Forward Intensity PD Model

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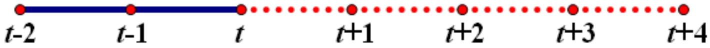
International Monetary Fund, Washington DC, July 17-18, 2017

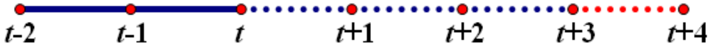
Outline

Questions of interest

- Single-obligor default prediction over different future periods (cumulative and forward)
  - Cumulative default probability



  - Forward default probability



- Portfolio credit analysis
  - Frequencies of defaults over some horizon
  - Exposure-weighted default distribution over some horizon

Factoring in other exits: default vs. merger/acquisition

Year	Active Firms	Defaults	(%)	Other Exit	(%)
1991	4012	32	0.80%	257	6.41%
1992	4009	28	0.70%	325	8.11%
1993	4195	25	0.60%	206	4.91%
1994	4433	24	0.54%	273	6.16%
1995	5069	19	0.37%	393	7.75%
1996	5462	20	0.37%	463	8.48%
1997	5649	44	0.78%	560	9.91%
1998	5703	64	1.12%	753	13.20%
1999	5422	77	1.42%	738	13.61%
2000	5082	104	2.05%	616	12.12%
2001	4902	160	3.26%	577	11.77%
2002	4666	81	1.74%	397	8.51%
2003	4330	61	1.41%	368	8.50%
2004	4070	25	0.61%	302	7.42%
2005	3915	24	0.61%	291	7.43%
2006	3848	15	0.39%	279	7.25%
2007	3767	19	0.50%	352	9.34%
2008	3676	59	1.61%	285	7.75%
2009	3586	67	1.87%	244	6.80%
2010	3396	25	0.74%	242	7.13%
2011	3224	21	0.65%	226	7.01%

Literature review

- Choosing between structural and reduced-form modeling approaches

- Discriminant analysis
  - Beaver (1966, 1968), Altman (1968), etc.
  - Model output: credit scores

- Binary response models: logit/probit regressions
  - Ohlson (1980), Zmijewski (1984), etc.
  - Model output: default probability in the next one period

  - Campbell, *et al* (2008): logit models for different periods ahead

Literature review (cont'd)

- Recent development: duration analysis
  - Shumway (2001), Chava and Jarrow (2004), etc.
- The model of Duffie, Saita and Wang (2007, *J of Financial Economics*):
  - Two Poisson processes (conditionally independent)
    - Default/bankruptcy
    - Other exit: merger and acquisition, etc.
  - Use spot intensities
    - Instantaneous rate of occurrence
    - Functions of the covariates (stochastic and deterministic)
  - Need to specify the time-series dynamics for the stochastic covariates

Literature review (cont'd)

- The model of Duan, Sun and Wang (2012, *J of Econometrics*):
  - Two Poisson processes (conditionally independent)
    - Default/bankruptcy
    - Other exit: merger and acquisition, etc.
  - Forward intensity
    - Instantaneous rate of occurrence
    - Functions of the covariates (stochastic and deterministic)
  - No need to specify the time-series dynamics for the stochastic covariates

Spot-intensity model

- Let instantaneous default and other exit intensities for the i-th firm at time t be λ_{it} and ϕ_{it}, respectively.
- Define two stopping times
  τ_{Di}: default time of the i-th firm
  τ_{Ci}: combined exit time of the i-th firm
- The default probability over $[t, t+\tau]$ becomes
  $E_t\left(\int_t^{t+\tau} e^{-(\lambda_{is}+\phi_{is})(s-t)}\lambda_{is}ds\right)$.

Spot-intensity model (cont'd)

- Model λ_{it} and ϕ_{it} as functions of state variables available at time t.
- $\lambda_{it} \geq 0$ and $\phi_{it} \geq 0$.
- $X_{it} = (x_{it,1}, x_{it,2}, \cdots, x_{it,k})$: the set of the state variables

$$\begin{aligned} \lambda_{it} &= \exp\{\lambda \exp[-\delta(t - t_B)] 1_{t > t_B} \\ &\quad + \alpha_0 + \alpha_1 x_{it,1} + \alpha_2 x_{it,2} + \cdots \alpha_k x_{it,k}\} \\ \phi_{it} &= \exp\left(\beta_0 + \beta_1 x_{it,1} + \beta_2 x_{it,2} + \cdots \beta_k x_{it,k}\right) \end{aligned}$$

  where t_B is August 2008 (Note that the US government bailed out AIG in September 2008)
- Discretize the model for empirical implementation

The likelihood function

$$\mathscr{L}(\alpha, \beta; \tau_C, \tau_D, X) = \prod_{i=1}^{N} \prod_{t=0}^{T-1} \mathscr{L}_{i,t}(\alpha, \beta)$$

$$\begin{aligned}\mathscr{L}_{i,t}(\alpha, \beta) =& 1_{\{t_{0i} \le t, \tau_{Ci} > t+\Delta t\}} P_t(\tau_{Ci} > t + \Delta t) \\ &+ 1_{\{t_{0i} \le t, \tau_{Di} = \tau_{Ci} = t+\Delta t\}} P_t(\tau_{Di} = \tau_{Ci} = t + \Delta t) \\ &+ 1_{\{t_{0i} \le t, \tau_{Di} \ne \tau_{Ci}, \tau_{Ci} = t+\Delta t\}} P_t(\tau_{Di} \ne \tau_{Ci} \& \tau_{Ci} = t + \Delta t) \\ &+ 1_{\{t_{0i} > t\}} + 1_{\{\tau_{Ci} \le t\}}\end{aligned}$$

- $P_t(\tau_{Ci} > t + \Delta t)$: probability of surviving both forms of exit over the next period
- $P_t(\tau_{Di} = \tau_{Ci} = t + \Delta t)$: probability that firm defaults in the next period
- $P_t(\tau_{Di} \ne \tau_{Ci} \& \tau_{Ci} = t + \Delta t)$: probability that firm exits in the next period due to other reasons

The likelihood function (cont'd)

$$P_t(\tau_{Ci} > t + \Delta t) = \exp\left[-(\lambda_{it} + \phi_{it})\Delta t\right]$$
$$P_t(\tau_{Di} = \tau_{Ci} = t + \Delta t) = 1 - \exp\left[-\lambda_{it}\Delta t\right]$$
$$P_t(\tau_{Di} \neq \tau_{Ci} \& \tau_{Ci} = t + \Delta t) = \exp[-\lambda_{it}\Delta t] - \exp\left[-(\lambda_{it} + \phi_{it})\Delta t\right]$$

with $\Delta t = 1/12$ (monthly data)

Note that the likelihood function is decomposable so that the parameters for the default and other exit intensity functions can be separately estimated.

Forward-intensity model

- Spot combined exit intensity: "average" rate of combined exit occurrence

$$\psi_{it}(\tau) \equiv -\frac{\ln(1 - F_{it}(\tau))}{\tau} = -\frac{\ln E_t\left[\exp\left(-\int_t^{t+\tau}(\lambda_{is} + \phi_{is})ds\right)\right]}{\tau}$$

$F_{it}(\tau)$: the time-t conditional distribution function of the combined exit time evaluated at $t + \tau$.
λ_{is}: instantaneous intensity for default.
ϕ_{is}: instantaneous intensity for other exit.

Forward-intensity model (cont'd)

- Forward exit intensity: forward rate of combined exit occurrence

$$g_{it}(\tau) \equiv \frac{F'_{it}(\tau)}{1 - F_{it}(\tau)} = \psi_{it}(\tau) + \psi'_{it}(\tau)\tau$$

- Forward default intensity: forward rate of default occurrence

$$\begin{aligned} f_{it}(\tau) &\equiv e^{\psi_{it}(\tau)\tau} \lim_{\Delta t \to 0} \frac{P_t(t+\tau < \tau_{Di} = \tau_{Ci} \le t+\tau+\Delta t)}{\Delta t} \\ &= e^{\psi_{it}(\tau)\tau} \lim_{\Delta t \to 0} \frac{E_t\left[\int_{t+\tau}^{t+\tau+\Delta t} \exp\left(-\int_t^s (\lambda_{iu} + \phi_{iu}) du\right) \lambda_{is} ds\right]}{\Delta t} \end{aligned}$$

τ_{Di}: default time of the i-th firm.
τ_{Ci}: combined exit time of the i-th firm.

- The default probability over $[t, t+\tau]$ becomes $\int_0^\tau e^{-\psi_{it}(s)s} f_{it}(s) ds$.

Forward-intensity model (cont'd)

- Model $f_{it}(\tau)$ and $g_{it}(\tau)$ directly as functions of state variables available at time t and the horizon of interest, τ.
- $g_{it}(\tau) \geq f_{it}(\tau) \geq 0$
- $X_{it} = (x_{it,1}, x_{it,2}, \cdots, x_{it,k})$: the set of the state variables

$$\begin{aligned} f_{it}(\tau) &= \exp\{\lambda(\tau)\exp[-\delta(\tau)(t-t_B)]1_{t>t_B} \\ &\quad +\alpha_0(\tau)+\alpha_1(\tau)x_{it,1}+\alpha_2(\tau)x_{it,2}+\cdots\alpha_k(\tau)x_{it,k}\} \\ g_{it}(\tau) &= f_{it}(\tau)+\exp\left(\beta_0(\tau)+\beta_1(\tau)x_{it,1}+\beta_2(\tau)x_{it,2}+\cdots\beta_k(\tau)x_{it,k}\right) \end{aligned}$$

where t_B is August 2008 (Note that the US government bailed out AIG in September 2008)

- Discretize the model for empirical implementation

The pseudo-likelihood function

$$\mathscr{L}_\tau(\alpha, \beta; \tau_C, \tau_D, X) = \prod_{i=1}^{N} \prod_{t=0}^{T-1} \mathscr{L}_{\tau,i,t}(\alpha, \beta)$$

$$\begin{aligned}\mathscr{L}_{\tau,i,t}(\alpha, \beta) =& 1_{\{t_{0i} \leq t, \tau_{Ci} > t+\tau\}} P_t(\tau_{Ci} > t + \tau) \\ &+ 1_{\{t_{0i} \leq t, \tau_{Di} = \tau_{Ci} \leq t+\tau\}} P_t(\tau_{Ci}; \tau_{Di} = \tau_{Ci} \leq t + \tau) \\ &+ 1_{\{t_{0i} \leq t, \tau_{Di} \neq \tau_{Ci}, \tau_{Ci} \leq t+\tau\}} P_t(\tau_{Ci}; \tau_{Di} \neq \tau_{Ci} \& \tau_{Ci} \leq t + \tau) \\ &+ 1_{\{t_{0i} > t\}} + 1_{\{\tau_{Ci} \leq t\}}\end{aligned}$$

- $P_t(\tau_{Ci} > t + \tau)$: probability of surviving both forms of exit over the defined interval
- $P_t(\tau_{Ci}; \tau_{Di} = \tau_{Ci} \leq t + \tau)$: probability that firm defaults at a particular period within the defined interval
- $P_t(\tau_{Ci}; \tau_{Di} \neq \tau_{Ci} \& \tau_{Ci} \leq t + \tau)$: probability that firm exits due to other reasons at a particular period within the defined interval

The pseudo-likelihood function (cont'd)

$$P_t(\tau_{Ci} > t+\tau) = \exp\left[-\sum_{s=0}^{\tau-1} g_{it}(s)\Delta t\right]$$

$$P_t(\tau_{Ci}; \tau_{Di} = \tau_{Ci} \leq t+\tau)$$

$$= \begin{cases} 1-\exp\left[-f_{it}(0)\Delta t\right], & \text{if } \tau_{Di} = t+1 \\ \exp\left[-\sum_{s=0}^{\tau_{Di}-t-2} g_{it}(s)\Delta t\right]\left\{1-\exp\left[-f_{it}(\tau_{Di}-t-1)\Delta t\right]\right\}, & \text{if } t+1 < \tau_{Di} \leq t+\tau \end{cases}$$

$$P_t(\tau_{Ci}; \tau_{Di} \neq \tau_{Ci} \& \tau_{Ci} \leq t+\tau)$$

$$= \begin{cases} \exp[-f_{it}(0)\Delta t] - \exp[-g_{it}(0)\Delta t], & \text{if } \tau_{Ci} = t+1 \\ \exp\left[-\sum_{s=0}^{\tau_{Ci}-t-2} g_{it}(s)\Delta t\right] \times & \\ \{\exp[-f_{it}(\tau_{Ci}-t-1)\Delta t] - \exp[-g_{it}(\tau_{Ci}-t-1)\Delta t]\}, & \text{if } t+1 < \tau_{Ci} \leq t+\tau \end{cases}$$

with $\Delta t = 1/12$ (monthly data)

Estimating the forward-intensity model

- It is an overlapped pseudo-likelihood function when the intended prediction horizon is greater than one basic time period (i.e., one month in our empirical implementation).
- The pseudo-likelihood function is decomposable so that estimation can be performed one forward period at a time.
- The pseudo-likelihood function continues to be decomposable to allow for separate estimations of the default intensity and the intensity for other form of exit.
- Because the numerical problem is non-sequential, it can be easily parallelized in computing.
- Note that the forward intensity function corresponding to $\tau = 0$ is the spot intensity function.

Data used in DSW (2012, *J of Econometrics*)

- Sample period: 1991-2011, monthly.
- Database:
  - Compustat
  - CRSP
  - Credit Research Initiative database
- 12,268 U.S. public companies (including financial firms) totaling 1,104,963 firm-month observations.

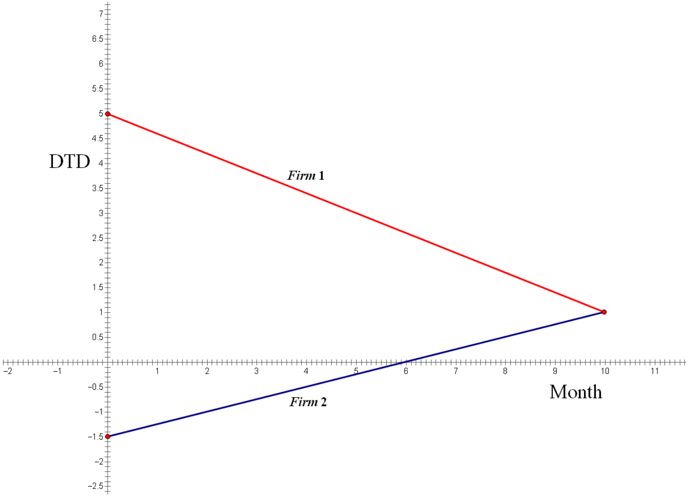
Covariates used in DSW (2012, *J of Econometrics*)

- An exponential decaying term to capture the US intervention effect
- 3-month treasury rate
- Trailing 1-year S&P500 return
- Distance to default
- Cash and short-term investments/Total assets
- Net income/Total assets
- Relative size
- Market to book ratio
- Idiosyncratic volatility

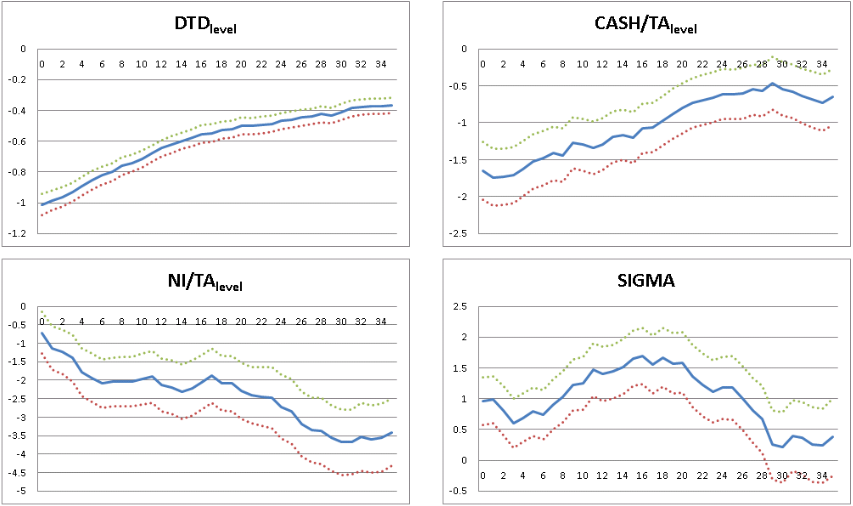
Note: see Duan and Wang (2012, *Global Credit Review*) for estimating DTDs of non-financial and financial firms.

Covariates (cont'd)

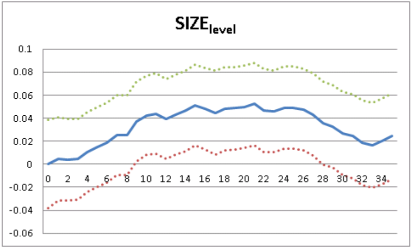
- Level and trend

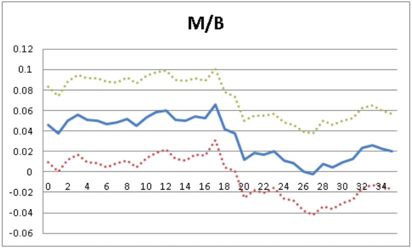


Parameter estimates reported in DSW (2012, *J of Econometrics*)

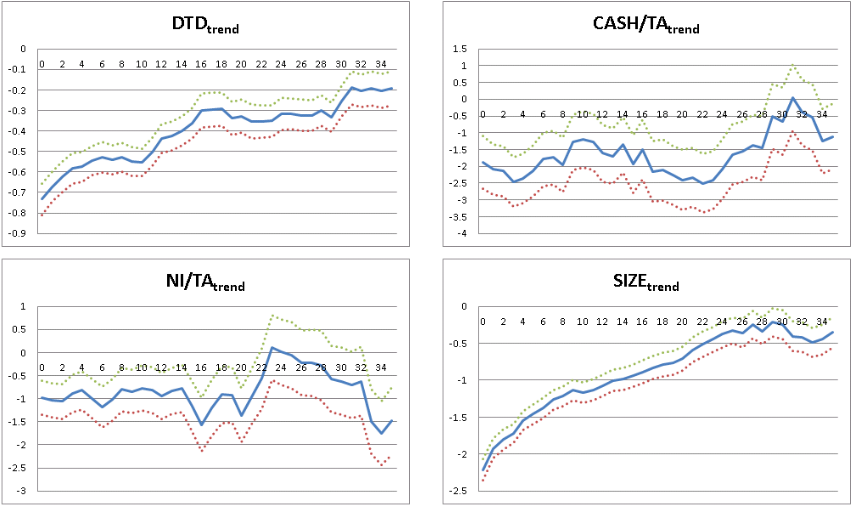


Parameter estimates (cont'd)

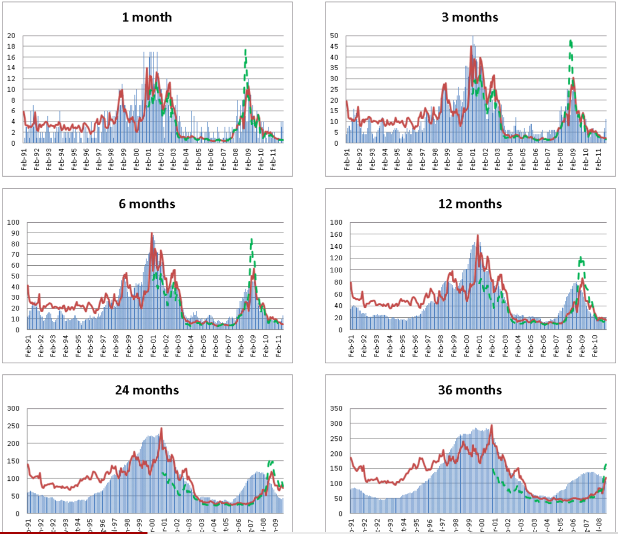




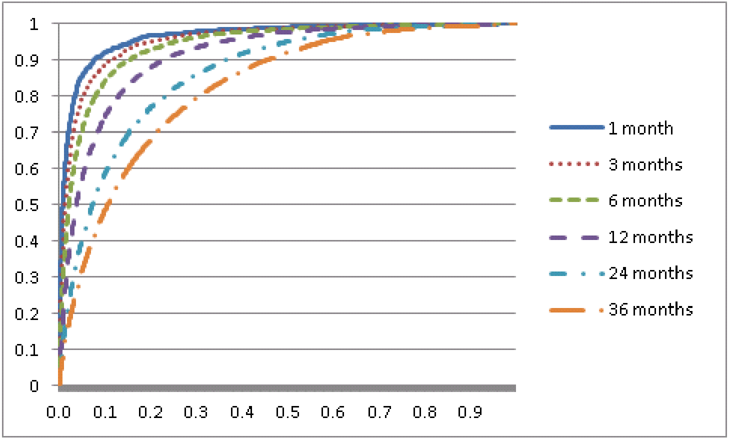
Parameter estimates (cont'd)



Predicted and realized # of defaults reported in DSW (2012, *J of Econometrics*)

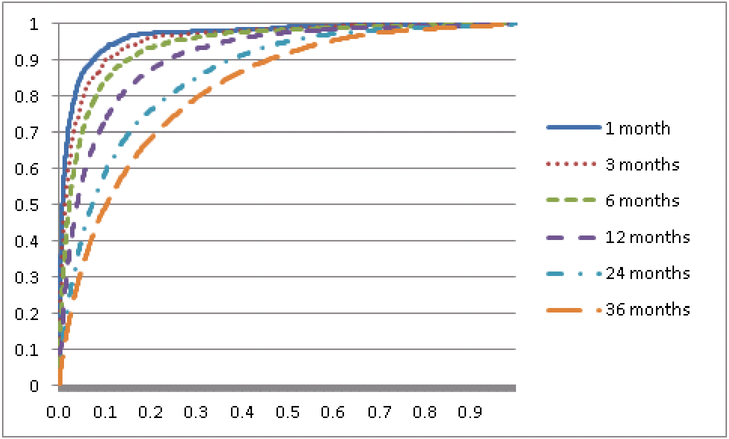


In-sample accuracy reported in DSW (2012, *J of Econometrics*)



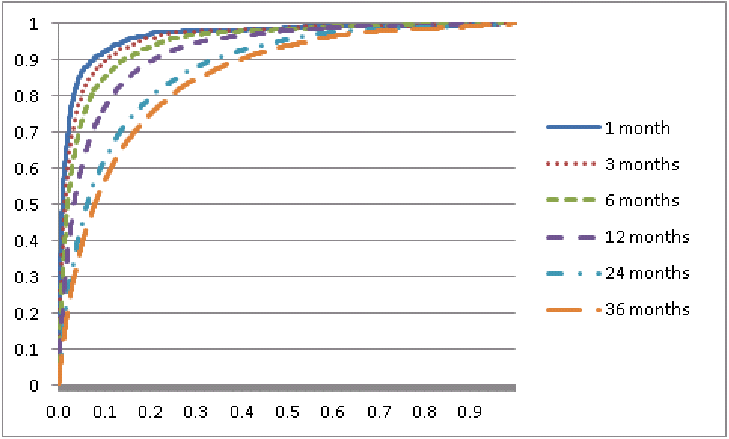
1 month	3 months	6 months	12 months	24 months	36 months
93.22%	91.30%	88.63%	83.52%	74.10%	66.67%

Out-of-sample (cross-section) accuracy reported in DSW (2012, *J of Econometrics*)



1 month	3 months	6 months	12 months	24 months	36 months
93.77%	91.74%	88.88%	83.36%	73.37%	65.47%

Out-of-sample (over time) accuracy reported in DSW (2012, *J of Econometrics*)



1 month	3 months	6 months	12 months	24 months	36 months
93.31%	91.81%	89.42%	85.16%	76.43%	72.45%

Parameter smoothing

- DSW (2012) demonstrated that their parameter estimates can be modeled by the Nelson-Siegel (1987) type of term structure function:

$$
\begin{aligned}
& h(\tau; \varrho_0, \varrho_1, \varrho_2, d) \\
= \quad & \varrho_0 + \varrho_1 \frac{1 - \exp(-\tau/d)}{\tau/d} + \varrho_2 \left[\frac{1 - \exp(-\tau/d)}{\tau/d} - \exp(-\tau/d) \right]
\end{aligned}
$$

- For the intercept and each of the 12 covariates, the NS function is fitted to 36 parameter estimates corresponding to 36 forward starting times (i.e., $0, 1, 2, \cdots, 35$ months).

Accuracy ratios with/without smoothing reported in DSW (2012, *J of Econometrics*)

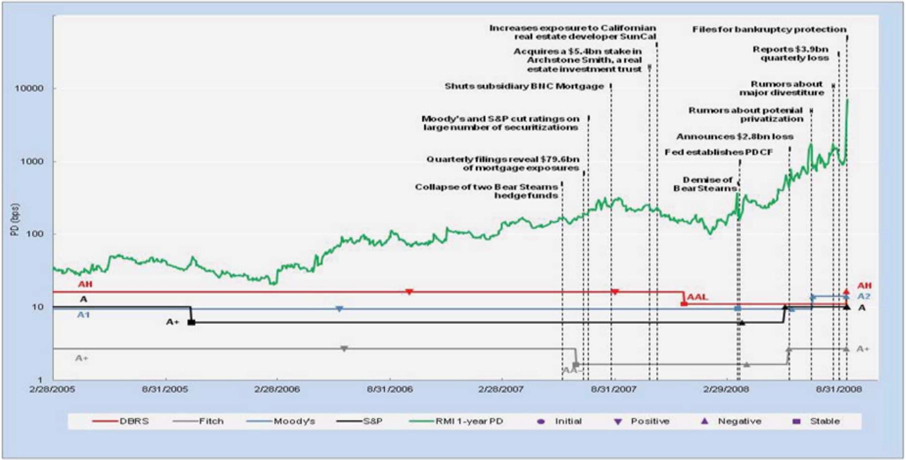
Panel A: In-sample result						
	1 month	3 months	6 months	12 months	24 months	36 months
Full sample	93.22%	91.30%	88.63%	83.52%	74.10%	66.67%
Full sample (smoothed)	93.29%	91.35%	88.65%	83.51%	74.07%	66.66%
Non-financial	93.21%	91.18%	88.32%	82.99%	73.96%	66.98%
Financial	93.03%	91.59%	90.57%	87.38%	74.18%	59.88%
Panel B: Out-of-sample (cross-section) result						
	1 month	3 months	6 months	12 months	24 months	36 months
Full sample	93.77%	91.74%	88.88%	83.36%	73.37%	65.47%
Full sample (smoothed)	93.61%	91.69%	88.88%	83.39%	73.33%	65.43%
Non-financial	93.69%	91.58%	88.50%	82.76%	73.16%	65.84%
Financial	91.28%	89.42%	88.70%	85.64%	71.65%	55.57%
Panel C: Out-of-sample (over time) result						
	1 month	3 months	6 months	12 months	24 months	36 months
Full sample	93.31%	91.81%	89.42%	85.16%	76.43%	72.45%
Full sample (smoothed)	93.51%	91.91%	89.37%	85.02%	76.52%	72.42%
Non-financial	93.73%	92.27%	89.80%	85.37%	77.11%	73.03%
Financial	92.70%	91.65%	91.06%	88.62%	78.04%	73.38%

DSW (2012) vs. Duffie, *et al* (2007)

Apply 4 covariates (trailing 1-year S&P 500 index return, 3-month treasury rate, firms' distance-to-default and firms' 1-year stock return)

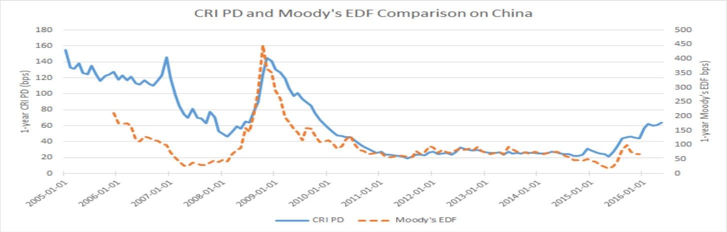
Panel A: In-sample result (1991-2011)						
	1 month	3 months	6 months	12 months	24 months	36 months
Duffie, *et al* (2007)	91.95%	90.06%	88.14%	85.37%	80.54%	77.22%
Duffie, *et al* (2007) (restricted)	91.95%	89.96%	87.24%	81.72%	71.28%	63.85%
Duan, *et al* (2012)	91.95%	89.63%	86.78%	81.43%	71.43%	64.01%
Panel B: In-sample result (2001-2011)						
	1 month	3 months	6 months	12 months	24 months	36 months
Duffie, *et al* (2007)	92.26%	91.08%	89.19%	86.58%	81.22%	77.58%
Duffie, *et al* (2007) (restricted)	92.26%	91.12%	88.91%	84.58%	75.04%	68.98%
Duan, *et al* (2012)	92.26%	90.85%	88.56%	84.68%	76.15%	70.39%
Panel C: Out-of-sample (over time) result (2001-2011)						
	1 month	3 months	6 months	12 months	24 months	36 months
Duffie, *et al* (2007)	91.97%	91.38%	87.43%	77.50%	60.33%	51.87%
Duffie, *et al* (2007) (restricted)	91.97%	90.80%	88.44%	83.52%	71.66%	65.04%
Duan, *et al* (2012)	91.97%	90.50%	88.04%	83.77%	74.67%	70.31%

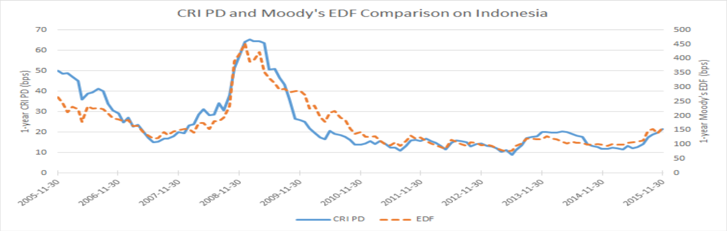
3 years leading up to Lehman Brothers' bankruptcy

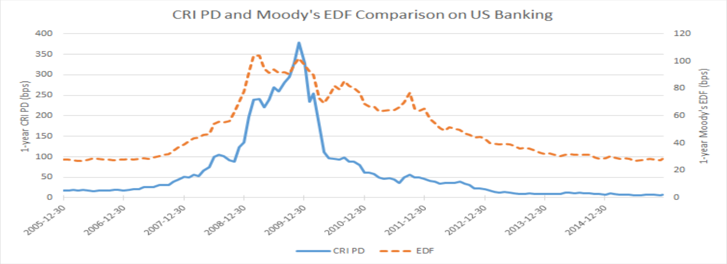


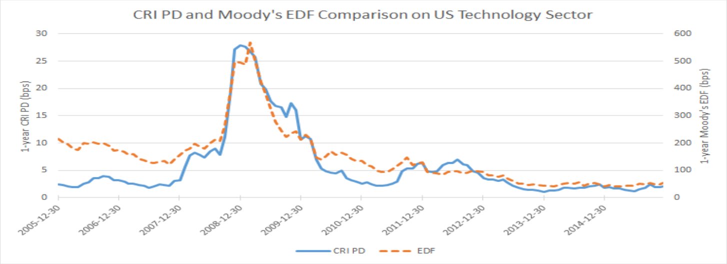
Source: "A Lead-Lag Investigation of RMI PD and CRA Ratings," *Global Credit Review* 2012, 169-188.

Median CRI-PD vs median Moody's EDF for different economies/sectors






CRI PD and Moody's EDF Comparison on US Banking
1-year CRI PD (bps)
1-year Moody's EDF (bps)
400
350
300
250
200
150
100
50
0
120
100
80
60
40
20
0
2005-12-30
2006-12-30
2007-12-30
2008-12-30
2009-12-30
2010-12-30
2011-12-30
2012-12-30
2013-12-30
2014-12-30
CRI PD
EDF

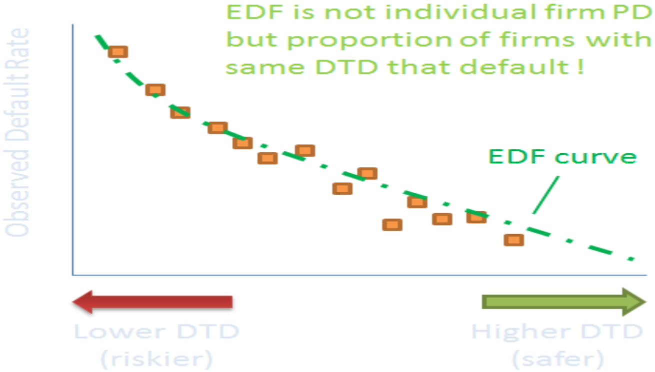

CRI PD and Moody's EDF Comparison on US Technology Sector
1-year CRI PD (bps)
1-year Moody's EDF (bps)
30
25
20
15
10
5
0
600
500
400
300
200
100
0
2005-12-30
2006-12-30
2007-12-30
2008-12-30
2009-12-30
2010-12-30
2011-12-30
2012-12-30
2013-12-30
2014-12-30
CRI PD
EDF

A few observations on EDF

- CRI PDs differ from Moody's EDFs. Typically
  - CRI PD < Moody's EDF (low, medium risk firms)
  - CRI PD > Moody's EDF (high risk firms)
- Four potential reasons (at least)
  - Moody's EDF does not model individual firm's PD explicitly
  - Omitted variables bias
  - Different calibration sample
  - Moody's EDF does not include other exits

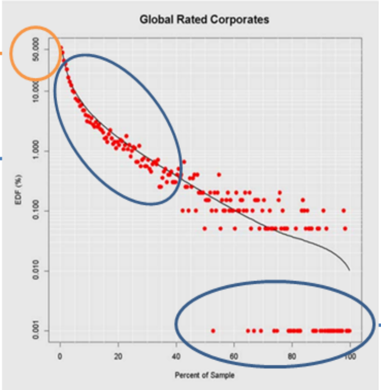
Moody's EDF calibration

1. Group firms in buckets with similar DTD
2. From past data, find the observed default rate of each bucket and plot against DTD
3. EDF = best fit curve



Maximum EDF is 50 percent (observed default rate,worst DTD bucket),

Good fit for risky firms but EDF cannot exceed 50 percent (may understate actual PD)



EDF may overstate default rate (and PD) of low-risk firms

Lower DtD (riskier)

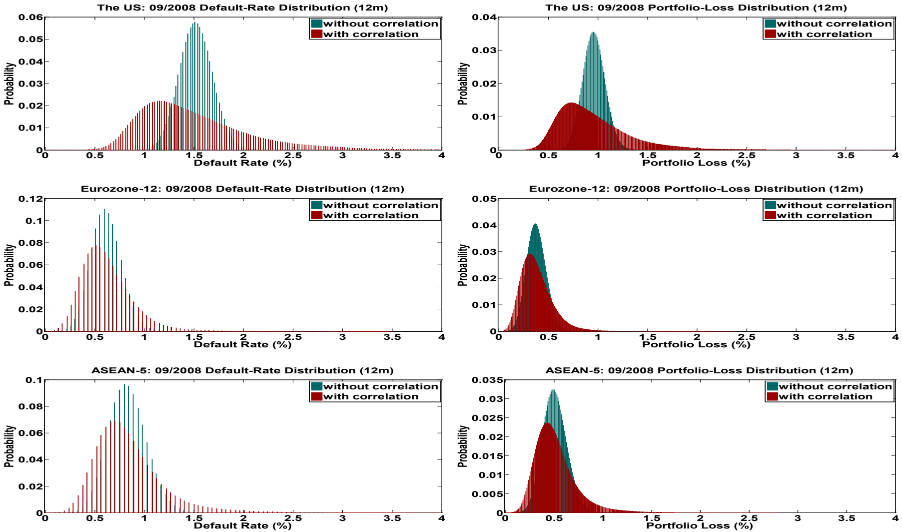
Higher DtD (safer)

Source: EDF 9: Introduction and Overview, Moody's Analytics

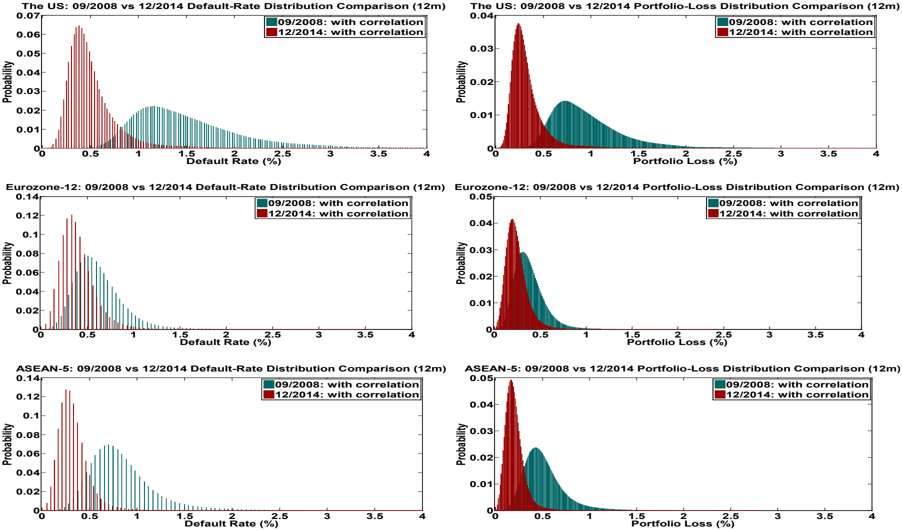
Modeling default correlations

- The spot-intensity approach of Duffie, et al (2007) provides natural default correlations (beyond the immediate period) through stochastic covariates, but modeling their joint time series dynamics poses a serious challenge.
- The forward-intensity approach of Duan, et al (2012) avoids modeling the time series dynamics of stochastic covariates, but it offers no default correlations beyond the immediate period.
- Duan and Miao (2016) models default correlations by constructing and estimating a factor model with sparse residual correlations for one-period probabilities of default (PDs) and probabilities of other exits (POEs). The factor model is further calibrated to term structure of PDs obligor by obligor. The calibrated factor model is then used to deduce correlated longer-term PDs for different obligors.

The results of the Duan and Miao (2016) method



Default correlations



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