List of finite differences for use in finite difference calculations

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The Camassa-Holm equation

The finite differences listed in this document pertains to the following equation, known as the Camassa-Holm equation (note: stationary reference frame):

$$u_t + 3uu_x - 2u_x u_{xx} - uu_{xxx} - u_{xxt} = 0 \tag{1}$$

Central in space, forward in time

The following are the finite differences for equation (1), using central differences in space, forward in time.

$$U_t = \frac{U_j^{n+1} - U_j^n}{k} \tag{2}$$

$$U_x = \frac{U_{j+1}^n - U_{j-1}^n}{2h} \tag{3}$$

$$U_{xx} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} \tag{4}$$

$$U_{xxx} = \frac{U_{j+2}^n - 2U_{j+1}^n + 2U_{j-1}^n - U_{j-2}^n}{2h^3} \tag{5}$$

$$U_{xxt} = \frac{(U_{j+1}^{n+1} - U_{j+1}^n) - 2(U_j^{n+1} - U_j^n) + (U_{j-1}^{n+1} - U_{j-1}^n)}{h^3} \tag{6}$$

The equation above can be rewritten as

$$
\begin{aligned}
\frac{U_j^{n+1}}{k} - \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^3} &= \\
\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} + \frac{\left(U_{j+1}^n - U_{j-1}^n\right)\left(U_{j+1}^n - 2U_j^n - U_{j-1}^n\right)}{h^3} & \\
-\frac{3U_j^n\left(U_{j+1}^n - U_{j-1}^n\right)}{2h} + \frac{U_j^n\left(U_{j+2}^n - 2U_{j+1}^n + 2U_{j-1}^n - U_{j-2}^n\right)}{2h^3} + \frac{U_j^n}{k} &
\end{aligned}
$$