21-259: Calculus in Three Dimensions Review Sheet

Jemmin Chang (jemmin.chang@cmu.edu)

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1 Derivatives and Integrals

1.1 Trigonometric Identities

- Inverses:

  $\csc x = \frac{1}{\sin x}$

  $\sec x = \frac{1}{\cos x}$

  $\cot x = \frac{1}{\tan x}$

- Pythagorean: $\sin^2 x + \cos^2 x = 1$
- Quotient: $\tan x = \frac{\sin x}{\cos x}$

- Double angle:

  $\sin(2x) = 2\sin x \cos x$

  $\cos(2x) = \cos^2 x - \sin^2 x$

All the useful identities can be derived easily from these. See Full Table of Trig Identities.

1.2 Derivatives

You should have this Table of Derivatives memorized, except hyperbolics.

- Product rule: $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + g'(x)f(x)$
- Quotient rule: $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$
- Chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$
- Implicit differentiation

1.3 Integrals

See Irina's Integration Practice and Solutions (Andrew login required).

- u-substitution (problem 1 above)
- Trig identity substitution (problem 3)
- Integration by parts (problems 2, 4)

  Can be applied multiple times. Note the "two-sided" technique in 4.
- Integration by partial fractions (problems 5, 6)
- Trig substitution (rare, but good to know)

2 Vectors and the Geometry of Space (Stewart's Chapter 12)

2.1 Vectors

- **Unit vector** in the direction of $\vec{v}$ is $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$
- **Orthogonal** means **perpendicular**

2.1.1 Dot Product

- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$
- Scalar projection of $\vec{b}$ onto $\vec{a}$: $comp_{\vec{a}}(b) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$
- Vector projection of $\vec{b}$ onto $\vec{a}$: $proj_{\vec{a}}(b) = comp_{\vec{a}}(b)\frac{\vec{a}}{|\vec{a}|}$

2.1.2 Cross Product

- $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$
- $\vec{a}$ and $\vec{b}$ are parallel iff $\vec{a} \times \vec{b} = \vec{0}$
- The area of the parallelogram determined by $\vec{a}$ and $\vec{b}$ is $A = |\vec{a} \times \vec{b}|$

2.1.3 Triple Product

- The volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$ is $V = \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

2.2 Lines

- **Parallel** lines have proportional direction vectors
- **Skew** lines neither intersect nor are parallel

  Check for intersection by setting the parametric equations equal and solving the system

2.2.1 Equations of Lines

- Vector equation: $\vec{r} = \vec{r_0} + t\vec{v}$, where r_0 is the vector from the origin to any point on the line, and v is the direction vector of the line
- Parametric equations: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

  a, b, c are the direction numbers of the line; i.e. $\vec{v} = \langle a, b, c \rangle$
- Symmetric equations: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$
- Line segment from $\vec{r_0}$ to $\vec{r_1}$: $\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$, $0 \le t \le 1$

2.3 Planes

- A plane is defined by a point $P(x_0, y_0, z_0)$ and a normal vector $\vec{n} = \langle a, b, c \rangle$
- **Parallel** planes have parallel (proportional) normal vectors

2.3.1 Equations of Planes

- Scalar equation: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
- Linear equation: $ax + by + cz + d = 0$

2.3.2 Distance between a Point and a Plane

This distance is equal to the scalar projection of a vector $\vec{b}$ from a point P_0 on the plane to the given point $P_1(x_1, y_1, z_1)$ onto the plane's normal vector $\vec{n} = \langle a, b, c \rangle$. So

- $D = |comp_{\vec{n}}\vec{b}| = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

2.4 Quadric Surfaces

2.4.1 Equations of Quadric Surfaces

Note: Everywhere we have x, y, z in these formulae we can replace with $x - h, y - k, z - p$ to shift the center of the surface from the origin to (h, k, p). Of course, we can also permute x, y, z to get orientations around different axes.

See Stewart's 7E page 830 for a chart with pictures.

- Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- Cone: $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

- Elliptic Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
- Hyperboloid of One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Hyperbolic Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
- Hyperboloid of Two Sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

3 Vector Functions (Stewart's Chapter 13)

3.1 Vector Functions and Space Curves

- A vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is **continuous at** a if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

- The set of points $(f(t), g(t), h(t))$ on an interval of the **parameter** t is a **space curve**
  $x = f(t), y = g(t), z = h(t)$ are the **parametric equations** of this space curve

3.2 Derivatives of Vector Functions

Let C be the curve defined by $\vec{r}$. Then

- $\vec{r'}(t)$ is the **tangent vector** to C
- The **tangent line** to C at a point P is the line through P and parallel to $\vec{r'}(t)$
- The **unit tangent vector** is $\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$
- The **unit normal vector** is $\vec{N}(t) = \frac{\vec{T'}(t)}{|\vec{T'}(t)|}$
- The **binormal vector** is $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

3.3 Arc Length

- The length of the space curve defined by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ on the interval $a \leq t \leq b$ is

$$L = \int_a^b |\vec{r'}(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

3.3.1 The Arc Length Function

- The arc length function of $\vec{r}(t)$ on the interval $a \leq t \leq b$ is

$$s(t) = \int_a^t |\vec{r'}(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

- Differentiating gives $\frac{ds}{dt} = |\vec{r'}(t)|$

3.3.2 Reparametrizing wrt Arc Length

- Use the arc length function to solve for t in terms of s. Then substitute $t(s)$ in for t in $\vec{r}(t)$.

3.4 Curvature

- A parametrization of a curve $\vec{r}(t)$ on an interval I is **smooth** if $\vec{r'}$ is continuous and $\vec{r'}(t) \neq 0$ on I.
- The **curvature** of a curve is

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T'}(t)|}{|\vec{r'}(t)|} = \frac{|\vec{r'}(t) \times \vec{r''}(t)|}{|\vec{r'}(t)|^3}$$

- For a plane curve $y = f(x)$, $\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$

4 Multivariable Functions (Stewart's Chapter 14)

4.1 Level Curves and Surfaces

- The **level curves** of a function $f(x, y)$ are the curves $f(x, y) = k$ for all constant k in the image of f (think contour map)
- The **level surfaces** of a function $f(x, y, z)$ are the curves $f(x, y, z) = k$ for all constant k in the image of f

4.2 Limits and Continuity

- Technical definition: Let D be the domain of a function f which includes points arbitrarily close to (a, b). Then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

iff

$$\forall(\epsilon > 0).\exists(\delta > 0).((x, y) \in D \wedge 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta) \implies |f(x, y) - L| < \epsilon$$

- A function is **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

4.2.1 Techniques for Finding and Proving Limits

See Irina's Limits Handout for examples of using these techniques (requires Andrew login).

- Direct substitution (when f is a rational function)
- Fancy algebra (factoring, cancelling, etc.)
- L'Hopital's rule (when direct substitution yields $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$)
- Squeeze theorem (bound the function on both sides, and show that the limits of these bounds are equal)
- Apply the definition

4.2.2 The Limit Does Not Exist!

Show that approaching from two different paths gives different limits.

4.3 Partial Derivatives

- f is **differentiable** at (a, b) if f_x and f_y exist and are continuous near (a, b)
- Clairaut's theorem: if f is defined on a disk D containing (a, b) and f_{xy} and f_{yx} are continuous on D, $f_{xy}(a, b) = f_{yx}(a, b)$

4.4 Tangent Planes

4.4.1 Finding an Equation of a Tangent Plane

- If f has continuous partial derivatives, then an equation of the **tangent plane** to the surface $z = f(x, y)$ at $P(x_0, y_0, z_0)$ is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

4.4.2 Linear Approximations

- The **linearization** of f at (a, b) is $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
- A **linear approximation** of f near (a, b) is $f(x, y) \approx L(x, y)$
- The **differential** dz is $dz = f_x(x, y)\, dx + f_y(x, y)\, dy$ (think error in z given error in x and y)

4.5 Chain Rule

- If $z = f(x(t), y(t))$, then $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$
- Implicit Function Theorem: under the logical conditions, when we have an equation $F(x, y) = 0$,
$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$
- Or if $F(x, y, z) = 0$, then
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

4.6 Directional Derivatives

- If $f(x, y)$ is differentiable, then the **directional derivative** of f in the direction of a unit vector $\vec{u} = \langle a, b\rangle$ is $D_u f(x, y) = f_x(x, y)a + f_y(x, y)b = \nabla f \cdot \vec{u}$

4.7 The Gradient Vector

- The **gradient** of $f(x, y, (z))$ is $\nabla f = \langle f_x, f_y, (f_z)\rangle$
- The direction of ∇f is the direction of fastest change in f

4.8 Local and Absolute Extrema

- A point (a, b) is a **critical point** of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$
- A local extremum must be a critical point, but not all critical points are extrema

4.8.1 The Second Derivatives Test

This test determines whether a critical point is a local extremum. If f has continuous second partial derivatives and (a, b) is a critical point of f, let

$$D = f_{xx}f_{yy} - f_{xy}^2$$

- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a **local minimum** point
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a **local maximum** point
- If $D(a, b) < 0$, then (a, b) is a **saddle point**
- When $D(a, b) = 0$, the test is inconclusive

4.8.2 Absolute Extrema

To find the absolute extrema of f on a closed set D, calculate the values of f at its critical points and along the boundaries of D (use the derivative and number line technique for finding extrema along boundary curves).

- The **absolute maximum** is the largest of these values
- The **absolute minimum** is the smallest of these values

4.9 Lagrange Multipliers

The Lagrange method maximizes a function $f(x, y, z)$ subject to one or two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$.

1. Set $\nabla f = \lambda\nabla g + \mu\nabla h$, yielding the system of equations:
   $f_x = \lambda g_x(+\mu h_x)$
   $f_y = \lambda g_y(+\mu h_y)$
   $f_z = \lambda g_z(+\mu h_z)$
   $g(x, y, z) = k$
   $(h(x, y, z) = c)$
2. Find the values of f at all points (x, y, z) which satisfy this system
3. The maximum of these is the absolute maximum value and the minimum of these is the absolute minimum value of f subject to the constraint(s)

5 Multiple Integrals (Stewart's Chapter 15)

Note: Remember that axes can be permuted, and the best order of integrals to take may change.

5.1 Double Integrals

Procedure:

1. Choose the most convenient coordinate system for the bounds
   - Rectangular (normal)

- Polar:

  $x = r\cos\theta$
  $y = r\sin\theta$

2. Evaluate the iterated integral $\iint_D f(x,y)\,dA$

   - Rectangular: $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$
   - Polar: $\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

5.1.1 Surface Area

The area of the surface $z = f(x,y)$, $(x,y) \in D$, where f_x and f_y are continuous, is

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

5.2 Triple Integrals

Procedure:

1. Choose the most convenient coordinate system for the bounds

   - Rectangular
   - Cylindrical: as polar above, plus $z = z$
   - Spherical:

     $z = \rho\cos\phi$
     $r = \rho\sin\phi$
     $\rho^2 = x^2 + y^2 + z^2$

2. Evaluate the iterated integral $\iiint_E f(x,y,z)\,dV$

   - Rectangular: $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$
   - Cylindrical: $\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$
   - Spherical: $\int_c^d \int_\alpha^\beta \int_a^b f(x,y,z)\,\rho^2\sin\phi\ d\rho\,d\theta\,d\phi$

6 Vector Calculus (Stewart's Chapter 16)

See Stewart's 7E page 1135 for a one page chart (with pictures) summary of how the main theorems of this chapter relate.

6.1 Vector Fields

- A **vector field** is a function $\vec{F}$ that assigns to each point (x,y,z) a vector $\vec{F}(x,y,z)$
- We can write a vector field in terms of its **component scalar functions**: $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

6.2 Conservative Fields

- A vector field $\vec{F}$ is **conservative** if it is the gradient of some scalar function f; i.e. $\vec{F} = \nabla f$
  f is called the **potential function** for $\vec{F}$

6.2.1 The Fundamental Theorem for Line Integrals

Let C be a smooth curve defined by $\vec{r}(t)$ on the interval $a \leq t \leq b$. If f is a differentiable function with a continuous gradient vector on C, then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

So the line integral of a conservative vector field over a curve C is simply the net change in the potential function between the endpoints of C; i.e. the integral is **independent of path**.

6.2.2 Determining Conservatism

- Suppose $\vec{F} = P\vec{i} + Q\vec{j}$ is a vector field on an open, simply-connected region D, and P and Q have continuous first-order derivatives. If
  $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
  throughout D, then $\vec{F}$ is conservative
- If $\vec{F}$ is conservative, then curl $\vec{F} = \vec{0}$. Further, if $\vec{F}$ is defined on all of $\mathbb{R}^3$ and its component functions have continuous partial derivatives, then the converse is true

6.2.3 Finding the Potential Function

Because $\vec{F} = \nabla f$, we have $f_x = P$, $f_y = Q$, and $f_z = R$. We use partial integration and differentiation, comparing with P, Q, and R, to solve for the function $f(x, y, z)$.

6.3 Line Integrals

6.3.1 Evaluating Line Integrals over Plane Curves

- For continuous f, the line integral (wrt arc length) over the curve C on the interval $a \leq t \leq b$ is
  $$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$
- With respect to x, $\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$
- With respect to y, $\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))\,y'(t)\,dt$

6.3.2 Evaluating Line Integrals over Space Curves

- For continuous f, the line integral (wrt arc length) over the curve C on the interval $a \leq t \leq b$ is
  $$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$
- With respect to x, $\int_C f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))\,x'(t)\,dt$

- With respect to y, $\int_C f(x, y, z)\, dy = \int_a^b f(x(t), y(t), z(t))\, y'(t)\, dt$
- With respect to z, $\int_C f(x, y, z)\, dz = \int_a^b f(x(t), y(t), z(t))\, z'(t)\, dt$

6.3.3 Line Integrals of Vector Fields

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a continuous vector field on the curve C defined by the vector $\vec{r}(t)$ on the interval $a \leq t \leq b$, the line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\, dx + Q\, dy + R\, dz$$

6.4 Green's Theorem

- The **positive orientation** of a simple, closed curve is a single *counterclockwise* traversal of the curve
- Let C be a positively-oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on D, then

$$\int_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

6.4.1 Vector Forms of Green's Theorem

- $\int_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \vec{k}\, dA$
- $\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \text{div } \vec{F}\, dA$ where $\vec{n}$ is the outward unit normal vector to C

6.5 Curl

Recall the definition $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$. Then the curl of a vector field $\vec{F}$ is

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

6.6 Divergence

The divergence of a vector field $\vec{F}$ is

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

assuming these partial derivatives exist. Note that div $\vec{F}$ is a **scalar field**.

- If P, Q, and R have continuous second-order partial derivatives, then div curl $\vec{F} = 0$

  It follows that if div $\vec{F} \neq 0$, $\vec{F}$ cannot be written as the curl of another vector field

6.7 Parametric Surfaces

- The set of points (x, y, z) traced out by $\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$ as (u, v) varies throughout a region D is a **parametric surface**
- Holding u or v constant gives the **grid curves** of $\vec{r}(u, v)$

6.7.1 Finding Equations of Parametric Surfaces

1. Choose a coordinate system where one of the variables is constant or can be written as a function of the other two

   Rectangular: e.g. when given $z = f(x, y)$

   Polar: e.g. when z can be written $z = f(r, \theta)$

   Spherical: e.g. when constant radius $\rho = a$

2. Write the vector function $\vec{r}(u, v)$ in terms of the two variables

6.7.2 Surfaces of Revolution

Example: given a function of the form $y = f(x)$, find parametric equations for the surface generated by rotating $y = f(x)$ about the x-axis. Take θ as the second parameter, and write $x = x$, $y = f(x)\cos\theta$, $z = f(x)\sin\theta$.

6.7.3 Tangent Planes to Parametric Surfaces

To find the tangent plane to a surface $\vec{r}(u, v)$ at $P(x, y, z)$,

1. Find the tangent vectors $\vec{r_u}$ and $\vec{r_v}$
2. Compute the normal vector $\vec{r_u} \times \vec{r_v}$
3. Find the point (u_0, v_0) which corresponds to (x, y, z)
4. Plug and chug into the scalar plane equation

6.7.4 Surface Area

If S is a **smooth** parametric surface ($\vec{r_u} \times \vec{r_v} \neq \vec{0}$) and is covered only once as (u, v) ranges throughout D, then the **surface area** of S is

$$A = \iint_D |\vec{r_u} \times \vec{r_v}|\, dA$$

6.8 Surface Integrals

The **surface integral** of f over a parametric surface S is

$$\iint_D f(\vec{r}(u, v))\, |\vec{r_u} \times \vec{r_v}|\, dA$$

where D is the region over which (u, v) ranges. **Note: we DON'T need to add the r or $\rho^2 \sin\phi$ when using polar or spherical coordinates in this case!**

For graphs, i.e. functions of the form $f(x, y, g(x, y))$ modulo axis permutation, this becomes

$$\iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

6.8.1 Surface Integrals of Vector Fields

If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S (the **flux** of $\vec{F}$ across S) is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\; dS = \iint_D \vec{F} \cdot (\vec{r_u} \times \vec{r_v})\, dA$$

6.9 Stokes' Theorem

Let S be an oriented, piecewise-smooth surface bounded by a simple, closed, piecewise-smooth curve C with positive orientation. Let $\vec{F}$ be a vector whose components have continuous partial derivatives on S. Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

6.10 Divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E with outward orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on E. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} \, dV$$

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