13. Riemann Curvature

As a feature of curvature, a vector field has a deficit angle ϕ obtained from integrating the covariant derivative of a vector field around a loop with area A. Of course $\phi \to 0$ as the loop gets smaller. We define the Riemann curvature as $K_R = \lim_{A \to 0} \frac{\phi}{A}$. This definition is intrinsic to the Riemannian manifold and does not depend on the embedding. We will see that this limit exists and $K_R = K_G$.

Vector Valued Differential Forms

While studying curvature, we considered the line integral of the covariant derivative of a vector field. To this end, it helps to extend our notion of scalar valued differential forms to vector valued differential forms. This extension follows a comfortable pattern. A vector valued 0-form is just a vector field. A vector valued 1-form is a field of linear maps from vectors to vectors. A vector valued k-form is a field of skew symmetric linear maps from k-tuples of vectors to vectors. Major operations on general vector valued k-forms are problematic, such as wedge product and exterior derivative.

We will use the general stokes theorem as a guiding principle for evaluating the loop integral of the derivative of a vector field. Stokes theorem extends to vector valued forms in a Cartesian space where we interpret a vector valued k-form using the individual components as scalar valued k-forms.

To use the general stokes theorem, we will need an extended version of the exterior derivative. Since the ordinary derivative of a vector field is not a tensor, We use the covariant derivative in the definition. The exterior derivative of a vector valued 1- form $\vec{\omega}(\vec{u})$ can be calculated in a Cartesian space as:

$$d\vec{\omega}(\vec{V},\vec{W}) = \nabla_{\vec{V}}\, \vec{\omega}(\vec{W}) - \nabla_{\vec{W}}\, \vec{\omega}(\vec{V}) - \vec{\omega}\left([\vec{V},\vec{W}]\right) \quad \text{(a tensor).}$$

In a Riemannian space the covariant derivative can be used to get the tensor $d\vec{\omega}(\vec{V},\vec{W}) = \delta_{\vec{V}}\, \vec{\omega}(\vec{W}) - \delta_{\vec{W}}\, \vec{\omega}(\vec{V}) - \vec{\omega}\left([\vec{V},\vec{W}]\right)$

There are problems using stokes theorem for vector valued forms. In the above discussion, the integral of the derivative along the path does not equal the difference of the vectors at the ends. We will address this next.

The Riemann Curvature Tensor

We wish to determine the covariant change in a vector field (disallow full circle rotations) around a small loop as: $\oint_{\text{loop}} \delta_{\dot{\vec{s}}} \vec{v}\, dt$. For the vector valued 1-form defined by $\vec{\omega}_{\vec{v}}(\dot{\vec{s}}) \equiv \delta_{\dot{\vec{s}}} \vec{v}$ in terms of the vector field $\vec{v}$. We would like to use the general stokes theorem and calculate: $\iint_{\text{loop area}} d\vec{\omega}_{\vec{v}}$.

Stokes Theorem does not extend to vector valued forms. However, we can work around this with scalar valued differential forms. Our scalar valued forms will represent rotation and we will define the Riemann curvature tensor.

Expressing Rotation

Consider two orthonormal vector fields $\hat{\alpha}$ and $\hat{\beta}$. The 1-form $\omega_{\alpha\beta}(\dot{s})=\langle \delta_{\dot{s}}\hat{\alpha},\hat{\beta}\rangle$ captures the idea of $\hat{\alpha}$ rotating while traversing a path s

Using an Orthogonal Coordinate System

Consider an orthogonal coordinate system with variables α and β with corresponding vector fields $\vec{\alpha}$ and $\vec{\beta}$ following the coordinate grid, and the normalized fields $\hat{\alpha}$ and $\hat{\beta}$.

We want to find the rotation of $\hat{\alpha}$ around a loop by integrating the 1-form $\omega_{\alpha\beta}$. Instead, we will integrate its exterior derivative over the interior area of the loop.

Use $d\omega(\vec{V},\vec{W})=\nabla_{\vec{V}}\,\omega(\vec{W})-\nabla_{\vec{W}}\,\omega(\vec{V})-\omega([\vec{V},\vec{W}])$ to get

$$d\omega_{\alpha\beta}(\vec{V},\vec{W})=\langle \delta_{\vec{V}}\delta_{\vec{W}}\hat{\alpha},\hat{\beta}\rangle+\langle \delta_{\vec{W}}\hat{\alpha},\delta_{\vec{V}}\hat{\beta}\rangle-\langle \delta_{\vec{W}}\delta_{\vec{V}}\hat{\alpha},\hat{\beta}\rangle-\langle \delta_{\vec{V}}\hat{\alpha},\delta_{\vec{W}}\hat{\beta}\rangle-\langle \delta_{[\vec{V},\vec{W}]}\hat{\alpha},\hat{\beta}\rangle$$

We will simplify the above expression by defining the Riemann curvature tensor by $\vec{R}(\vec{U},\vec{W})\vec{v}\equiv \delta_{\vec{U}}\delta_{\vec{W}}\vec{v}-\delta_{\vec{W}}\delta_{\vec{U}}\vec{v}-\delta_{[\vec{U},\vec{W}]}\vec{v}$.

Traditionally, this tensor is encountered by examining derivatives of the metric tensor. Here, we encountered it through analyzing vector field rotation and Stokes theorem. This tensor has symmetry properties that are useful to us.

Now use $\vec{R}(\vec{\alpha},\vec{\beta})\hat{\alpha}=\delta_{\vec{\alpha}}\delta_{\vec{\beta}}\hat{\alpha}-\delta_{\vec{\beta}}\delta_{\vec{\alpha}}\hat{\alpha}-\delta_{[\vec{\alpha},\vec{\beta}]}\hat{\alpha}$ to get

$$d\omega_{\alpha\beta}(\vec{\alpha},\vec{\beta})=\langle \vec{R}(\vec{\alpha},\vec{\beta})\vec{\alpha},\vec{\beta}\rangle+\langle \delta_{\vec{\beta}}\vec{\alpha},\delta_{\vec{\alpha}}\vec{\beta}\rangle-\langle \delta_{\vec{\alpha}}\vec{\alpha},\delta_{\vec{\beta}}\vec{\beta}\rangle$$

We have $\langle \delta_{\vec{\beta}}\hat{\alpha},\delta_{\vec{\alpha}}\hat{\beta}\rangle=0$ and $\langle \delta_{\vec{\alpha}}\hat{\alpha},\delta_{\vec{\beta}}\hat{\beta}\rangle=0$, because $\hat{\alpha}$ and $\hat{\beta}$ are orthonormal.

So $d\omega_{\alpha\beta}(\vec{\alpha},\vec{\beta})=\langle \vec{R}(\vec{\alpha},\vec{\beta})\hat{\alpha},\hat{\beta}\rangle=\dfrac{1}{\|\vec{\alpha}\|\|\vec{\beta}\|}\langle \vec{R}(\vec{\alpha},\vec{\beta})\vec{\alpha},\vec{\beta}\rangle$.

In our case of an orthogonal coordinate system , we can calculate the deficiency angle as $-\phi = \iint\limits_{\text{loop interior}} \frac{1}{\|\vec{\alpha}\|\|\vec{\beta}\|}\langle \vec{R}(\vec{\alpha},\vec{\beta})\vec{\alpha},\vec{\beta}\rangle \, d\alpha \, d\beta$

Calculating Riemann Curvature

In our orthogonal coordinates case,

$$K_R = \lim_{A\to 0} \frac{\phi}{A} = -\lim_{A\to 0} \frac{\iint\limits_{\text{loop interior}} \frac{1}{\|\vec{\alpha}\|\|\vec{\beta}\|}\langle \vec{R}(\vec{\alpha},\vec{\beta})\vec{\alpha},\vec{\beta}\rangle \, d\alpha \, d\beta}{\iint\limits_{\text{loop interior}} \|\vec{\alpha}\|\|\vec{\beta}\| \, d\alpha \, d\beta} = -\frac{\langle \vec{R}(\vec{\alpha},\vec{\beta})\vec{\alpha} \, , \, \vec{\beta}\rangle}{\|\vec{\alpha}\|^2\|\vec{\beta}\|^2}$$

Generalizing Riemann Curvature

What happens if we use different vector fields $\vec{\alpha}$ and $\vec{\beta}$? Our calculations are based on a particular choice of independent vector fields $\vec{\alpha}$ and $\vec{\beta}$ and their total rotation. Any other pair of vector fields should experience the same total rotation for a given loop.

To deal with a non-orthogonal coordinate system we conjecture the generalization

$$K_R = -\frac{\langle \vec{R}(\vec{\alpha},\vec{\beta})\vec{\alpha} \, , \, \vec{\beta}\rangle}{\|\vec{\alpha}\|^2\|\vec{\beta}\|^2 - \langle \vec{\alpha},\vec{\beta}\rangle^2}$$

The independence of K_R on $\vec{\alpha}$ and $\vec{\beta}$ can be shown by considering replacing $\vec{\alpha}$ and $\vec{\beta}$ with independent linear combinations of these two and find that this reduces to the original expression. This reduction uses linearity and these symmetry identities for the Riemann curvature tensor:

$\vec{R}(\vec{u},\vec{v})\vec{w} = -\vec{R}(\vec{v},\vec{u})\vec{w}$ and $\langle \vec{R}(\vec{u},\vec{v})\vec{w},\vec{z}\rangle = \langle \vec{R}(\vec{w},\vec{z})\vec{u},\vec{v}\rangle$.

Relating Gaussian and Riemann Curvature

For a 2-d manifold embedded in 3-d, we have two definitions of curvature that turn out to be the same. To show this, we explore the relationship between the tipping of the embedded normal with the rotation of vector fields within the surface. We will build a field of 3-d axes over the embedded surface and examine its behavior around a loop. First we consider a small region of the manifold and the vector field of normals $\vec{\gamma}$. With some exertion, we can choose a coordinate system with parameters α and β where the local basis vectors $\vec{\alpha}$ and $\vec{\beta}$ are orthogonal eigenvectors of the derivative of $\hat{\gamma}$. In the 3-d embedding space, $\nabla_{\vec{\alpha}}\hat{\gamma} = \lambda_\alpha \vec{\alpha}$ and $\nabla_{\vec{\beta}}\hat{\gamma} = \lambda_\beta \vec{\beta}$. Also, we can have

$\|\vec{\alpha}\|\|\vec{\beta}\| \, \hat{\gamma} = \vec{\alpha}\times\vec{\beta}$,with $\vec{\alpha} = \frac{\|\vec{\alpha}\|}{\|\vec{\beta}\|}\vec{\beta}\times\hat{\gamma}$ and $\vec{\beta} = \frac{\|\vec{\beta}\|}{\|\vec{\alpha}\|}\hat{\gamma}\times\vec{\alpha}$

We will show the relationship between the eigenvalues of the derivative of $\vec{\gamma}$ and the rotation of the coordinate vectors. To this end, we follow the triple of vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ along a path s(t) and view its 3-d rotational velocity as an axial vector $\vec{\omega}$. For any vector field $\vec{v}$ that remains fixed with respect to the $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ along the path, the velocity due to rotation is given by $\vec{\omega} \times \vec{v}$,

At any point in the path, we can express the rotation velocity as the sum of a rotation tipping away from $\hat{\gamma}$ about an axis tangent to the surface, and a rotation about $\hat{\gamma}$ which reflects the rotation involved with Riemannian curvature:

$$\vec{\omega}(\dot{\vec{s}}) = \hat{\gamma} \times \nabla_{\dot{\vec{s}}} \hat{\gamma} + \frac{1}{\|\vec{\alpha}\| \|\vec{\beta}\|} \left(\nabla_{\dot{\vec{s}}} \vec{\alpha} \cdot \vec{\beta} \right) \hat{\gamma} = \vec{\omega}_{//}(\dot{\vec{s}}) + \vec{\omega}_{\perp}(\dot{\vec{s}})$$

If we follow the vector field $\vec{v}$ around a loop, we find that the net change is $\vec{0}$.
So, the loop integral must satisfy:

$$\oint \vec{\omega}(\dot{\vec{s}}) \times \vec{v} \, dt = \vec{0}$$

We can choose a vector valued 1-form defined as $\vec{\omega}_\beta(\dot{\vec{s}}) \equiv \vec{\omega}(\dot{\vec{s}}) \times \hat{\beta}$
and note that $\vec{\omega}_\beta = \vec{\omega}_{\beta //}(\dot{\vec{s}}) + \vec{\omega}_{\beta \perp}(\dot{\vec{s}})$
In summary $\vec{\omega}$ is the rotational speed resulting from traversing the path.
And we have vector valued 1-forms $\vec{\omega}_\beta$, $\vec{\omega}_{\beta //}$, and $\vec{\omega}_{\beta \perp}$ that depend on $\vec{\omega}$ and $\hat{\beta}$.

We can evaluate $\oint \vec{\omega}_\beta(\dot{\vec{s}}) dt = \vec{0}$ on an arbitrarily small loop using Stokes Theorem on the 3-d Cartesian embedding space. Then we find that $d\vec{\omega}_{\beta //}(\vec{\alpha}, \vec{\beta}) = -d\vec{\omega}_{\beta \perp}(\vec{\alpha}, \vec{\beta})$.

Evaluating $d\vec{\omega}_{\beta //}(\vec{\alpha}, \vec{\beta})$ results in an expression with the required eigenvalues because it contains terms like $\nabla_{\dot{\vec{\alpha}}} \hat{\gamma} = \lambda_\alpha \vec{\alpha}$ Evaluating $d\vec{\omega}_{\beta \perp}(\vec{\alpha}, \vec{\beta})$ will result in an expression containing $\vec{R}(\vec{\alpha}, \vec{\beta})\vec{\alpha}$. After comparing corresponding components of $d\vec{\omega}_{\beta //}$ and $d\vec{\omega}_{\beta \perp}$ we find that $K_R = K_G$.