The Twin Paradox

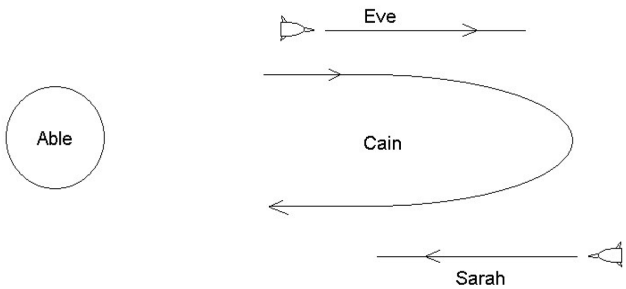
Able and Cain are twins. Able watches Cain go on a long fast trip and returns when Able is old, but Cain is still young. You might think Cain sees Able go on a trip and returns young. This is impossible. Why don't we have time symmetry?

For starters, the situation isn't symmetric because Cain changes his velocity. A more involved analysis is required. We will examine a simple trip for Cain where he travels strait out and back at the same speed (v), with infinite acceleration at the turn-around point.

Eve and Sarah

There will be two observers (Eve and Sarah), traveling at constant velocities, matching Cain's out and back legs of the trip. Eve passes by Able at the start of Cain's trip. Eve meets Sarah going the opposite direction at Cains turn-around point. Sarah passes by Able at the end of Cain's trip.

We make Cain's trip even simpler by having him ride with Eve on the way out, and Sarah on the way back.



Riding with Eve and Sarah

It is important that Eve and Sarah always travel at a constant velocity, so that the rules of special relativity apply to their views of Able and Cain. The three reference frames of Able, Eve, and Sarah have no trouble knowing when and where events happen, via various mechanisms such as radar. However, time and distance increments for the same events will be different in their respective reference frames. In particular, they can know the readings on Able's clock in terms of the readings on their own clock.

Now, Cain initiates his trip by jumping from Abel's reference frame onto Eves ship as she passes by. At the turn-around point, Cain jumps onto Sarah's ship for the return leg. Paradoxically, Sarah, Eve, and Cain all see Able's clock running slower during each leg of the trip. How can Able's clock be ahead when Cain returns? The key is that Sarah sees a later absolute time on Able's clock at the turn-around event than Eve does. This is a case of non-simultaneity of events from different reference frames. Cain's view of Able's clock is always the same as Eve's or Sarah's, according to who he is riding with.

Trip Views
Lets imagine Cain's trip when viewed from different reference frames.

In Able's reference frame, Cain's trip is simply out to the right and back, leaving with Eve and returning with Sarah.

In Eve's reference frame, Able is moving to the left. Cain begins his trip by stopping and waiting with Eve until Sarah comes by and chases quickly after Able to the left.

In Sarah's reference frame, Able and Cain are approaching from the left with Cain running ahead. When Cain reaches Sarah, he stops and waits for Able to catch up.

We will find that Able, Eve, and Sarah all see different times on Able's clock at the turn-around event. According to Eve, the turn-around is earlier than for Able which is earlier than for Sarah.

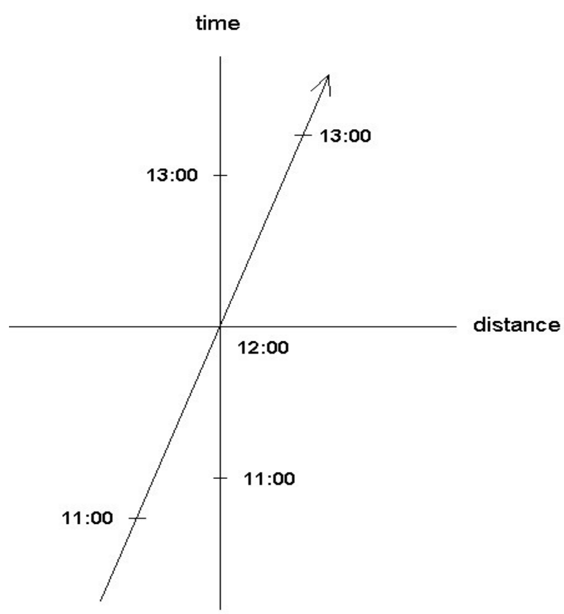
As a result of the extra observers (Eve and Sarah), the turn-around point seems to expand into a time interval. We will interpret the turn-around time interval as equivalent to Cain going into hibernation, and arriving home younger than Able.

Minkowski Diagrams
Special relativity discussions typically use space time graphs with a vertical time axis and spacial coordinates on the horizontal axis. A stationary point is represented as a vertical line.

Reading a Moving Clock
It is reasonable to talk about reading clock dials in moving reference frames even though our sightings may be late because of the speed of light. This is a little like astronomers viewing a distant supernova and stating that it happened millions of years ago. So, we can speak of the position of moving clock hands relative to our own clock after adjustment for distance and the speed of light.

Viewing moving clocks is especially interesting if a moving clock is synchronized with ours when it passes through our space origin. The moving clock has longer hour increments.

Our view of the moving clock shows a later time when it is approaching and an earlier time when it is leaving.



Algebraic Details

We will be using the Lorentz scale factor for relating time and distance increments between moving reference frames. $\gamma=\frac{1}{\sqrt{1-v^2/c^2}}=\frac{1}{\sqrt{1-\beta^2}}>1$.

Here, v is the relative speed between Able and Eve or Able and Sarah.

If Eve or Sarah watches a 2 hr movie on her space ship, then Able watching the same movie through a telescope sees the movie last 2γ hours (ignoring Doppler effect). If Eve measures the length of her space ship to be 50 meters, Able sees the ship to be $50/\gamma$ meters long.

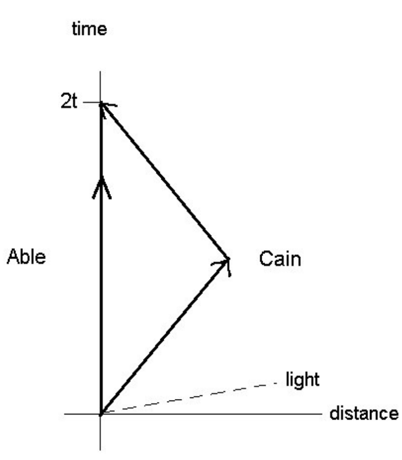
We will also use the conversion formula for velocity in a moving reference frame

$$u=\frac{v+u'}{1+vu'/c^2}$$.

If Eve throws a ball to the right at 100 meters per second (as measured by Eve), then Able sees the ball traveling $\frac{v+100}{1+(100\,v)/c^2}$ meters per second.

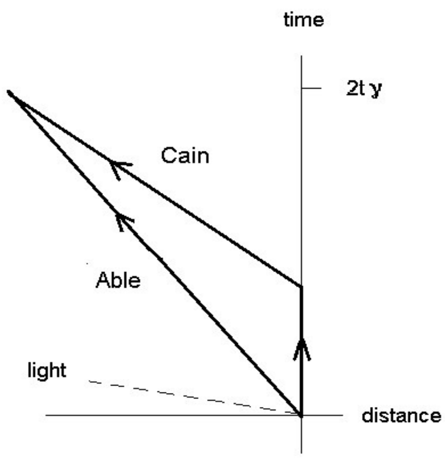
The tricky part is finding Cain's wait and travel times and distance. This is resolved by knowing when and how fast Cain leaves and returns in Able's reference frame.

Able's View, Out and Back



This is a simple case of applying the conversion factor γ . Able watches Cain travel at velocity v to the turn-around at distance vt during a time t . The total trip occurs during a time 2t. But Cain's elapsed clock time is $2t\frac{1}{\gamma}$.

Eve's View, Wait and Chase



From Eve's view, Cain waits with her and then chases after Able.

Cain ends his total trip at at time $2t\gamma$, while Able traveled a distance $2tv\gamma$.
Cain's chase velocity is $v_C = \dfrac{2v}{1+v^2/c^2} = \dfrac{2v}{1+\beta^2}$, using velocity conversion.

His chase time is $2tv\gamma\dfrac{1}{v_C} = t\left(1+\beta^2\right)\gamma$, using distance/speed.

His wait time is $2t\gamma - t\left(1+\beta^2\right)\gamma$, subtracting chase time from total time.

Cain's Lorenz factor during the chase is

$$\gamma_C = \frac{1}{\sqrt{1-v_C^2/c^2}} = \frac{1}{\sqrt{1-\dfrac{4\beta^2}{\left(1+\beta^2\right)^2}}} = \frac{\left(1+\beta^2\right)}{\sqrt{1+2\beta^2+\beta^4-4\beta^2}} = \frac{\left(1+\beta^2\right)}{\sqrt{1-2\beta^2+\beta^4}} = \frac{\left(1+\beta^2\right)}{\left(1-\beta^2\right)}$$

At the end of the trip, Able's clock shows his aging at $2t$

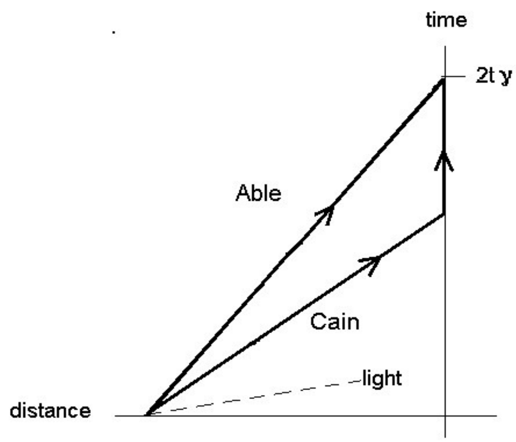
Meanwhile Cain's total aging on the wait-and-chase legs of the trip is given by

$$\left[2t\gamma - t\left(1+\beta^2\right)\gamma\right] + \left[t\gamma\frac{\left(1+\beta^2\right)}{\gamma_C}\right] = t\gamma\left[2-\left(1+\beta^2\right)+\frac{\left(1+\beta^2\right)\left(1-\beta^2\right)}{\left(1+\beta^2\right)}\right] = t\gamma\left[1-\beta^2+1-\beta^2\right] = 2t\,\frac{1}{\gamma}$$

Note that there is no Lorentz factor for Cain while he is waiting.

We conclude that Eve sees Cain aging less than Able by the same $1/\gamma$ factor.

Sarah's View, Run Ahead and Wait



From Sarah's view, Able and Cain are moving towards her, with Cain running ahead. Then, Cain stops and waits with Sarah until Able catches up.

The algebra is similar Eve's wait and chase case. The order is slightly changed.

In this case, Cain's run-ahead velocity is $v_C = \dfrac{2v}{1+v^2/c^2} = \dfrac{2v}{1+\beta^2}$.

Then Cain stops and waits for Able to catch up.
They meet at time $2t\gamma$ and distance $2vt\gamma$.

Cain's run-ahead travel time is $2tv\gamma\dfrac{1}{v_C} = t\left(1+\beta^2\right)\gamma$

Cain's wait time is $2t\gamma - t\left(1+\beta^2\right)\gamma$

Cain's Lorentz factor is $\gamma_C = \dfrac{\left(1+\beta^2\right)}{\left(1-\beta^2\right)}$

During the trip, Able has aged $2t$.

Meanwhile Cain's ageing on the run-ahead-and-wait legs of the trip is given by

$\left[t\gamma\dfrac{\left(1+\beta^2\right)}{\gamma_C}\right] + \left[2t\gamma - t\left(1+\beta^2\right)\gamma\right] = 2t\ \dfrac{1}{\gamma}$, just as before.

What Cain Sees

We have ignored what Cain see's because he is not in a constantly moving reference frame.

Evidently, when Cain jumps to Sarah's ship, he sees Able's slower clock jump ahead. Able's clock remains ahead of Cain's until the end of the trip. Cain might think that he fell asleep during his jump, and his clock froze during that time.

Cain's infinite acceleration at the turn-around is unrealistic. But, if he undergoes a finite but extreme acceleration, he will see Able's clock running faster. This is a typical general relativity effect, where an observer that is accelerating or in a gravitational field sees the un-accelerated world evolving faster.

The case with Cain undergoing finite acceleration could be analyzed with a continuum of observers matching Cain`s velocity as he changes transportation between Eve and Sarah.

Side Note about the Doppler Shift

In the previous discussion, I have described time observations after the observers have accounted for distance with appropriate calculations. I did not discuss the pre-calculation observations that include a time shift and Doppler effect. A Doppler shift analysis can replace the Lorentz scale factor analysis. However, this does not address the coordinated clocks issue when Cain jumps ships from Eve to Sarah.