2. Physics of Space Plasma

2.2 SINGLE-PARTICLE MOTION

$$\mathbf{F}_{\mathrm{L}} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \qquad \text{(Lorentz-force law)} \tag{2.1}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} + \mathbf{F}_g \tag{2.2}$$

Let Uniform B, E=0, B in the z direction

$$m\dot{v}_x = qv_y B; \qquad m\dot{v}_y = -qv_x B \tag{2.3}$$

By substitution,

$$\ddot{v}_j = -(qB/m)^2 v_j = -\Omega_c^2 v_j \quad \text{or} \quad \ddot{x}_j = -\Omega_c^2 x_j \tag{2.4}$$

TABLE 2.1. Maxwell's Equations in Different Systems of Units

SI Units	Gaussian Units	
$\nabla \cdot \mathbf{D} = \rho_q$	$\nabla \cdot \mathbf{D} = 4\pi\rho_q$	Poisson's equation
$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$	$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	Ampère's law
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$	Divergenceless **B**
$\mathbf{F} = q[\mathbf{E} + \mathbf{u} \times \mathbf{B}]$	$\mathbf{F} = q[\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B}]$	Lorentz-force law

Gyro frequency, cyclotron frequency

$$\Omega_c = qB/m \tag{2.5}$$

Gyro radius, cyclotron radius, Larmor radius

$$\rho_c = \frac{v_\perp}{\Omega_c} = \frac{mv_\perp}{qB} \tag{2.6}$$

$$m\,\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \frac{d(\frac{1}{2}mv^2)}{dt} = q\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}) = 0 \tag{2.7}$$

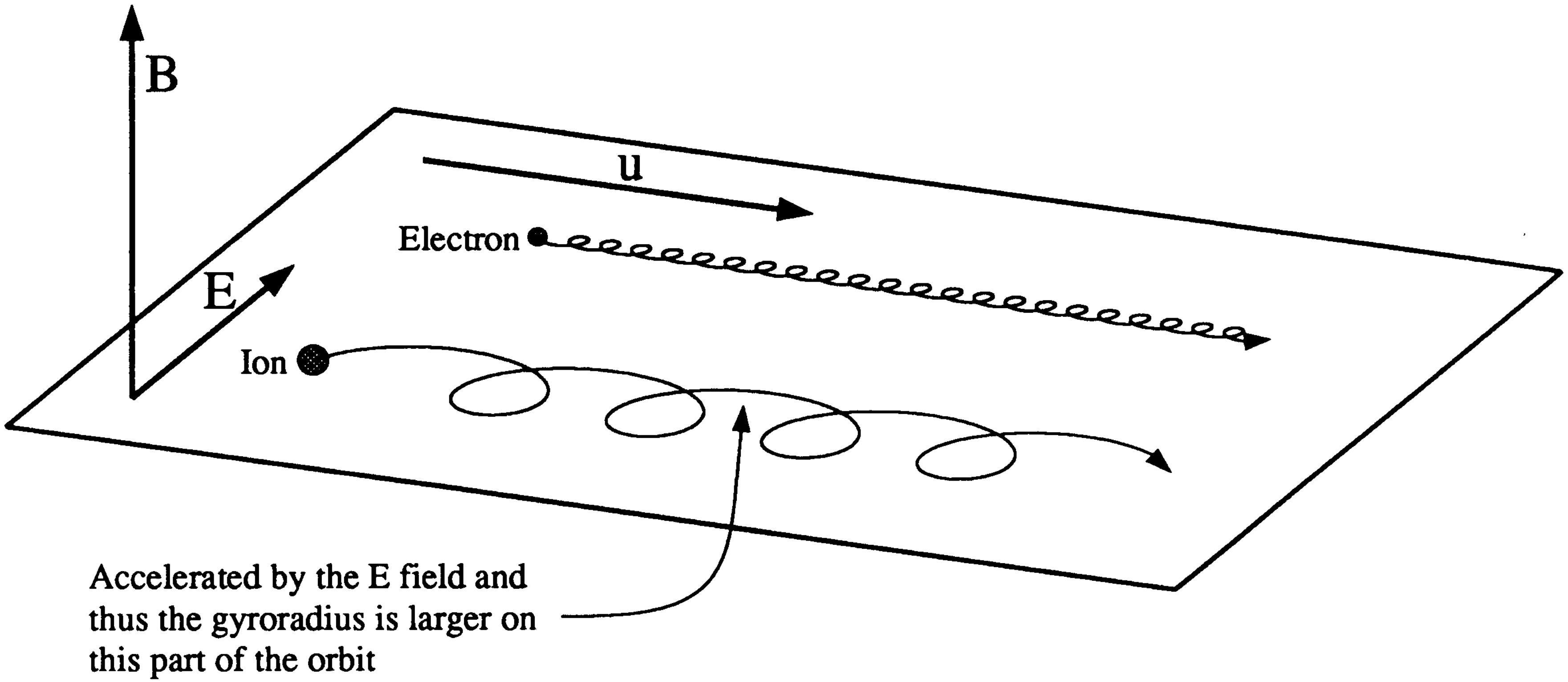


FIG. 2.1. Schematic showing the motions of ions (charge e) and electrons (charge $-e$) in a uniform magnetic field **B** in the presence of an electric field **E** perpendicular to **B**. The diagram represents motion in a plane perpendicular to the magnetic field. For both signs of the charge, the motion along the magnetic field is at constant velocity, unaffected by the presence of the fields.

ExB drift No current

$$\mathbf{u}_E = \mathbf{E} \times \mathbf{B}/B^2 \tag{2.8}$$

External force drift Current

$$\mathbf{u}_F = \mathbf{F} \times \mathbf{B}/qB^2 \tag{2.9}$$

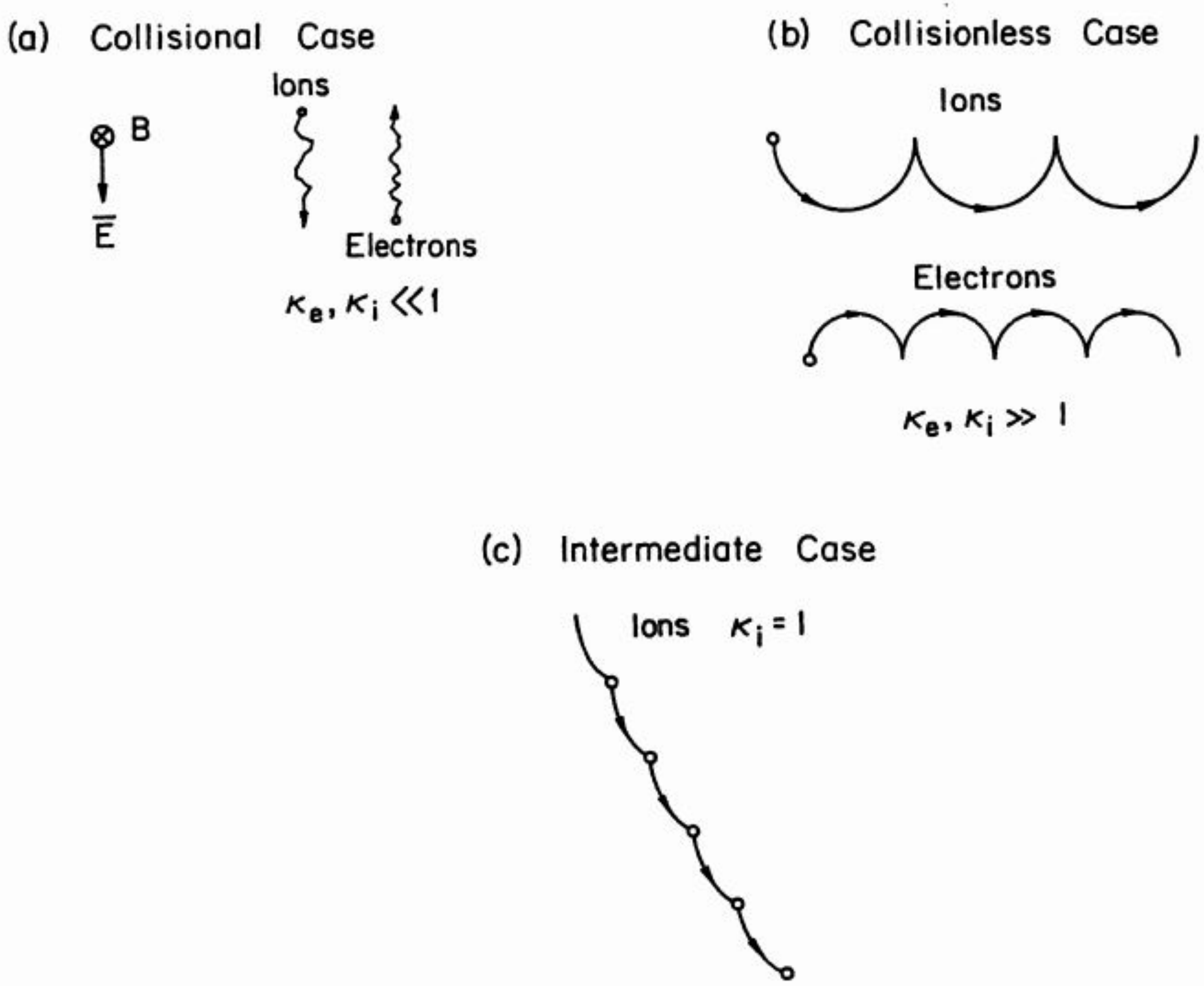


Fig. 2.4. Ion and electron trajectories for various values of κ.

Gradient B drift

$$\mathbf{u}_g = \tfrac{1}{2} m v_\perp^2 \mathbf{B} \times \nabla \mathbf{B} / q B^3 \tag{2.10}$$

$$\frac{\hat{\mathbf{n}}}{R_c} = -(\hat{\mathbf{b}} \cdot \nabla)\hat{\mathbf{b}} \tag{2.11}$$

where $\hat{\mathbf{b}} = \mathbf{B}/B$, and $\hat{\mathbf{n}}$ is a unit vector perpendicular to $\mathbf{B}$ that points away from the center of curvature, one can express the curvature drift velocity $\mathbf{u}_c$ as

$$\mathbf{u}_c = \frac{m v_\parallel^2 \mathbf{B} \times (\hat{\mathbf{b}} \cdot \nabla)\hat{\mathbf{b}}}{q B^2} = -\frac{m v_\parallel^2 \mathbf{B} \times \hat{\mathbf{n}}}{R_c \, q B^2} \tag{2.12}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \text{(Faraday's law)} \tag{2.13}$$

The first adiabatic invariant

μ is called a magnetic moment, $\boldsymbol{\mu} = \frac{1}{2}\mathbf{r} \times \mathbf{j}$

$$\mu = \frac{\frac{1}{2}mv_{\perp}^2}{B} \tag{2.14}$$

$$\mathbf{F} = \boldsymbol{\mu} \cdot \nabla B = -\mu \frac{dB}{dz} \tag{2.15}$$

The mirror point

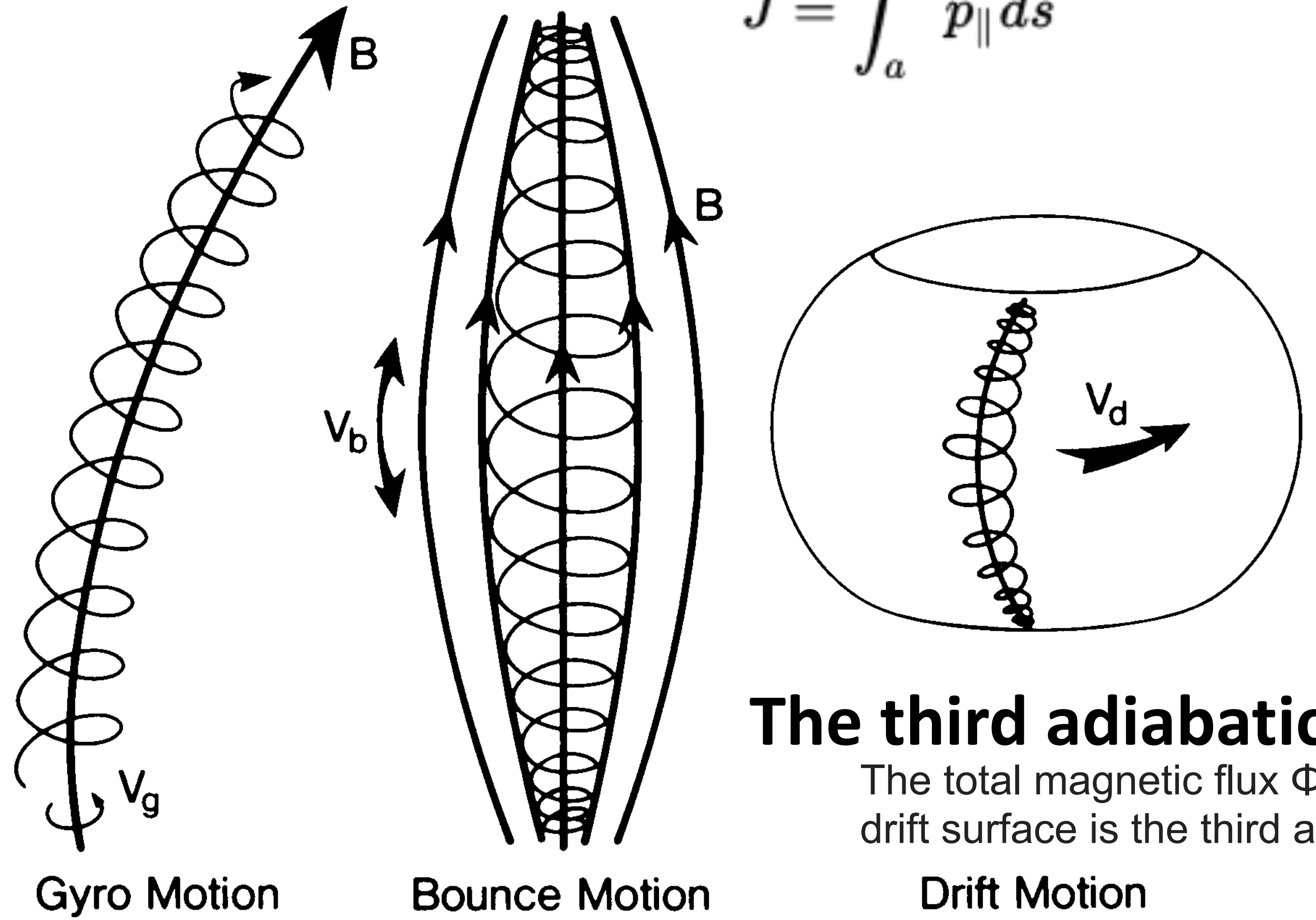
$$B = \tfrac{1}{2}mv^2\mu$$

The first adiabatic invariant

$$\mu = \frac{\frac{1}{2} m v_{\perp}^2}{B}$$

The second adiabatic invariant

$$J = \int_a^b p_{\parallel} ds$$



The third adiabatic invariant

The total magnetic flux Φ enclosed by a drift surface is the third adiabatic invariant

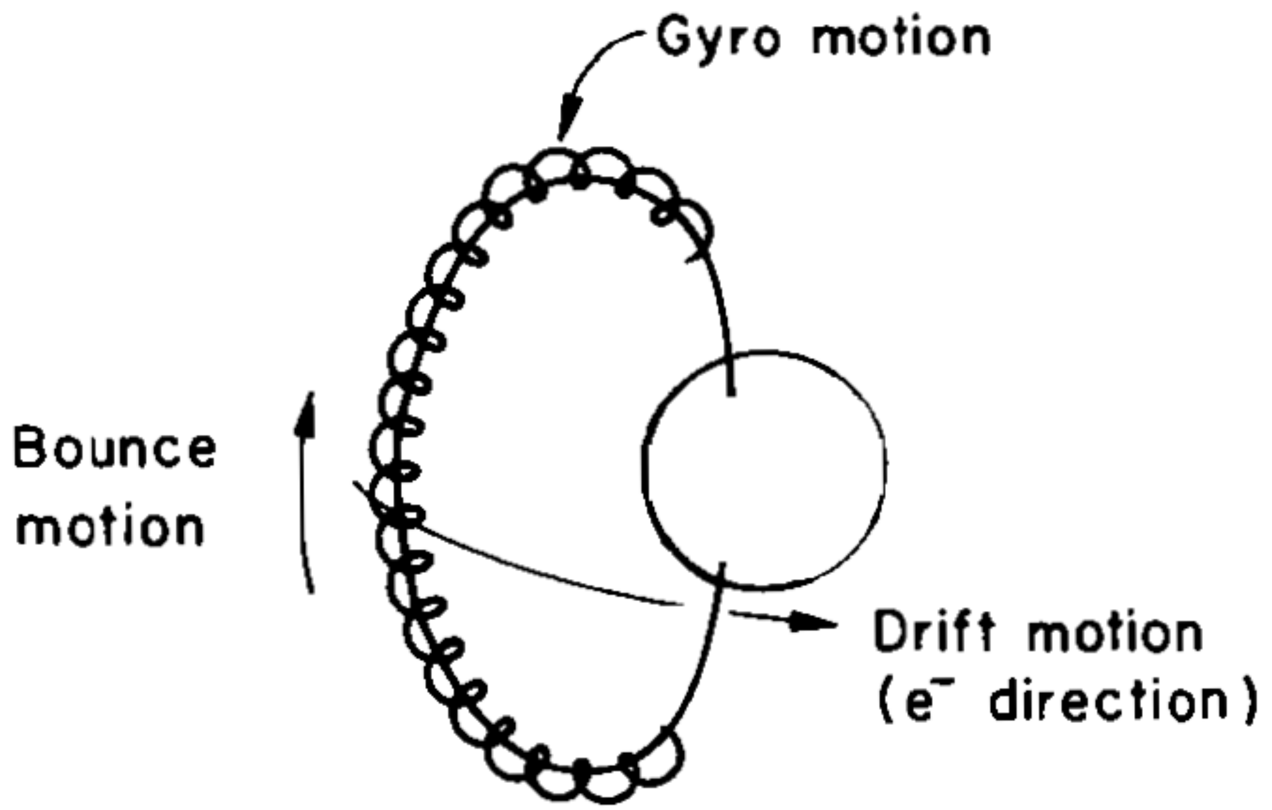


Fig. 2.11. The three oscillatory motions in the earth's magnetic field which are associated with the three adiabatic invariants.

2.3 COLLECTIONS OF PARTICLES

- Characteristics of plasma: Quasi neutrality, collective behavior, good conductor.
- Phase space: 3D displacement + 3D velocity
- The phase-space density function $f(\mathbf{r}, \mathbf{v}, t)$ is also called the single-particle distribution function.
- Volume = $d\mathbf{v}\, d\mathbf{r} = dv_x\, dv_y\, dv_z\, dx\, dy\, dz$
- The differential phase-space volume = $f(\mathbf{r}, \mathbf{v}, t)\, d\mathbf{v}\, d\mathbf{r}$

The zeroth moment of the distribution

$$n_s(\mathbf{r}, t) = \int d\mathbf{v} f_s(\mathbf{r}, \mathbf{v}, t) \qquad \rho_s(\mathbf{r}, \mathbf{v}, t)/m_s \tag{2.16}$$

The first moment of the distribution

$$\mathbf{u}_s(\mathbf{r}, t) = \int d\mathbf{v}\, \mathbf{v} f_s(\mathbf{r}, \mathbf{v}, t) \Big/ \int d\mathbf{v}\, f_s(\mathbf{r}, \mathbf{v}, t) \tag{2.17}$$

The average kinetic energy per particle of s type in a unit volume of the plasma

$$<\tfrac{1}{2}m_s(\mathbf{v}-\mathbf{u}_s)^2> = \int d\mathbf{v}\ \tfrac{1}{2}m_s(\mathbf{v}-\mathbf{u}_s)^2 f_s(\mathbf{r},\ \mathbf{v},\ t) \Big/ \int d\mathbf{v} f_s(\mathbf{r},\ \mathbf{v},\ t) \qquad (2.18)$$

The hydrostatic partial pressure

$$p_s/n_s = (2/N) < \tfrac{1}{2}m_s(\mathbf{v}-\mathbf{u}_s)^2 > \qquad (2.19)$$

where *N* is the number of independent components of velocity, *Ps* is in Pascale (Pa), 1 Pa= 1N/m^2.

For the system in equilibrium

$$n_s(\mathbf{r}, v) = \int dt\, f_s(\mathbf{r},\ \mathbf{v},\ t)$$

Ideal Gas Law: *P=nKT*

Maxwellian distribution

For systems in equilibrium, the phase-space distribution is the Maxwellian distribution, the equilibrium distribution function for particles of type s flowing with velocity $\mathbf{u}_s$. It is given by

$$f_s(\mathbf{r},\, \mathbf{v}) = A_s \exp\left[-\frac{\frac{1}{2}m_s(\mathbf{v}-\mathbf{u}_s)^2}{kT_s} \right] \qquad \text{(Maxwellian distribution)} \quad (2.20a)$$

Here, k is the Boltzmann constant, 1.3807×10^{-23} joules per degree Kelvin ($\mathrm{J} \cdot \mathrm{K}^{-1}$), and A_s is a constant that is related to the number density such that equation (2.16) holds: $A_s = n_s(m/2\pi kT)^{\frac{3}{2}}$.

$$n_s(\mathbf{r}, v) = \int dt\, f_s(\mathbf{r},\, \mathbf{v},\, t)$$

the ideal-gas law, $p_s = n_s kT_s$

For **U**s=0

$$\int_{\text{directions}} f_s(\mathbf{r}, \mathbf{v}, t)\, d\Omega_v v^2\, dv = [4\pi f_s(\mathbf{r}, |v|, t) v^2] dv = g(\mathbf{r}, v)\, dv \qquad (2.20b)$$

For monatomic particles, the temperature, properly defined only for an equilibrium particle distribution, is related to the average energy of random motion by equation (2.18), which yields

$$< \tfrac{1}{2} m_s (\mathbf{v} - \mathbf{u}_s)^2 > \, = NkT_s/2 \qquad (2.21)$$

Equation (2.20a) shows that in a Maxwellian distribution, the proportion of particles with large random velocities $|\mathbf{v} - \mathbf{u}_s|$ increases with T_s. This point can be conveniently expressed by defining the most probable thermal speed of the plasma, v_{Ts}:

$$v_{Ts} = (2kT_s/m_s)^{\frac{1}{2}} \qquad \text{(most probable thermal speed)} \qquad (2.22)$$

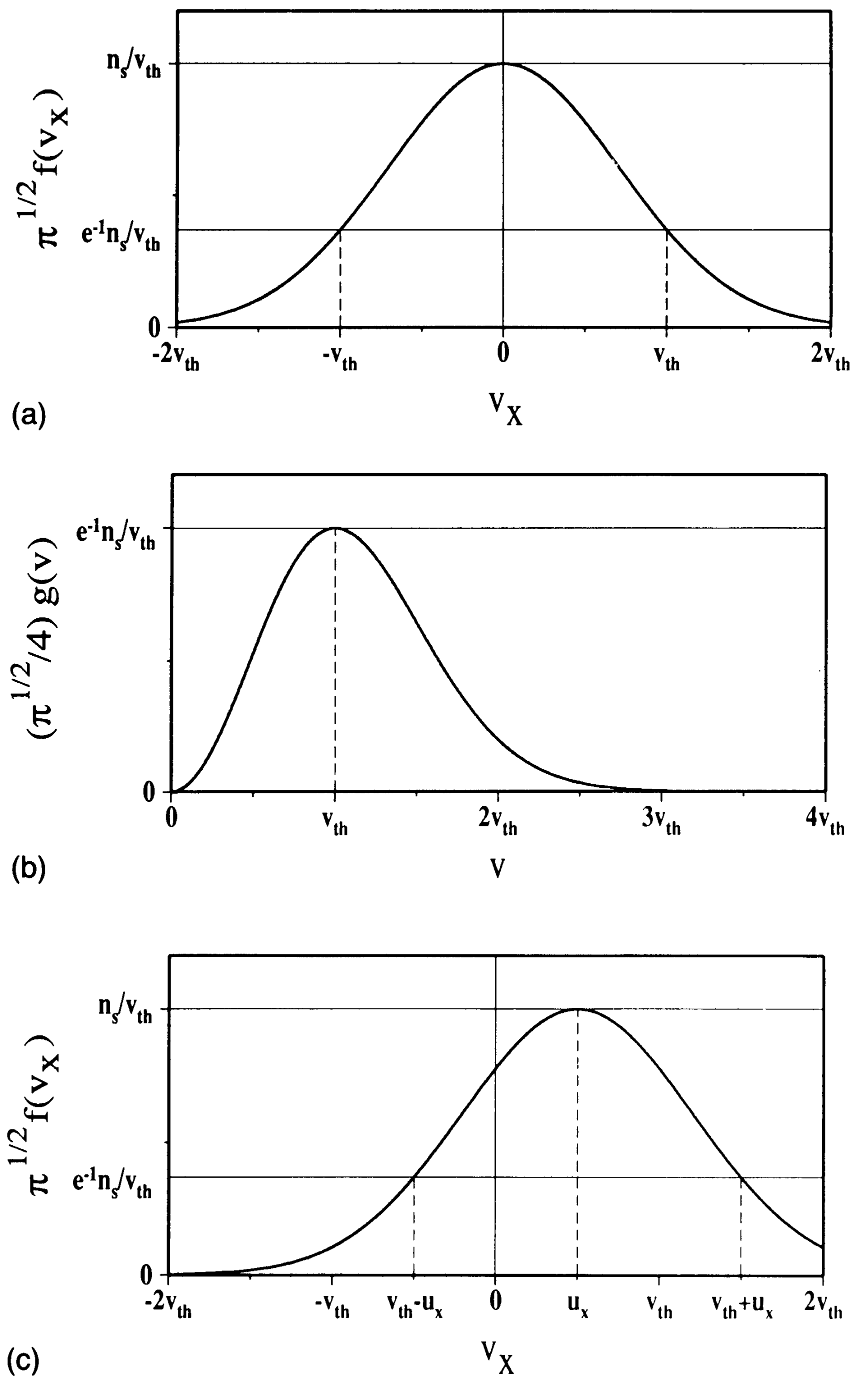


FIG. 2.3. Maxwellian distributions: In (a) and (b) the plasma is at rest. Part (a) shows the dependence of a three-dimensional Maxwellian distribution on one component of the velocity, here taken as v_x. The dependence on the other components of velocity has been removed by integration. Part (b) shows the dependence of $g(\mathbf{r}, v)$ on the speed, v; $v_{th} = (2kT_s/m_s)^{1/2}$ is the thermal speed. In (c) the plasma is flowing toward positive x at a velocity u_x. Notice that in (a) and (c), the position of the maximum is unaffected by changes of temperature. As the temperature increases, v_{th} increases, and the height of the peak of the distribution drops, whereas the width of the distribution increases. In (b), the position of the maximum shifts to the right as temperature increases. Its height increases.

$$f_s(\mathbf{r},\ \mathbf{v}) = A_s' \exp\left[-\frac{\frac{1}{2}m_s(v_\| - u_{\|s})^2}{kT_{\|s}}\right] \exp\left[-\frac{\frac{1}{2}m_s(\mathbf{v}_\perp - \mathbf{u}_{\perp s})^2}{kT_{\perp s}}\right] \tag{2.23}$$

(bi-Maxwellian distribution)

$$<\tfrac{1}{2}m_s(v_\| - u_{\|s})^2> = \tfrac{1}{2}kT_{\|s} \quad \text{and} \quad <\tfrac{1}{2}m_s(\mathbf{v}_\perp - \mathbf{u}_{\perp s})^2> = kT_{\perp s}$$

Sometimes the Maxwellian form is followed quite closely up to some energy, but the number of particles at high energies follows a power-law decrease rather than the exponential decrease of the Maxwellian distribution. The kappa distribution was devised to represent this situation in a convenient analytical form. For a single species, it is given by

$$f_s(\mathbf{r},\ \mathbf{v}) = A_{\kappa s}\left[1 + \frac{\frac{1}{2}m_s(\mathbf{v} - \mathbf{u}_s)^2}{\kappa E_{T_s}}\right]^{-\kappa - 1} \qquad \text{(kappa distribution)} \tag{2.24}$$

Although the phase-space distribution function is central to a theoretical interpretation of the behavior of collections of particles, it is measured only indirectly. Measurements provide the differential directional flux of particles within a range of solid angles $d\Omega$ and within an energy band of width dW about W, where W is the kinetic energy, $\frac{1}{2}m_s v^2$. Representing this flux as $\partial^2 J/\partial\Omega\, \partial W$, we find

$$f(\mathbf{r}, \mathbf{v}) = [m^2/2W]\, \partial^2 J/\partial\Omega\, \partial W \tag{2.25}$$

$$f_s(\mathbf{r}, \mathbf{v}) = A_s \exp\left[-\frac{\frac{1}{2}m_s(\mathbf{v}-\mathbf{u}_s)^2}{kT_s} \right] \qquad \text{(Maxwellian distribution)} \tag{2.20a}$$

$$f_s(\mathbf{r}, \mathbf{v}) = A_{\kappa s} \left[1 + \frac{\frac{1}{2}m_s(\mathbf{v}-\mathbf{u}_s)^2}{\kappa E_{T_s}} \right]^{-\kappa-1} \qquad \text{(kappa distribution)} \tag{2.24}$$

2.4 THE PASMA STATE

The electrostatic potential of an isolated particle (assumed to be an ion of charge q in this example) is $\phi = q/4\pi\epsilon_0 r$.

$$\phi = qe^{-r/\lambda_D}/(4\pi\epsilon_0 r) \qquad (2.26)$$

where λ_D is called the Debye length. In an electron-proton plasma,

$$\lambda_D = (\epsilon_0 kT/ne^2)^{\frac{1}{2}} \qquad \text{(Debye length)} \qquad (2.27)$$

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{n_e e^2}} = 69\sqrt{\frac{T}{n_e}}$$

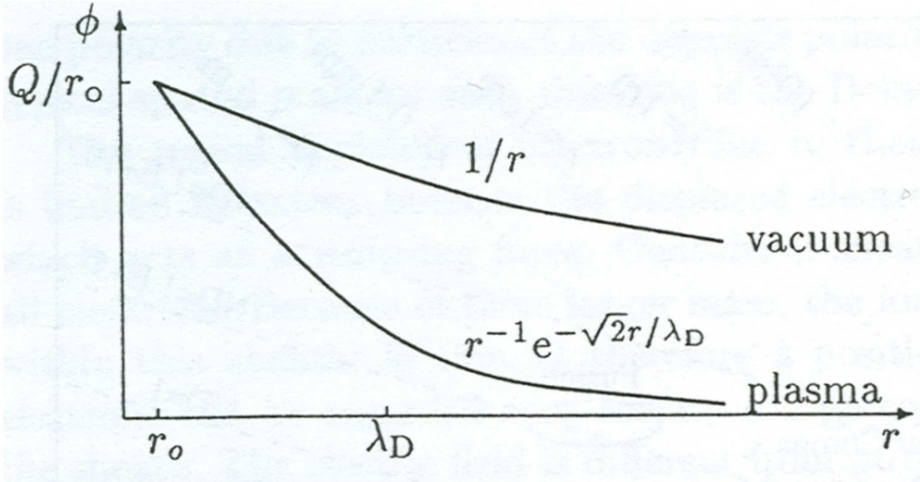
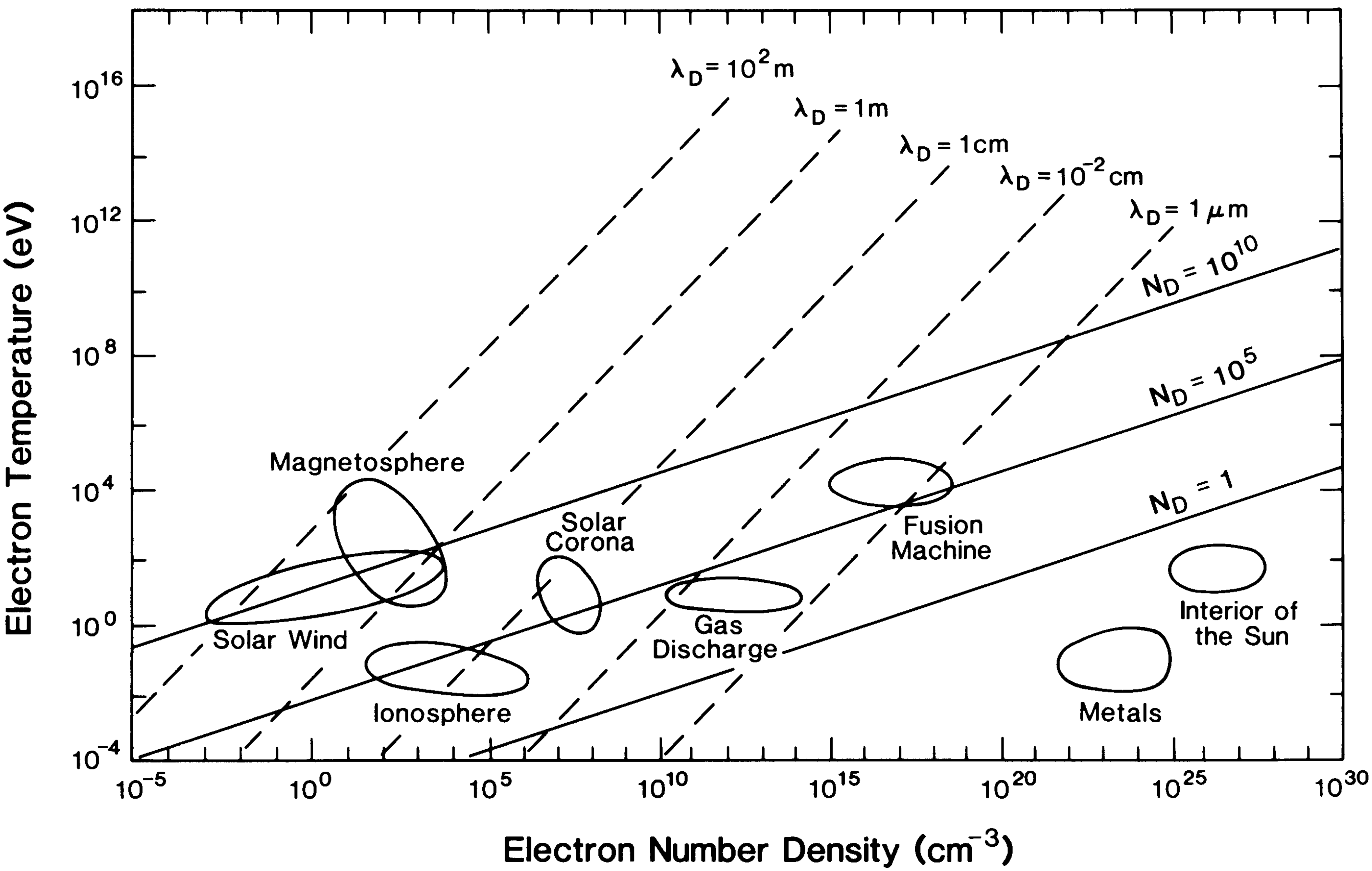


Fig. 3.14. The electric potential of a test charge is reduced by the surrounding plasma



Electron Temperature (eV)
Electron Number Density (cm^{-3})
10^{16}
10^{12}
10^{8}
10^{4}
10^{0}
10^{-4}
10^{-5}
10^{0}
10^{5}
10^{10}
10^{15}
10^{20}
10^{25}
10^{30}
$\lambda_D = 10^2 m$
$\lambda_D = 1 m$
$\lambda_D = 1 cm$
$\lambda_D = 10^{-2} cm$
$\lambda_D = 1 \mu m$
$N_D = 10^{10}$
$N_D = 10^{5}$
$N_D = 1$
Magnetosphere
Solar Wind
Ionosphere
Solar Corona
Gas Discharge
Fusion Machine
Interior of the Sun
Metals

Debye Shielding

Plasma	**Density** $n_e(\mathrm{m}^{-3})$	**Electron temperature** $T(\mathrm{K})$	**Magnetic field** $B(\mathrm{T})$	**Debye length** $\lambda_D(\mathrm{m})$
Gas discharge	10^{16}	10^{4}	--	10^{-4}
Tokamak	10^{20}	10^{8}	10	10^{-4}
Ionosphere	10^{12}	10^{3}	10^{-5}	10^{-3}
Magnetosphere	10^{7}	10^{7}	10^{-8}	10^{2}
Solar core	10^{32}	10^{7}	--	10^{-11}
Solar wind	10^{6}	10^{5}	10^{-9}	10
Interstellar medium	10^{5}	10^{4}	10^{-10}	10
Intergalactic medium	1	10^{6}	--	10^{5}

Source: Chapter 19: *The Particle Kinetics of Plasma*
http://www.pma.caltech.edu/Courses/ph136/yr2002/

TABLE 2.2. Properties of Typical Plasmas

Plasma Type	Density (m^{-3})	Temperature (eV)	Debye Length (m)	N_D
Interstellar	10^6	10^{-1}	1	10^6
Solar wind	10^7	10	10	10^{10}
Solar corona	10^{12}	10^2	10^{-1}	10^9
Solar atmosphere	10^{20}	1	10^{-6}	10^2
Magnetosphere	10^7	10^3	10^2	10^{13}
Ionosphere	10^{12}	10^{-1}	10^{-3}	10^4
Gas discharge	10^{20}	1	10^{-6}	10^2
Fusion machine	10^{22}	10^5	10^{-5}	10^7

Spacecraft (satellite) potential

The Debye length is helpful in understanding how a spacecraft affects the space plasmas that surround it. In a collisionless plasma, a spacecraft can develop a net negative charge, because for equal ion and electron temperatures, the electron flux is $(m_i/m_e)^{\frac{1}{2}}$ larger than the ion flux, and the spacecraft potential becomes negative. Solar radiation can liberate photoelectrons from the surface of the spacecraft, often producing a sufficiently large negative current that the spacecraft potential will become positive. Any net charge will perturb the plasma in the immediate vicinity of the spacecraft, a region referred to as a "plasma sheath." The scale size of the perturbed region will be λ_D. At distances large compared with λ_D, the plasma will be completely unaffected by the presence of the spacecraft.

It is useful to consider a sphere of radius λ_D centered on a plasma ion. This is referred to as the Debye sphere. The number of particles within the Debye sphere, $N_D = 4\pi n\lambda_D^3/3$, is proportional to $T^{\frac{3}{2}}/n^{\frac{1}{2}}$ and should be large for the expected shielding to occur. We shall generally assume that the number is large and that the expected shielding occurs;

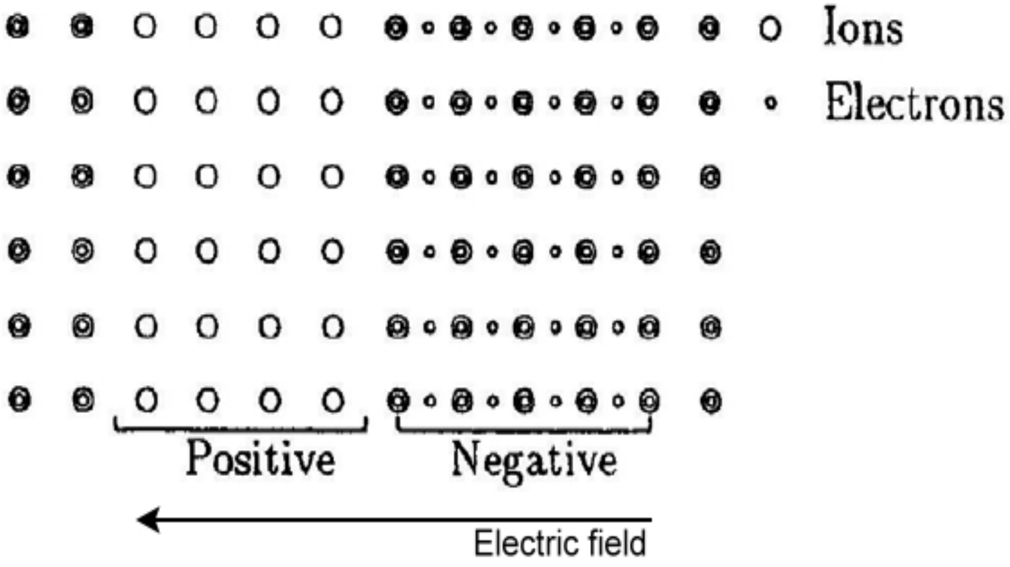
N_D>1 => g<<<1 (high temperature, low density)

characteristic time, characteristic length, collision frequency, mean free path, gyro frequency, gyro radius, plasma frequency, Debye length

$$\omega_{ps} = (n_s e^2/\epsilon_0 m_s)^{\frac{1}{2}} \qquad \text{(plasma frequency)} \tag{2.28}$$

F_p = 9000 $N^{1/2}$, plasma frequency F_p in Hz, electron density N in #/cm^3

Plasma frequency



$$m_e \frac{\partial^2 x}{\partial t^2} = -e\vec{E}$$

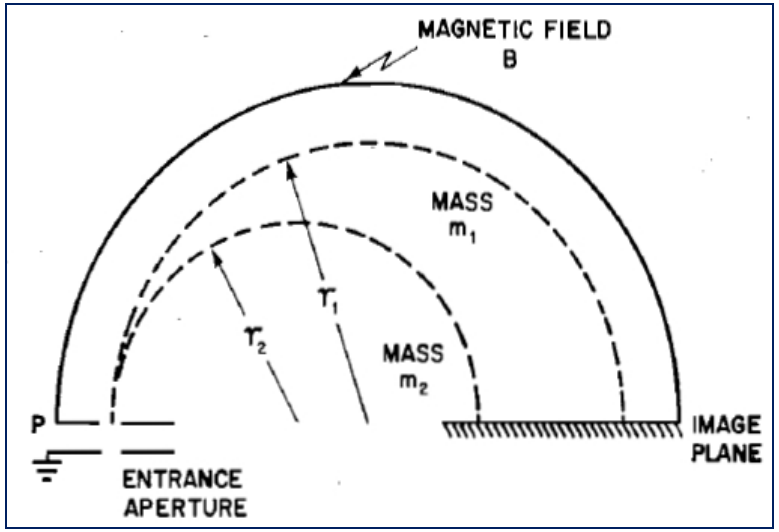
$$E = \frac{n_e e x}{\varepsilon_0}$$

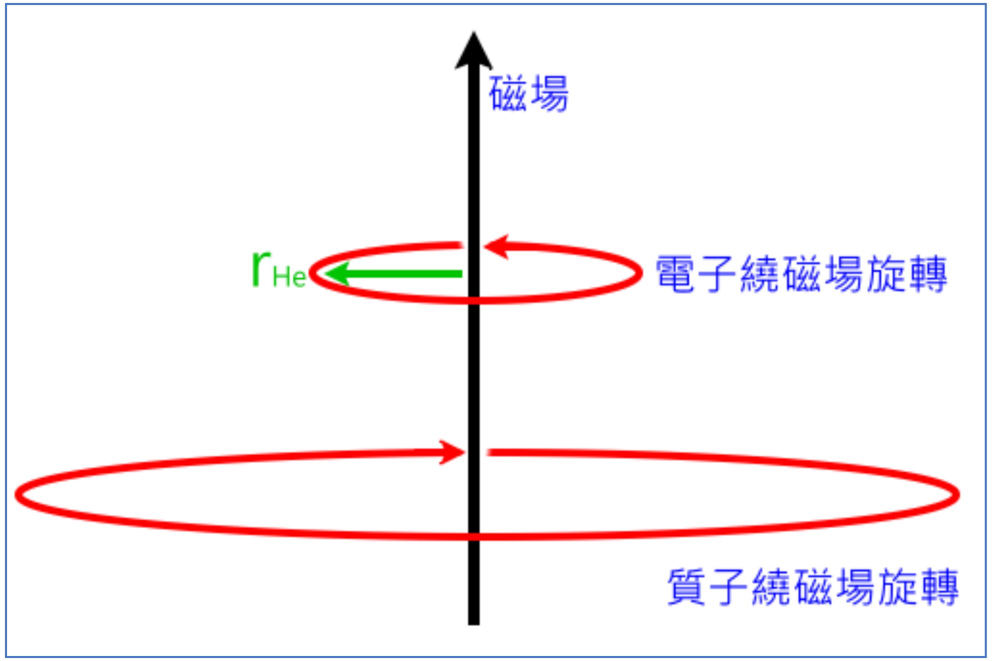
$$\frac{\partial^2 x}{\partial t^2} = -\frac{n_e e^2}{\varepsilon_0 m_e} x = -\omega_{pe}^2 x$$

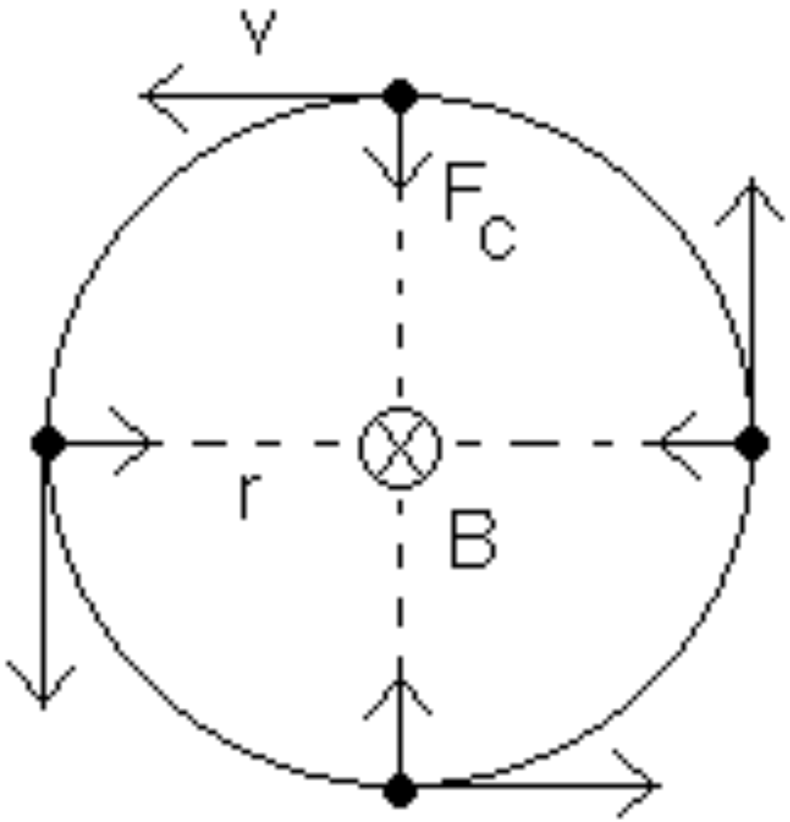
$$\omega_{pe} = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}$$

$$\boxed{f = 9000\sqrt{n_e}}$$

Gyro-motion


MAGNETIC FIELD
B
MASS
m_1
r_1
r_2
MASS
m_2
P
ENTRANCE
APERTURE
IMAGE
PLANE


磁場
r_{He}
電子繞磁場旋轉
質子繞磁場旋轉



$$F_B = \frac{mV^2}{r_H} = qVB$$

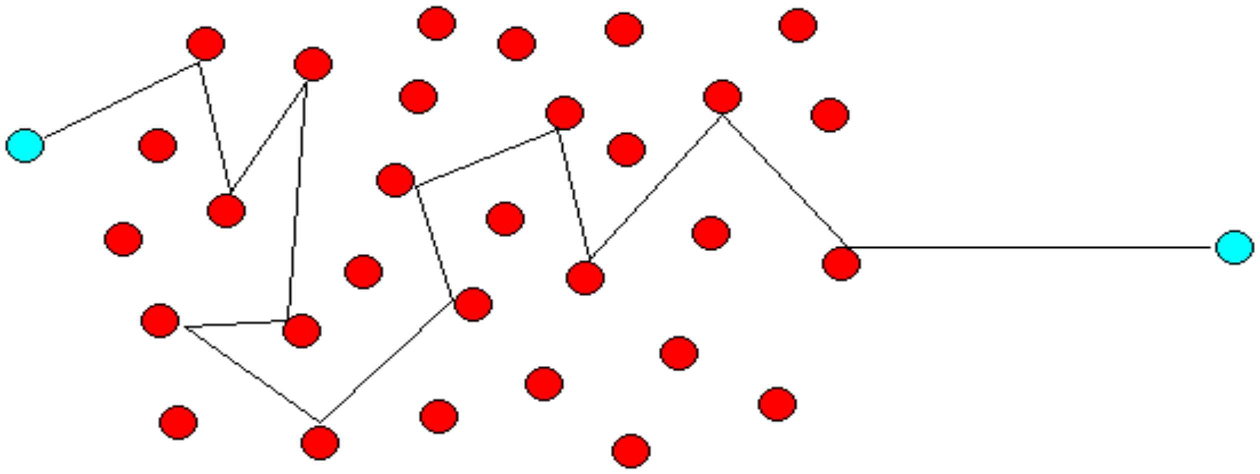
$$r_H = \frac{mV}{qB}$$ Gyro-radius

$$\omega_H = \frac{qB}{m}$$ Gyro-frequency

Mean free path

Mean Free Path

all particles, including photons, suffer from collisions with other particles such that their path through space is very short the higher the densities. This typical path length is called the mean free path.



the Universe is opaque at high densities (the mean free path of a photon is very short), as the density drops with time, the Universe becomes transparent (the mean free path of a photon becomes very large).

$$l = \frac{V}{v} = \frac{1}{\sigma n}$$

$l \gg r_H$ ($\nu \ll \omega$), Collision less plasma

$l \ll r_H$ ($\nu \gg \omega$), Collision plasma

http://abyss.uoregon.edu/

Scale height

壓力隨高度的變化可以用流體靜力方程式(hydrostatic equilibrium equation)加以描述，

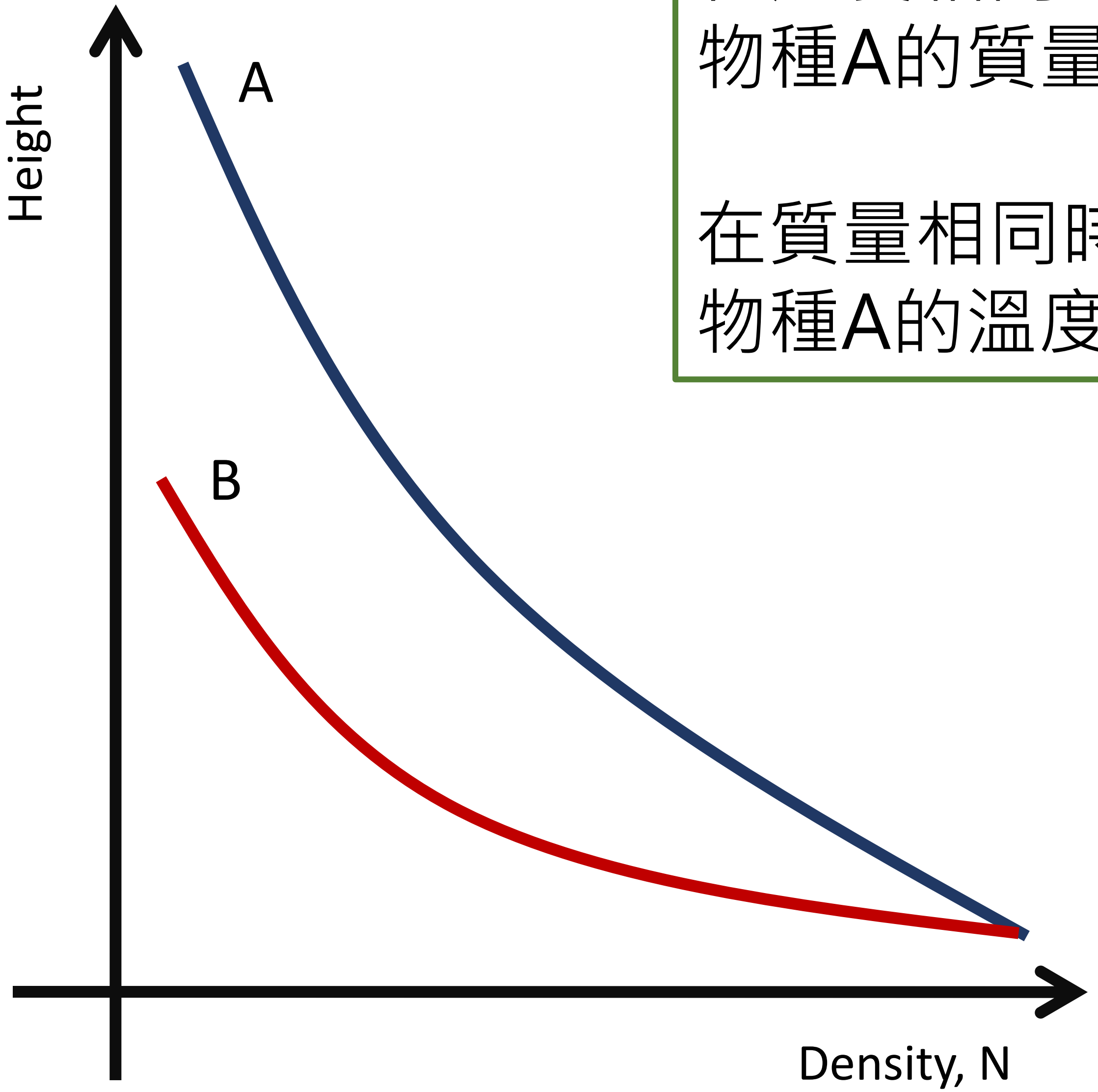
$$dp = -\rho g dh \tag{4.05}$$

由理想氣方程式可以得知 pv=nkT，故可改寫靜力方程式成為：

$$\frac{dp}{p} = -\frac{nmg}{nkT}dh = -\frac{dh}{H} \tag{4.06}$$

故可以得到

$$H = \frac{kT}{mg} \tag{4.07}$$



在溫度相同時，
物種A的質量___物種B的質量

在質量相同時，
物種A的溫度___物種B的溫度

在電漿中，同時考慮電子與質子兩種不同的物質

$$H = \frac{k(T_i + T_e)}{(m_i + m_e)g}$$

經過簡化後可得到標尺高為：

$$H = \frac{2kT}{m_i g}$$

表4-1 物理參數的空間比較

	德拜半徑 λ_D (cm)	標尺高 H_i (cm)	旋繞半徑 r_{Hp} (cm)	旋繞半徑 r_{He} (cm)	平均自由程 L (cm)	旋繞頻率 f_{He} (Hz)	電子溫度 T (k)	電漿密度 N (el/cm^3)	磁場強度 B (nT)
日冕附近 (corona)	1	10^{10}	10^3	10	10^8	10^8	10^8	10^8	1
地球軌道 (1 AU)	10^3	10^{12}	10^7	10^5	10^{13}	10^4	10^5	10	5×10^{-5}

2.4 THE FLUID DESCRIPTION OF A PLASMA

Individual particle motions vs. Properties of a collection of many particles

Hydrodynamics vs. Magnetohydrodynamics (MHD)

$$\frac{\partial n_s}{\partial t} + \nabla \cdot n_s \mathbf{u}_s = S_s - L_s \qquad \text{(continuity equation)} \tag{2.29a}$$

Here n_s, $\mathbf{u}_s$, S_s, and L_s may be functions of position and time. $S_s - L_s$ gives the net rate at which particles of type s are added per unit volume. Plasma ions and electrons may be added (i.e., S_s may be nonvanishing), If S_s and L_s vanish, equation (2.29a) implies mass conservation. To prove this, multiply equation (2.29a) by m_s, integrate over a fixed volume, and identify $\int dx\ \rho_s = M_s$ as the total mass of species s in the volume. Here, $\rho_s = m_s n_s$. Then

$$\frac{\partial M_s}{\partial t} + \int_{\text{volume}} \nabla \cdot (\rho_s \mathbf{u}_s)\, d\mathbf{r} = \frac{\partial M_s}{\partial t} + \int_{\text{surface}} d\mathbf{S} \cdot \rho_s \mathbf{u}_s = 0 \tag{2.30}$$

where $d\mathbf{S}$ is an element of the surface bounding the volume of interest

$$\rho_s\left(\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s\right) + m_s \mathbf{u}_s (S_s - L_s) = \tag{2.31b}$$

$$-\nabla p_s + \rho_{qs}\mathbf{E} + \mathbf{j}_s \times \mathbf{B} + \rho_s \mathbf{F}_g / m_s \qquad \text{(momentum equation)}$$

The electric field obeys Poisson's equation:

$$\nabla \cdot \mathbf{E} = \rho_q / \epsilon_0 \qquad \text{(Poisson's equation)} \tag{2.34}$$

where ρ_q is the net charge density. The electric field also enters into Faraday's law, introduced in equation (2.13):

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \text{(Faraday's law)} \tag{2.35}$$

To this must be added Ampère's law, which relates the magnetic field to the net current $\mathbf{j}$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \qquad \text{(Ampère's law)} \tag{2.36a}$$

and the requirement that $\mathbf{B}$ be divergenceless:

$$\nabla \cdot \mathbf{B} = 0 \tag{2.37}$$

$$|\nabla \times \mathbf{B}| \approx \frac{B}{L}, \qquad \mu_0\epsilon_0\left|\frac{\partial \mathbf{E}}{\partial t}\right| \approx \mu_0\epsilon_0\frac{E}{\tau} = \frac{E}{c^2\tau}$$

where L and τ are characteristic MHD length and time scales, respectively, and $c = (\mu_0\epsilon_0)^{-\frac{1}{2}}$ is the velocity of light. For the slow changes over long distances and the nonrelativistic flows required in MHD, the inequality

$$\frac{\mu_0\epsilon_0\left|\dfrac{\partial \mathbf{E}}{\partial t}\right|}{|\nabla \times \mathbf{B}|} \approx \frac{E}{Bc}\frac{L}{c\tau} \approx \frac{v}{c}\frac{L}{c\tau} \ll 1$$

is very strongly satisfied, and the second term on the right-hand side of equation (2.36a) can be dropped to give

$$\nabla \times \mathbf{B} = \mu_0\mathbf{j} \qquad \text{(Ampère's law in the MHD limit)} \tag{2.36b}$$

This form of Ampère's law may be more familiar expressed as an integral:

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int_S \mathbf{j} \cdot d\mathbf{S} \tag{2.40}$$

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{j} = 0 \qquad \text{(current continuity)} \tag{2.39}$$

$$\nabla \cdot \mathbf{j} = 0 \qquad \text{(divergenceless current density)} \tag{2.41}$$

This equation implies that all currents in MHD systems must close on themselves. There are neither sources of charge nor sinks. Equation (2.37), $\nabla \cdot \mathbf{B} = 0$, remains unchanged in the MHD approximation, as does Faraday's law:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \text{(Faraday's law)} \tag{2.35}$$

Poisson's equation cannot be simplified, because both sides are equally small, but as the charge density does not enter into any of the other equations, this equation is not needed.

there are 14 unknowns: **E**, **B**, **j**, **u**, ρ, and p.

$$\frac{\partial n_s}{\partial t} + \nabla \cdot n_s \mathbf{u}_s = S_s - L_s \qquad \text{(continuity equation)} \tag{2.29a}$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + U\right) + \nabla \cdot \left[\left(\tfrac{1}{2}\rho u^2 + U\right)\mathbf{u} + p\mathbf{u} + \mathbf{q}\right] = \mathbf{j} \cdot \mathbf{E} + \rho \mathbf{u} \cdot \mathbf{F}_g / m \tag{2.42}$$

$$\nabla \cdot \mathbf{E} = \rho_q / \epsilon_0 \qquad \text{(Poisson's equation)} \qquad \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \tag{2.34}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \text{(Faraday's law)} \tag{2.35}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \qquad \text{(Ampère's law in the MHD limit)} \tag{2.36b}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{j} = 0 \qquad \text{(divergenceless current density)} \tag{2.37}$$

Many treatments avoid introducing energy conservation explicitly. Instead, they obtain an additional equation from the assumption that there is no change in the entropy of a fluid element as it moves through the system. This means that the pressure and the density are related by

$$p\rho^{-\gamma}=\text{constant} \quad \text{or} \quad \frac{\partial p}{\partial t}+\mathbf{u}\cdot\nabla p=c_s^2\left(\frac{\partial \rho}{\partial t}+\mathbf{u}\cdot\nabla\rho\right) \tag{2.43}$$

where c_s is the speed of sound, defined by

$$c_s^2=\gamma p/\rho \tag{2.44}$$

and γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume, often referred to as the polytropic index. If the fluid is in thermodynamic equilibrium and the relation (2.21) is valid, then $\gamma=(N+2)/N$, or, in a three-dimensional system, $\gamma=\frac{5}{3}$. Measurements in space plasmas often yield values of γ that differ from $\frac{5}{3}$, which indicates that the idealized approximations used in obtaining an expression for γ are not always valid. We have noted that Ohm's law gives a

$$\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{V}) = 0$$

the equation of motion,

$$\rho (d\mathbf{V} \ dt) = -\nabla p + \rho \mathbf{g} + \eta \nabla^2 \mathbf{V} + \mathbf{J} \times \mathbf{B}$$

and Maxwell's equations,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{E} = \rho_c / \varepsilon_0$$

For a fluid with a high heat conductivity we could use isothermal conditions

$$dT/dt = d(p/\rho)/dt = 0$$

or for an adiabatic fluid

$$d(p^\gamma/\rho^\gamma)/dt = 0$$

The MHD equations can now be written

$$\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{2.60a}$$

$$\nabla \cdot \mathbf{B} = 0 = \nabla \cdot \mathbf{E} \tag{2.60b}$$

$$\rho (d\mathbf{V}/dt) = -\nabla (p + B^2/2\mu_0) + (\mathbf{B} \cdot \nabla)\mathbf{B}/\mu_0 \tag{2.60c}$$

$$\partial \mathbf{B}/\partial t = \nabla \times (\mathbf{V} \times \mathbf{B}) + (1/\sigma \mu_0)\nabla^2 \mathbf{B} \tag{2.60d}$$

One of the equations of state completes the system.

Frozen-in flux

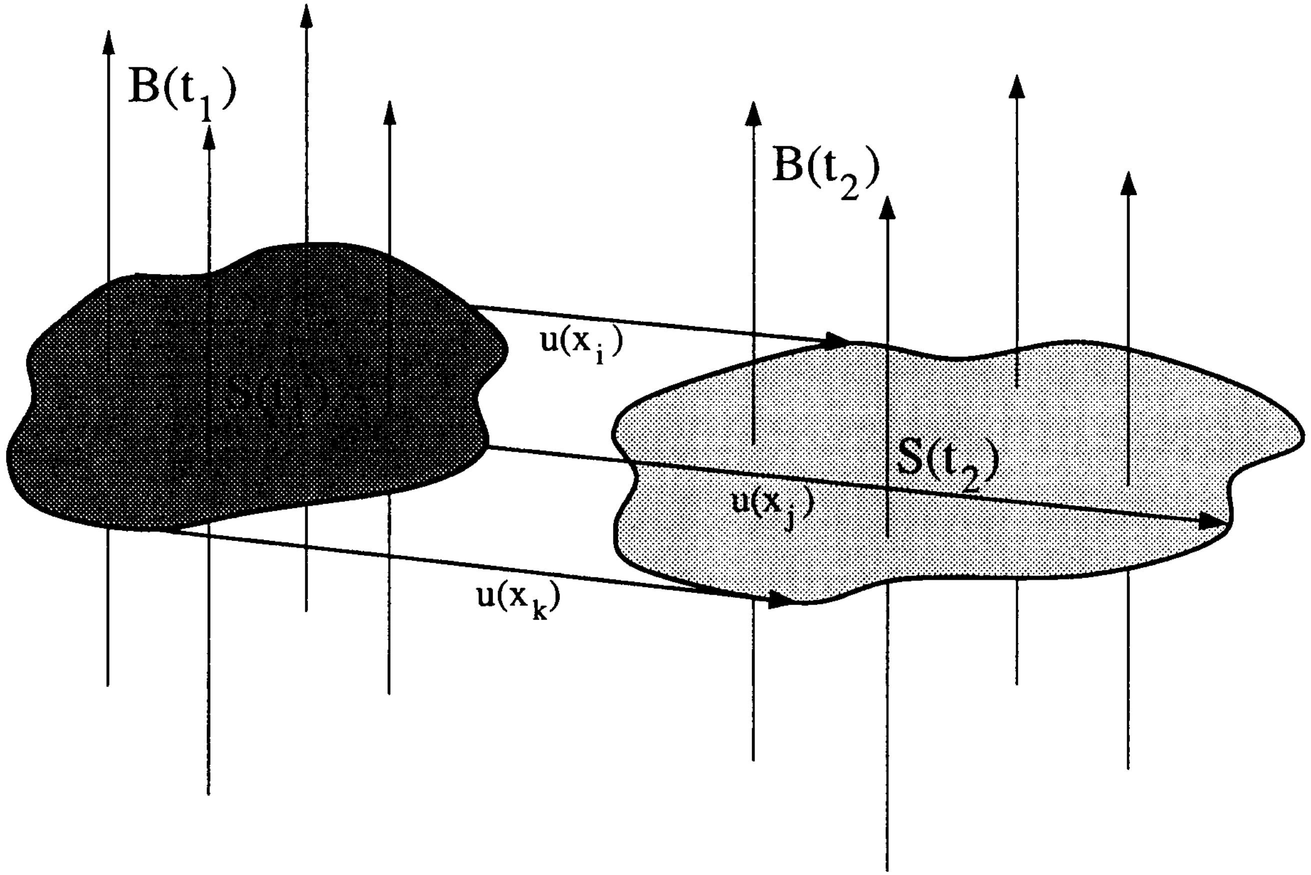


FIG. 2.5. Schematic illustration of frozen-in flux. The fluid that lies on surface $S(t_1)$ threaded by a magnetic field with a component B_{n1} normal to the surface at time t_1 flows through the system and lies on surface $S(t_2)$ threaded by a field with normal component B_{n2} at time t_2. The frozen-in flux condition requires that $B_{n2} = B_{n1} S(t_1)/S(t_2)$.

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S}$$

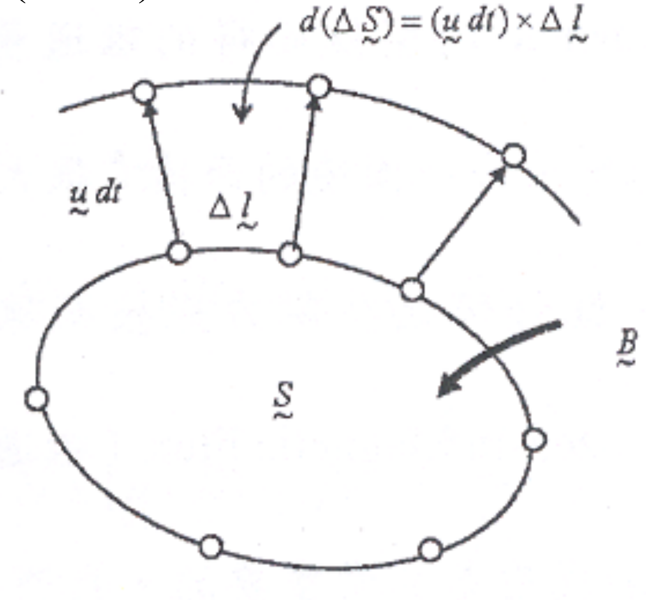
Frozen in magnetic field

Magnetic flux $\phi = \oiint \vec{B} \cdot d\vec{s}$

$$\frac{d\phi}{dt} = \frac{d}{dt} \oiint \vec{B} \cdot d\vec{s} = \oiint \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oiint \vec{B} \cdot \frac{d(\Delta \vec{s})}{dt} = \oiint \left[\nabla \times (\vec{u} \times \vec{B}) \right] \cdot d\vec{s} + \oiint \vec{B} \cdot \frac{d(\Delta \vec{s})}{dt}$$

$$\frac{d\phi}{dt} = \oiint \left[\nabla \times (\vec{u} \times \vec{B}) \right] \cdot d\vec{s} + \oiint \vec{B} \cdot \frac{d(\Delta \vec{s})}{dt} = \oint (\vec{u} \times \vec{B}) \cdot d\vec{l} + \oint \vec{B} \cdot (\vec{u} \times d\vec{l}\,) = 0$$

$$\frac{d\phi}{dt} = 0$$



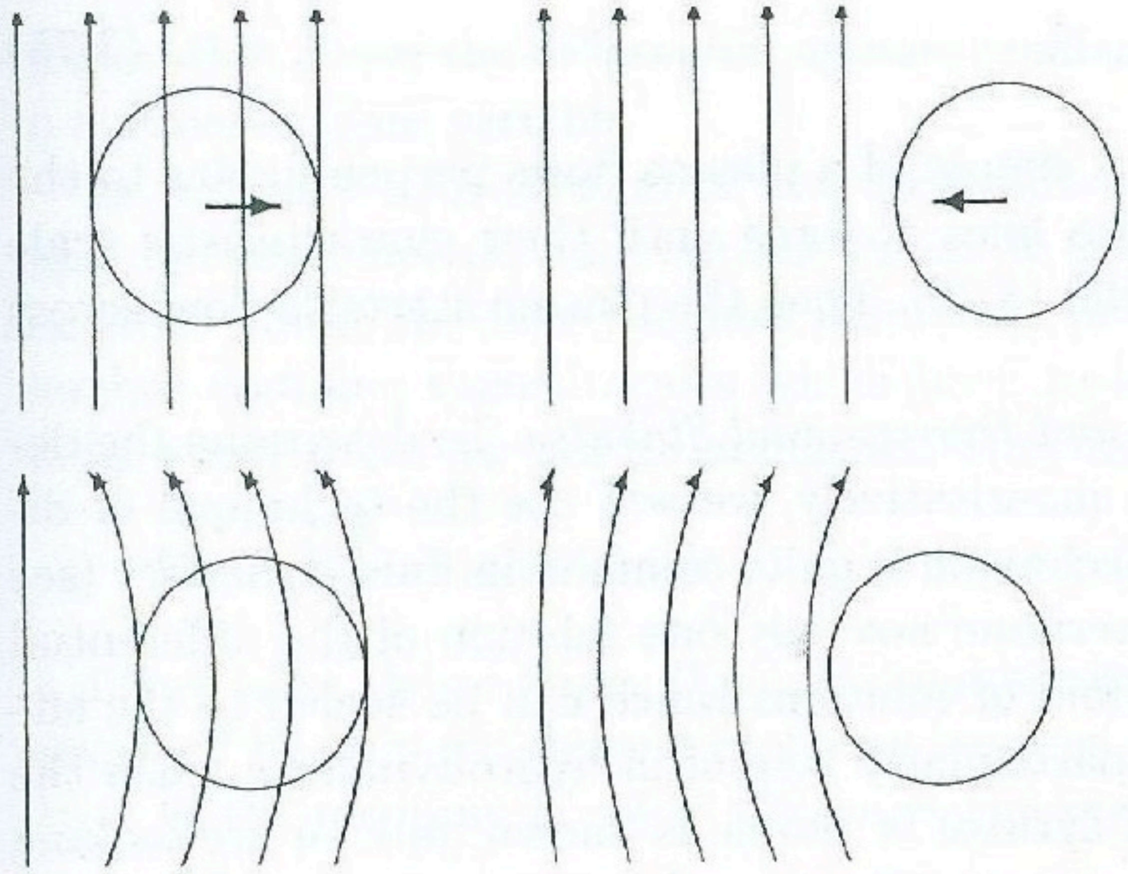


Fig. 3.7. Frozen-in (*left*) and frozen-out (*right*) magnetic fields

$$\mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla B^2/2\mu_0 + (\mathbf{B} \cdot \nabla)\mathbf{B}/\mu_0 \tag{2.47}$$

a magnetic pressure

$$p_B = B^2/2\mu_0$$

$$\beta = \frac{p}{B^2/2\mu_0} \tag{2.49}$$

A plasma is called "cold" if $\beta \ll 1$; for $\beta \geq 1$, the plasma is called "warm,"

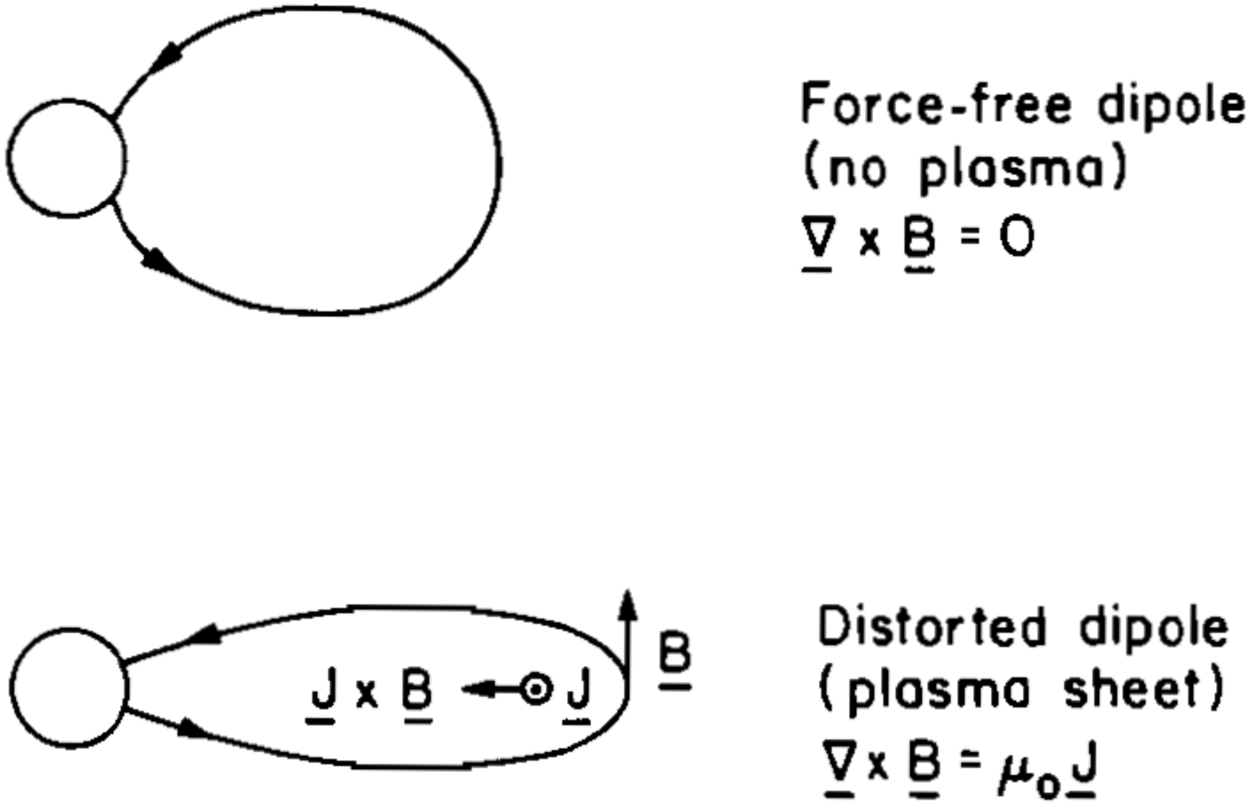
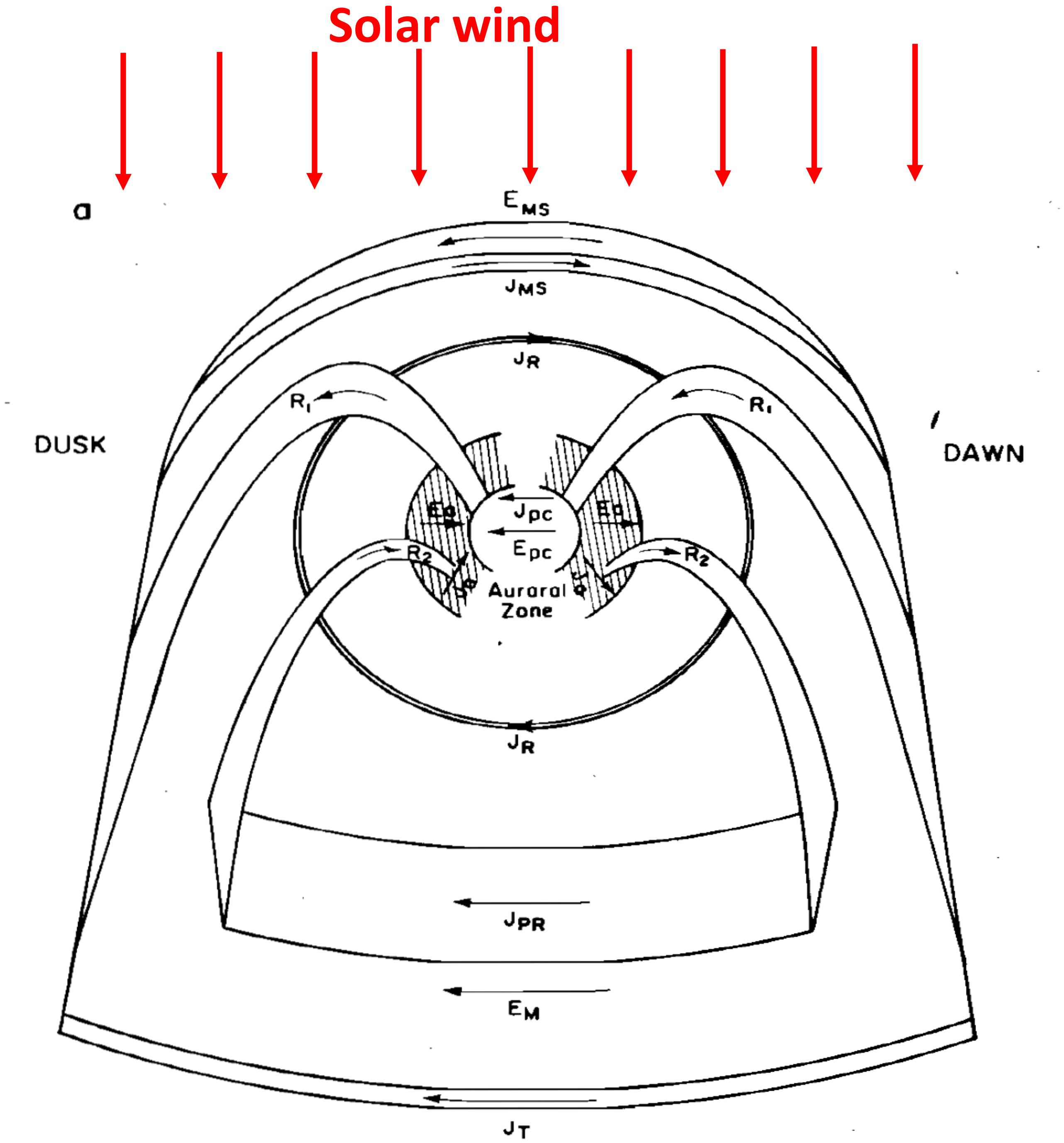


Fig. 2.12. Illustration of the difference between a force-free dipole and a distorted field configuration. The magnetic tension force is related to the $\mathbf{J} \times \mathbf{B}$ force shown.



Solar wind
a
E MS
J MS
J R
R1
R1
DUSK
DAWN
J pc
E pc
R2
R2
Auroral
Zone
J R
J PR
E M
J T