Structural vs. Reduced-Form Credit Risk Models

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Types of credit risk

- Default risk: The risk that the counterparty to a financial contract does not honor the terms of a contract (obligor-based).
- Recovery risk: The risk that a defaulting instrument gets a random fraction of the promised payment (instrument-specific).
- Downgrade risk: The risk that a recognized credit rating agency reduces the credit rating of a debt issuer.
- Credit spread risk: The risk that the spread over a reference rate for a debt obligation may increase or decrease.

Note: Default risk does not refer only to the risk of a debt issuer defaulting, but also to counterparty default in connection with derivative contracts and failure to collect account receivables in typical business dealings.

Default and insolvency

- Default is defined as an obligor's failure to honor its payment or delivery obligations.
- Insolvency (balance-sheet) means that an obligor's asset value falls below its liabilities. Balance-sheet insolvency does not necessary lead to default.
- When a balance-sheet solvent obligor defaults due to illiquidity, it is known as cash-flow insolvency.

Types of default

- Payment defaults:
  - Inability to pay within the grace period
  - Repudiation: Refusal to accept a claim as valid
  - Moratorium: Stoppage of payments for some period of time

  The last two items are only for sovereign obligors due to the protection of sovereign immunity.
- Bankruptcy or bankruptcy protection (US: Chapters 7 and 11): Formal legal procedure to attempt fair treatment of all creditors of a defaulted obligor or to retain the going-concern value.
- Involuntary debt organization
- Supply chain credit relationships are typically quite loosely defined/executed, i.e., the grace period is stretchable.
- Others such as conservatorship (Freddie Mac and Fannie Mae in 2008)

Equity as a call option

Call option analogy:

For a limited liability corporate, equity holders are like call option holders, because when the corporate performs well, pay off the debt and keep the residual value, i.e., exercise the call option. If it turns out to be the opposite, limited liability allows for walking away, i.e., default.

Key components of this analogy:

- Current market value of the corporate (i.e., the underlying asset value) and its stochastic dynamics (i.e., volatility)
- Maturity of the debt (i.e., option maturity)
- Promised debt payment or default point (i.e., strike price)

Driver of default: Only insolvency matters and liquidity plays no role.

Recovery rate: Endogenously determined by asset value alone.

The Merton (1974) model

Assumes the firm's unobserved asset value, V_t, follows a geometric Brownian motion:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dB_t$$

where B_t is a standard Brownian motion.

Let F_t denote the time-t assumed default point applicable at time T. The equity value at time t becomes:

$$E_t = V_t \Psi(d_t) - F_t e^{-r(T-t)} \Psi(d_t - \sigma\sqrt{T-t})$$

where

$$d_t = \frac{\ln\left(\frac{V_t}{F_t}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

and $\Psi(\cdot)$ the standard normal cumulative distribution function.

The time-t probability of default equals $\Psi(-DTD_t)$, where

$$DTD_t = \frac{\ln\left(\frac{V_t}{F_t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

To avoid excessive sampling error associated with the estimator for μ, one should use

$$DTD_t^* := \frac{\ln\left(\frac{V_t}{F_t}\right)}{\sigma\sqrt{T-t}}$$

Reduced-form models

Forgo the underlying structure leading to a corporate default, and instead look for a statistical relationship by linking observed defaults to a set of default predictors. Reduced-form models differ in two aspects:

- Employ different sets of default predictors (macroeconomic/macrofinancial variables and individual firm attributes). The choice is mainly based on economic intuition and to some degree data mining. Examples of macro variables are interest rate, term spread, market volatility, market liquidity, credit cycle indices, etc. Individual firm attributes are, for example, DTD, liquidity, volatility, profitability, size, etc.
- Use different statistic tools for classification, for example, logistic regressions or more appropriately multinomial logistic regression. (Why multinomial logistic regression?) Methods abound, but the forward intensity model adopted by the CRI system is far superior.

Challenges in implementing structural models

Technical implementation difficulties:

- Asset values are not directly observable
- Parameters (μ and σ) are for the unobserved asset value process

Conceptual implementation difficulties:
A typical firm's capital structure is much more complex than the model's description (maturity, coupon, priority, etc).

Ad hoc approach 1: market value proxy

- Market value of equity plus book value of liabilities
- Using the market value proxy to estimate parameters is a common practice in empirical credit risk analyses and corporate finance applications. (Brockman and Turtle (2003, *J of Financial Economics*), Eom, Helwege and Huang (2004, *Review of Financial Studies*), Lang and Stulz (1994, *J of Political Economy*), Berger and Ofek (1995, *J of Financial Economics*), etc.

Problems with using the market value proxy

- The market value is an upward biased estimate.
- The magnitude of bias is related to the nature of the firm such as volatility. Thus, it changes the ordinal rankings of the asset value for a sample of firms.
- It is completely silent about the parameters governing the contractual aspect of corporate securities; for example, (1) the financial distress level in some structural credit risk model and (2) forbearance for regulated firms.

Ad hoc approach 2: Volatility restriction

Jones, Mason and Rosenfeld (1984, *J of Finance*) and Ronn and Verma (1986, *J of Finance*)

$$
\begin{aligned}
E_t &= V_t\Psi(d_t) - F_t e^{-r(T-t)}\Psi(d_t - \sigma\sqrt{T-t}) \\
\sigma_{E_t} &= \frac{V_t\Psi(d_t)\sigma}{V_t\Psi(d_t) - F_t e^{-r(T-t)}\Psi(d_t - \sigma\sqrt{T-t})}
\end{aligned}
$$

- The left-hand-side variables are plugged in with equity capitalization and sample equity return volatility. Use the two equations to solve for two unknowns on the right-hand-side: V_t and σ.
- The method is widely used in the credit risk and deposit insurance literature, and is sometimes mistaken as the KMV method.

Problems with the volatility restriction method: (Duan, 1994, *Mathematical Finance*)

- The volatility relationship comes from the first derivative of the equity valuation equation. It does not provide an additional restriction for identification.
- The solution is incorrectly obtained by setting the equity volatility to a constant.
- It cannot produce an estimate for μ or any other unknown parameters in more complex models.

The KMV method

(Crosbie and Bohn, 2003, Moody's KMV Technical Document, p17)

- Assume 1-year maturity for all debts and the maturity remains unchanged from one time point to another.
- Assume an overall debt level using the following default point formula:

$$F = F_{\text{short term}} + 0.5 \times F_{\text{long term}}$$

- Apply an iterative procedure to find the "optimal" σ and μ. (1) Start with some guessed value of σ, (2) obtain the implied asset value time series by inverting equity value with the option pricing formula, (3) compute the sample mean and variance using the implied asset value time series, and (4) repeat until convergence is obtained.

Issues with the KMV method:

- The KMV method provides point estimates but no sampling errors.
- The KMV method can only obtain parameter estimates for the underlying dynamics of the risk factors, but not for the unknown parameters related to capital structure.

  An example: As of December 31, 2007, AIG's total equity market capitalization was \$147.863 billion. The debt size using the KMV formula would become \$92.279 billion (much smaller than equity). AIG's other liabilities stood at \$799.445 billion, but it would be completely left out of the KMV default point.
- If one wants to add AIG's other liabilities to the default point, what should be the unknown haircut rate?

A modified default point formula

- It makes more sense to modify the KMV default point formula to

$$F = F_{\text{short term}} + 0.5 \times F_{\text{long term}} + \delta \times F_{\text{other}}$$

  (see Duan, Sun and Wang, 2012, *J of Econometrics*)

- Using the modified default point formula makes equity a function of unknown δ in addition to σ. The iterative KMV method can update σ, but is incapable of updating δ.

The MLE solution

Transformed data method: (Duan, 1994 and 2000, *Mathematical Finance*)

- A dynamic model is assumed for the unobserved asset price and/or risk factors.
- A time series of prices on some derivative is available. Want to assess default probability and/or price other derivatives.
- The likelihood function is the product of (1) the likelihood value evaluated at the implied values for the asset price or state variable and (2) the Jacobian of the transformation from the observed times series to the unobserved time series.

A scaling assumption (Duan, Sun and Wang, 2012, *J of Econometrics*)

- A firm's asset value can significantly change with a major investment and financing action.
- It makes more sense to standardize a firm's market value of assets by the book value so that the pure scaling effect will not distort the parameter estimates.
- Use the asset value scaled by the book value in the log-likelihood function.

The log-likelihood of a sample of equity values:

$$
\begin{aligned}
& \ln \mathcal{L}(\mu, \sigma, \delta; \boldsymbol{E}_{1:T}) \\
= \quad & -\frac{(T-1)}{2} \ln(2\pi) - (T-1) \ln \sigma - \frac{1}{2\sigma^2} \sum_{t=2}^{T} \left(\Delta \hat{W}_t - \mu \right)^2 \\
& - \sum_{t=2}^{T} \hat{W}_t - \sum_{t=2}^{T} \ln \Psi \left(d_t(\hat{V}_t(\sigma, \delta), F_t(\delta), \sigma) \right)
\end{aligned}
$$

where $\hat{V}_t(\sigma, \delta)$ is the implied asset value, $\hat{W}_t = \ln(\hat{V}_t(\sigma, \delta)/A_t)$ and A_t is book asset value at time t.

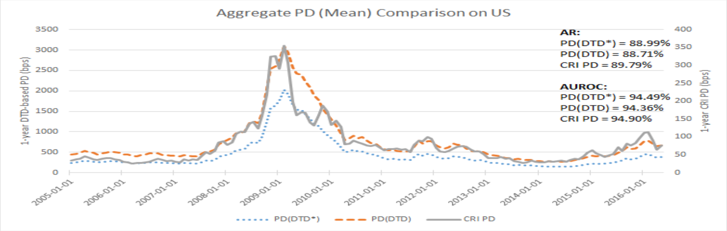
Implementation decision

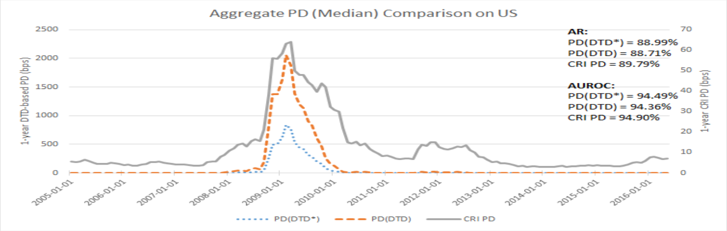
- The haircut to other liabilities, i.e., δ, can be identified only when balance sheet items experience changes. In the typical implementation of using one year of daily time series (250 trading days), balance sheets (quarterly) change only three times. To obtain any reasonable precision, one needs to assume a common haircut rate for all firms in a group, say, the banking sector.
- The current CRI implementation adopts a two-stage estimation approach. (1) Estimate μ, σ and δ firm by firm, (2) average δ across firms, and (3) apply the average δ to restimate μ and σ firm by firm.
- The two-stage estimation approach can actually be improved to become a joint estimation, see Duan and Wang (2017)[1].

[1] Estimating Distance-to-Default with a Sector-Specific Liability Adjustment via Sequential Monte Carlo, J.C. Duan and C. Wang (2017), "Applied Quantitative Finance", eds. Hardle, Chen and Overbeck, Springer.

Default prediction using DTD

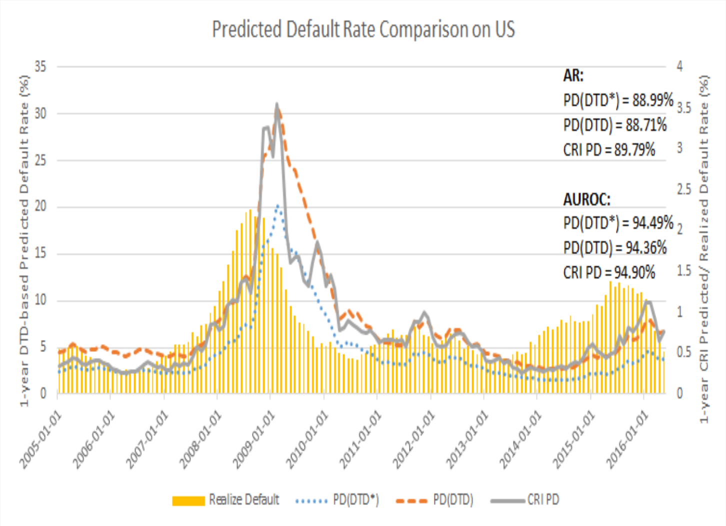
DTD can be directly used to estimate the physical default probability, but it is known to perform rather poorly.



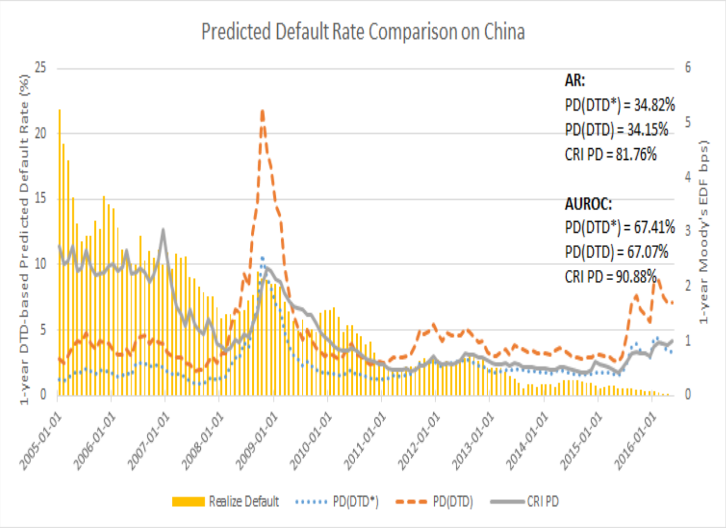


Model PDs vs realized default rates

1-year PD vs realized default rate in the subsequent year (US firms)



1-year PD vs realized default rate in the subsequent year (Chinese firms)



How can DTD be properly used?

- DTD can be quite informative as an explanatory variable in empirical estimation/calibration.
- Ironically, DTD which is generated by a structural model must be used as part of some reduced-form model in order to generate sensible results in practical applications.
- Better estimated DTD will be more informative in ranking credit risks of different corporates , i.e., a better default predictor in a reduced-form model.
- DTD can be understood as a volatility-adjusted leverage, i.e., an improved leverage measure allowing for comparison across industries, for example, financial firms vs non-financial corporates.

Market value proxy and volatility restriction (Duan & Wang, 2012, *Global Credit Review*)

	IBM (million USD)	Barclays (million GBP)	Tokio Marine (million JPY)
The Market Value Proxy Method			
μ	14.91%	-6.94%	-7.53%
σ	16.06%	6.67%	4.84%
Asset value (12/2011)	306,988	1,452,410	15,510,801
DTD (12/2011)	8.4697	-0.8495	0.3326
DTD* (12/2011)	7.6215	0.2233	1.9147
The Volatility Restriction Method			
μ	17.03%	-7.69%	-27.51%
σ	18.19%	3.45%	12.74%
Asset value (12/2011)	278,482	448,327	3,415,782
DTD (12/2111)	10.2009	5.6874	2.0445
DTD* (12/2011)	9.2645	7.8173	4.2032

KMV and MLE (with the KMV assumption)

	IBM (million USD)	Barclays (million GBP)	Tokio Marine (million JPY)
The KMV Method			
μ	17.09%	-7.90%	-26.90%
σ	18.51%	6.09%	16.84%
Asset value (12/2011)	267,464	359,291	3,352,857
DTD (12/2011)	9.8056	-0.4710	1.4360
DTD* (12/2011)	8.9748	0.8563	3.1174
The Transformed-Data MLE Method (with the KMV assumption)			
μ	13.06%	-1.58%	-20.33%
σ	18.47%	6.91%	16.83%
Asset value (12/2011)	267,464	358,293	3,352,858
DTD (12/2011)	9.6054	0.4517	1.8285
DTD* (12/2011)	8.9911	0.7148	3.1208

MLE and improved MLE

	IBM (million USD)	Barclays (million GBP)	Tokio Marine (million JPY)
The Transformed-Data MLE Method (with the KMV assumption)			
μ	13.06%	-1.58%	-20.33%
σ	18.47%	6.91%	16.83%
Asset value (12/2011)	267,464	358,293	3,352,858
DTD (12/2011)	9.6054	0.4517	1.8285
DTD* (12/2011)	8.9911	0.7148	3.1208
The Transformed-Data MLE Method (including other liabilities)			
μ	10.70%	-1.02%	-5.08%
σ	18.02%	1.54%	5.17%
δ	45.78%	61.83%	60.78%
Asset value (12/2011)	280,498	979,658	10,696,331
DTD (12/2011)	8.7478	0.5292	1.6400
DTD* (12/2011)	8.2132	1.1915	2.6339

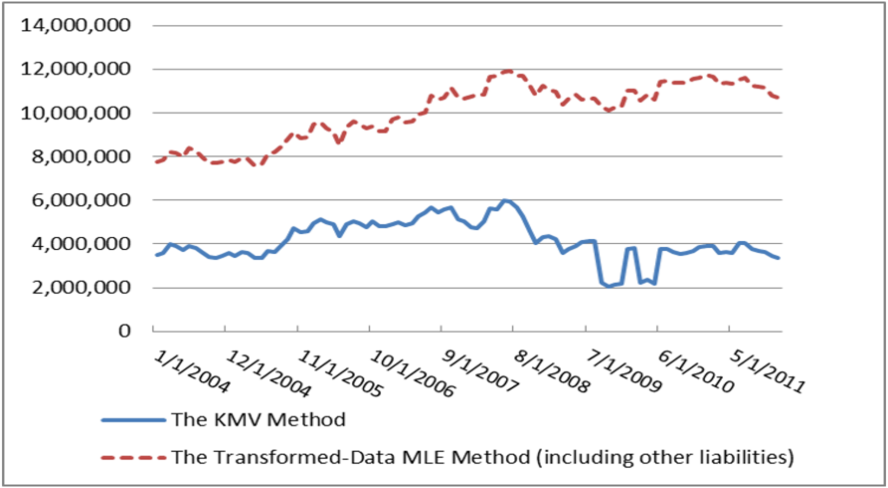
Banks: KMV and improved MLE

	Bank of America (million USD)	Barclays (million GBP)	DBS (million SGD)
Panel A: Input Variables			
Market cap	56,355	21,477	26,976
Short-term debt	617,218	255,193	47,696
Long-term debt	383,517	171,657	18,940
Other liabilities	1,038,408	1,004,083	210,572
Panel B: The KMV Method			
μ	-20.41%	-7.90%	-0.94%
σ	9.41%	6.09%	25.85%
Asset value (12/2011)	849,796	359,291	83,381
DTD (12/2011)	-1.6927	-0.4710	1.2948
DTD* (12/2011)	0.5232	0.8563	1.4604
Panel C: The Transformed-Data MLE Method (including other liabilities)			
μ	-6.45%	-1.02%	-3.88%
σ	3.39%	1.54%	4.51%
δ	57.40%	61.83%	67.24%
Asset value (12/2011)	1,456,323	979,658	224,990
DTD (12/2011)	-0.8484	0.5292	1.8795
DTD* (12/2011)	1.0579	1.1915	2.7491

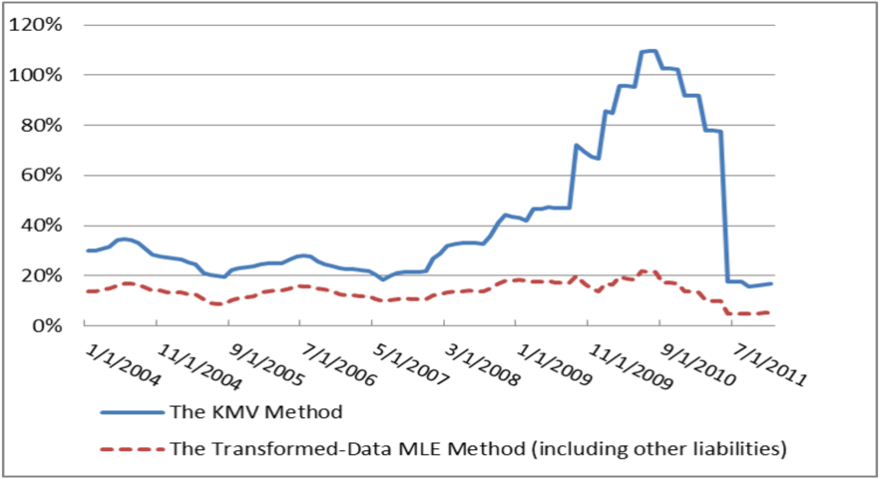
Insurers: KMV and improved MLE

	Sun Life (million CAD)	AXA SA (million EUR)	Tokio Marine (million JPY)
Panel A: Input Variables			
Market cap	11,046	23,678	1,371,714
Short-term debt	694	103,590	1,922,395
Long-term debt	4,889	9,601	121,673
Other liabilities	188,766	543,661	12,095,019
Panel B: The KMV Method			
μ	-0.4061	-0.2891	-0.2690
σ	0.2389	0.4078	0.1684
Asset value (12/2011)	14,154	118,351	3,352,857
DTD (12/2011)	4.4863	-0.6972	1.4360
DTD* (12/2011)	6.3062	0.2156	3.1174
Panel C: The Transformed-Data MLE Method (including other liabilities)			
μ	-0.0365	-0.0365	-0.0508
σ	0.0350	0.0784	0.0517
δ	0.5828	0.6172	0.6078
Asset value (12/2011)	122,856	460,403	10,696,331
DTD (12/2011)	1.3611	-0.0341	1.6400
DTD* (12/2011)	2.4077	0.4645	2.6339

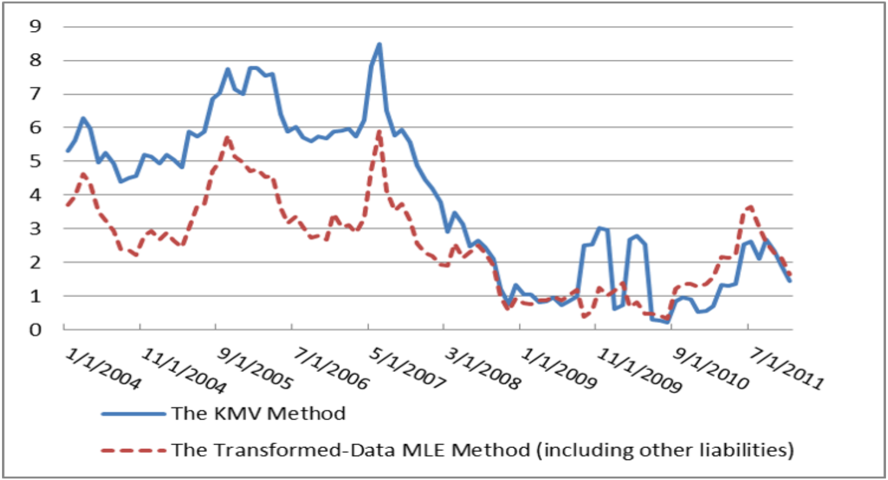
Tokio Marine: implied asset values



Tokio Marine: asset volatilities



Tokio Marine: DTDs



Tokio Marine: DTD*s