

2. Hausdorff Distance by Wikipedia `https://en.wikipedia.org/wiki/Hausdorff_distance#:~:text=it%20is%20the%20greatest%20of%20all%20the%20distances%20from%20a%20point%20in%20one%20set%20to%20the%20closest%20point%20in%20the%20other%20set.`

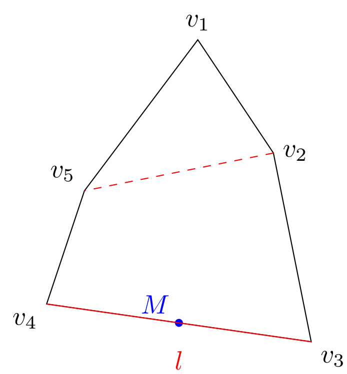
3. The II by Robert Osserman (University of Washington) `https://sites.math.washington.edu/~toro/Courses/20-21/MSF/osserman.pdf`

4. The II by Simon Brendle and Michael Eichmair (American Mathematical Society) `https://www.ams.org/journals/notices/202406/rnoti-p708.pdf`

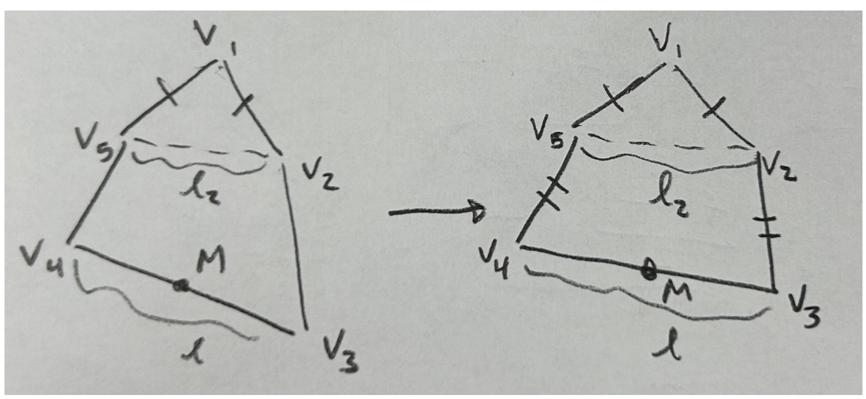
5. Inequalities that Imply the II by Andrejs Treibergs (University of Utah) `https://www.math.utah.edu/~treiberg/isoperim/isop.pdf`

Acknowledgments

Look's like you made it to the end! Thank you for taking the time to read this paper. I would really like to thank Dr. Alice Chang, Dr. Kate Okikiolu, and Shouda Wang for being extremely patient with me during this exposition process and guiding me in the right directions to think when puzzled. I would like to thank my family and friends for their continued support throughout my Princeton education.



(a) Slicing Step



(b) Symmetrizing Step

Figure 6: Re-engineered Steiner Symmetrization

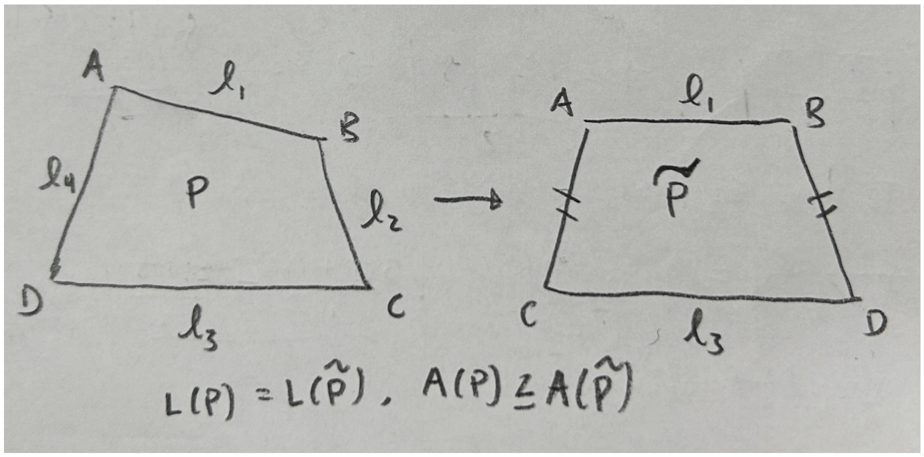


Figure 7: Maximizing the area of a quadrilateral when a pair of opposite sides are fixed

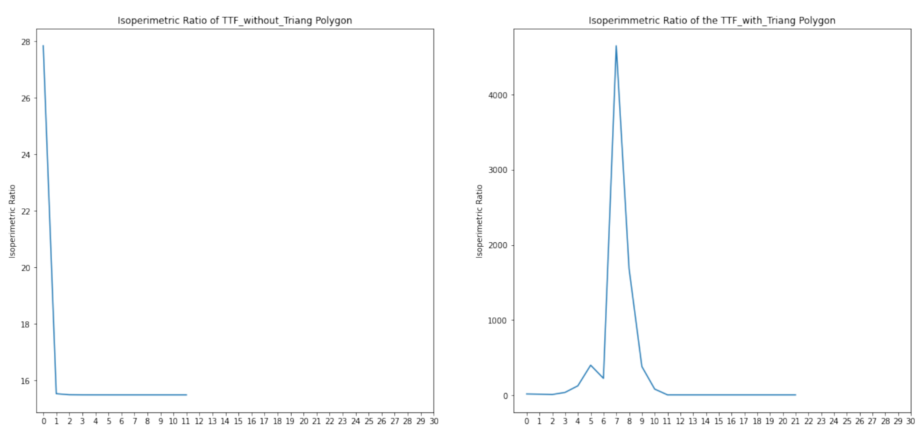
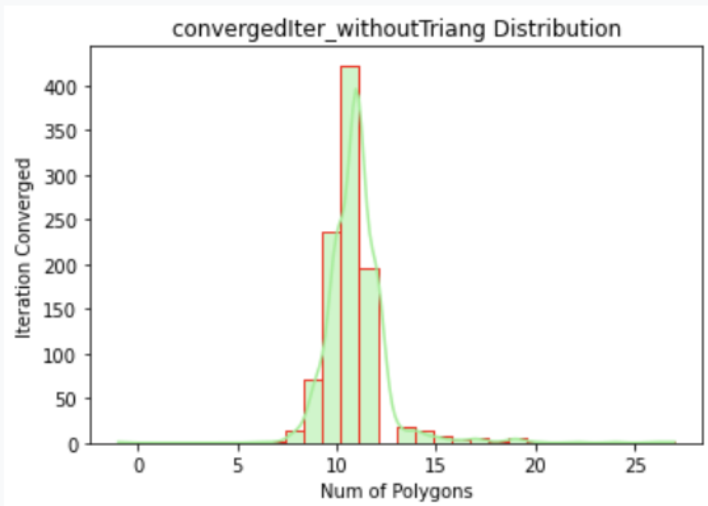


Figure 8: Converging Iteration Mark for a single polygon with 30 equal edge iterations, and 30 trap/tri fitting iterations

```
Mean convergedIter_withoutTriang:  11.016

std convergedIter_withoutTriang:  1.6744384133195225

variance convergedIter_withoutTriang:  2.803744
```




```
Mean convergedIter_withTriang:  10.255

std convergedIter_withTriang:  1.2930487229799195

variance convergedIter_withTriang:  1.6719750000000004
```

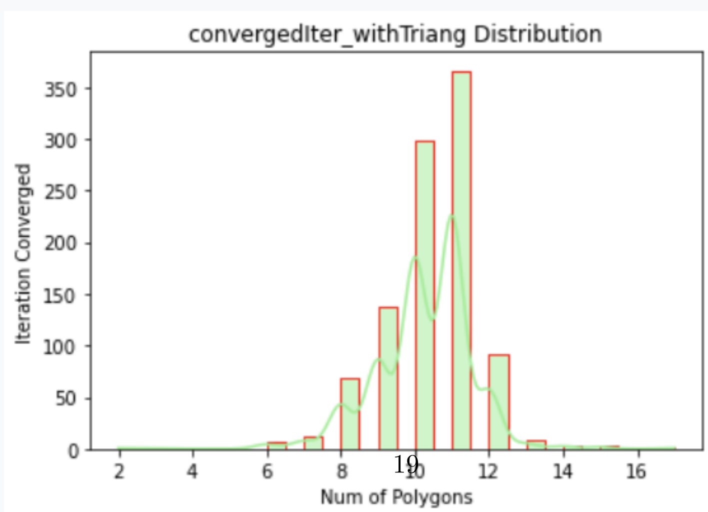


Figure 9: IR Convergence Distributions of 1000 random polygons with 30 equal edge iterations, and 30 trap/tri fitting iterations