Lab 6

AC-Circuit Theory and Complex Numbers

In preparation for future labs, we provide here some basics about AC circuits. (AC stands for alternating current.) We also explain how exponentials of imaginary numbers are equivalent to trigonometric functions. These powerful techniques make analysis of alternating currents much easier. The exercises in this 'lab' do not require equipment; they are traditional homework problems.

6.1   Resistors, Capacitors, and Inductors

Resistance is commonly denoted by the letter R; the SI unit for resistance is the ohm ($\Omega \equiv$ volt/amp). To cause a current I (SI unit: amp $\equiv$ coul/s) to flow through a resistor, you must apply a *voltage* or *potential* (SI unit: volt$\equiv$ J/coul). This is governed by Ohm's law:

$$V_R = IR \tag{6.1}$$

Capacitance is commonly denoted by the letter C; the SI unit for capacitance is the farad (F $\equiv$ coul/volt). If you apply a voltage to the capacitor, you can cause a charge Q to accumulate according to

$$V_C = \frac{Q}{C} \tag{6.2}$$

This really means that you cause a current to flow, which charges up the capacitor. Since current is charge per time that flows into the capacitor, the accumulated charge is simply

$$Q = \int I\,dt \tag{6.3}$$

Inductance is commonly denoted by the letter L; the SI unit for inductance is the henry (H $\equiv$ volt$\cdot$s/amp). An inductor is essentially a coil of wire that produces a magnetic field when current flows through it. It takes energy to establish magnetic fields (returned when the field turns off), which causes inductors to oppose

changes in current. The voltage across an inductor is related to the change in current flowing through it according to

$$V_L = L\frac{dI}{dt} \tag{6.4}$$

6.2 AC Driven Series LRC Circuit

Now consider a series circuit comprised of a resistor R, a capacitor C, and an inductor L, as depicted in Fig. 6.1. If you apply a voltage V to the whole circuit, it will be balanced by the sum of the voltages across the individual elements:[1]

$$V = V_R + V_C + V_L \tag{6.5}$$

Substitution of (6.1)-(6.4) into (6.4) yields

$$V = RI + \frac{1}{C}\int I dt + L\frac{dI}{dt} \tag{6.6}$$

Now suppose that suppose that your applied voltage is a sinusoidal waveform

$$V(t) = V_0 \cos\omega t \tag{6.7}$$

You might like to know what current $I(t)$ flows in the circuit as a result of the applied voltage. In this case, you will be interested to know that the solution to (6.5) has the form

$$I(t) = I_0 \cos\left(\omega t - \phi_I\right) \tag{6.8}$$

The current also oscillates with frequency ω. The amplitude I_0 and phase [2] ϕ both depend on R, C, L, and ω. A different phase means that the oscillating current is shifted in time relative to the applied voltage, as depicted in Fig. 6.2.

It turns out that finding I_0 and ϕ_I is a bit of work. Since it involves the cosine, not surprisingly, there is trigonometry involved. If you are interested, the solution is worked out for you using brute force in appendix 6.A, but first we will show you a much better way. (You might want to glance briefly at appendix 6.A to better appreciate the alternative.) It turns out that complex numbers come in handy for managing the trigonometry involved in solving sinusoidally driven LRC circuits. The more complicated the circuit, the more powerful the technique becomes. Because complex-number notation is used so pervasively in physics, it is worth making a significant investment at this time.
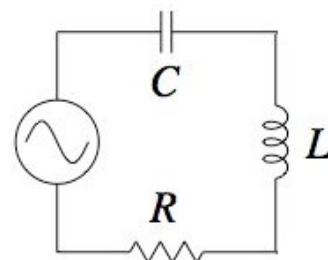

Figure 6.1 Driven series LRC circuit. The symbol at the left of the circuit is the AC voltage source that drives the circuit.
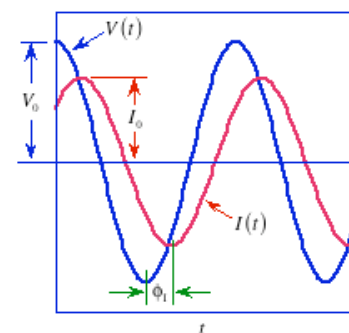

Figure 6.2 Voltage and current for an LRC circuit.

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[1] It doesn't matter what order you add these voltages up, so you can hook up the elements in any order.

[2] We have followed the convention of the Physics 220 textbook, where the phase ϕ_I is introduced with a minus sign. Different conventions do not change the physics, although they have the potential to lead to confusion.

6.3   Complex Numbers

As we saw in the previous section, AC circuits involve functions of the form $V = V_0 \cos \omega t$ and $I = I_0 \cos(\omega t - \phi_I)$, where V and I represent voltage and current. The phase term ϕ in the cosine means that the sine function is also intrinsically present through the identity

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \qquad (6.9)$$

With a good background in trigonometry, one can solve AC-circuit problems. However, (6.9) and similar trig identities can be cumbersome to use. Fortunately, complex notation offers an equivalent approach that is far more convenient. The modest investment needed to become comfortable with complex notation will definitely be worth it.

Hopefully, you have encountered complex numbers before, where $i \equiv \sqrt{-1}$. Generically, a *complex number* is written in the form $z = a + ib$, where a and b are real numbers. The real part, denoted Re$\{z\}$, is a, and the imaginary part, denoted Im$\{z\}$, is b. As an example, if $z = 2 - i5$, then Re$\{z\} = 2$ and Im$\{z\} = -5$.

A common way of obtaining the real part of an expression is simply by adding the complex conjugate and dividing the result by 2:

$$\text{Re}\{z\} = \frac{1}{2}\left(z + z^*\right) = \frac{1}{2}(a + ib) + \frac{1}{2}(a - ib) = a \qquad (6.10)$$

The complex conjugate of $z = a + ib$, denoted with an asterisk, is $z^* = a - ib$. A number times its own complex conjugate is guaranteed to be real and non-negative:

$$z^* z = (a - ib)(a + ib) = a^2 - iab + iab + b^2 = a^2 + b^2 \qquad (6.11)$$

The complex conjugate is useful for eliminating complex numbers from the denominator of expressions. Multiply and divide any expression by the complex conjugate of the denominator:

$$\frac{a + ib}{c + id} = \frac{(a + ib)}{(c + id)}\frac{(c - id)}{(c - id)} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2} \qquad (6.12)$$

The absolute value (sometimes called modulus or magnitude) of a complex number is the square root of this product:

$$|z| \equiv \sqrt{z^* z} = \sqrt{a^2 + b^2} \qquad (6.13)$$

6.4   Euler's Equation

So how do complex numbers relate to trigonometric functions? The connection is embodied in Euler's formula:

$$e^{i\alpha} = \cos \alpha + i \sin \alpha \qquad (6.14)$$

Euler's formula can be proven using a Taylor's series expansion (about the origin):

$$f(x) = f(0) + \frac{x}{1!}\frac{df}{dx}\bigg|_{x=0} + \frac{x^2}{2!}\frac{d^2 f}{dx^2}\bigg|_{x=0} + \frac{x^3}{3!}\frac{d^3 f}{dx^3}\bigg|_{x=0} + \cdots \qquad (6.15)$$

Expanding each function appearing in (6.14) about the origin yields

$$\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots \qquad (6.16)$$

$$i\sin\alpha = i\alpha - i\frac{\alpha^3}{3!} + i\frac{\alpha^5}{5!} - \cdots \qquad (6.17)$$

$$e^{i\alpha} = 1 + i\alpha - \frac{\alpha^2}{2!} - i\frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + i\frac{\alpha^5}{5!} - \cdots \qquad (6.18)$$

One can quickly verify that (6.18) is the sum of (6.16) and (6.17), which completes the proof.

By inverting Euler's formula (6.14) (i.e. use algebra on the formula written with α and again with $-\alpha$), we obtain the following representation of the cosine and sine:

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \quad \text{and} \quad \sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \qquad (6.19)$$

Example 6.1

Using (6.19), prove (6.9).

Solution:

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{e^{i\alpha}+e^{-i\alpha}}{2}\frac{e^{i\beta}+e^{-i\beta}}{2} + \frac{e^{i\alpha}-e^{-i\alpha}}{2i}\frac{e^{i\beta}-e^{-i\beta}}{2i}$$
$$= \frac{e^{i(\alpha+\beta)}+e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}+e^{-i(\alpha+\beta)}}{4} - \frac{e^{i(\alpha+\beta)}-e^{i(\alpha-\beta)}-e^{-i(\alpha-\beta)}+e^{-i(\alpha+\beta)}}{4}$$
$$= \frac{e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}}{2} = \cos(\alpha-\beta)$$

6.5 Polar Format of Complex Numbers

With the aid of Euler's formula, it is possible to transform any complex number $z = a + ib$ into the form $\rho e^{i\phi}$ where a, b, ρ, and ϕ are all real. From (6.19), we see that the required connection between (ρ, ϕ) and (a, b) is

$$\rho e^{i\phi} = \rho\cos\phi + i\rho\sin\phi = a + ib \qquad (6.20)$$

The real and imaginary parts of this equation must separately be equal, which forces $a = \rho\cos\phi$ and $b = \rho\sin\phi$. These two equations can be solved together to yield

$$\rho = \sqrt{a^2 + b^2} = |z| \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{b}{a}\right) \quad (a > 0) \qquad (6.21)$$

Note that ρ is just the absolute value of a complex number as defined in (6.13).

The transformations (6.21) have a clear geometrical interpretation in the complex plane, and this makes them easier to remember. They are just the usual connections between Cartesian and polar coordinates. As seen in Fig. 6.3, ρ is the hypotenuse of a right triangle having legs with lengths a and b, and ϕ is the angle that the hypotenuse makes with the x-axis.

Example 6.2

Write $-3 + 4i$ in polar format.

Solution: We must be careful with the negative real part since it indicates quadrant III, which is outside of the domain of the inverse tangent (quadrants I and IV). Best to factor the negative out and deal with it separately.[3]

$$-3 + 4i = -(3 - 4i) = -\sqrt{3^2 + (-4)^2}\, e^{i\tan^{-1}\frac{(-4)}{3}} = e^{i\pi} 5 e^{-i\tan^{-1}\frac{4}{3}} = 5 e^{i2.214}$$

Here, we used the fact that $e^{i\pi} = \cos\pi + i\sin\pi = -1$.

Figure 6.3 A number in the complex plane can be represented either by Cartesian or polar representation.

The phase ϕ in $e^{i\phi}$ is assumed to be expressed in radians (rather than degrees) unless otherwise specified, which is uncommon.

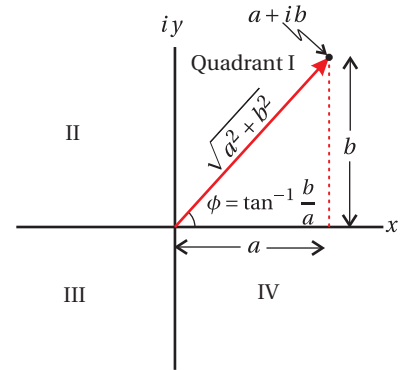## 6.6 Real vs Complex AC Quantities

Here are a couple of tricks worth mentioning.

Hiding the Phase in a Complex Amplitude
An AC quantity like current can either be written as a real function, $I_0 \cos(\omega t - \phi_I)$, or as the real part of a complex function, $\tilde{I}_0 e^{i\omega t}$, where the phase ϕ_I is conveniently hidden in the complex factor $\tilde{I}_0 \equiv I_0 e^{-i\phi_I}$, where I_0 is real. These two expressions can be reconciled using Euler's formula (6.14), showing that

$$I_0 \cos(\omega t - \phi_I) = \text{Re}\left\{ \tilde{I}_0 e^{i\omega t} \right\} \tag{6.22}$$

Fast and Loose with Re{ }
As a reminder, the operation Re{} retains only the real part of the argument without regard for the imaginary part. It is common (even conventional) in physics to omit the explicit writing of Re{}. Thus, physicists participate in a secret conspiracy (don't let the mathematicians find out) where $\tilde{I}_0 e^{i\omega t}$ actually means $I_0 \cos(\omega t - \phi_I)$. This laziness is permissible because it is possible to perform linear operations such as addition, differentiation, or integration while procrastinating the taking of the real part until the end. That is

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[3] We will not face this situation in LRC circuits because resistance is non negative.

$$\text{Re}\{f\} + \text{Re}\{g\} = \text{Re}\{f + g\}$$
$$d\text{Re}\{f\}/dx = \text{Re}\{df/dx\}$$
$$\int \text{Re}\{f\}\, dx = \text{Re}\left\{\int f\, dx\right\} \tag{6.23}$$

As an example, note that $\text{Re}\{1 + 2i\} + \text{Re}\{3 + 4i\} = \text{Re}\{(1 + 2i) + (3 + 4i)\} = 4$.

You must be careful, however, when multiplying complex quantities. When calculating the power (P) dissipated in a circuit, which is voltage times current, it is essential to take the real parts of both voltage and current *before* multiplying. That is

$$P(t) = \text{Re}\left\{V_0 e^{i\omega t}\right\} \times \text{Re}\left\{\tilde{I}_0 e^{i\omega t}\right\} \neq \text{Re}\left\{V_0 e^{i\omega t} \times \tilde{I}_0 e^{i\omega t}\right\} \tag{6.24}$$

As an example, $\text{Re}\{1 + 2i\} \times \text{Re}\{3 + 4i\} = 3$, but $\text{Re}\{(1 + 2i)(3 + 4i)\} = -5$.

6.7 LRC Circuits in Complex Notation

We return to the AC driven circuit considered previously in section 6.2. We write the applied voltage (6.7) as

$$V(t) = V_0 e^{i\omega t} \tag{6.25}$$

In keeping with our conspiracy, we don't bother writing explicitly $\text{Re}\{\}$. Secretly, we really mean $\text{Re}\{V(t)\}$, which according to (6.14) is just $V_0 \cos\omega t$.

We expect the current flowing in the circuit to oscillate at the same frequency ω, so we write

$$I(t) = \tilde{I}_0 e^{i\omega t} \tag{6.26}$$

In this case, $\tilde{I}_0$ must be regarded as a complex number to account for the fact that it can oscillate with a phase different from the applied voltage. As before, taking the real part is implied, so when you look at (6.26), you should be thinking (6.22).

When we plug the above voltage and current into our series AC circuit equation (6.6), $V = RI + \frac{1}{C}\int I\, dt + L\frac{dI}{dt}$, we get

$$V_0 e^{i\omega t} = R\tilde{I}_0 e^{i\omega t} + \frac{1}{C}\int \tilde{I}_0 e^{i\omega t}\, dt + L\frac{d}{dt}\tilde{I}_0 e^{i\omega t} = \tilde{I}_0 e^{i\omega t}\left(R + \frac{1}{i\omega C} + i\omega L\right) \tag{6.27}$$

Therefore,

$$\tilde{I}_0 = \frac{V_0}{R + \frac{1}{i\omega C} + i\omega L} \tag{6.28}$$

We are essentially done. We have the solution for the current. The only thing left to do is to rewrite the complex number in the more convenient polar format before taking its real part. (Only the real part is physical.)
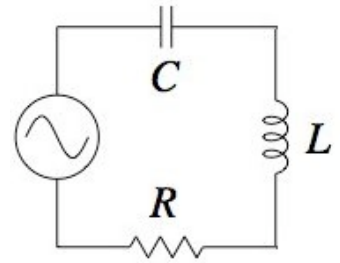

Figure 6.4 Driven series LRC circuit.

> **Example 6.3**
>
> Find the amplitude and phase of the current in a series LRC circuit driven with AC voltage having peak amplitude 10 V and frequency $f = 60$Hz, if $R = 50\Omega$, $C = 60\mu$F, and $L = 40$mH.
>
> **Solution:** The angular frequency is $\omega = 2\pi f = 2\pi 60\text{s}^{-1} = 377\text{s}^{-1}$. The complex current amplitude (6.27) is
>
> $$\tilde{I}_0 = \frac{10 \text{ V}}{50\Omega + \frac{1}{i(377\text{s}^{-1})(6\times 10^{-5}\text{F})} + i(377\text{s}^{-1})(4 \times 10^{-2}\text{H})} = \frac{10\text{V}}{(50 - i29.1)\,\Omega} = 0.173\text{Ae}^{+i0.527}$$
>
> The overall current is then
>
> $$I(t) = \text{Re}\{\tilde{I}_0 e^{i\omega t}\} = I_0\cos(\omega t - \phi) = 173 \text{ mA } \cos\left[(377\text{s}^{-1})t - (-0.527)\right]$$
>
> Because ϕ is negative, the current *leads* the voltage by $(0.527 \text{ rad})(180°/(\pi \text{ rad})) = 30.2°$.

6.8 Equipment

No equipment needed.

Appendix 6.A Solving an LRC Circuit the Hard Way: with Sine and Cosine

If you read this appendix, you are a real nerd, but here goes. We solve the series LRC circuit (6.6), $V = RI + \frac{1}{C}\int I dt + L\frac{dI}{dt}$, without complex notation. As usual, we assume the circuit is driven with voltage $V = V_0\cos\omega t$. Since we expect the current to oscillate with the same frequency but at a different phase, we need to allow the solution to include both cosine and a sine:

$$I(t) = A\cos\omega t + B\sin\omega t \tag{6.29}$$

We need to plug (6.29) into (6.6) to see 1) if it works, and, if it does, 2) what A and B should be. Plugging in, we get

$$V = R(A\cos\omega t + B\sin\omega t) + \frac{1}{C}\int (A\cos\omega t + B\sin\omega t)\,dt + L\frac{d}{dt}(A\cos\omega t + B\sin\omega t)$$

or

$$V_0\cos\omega t = R(A\cos\omega t + B\sin\omega t) + \frac{1}{\omega C}(A\sin\omega t - B\cos\omega t) + \omega L(-A\sin\omega t + B\cos\omega t)$$

For this equation to be true, we need the coefficients on the cosine and the sine to separately balance. The requirement is

$$V_0 = RA - B\frac{1}{\omega C} + \omega LB \quad \text{and} \quad RB + \frac{A}{\omega C} - A\omega L = 0 \tag{6.30}$$

Now for a little algebra on these two equations. From the latter equation, we have $B = \frac{A}{R}\left(\omega L - \frac{1}{\omega C}\right)$, which we plug into the former equation to find

$$A = \frac{V_0 R}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{6.31}$$

and, after back substituting,

$$B = \frac{V_0 \left(\omega L - \frac{1}{\omega C}\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{6.32}$$

We can put our trial solution (6.29) into the form (6.8) by factoring out $\sqrt{A^2 + B^2}$ as follows:

$$I(t) = \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}}\cos\omega t + \frac{B}{\sqrt{A^2 + B^2}}\sin\omega t\right) \tag{6.33}$$

Referring to Fig. 6.5, we can think of $A/\sqrt{A^2 + B^2}$ and $B/\sqrt{A^2 + B^2}$ as the cosine and sine of an angle

$$\phi_I = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \tag{6.34}$$

Then we may write

$$I(t) = \sqrt{A^2 + B^2}\left(\cos\phi_I \cos\omega t + \sin\phi_I \sin\omega t\right) = \sqrt{A^2 + B^2}\cos\left(\omega t - \phi_I\right) \tag{6.35}$$

where we have used the angle-addition formula for cosine. To make this match the form (6.8), $I(t) = I_0 \cos\left(\omega t - \phi_I\right)$, we assign

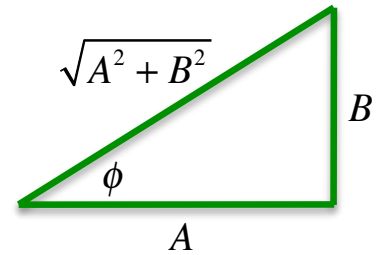$$I_0 = \sqrt{A^2 + B^2} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{6.36}$$

Figure 6.5 Right triangle with legs A and B.

Exercises

A. Complex Arithmetic

L6.1   Compute
(a) $(-1 + 2.7\,i) + (13.9 - 0.5\,i)$
(b) $\left(\sqrt{3} + 8\,i\right) + \left(-2\sqrt{3} - 7\,i\right)$
(c) $\frac{z + z^*}{2}$, where $z = 2 - 3i$
Example: $(5 + 3\,i) + (-2 + 7\,i) = 3 + 10\,i$

L6.2   Compute
(a) $i(-1 + i)$
(b) $(3 + 4\,i)\,(3 - 4i)\,/25$
(c) $1 + i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7$
Example: $(2 + 4\,i)\,(3 - 2\,i) = 6 - 4i + 12\,i + 8 = 14 + 8i$

L6.3   Compute
(a) $1/i$
(b) $2/(1 - i)$
(c) $(5.2 - 4.8\,i)\,/\,(0.6 + 0.8\,i)$
Example: $\frac{(5 + 10\,i)}{(1 - 2\,i)} = \frac{(5 + 10\,i)}{(1 - 2\,i)}\frac{(1 + 2\,i)}{(1 + 2\,i)} = \frac{-15 + 20\,i}{5} = -3 + 4\,i$

L6.4   Compute
(a) $|i|$
(b) $|4 + 3\,i|$
(c) $|\cos\alpha + i\,\sin\alpha|$
Example: $|3 + 4\,i| = \sqrt{3^2 + 4^2} = 5$

L6.5   Convert to polar format:
(a) $-i$
(b) $5.71 - \sqrt{3}\,i$
(c) $-3 - 2\,i$
Example: $3 - 3\,i = \sqrt{3^2 + (-3)^2}\,e^{i\tan^{-1}(-3/3)} = 3\sqrt{2}\,e^{-i\pi/4}$

L6.6   Convert to standard form:
(a) $2\,e^{i\pi/6}$
(b) $8.9\,e^{-100\,i}$ (Be careful: 100 expresses radians – not degrees.)
(c) $e^{e^{i\pi/2}}$
Example: $4.5\,e^{-1.2\,i} = 4.5\,[\cos(-1.2) + i\,\sin(-1.2)] = 1.63 - 4.19\,i$

B. Complex Functions

L6.7   Which is not true?
(a) The real part of the sum of two functions is the same as the sum of the real parts of the functions.
(b) The real part of the derivative of a function is the same as the derivative of the real part.

(c) The real part of the integral of a function is the same as the integral of the real part.

(d) The real part of the product of two functions is the same as the product of the real parts of the functions.

L6.8   Recast $u = 1 + i/\omega$ into the form $u = |u|e^{i\phi}$, assuming ω is a real. Plot both $|u|$ and ϕ as functions of ω over the range -3 to 3. You may use a computer to graph the curves or draw them by hand.

L6.9   Recast $v = 2e^{i\omega}$, where ω is real, into standard form $\mathrm{Re}\{v\} + i\,\mathrm{Im}\{v\}$. Plot both $\mathrm{Re}\{v\}$ and $\mathrm{Im}\{v\}$ as functions of ω over the range $-\pi$ to π. You may use a computer to graph the curves or draw them by hand.

L6.10   Following the method used in example 6.1, prove
$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

L6.11   Following the method used in example 6.1, prove
$\cos^2\alpha + \sin^2\alpha = 1$

C. Series AC Circuits

L6.12   Show that $1/(\omega C)$ and ωL both have units of ohms (Ω). (See section 6.1.)

L6.13   The LRC circuit in Example 6.3 ($R = 50\Omega$, $C = 60\mu F$, $L = 40$ mH) was driven at $f = 60$ Hz. Plot voltage and the current as functions of time on the same graph over a period of two oscillations. Don't worry about amplitudes and units on the vertical axis; but be sure to accurately portray the relative phase.

L6.14   Repeat the previous exercise for $f = 600$ Hz. You will need to recompute the phase of the current at this frequency.

L6.15   At what frequency will the voltage and current in Example 6.3 be in phase? Hint: Set $\phi = 0$ and solve for f.