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ESTIMATION

Program 1: The Sophisticated Investor.

The objective of the program is to ascertain how well the pattern of information acquisition by a specific investor concerning the valuation of his portfolio (decisions to "look" or "not look") can be replicated by the actions of a hypothetical rational investor with loss averse preferences and rational expectations in the context of a simple dynamic economic model. The notion of rational expectations is the (very strong) assumption that the model agent knows the precise probability distributions governing all economic quantities of interest to him.[1]

As a rational investor, he uses this knowledge to make information acquisition decisions in order to maximize the discounted present value of future utility where, as noted in the introduction to the text, his period utility function is defined over portfolio gains and losses relative to a benchmark rather than the more customary consumption. The present note describes the strategy employed in the numerical solution to this problem.

Underlying our numerical strategy is the perspective that the individual investor whose behavior we wish to match has held his Vanguard account for a very long time and plans to continue this status for the indefinite future. In effect we are attempting to match the "slice-out-of-time" information acquisition behavior for a specific investor, whom we view as an infinite horizon utility maximizer, for the period 2007-2008.

The structure of the numerical algorithm is first to use straightforward dynamic programming to compute the optimal decision rule for the hypothetical investor. When the hypothetical investor is confronted with the same rate of return phenomenon as his actual counterpart, and applies his derived optimal decision rule, he generates a series of hypothetical "look" and "not look" actions which can be compared to the actual observed decisions of the investor under study. A simple goodness of fit measure is proposed. Some basic knowledge of dynamic programming is assumed.

The dynamic programming portion of Program 1 is simply the numerical manifestation of the principle of optimality which holds that an investor's period by period decisions over his lifetime must, if they are to be inter-temporally optimal, satisfy the following criterion: the investor's optimal decision to "look" or "not look" in period t is conditional on the assumption that he will act optimally in all future time periods. Accordingly, the investor must, in period t, know the conditional decisions he will make in all feasible future states and dates and their utility consequences before he makes his period t decision. Our algorithm must thus employ some backward recursion feature, and this note principally describes how this procedure is implemented.

Suppose a specific actual investor has been selected.

[1] We implicitly assume the specific investor has this ability as well.

1.1 The required input data.

Since we have assumed rational expectations on the part of the model investor, we need to take a stand on what he knows. In particular, we assume the model investor, at any time, knows the probability density over his actual wealth, which he does not observe prior to his look/don't look decision, as conditioned by what he does observe, $\overline{W}_t$ and W_{-T_t}. The former quantity is his approximate wealth as determined largely by $\{r_{m,t}\}$, while the latter is his true wealth when he last looked, T_t periods ago. Because both $\tilde{r}_{m,t}$ and $\tilde{r}_{p,t}$ are individually and jointly log-normally distributed in the data, we will assume this relationship in the model as well. When the model investor draws near the end of a period and is confronted with the look/don't look decision we thus assume he knows the joint density $f\left(W_t \,|\, \overline{W}_t, W_{-T_t}, \mu_p, \sigma_p, \mu_m, \sigma_m, \rho_{r_m r_p}, \beta_{r_m r_p}\right)$ which, as noted, is condition not only by $\overline{W}_t$ and W_{-T_t}, but all the moments, correlation, and beta relevant to the precise density identification. Knowledge of this density is necessary for the model investor to compute, among other quantities, his expected utility of looking that period. Since the model investor's environment must be (subject to the limitations of our model) coincident with the actual investor's environment, the $\mu_p, \sigma_p, \mu_m, \sigma_m, \rho_{\tilde{r}_m \tilde{r}_p}$, and $\beta_{\tilde{r}_m \tilde{r}_p}$ quantities in the model must match their counterparts in the data[2]. Accordingly, our basic input data is:

(1) $\{r^c_{m,t}\}$, $\hat{\mu}^c_m$, $\hat{\sigma}^c_m$ (the actual sample daily realizations of market returns, continuously compounded and their statistical summary; this data is common to all investors and their hypothetical counterparts), and,

(2) $\{r^c_{p,t}\}, \{W_t\}$, $\hat{\mu}^c_p$, $\hat{\sigma}^c_p$, $\hat{\rho}_{r^c_{m,t}, r^c_{p,t}}$, $\hat{\beta}_{r^c_{m,t}, r^c_{p,t}}$ (the actual daily portfolio returns and normalized wealth series for the specific investor under study and his return series statistical summary inclusive of its statistical relationship to the market's returns; these return series are also continuously compounded). The length of the series will be noted as $\hat{T}$.

We need $\{r^c_{p,t}\}$ and $\{W_t\}$, in particular, because when the model investor looks, we want him to see what the actual investor sees; i.e., the actual investor's true wealth.

[2] Effectively we assume the actual investor knows the conditional density himself, and can make the same calculations. This may be a stretch. But we don't know and, in fact, don't need to know how the actual investor processes this data. Our model is a good one if it replicates his behavior. This is the (limited) sense of a "good model" in the content of economic science.

1.2 An Important Approximation.

In the execution of the about-to-be described algorithm, we will need a discrete approximation to the log-normal distributions which govern the evolution of $\{\overline{W}_t\}$ and of $\{W_t\}$. We outline below how this is accomplished, illustrated in the context of $\{\overline{W}_t\}$.

Suppose at some time period t, the investor's approximate wealth signal is $\overline{W}_t$, and he elects not to look. In making this decision, as we will see, he will need to know a lot about the probability distribution governing $\overline{W}_{t+1}$

While the investor knows:

$\tilde{\overline{W}}_{t+1} = \overline{W}_t e^{\tilde{r}^c_{m,t}}$, where $\tilde{r}^c_{m,t} \sim N\left(\hat{\mu}^c_m, \hat{\sigma}^c_m\right)$,

it is information that is difficult to work with when packaged as a continuous density. A discrete approximation to the continuous probability distribution governing the investor's approximate wealth $\overline{W}_{t+1}$ is needed.

To construct such an approximation it is customary to divide the time period into N sub periods, and track the evolution of $\overline{W}_t$ through this tree of sub-periods. In each sub-period,

$\tilde{r}^c_{m,t} \sim N\left(\frac{\hat{\mu}^c_m}{N}, \hat{\sigma}^c_m\sqrt{1/N}\right)$.

The binomial approximation to this continuous density is to postulate that in any sub-period $\overline{W}_t$ can either increase or decrease in value according to:

a. $u = e^{\hat{\sigma}^c_m\sqrt{1/N}}$,

b. $d = 1/u$,

c. $\pi = \dfrac{e^{\hat{\mu}^c_m\left(\frac{1}{N}\right)} - d}{u-d}$,

where u denotes the gross rate of return on the market in the "up-market" state, while d denotes the gross rate of return in the "down-market" state; π is the probability of the former occurrence, $1-\pi$ of the latter[3]. With these identifications, the evolution of $\overline{W}_t$ through the many sub-periods is as follows:

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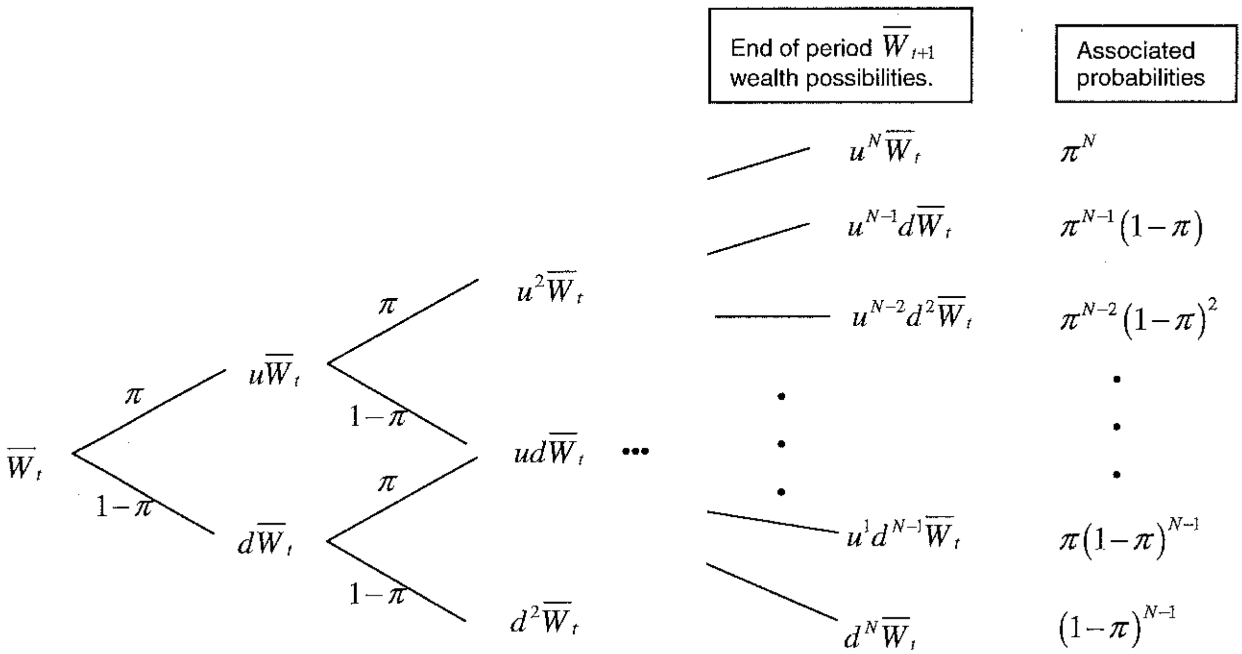
[3] Identifications a., b., and c. uniquely guarantee that $\hat{\mu}^c_m\left(\frac{1}{N}\right) = \pi(u-1) + (1-\pi)(d-1)$, and

$\hat{\sigma}^c_m\sqrt{\frac{1}{N}} = \sqrt{\pi\left(u-1-\frac{1}{N}\hat{\mu}^c_m\right)^2 + (1-\pi)\left(d-1-\frac{1}{N}\hat{\mu}^c_m\right)^2}$; i.e., the continuous distribution and its approximation agree on the critical moments.



The $N+1$ discrete $\overline{W}_{t+1}$ possibilities so generated, in conjunction with their cumulative probabilities, will constitute the basis of our discrete approximation to $\tilde{\tilde{W}}_{t+1}$ under the log-normal distribution: $\overline{W}_{t+1} \sim \log normal\left(\hat{\mu}_m^c \overline{W}_t,\ \hat{\sigma}_m^c \sqrt{\overline{W}_t}\right),\ 0 < \tilde{\tilde{W}}_{t+1} < \infty$. Accordingly, all possible wealth levels must be assigned one of the discrete probabilities. Such identification is accomplished in the following way:

if $\widetilde{W}_{t+1} \in B\left(\overline{W}_t, 0\right) = \left\{ W_{t+1} : \frac{1}{2}\left(u^N\overline{W}_t + u^{N-1}d\overline{W}_t\right) \le W_{t+1} < \infty \right\}$, then

$prob\left(\widetilde{W}_{t+1}\right) = \pi^N$;

if $\widetilde{W}_{t+1} \in B\left(\overline{W}_t, 1\right) = \left\{ W_{t+1} : \frac{1}{2}\left(u^{N-1}dW_t + u^{N-2}d^2W_t\right) \le W_{t+1} < \frac{1}{2}\left(u^NW_t + u^{N-1}dW_t\right) \right\}$, then

$prob\left(\widetilde{W}_{t+1}\right) = \pi^{N-1}(1-\pi)$;

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if $\widetilde{W}_{t+1} \in B\left(\overline{W}_t, j\right) = \left\{ W_{t+1} : \frac{1}{2}\left(u^jd^{N-j}\overline{W}_t + u^{j-1}d^{N-j+1}\overline{W}_t\right) \le W_{t+1} < \frac{1}{2}\left(u^jd^{N-j}\overline{W}_t + u^{j+1}d^{N-j+1}\overline{W}_t\right) \right\}$, then

$prob\left(\widetilde{W}_{t+1}\right) = \pi^j(1-\pi)^{N-j}$.

Lastly,

if $\widetilde{W}_{t+1} \in B\left(\overline{W}_t, 0\right) = \left\{0 < W_{t+j} \frac{1}{2}\left(d^N \overline{W}_t + u d^{N-1} \overline{W}_t\right)\right\}$, then

$prob\left(\widetilde{W}_{t+1}\right) = (1-\pi)^N$.

This procedure amounts to constructing non overlapping half open intervals around the $N+1$ possible discrete wealth levels and assigning to each one a specific probability value. These intervals cover the entire range of potential $\widetilde{W}_{t+1}$ wealth levels, $(0;\infty)$. If W_{t+1} is any value in interval $B\left(\overline{W}_t, j\right)$ it is assigned probability $\pi^j (1-\pi)^{N-j}$.

1.3 Parameter Choices.

Our objective is to assess what particular combination of parameters $(\alpha, \beta, \gamma_0, \theta, \delta)$, when assigned to the model's artificial agent, allows his choices to best replicate the actually observed series of "looks" and "not looks" undertaken by the investor under study.
Sets A,B,G,T, and D, each a subset of R_1^+ must be chosen so that the eligible $(\alpha, \beta, \gamma_0, \theta, \delta) \in A \times B \times G \times T$. In what follows, let a particular candidate $(\alpha', \beta', \gamma_0', \theta', \delta')$ be chosen. The next sections describe how an artificial agent, with parameter assignment $(\alpha', \beta', \gamma_0', \theta', \delta')$, constructs his sequence of "look" or "does not look" decisions.

1.4 The State Variables and their range.

The state vector for the artificial agent in the model economy is $(\overline{W}_t, b_t, T_t, W_{-T_t})$ with the meaning assigned to each variable the same as in the text. Each argument of this vector is important. First, b_t is important as it enters directly as an argument in both $u^L(\ )$ and $u^{DL}(\ )$. The remaining arguments $\overline{W}_t, T_t, \text{and } W_{-T_t}$ are important as they condition the period t probability distribution on the agents actual wealth level. This conditioning follows because $\tilde{r}_p$ and $\tilde{r}_m$ are linearly related.

It will be necessary to circumscribe the overall subset of R_4^+ in which eligible state variable combinations lie.

Let $\mu^* = \max\left\{\hat{\mu}_p^c, \hat{\mu}_m^c\right\}$

$\sigma^* = \max\left\{\hat{\sigma}_p^c, \hat{\sigma}_m^c\right\}$.

For each t, we constrain the eligible $(\overline{W}_t, b_t, T_t, W_{-T_t})$ to lie in $P^N_{[W^L_t, W^U_t]} \times P^N_{[W^L_t, W^U_t]} \times \mathscr{T}_t \times P^N_{[W^L_t, W^U_t]} \equiv S^N_t$, where $P^N_{[W^L_t, W^U_t]}$ is an $N+1$ element, equally-spaced partition of $[W^L_t, W^U_t] = \left[W_0 e^{(\mu^* - 2\sigma^*)\hat{T}}, W_0 e^{(\mu^* - 2\sigma^*)\hat{T}}\right]$.

Given μ^* and σ^*, $[w_t^L, w_t^U]$ is constructed in this way to reflect the fact that there is a 95% probability that W_t, $\overline{W}_t$, W_{-T_t}, and b_T each, individually, lies in this region. The fineness of the partition, as measured by N, then becomes an additional parameter to be chosen prior to program initiation.

The time index T_t, which measures the number of time periods prior to t that the investor last looked lies in the set $\mathcal{T}_t = \{1,2,3,...,t\}$, where t is the date to which the decision process has then evolved. In particular, the model agent cannot have looked more distantly in the past than the length of time his decision process has been active.
All calculations will be undertaken with regard only to S_t^N in a manner that will be made clear.

1.5 Determination of the model agent's choice function.

Denote the agent's decision, when he finds himself in state $(\overline{W}_t, b_t, T_t, W_{-T_t})$, by $A_t = A(\overline{W}_t, b_t, T_t, W_{-T_t}) \in \{0,1\}$ with "1" signifying the decision to look and "0" the decision not to look. Dynamic programming, as operationalyzed by backward induction, is the standard technique to estimate A_t (that is, to define A_t everywhere on the state space S_t^N). The notion behind backward induction is first to assume a terminal date for the agent's decision process and determine his action at that terminal date. We then solve for A_t under the assumption that the investor will employ $\{A_s : s > t\}$ in all periods leading up to the terminal date. In this way the investor's decision (whether to look or not) is influenced by the knowledge of how his welfare and decisions in the future dates are determined by his action in period t. These latter actions have continued influence because of the evolution of the benchmark, and the resetting of the $\{\overline{W}_t\}$ for any period in which the investor checks his portfolio.

To describe this procedure we use the same notation as in text equations (I), (II), and (III) and similarly adopt the convention that "j" will index "backward time"; e.g., $j = 0$ will now correspond to the final period of the model investor's time horizon.

As we work backwards, the influence of the decisions made in the artificial final period ($j = 0$) on the model investor's expected utility, is less and less significant as its immediate utility contribution ends up being increasingly discounted. As a consequence, as j increases eventually we will start to see no change in the $A_j(\ )$ as j continues to increase. When no change is observed in the action rule from j to $j+1$, for any set of state variables, we will take the decision rule at the j^{th} iteration as representing the decision rule of the "infinitely lived" permanent investor. The $A_j(\ )$ thereby achieved constitutes the "slice of life" decision function.

1.5.1 Consider the agent's behavior at $j=0$. His choice of action at $j=0$, when he finds himself in some state $(\overline{W}_0, b_0, T_0, W_{-T_0}) \in S_{\hat{T}}^N$, is determined according to the following decision problem:

$$J_0(\overline{W}_0, b_0, T_0, W_{-T_0}) = \max_{A_0 \in \{0,1\}} \left\{ E_{W_0/\overline{W}_0, W_{-T_0}} u^L(W_0, b_0), u^{DL}(\overline{W}_0, b_0) \right\},$$

Where,

if $J_0(\overline{W}_0, b_0, T_0, W_{-T_0}) = E_{W_0/\overline{W}_0, W_{-T_0}} u^L(W_0, b_0)$,

then $A_0(\overline{W}_0, b_0, T_0, W_{-T_0}) = 1$ (he looks); and

if $J_0(\overline{W}_0, b_0, T_0, W_{-T_0}) = u^{DL}(\overline{W}_0, b_0)$,

then $A_0(\overline{W}_0, b_0, T_0, W_{-T_0}) = 0$ (he does not look).

In this way, $J_0(\overline{W}_0, b_0, T_0, W_{-T_0}) : S_{\hat{T}}^N \to R$ and $A_0(\overline{W}_0, b_0, T_0, W_{-T_0}) : S_{\hat{T}}^N \to \{0,1\}$.

Notice that these functions are defined over a partition of all possible states in which the actual investor might find himself after the passage of $\hat{T}$ days.

To be more precise,

$$E_{W_0/\overline{W}_0, W_{-T_0}} u^L(W_0, b_0) = \sum_{W_0(k) \in P^N_{[W^L_{\hat{T}}, W^H_{\hat{T}}]}} u(W_0(k_0), b_0)\, prob(W_0(k) \mid \overline{W}_0, W_{-T_0})$$

Where

$$prob(W_0(k) \mid \overline{W}_0, W_{-T_0}) = \frac{1}{\sqrt{2\pi}\sqrt{1-\hat{\rho}^{\,2}_{r^c_{m,t}, r^c_{p,t}}}\ \hat{\sigma}^c_m} \cdot e^{-\frac{1}{2}\left[\frac{\ln\left(\frac{W_0(k)}{W_{-T_0}}\right) - T_0\hat{\mu}^c_p - \left(\hat{\sigma}^c_p \big/ \hat{\sigma}^c_m\right)\hat{\rho}_{r^c_{m,t}, r^c_{p,t}}\left(\ln\left(\frac{\overline{W}_0}{W_{-T_0}}\right) - T_0\hat{\mu}^c_m\right)}{\sqrt{1-\hat{\rho}^{\,2}_{r^c_{m,t}, r^c_{p,t}}}\ \hat{\sigma}^c_m}\right]^2}$$

These conditional probabilities formally capture the fact that the evolution of the investor's approximate wealth measure, since he last looked, and his actual wealth as seen at that date, together provide information as to the probability distribution governing his actual wealth at $j=0$, under the distributional assumptions made throughout this paper.

We retain $A_0(\overline{W}_0, b_0, T_0, W_{-T_0})$ and $J_0(\overline{W}_0, b_0, T_0, W_{-T_0})$ for all $(\overline{W}_0, b_0, T_0, W_{-T_0}) \in S_{\hat{T}}^N$ for later use.

Our description of the model investor's decision process at $j=0$ suggests he behaves myopically at this time whereas the actual investor, whose behavior we seek to replicate, most likely does not: although our data description of his behavior (choices) ends at this point, there is no evidence he terminates his account at the same date. As a result, our $j=0$ convention introduces bias (as would any arbitrary convention we might propose).

By searching for the stationary action rule of an infinitely-lived investor, however, this initial bias is irrelevant: as we work backwards, the effect of discounting leads its utility consequences for the form of the stationary action rule to become inconsequential. By the very nature of its construction, therefore, the form of action rule is invariant to any assumption about actions at the backward induction terminal date. Since this action rule is then applied to the same series of state vectors as experienced by the actual investor, the model investor and the actual investor make their decisions within the context of the same environment, and thus can be legitimately compared.

1.5.2 Next consider the agent's behavior at $j=1$: the choice of $A_1(\overline{W}_1,b_1,T_1,W_{-T_1})$ and the determination of $J_1(\overline{W}_1,b_1,T_1,W_{-T_1})$ for all $(\overline{W}_1,b_1,T_1,W_{-T_1})\in S_{\hat{T}}^N$. In terms of backward time the model agent is in the next-to-last period of his decision horizon.

$$(1)\quad J_1(\overline{W}_1,b_1,T_1,W_{-T_1})=\max_{A_1\in\{0,1\}}\begin{Bmatrix} E_{W_1/\overline{W}_1,W_{-T_1}}u^L(W_1,b_1)+\beta E_{\overline{W}_0/\overline{W}_1,W_{-T_1}}J_0(\overline{W}_0,b_0^L,T_0^L,W_{-T_0}^L), \\ u^{DL}(\overline{W}_1,b_1)+\beta E_{\overline{W}_0/\overline{W}_1,W_{-T_1}}J_0(\overline{W}_0,b_0^{DL},T_0^{DL},W_{-T_0}^{DL}) \end{Bmatrix}.$$

For ease of exposition, let the first term in the brackets defining (1) be denoted by U_1^L, and the second, U_1^{DL}. If $U_1^L>U_1^{DL}$, the model agent looks; otherwise he does not. For additional clarity we also add superscripts L and DL to the arguments of $J_0(\ )$ to clarify which term we are discussion.

a. Computation of U_1^L.

(i) $\quad E_{W_1/\overline{W}_1,W_{-T_1}}u^L(W_1,b_1)$.

At $j=1$, the agent must decide to look or not based on the information available to him $\overline{W}_1$, T_1, and W_{-T_1}, which does not include knowledge of his true wealth at that time, W_1. As a result, his $j=1$ utility increment must be present in expected form, an expectation that is nearly identical in structure to the analogous $j=0$ case:

$$E_{W_1/\overline{W}_1,W_{-T_1}} u^L(W_1,b_1) = \sum_{W_1(k)\in P^N_{[W^L_{\hat{T}},W^u_{\hat{T}}]}} u^L(W_1(k),b_1)\, prob(W_1(k)\,|\,\overline{W}_1,W_{-T_1})$$

$$= \sum_{W_1(k)\in P^N_{[W^L_{\hat{T}},W^u_{\hat{T}}]}} u^L(W_1(k),b_1)\cdot \frac{1}{\sqrt{2\pi}\sqrt{1-\hat{\rho}^2}\,\hat{\sigma}^c_m}$$

$$\cdot e^{-\frac{1}{2}\left[\frac{\ln\left(\frac{W_1(k)}{W_{-T_1}}\right)-T_1\hat{\mu}^c_p-\left(\hat{\sigma}^c_p/\hat{\sigma}^c_m\right)\hat{\rho}^c_p\left(\ln\left(\frac{\overline{W}_1}{W_{-T_1}}\right)-T_1\hat{\mu}^c_m\right)}{\sqrt{1-\hat{\rho}^2_p}\,\hat{\sigma}^c_m}\right]^2},$$

with $\hat{\beta}^c_p = \hat{\beta}_{r^c_{m,t},r^c_{p,t}}$ and $\hat{\rho}^c_p = \hat{\rho}_{r^c_{m,t},r^c_{p,t}}$ (we will maintain this identification going forward to simplify our notation). Information available to the investor is captured by the dependence of the aforenoted conditional density on the known quantities $\overline{W}_1$, T_1, and W_{-T_1}.

(ii) $\beta E_{\overline{W}_0/\overline{W}_1,W_{-T_1}} J_0(\overline{W}_0,b^L_0,T^L_0,W^L_{-T_0})$.

This expression defines the agent's continuation expected utility conditional on his having decided to look at $j=1$, but before he actually does look. We first note that:

$T^L_0 = 1$ (he looked at j=1),
$W^L_{-T_0} = W_1$ (when he looked at j=1, he saw W_1, whatever that might be), and
$b^L_0 = \delta W_1 + (1-\delta)b_1$

Both W_1 and b^L_0 are rounded to the nearest partition element in $P^N_{[W^L_{\hat{T}},W^L_{\hat{T}}]}$. Note that the $j=0$ benchmark is updated from $j=1$ in a manner consistent with the investor's having looked at $j=1$. A more detailed representation of (ii) is as follows:

$$E_{W_0/\overline{W}_1,W_{-T_1}} J_0(\overline{W}_0,b^L_0,T^L_0,W^L_{-T_0}) = \sum_{\overline{W}_0(l)\in P^N_{[W^L_{\hat{T}},W^u_{\hat{T}}]}}\ \sum_{W_1(k)\in P^N_{[W^L_{\hat{T}},W^u_{\hat{T}}]}} J_0(\overline{W}_0,b^L_0,T^L_0,W^L_{-T_0})$$

$$\cdot prob\left(\overline{W}_0(l)\,|W_1(k)\right) prob\left(W_1(k)\,|\,\overline{W}_1,W_{-T_1}\right)$$

Note that if the agent elects to look at $t=1$ and sees $W_1(k)$, then his $j=1$ benchmark becomes $W_1(k)$, and thus $\overline{W}_0 = W_1(k)e^{r^c_{m,0}}$ for some realization $r^c_{m,0}$ drawn from $N(\hat{\mu}^c_m,\hat{\sigma}^c_m)$.

Because the model investor must make this calculation before he looks, there is a double expectation taken first over the possible W_1 wealth levels the investor might find when he looks and the expectation over possible W_0 values that could follow from whatever W_1 he does actually observe.

While $prob\left(W_1(k)\,|\,\overline{W_1},W_{-T_1}\right)$, for each $W_1(k)\in P^N_{\left[W^L_{\hat{T}},\,W^U_{\hat{T}}\right]}$ is defined in a manner directly analogous to their computation in the calculation of $E_{W_1/\overline{W_1},W_{-T_1}}u^L(W_1,b_1)$, $prob\left(\overline{W}_0(l)\,|\,W_1(k)\right)$ is computed as per the discussion in section 1.2.

b. Computation of U_1^{DL}.

(i) $u^{DL}(W_1,b_1)$ is computed directly from the definition of $u^{DL}(\cdot,\cdot)$.

(ii) $\beta E_{\overline{W}_0/\overline{W_1},W_{-T_1}}J_0(\overline{W}_0,b_0^{DL},T_0^{DL},W^{DL}_{-T_0})$

Three of the arguments of $J_0(\cdot)$ are deterministic and related to the state vector $(\overline{W}_1,b_1,T_1,W_{-T_1})$; by earlier specifications:

$b_0^{DL}=\theta\overline{W}_1+(1-\theta)b_1$

$T_0^{DL}=T_1+1$

$W^{DL}_{-T_0}=W_{-T_1}$.

The expression

$$E_{\overline{W_0}/\overline{W_1},W_{-T_1}}J_0(\overline{W}_0,b_0^{DL},T_0^{DL},W^{DL}_{-T_0})=\sum_{\overline{W}_0(l)\in P^N_{[W^L_{\hat{T}},W^u_{\hat{T}}]}}J_0(\overline{W}_0(l),b_0^{DL},T_0^{DL},W^{DL}_{-T_0})\cdot prob\left(\overline{W_0}(l)\,|\,\overline{W_1}\right)$$

Where $prob\left(\overline{W_0}(l)\,|\,\overline{W_1}\right)$ is computed as per section 1.2.

The program then retains $A_1(\overline{W}_1,b_1,T_1,W_{-T_1})$ and $J_1(\overline{W}_1,b_1,T_1,W_{-T_1})$ for all $(\overline{W}_1,b_1,T_1,W_{-T_1})\in S^N_{\hat{T}}$.

1.5.3 Computing $A_{j+1}(\overline{W}_{J+1},b_{j+1},T_{j+1},W_{-T_{j+1}})$ and $J_{j+1}(\overline{W}_{j+1},b_{j+1},T_{j+1},W_{-T_{j+1}})$ for all $(\overline{W}_{j+1},b_{j+1},T_{j+1},W_{-T_{j+1}})\in S^N_{\hat{T}}$, given $J_j(\overline{W}_j,b_j,T_j,W_{-T_j})$ defined, by prior constructions, for all $(\overline{W}_j,b_j,T_j,W_{-T_j})\in S^N_{\hat{T}}$. The procedure is nearly identical to the one used to compute $(\overline{W}_1,b_1,T_1,W_{-T_1})$.

We define

$$J_{j+1}(\overline{W}_{j+1},b_{j+1},T_{j+1},W_{-T_{j+1}})=$$

$$\max_{A_{j+1}(\overline{W}_{j+1},b_{j+1},T_{j+1},W_{-T_{j+1}})\in\{0,1\}}\left\{\begin{array}{l} E_{W_{j+1}|\overline{W}_{j+1},W_{-T_{j+1}}}u^{L}\left(W_{j+1},b_{j+1}\right)+\beta E_{\overline{W}_j|\overline{W}_{j+1},W_{-T_{j+1}}}J_j(\overline{W}_j,b_j^L,T_j^L,W_{-T_j}^L), \\ u^{DL}\left(\overline{W}_{j+1},b_{j+1}\right)+\beta E_{\overline{W}_j|\overline{W}_{j+1},W_{-T_{j+1}}}J_1(\overline{W}_j,b_j^{DL},T_j^{DL},W_{-T_j}^{DL}) \end{array}\right\},$$

and let U_{j+1}^{L} and U_{j+1}^{DL} denote, respectively, the first and second terms in the brackets. If $U_{j+1}^{L}>U_{j+1}^{DL}$, then $A_{j+1}=1$. Otherwise, $A_{j+1}=0$.

a. Computation of U_{j+1}^{L}.

(i)

$$E_{W_{j+1}/\overline{W}_{j+1},W_{-T_{j+1}}}u^{L}(W_{j+1},b_{j+1})$$

$$=\sum_{W_{j+1}(l)\in P_{[W_{\hat{T}}^L,W_{\hat{T}}^u]}}u^{L}(W_{j+1}(l),b_{j+1})\cdot\left(\frac{1}{\sqrt{2\pi}\sqrt{1-\hat{\rho}_p^{2}}\,\hat{\sigma}_m^{c}}\right)$$

$$\cdot e^{-\frac{1}{2}\left[\frac{\ln\left(\frac{W_{j+1}(l)}{W_{-T_{j+1}}}\right)-T_{j+1}\hat{\mu}_p^{c}-\left(\hat{\sigma}_p^{c}/\hat{\sigma}_m^{c}\right)\hat{\rho}_p^{c}\left(\ln\left(\frac{\overline{W}_{j+1}}{W_{-T_{j+1}}}\right)-T_{j+1}\hat{\mu}_m^{c}\right)}{\sqrt{1-\hat{\rho}_p^{2}}\,\hat{\sigma}_m^{c}}\right]^2}$$

Note that the quantities T_{j+1} and $W_{-T_{j+1}}$ come from the $j+1$ state vector while $\hat{\mu}_m^{c},\hat{\sigma}_m^{c},\hat{\beta}_p,\hat{\rho}_p$ etc. are known to the agent. With index updates, expression (i) is identical to the one defining the first term of $J_1(\ )$.

(ii)

$$E_{\overline{W}_j/\overline{W}_{j+1},W_{-T_{j+1}}}J_j(\overline{W}_j,b_j^L,T_j^L,W_{-T_j}^L)$$

$$=\sum_{W_j(l)\in P^N_{[W_{\hat{T}}^L,W_{\hat{T}}^u]}}\ \sum_{W_{j+1}(k)\in P^N_{[W_{\hat{T}}^L,W_{\hat{T}}^u]}}J_0\left(\overline{W}_j(l),\delta b_{j+1},(1-\delta)W_{j+1}(k),1,W_{j+1}(k)\right)$$

$$\cdot prob\left(\overline{W}_j(l)\,|\,W_{j+1}(k)\right)\cdot prob\left(W_{j+1}(k)\,|\,\overline{W}_{j+1},W_{-T_{j+1}}\right)$$

where $prob\left(W_j(l) \mid W_{j+1}(k)\right)$ is defined per section 1.2 and $prob\left(W_{j+1}(k) \mid \overline{W}_{j+1}, W_{-T_{j+1}}\right)$ is defined as per calculation (i).

b. Computation of U^{DL}_{j+1}.

(i) $U^{DL}\left(\overline{W}_{j+1}, b_{j+1}\right)$ is again calculated directly from the definition of $U^{DL}(\cdot)$.

(ii) The continuation value

$$E_{\overline{W}_j/\overline{W}_{j+1}, W_{-T_{j+1}}} J_j(\overline{W}_j, b_j^{DL}, T_j^{DL}, W_{-T_j}^{DL})$$

$$= \sum J_j\left(\overline{W}_j(l), \theta\overline{W}_{j+1}, (1-\theta)b_j, T_{j+1}+1, W_{-T_{j+1}}\right) \cdot prob\left(\overline{W}_j(l) \mid W_{j+1}, W_{-T_{j+1}}\right)$$

where the probabilities are again computed as per section 1,2 etc.

The process continues iteratively for $j+1, j+2, \ldots$

1.5.4 Convergence

For every 4-vector $(\overline{W}, b, T, W_{-T}) \in S_{\hat{T}}^N$, we have constructed a series of numbers $\left\{A_j = A_j(\overline{W}, b, T, W_{-T}) : j = 1, 2, \ldots, A_j \in \{0,1\}\right\}$ such that A_j solves $J_j(\overline{W}, b, T, W_{-T})$.
We continue this iteration procedure until, for some $\hat{j}$ and all $j' \geq \hat{j}+1$, $\sum_{(\overline{W}, b, T, W_{-T}) \in S_{\hat{T}}^N} \left|A_{j'}(\overline{W}, b, T, W_{-T}) - A_{j'-1}(\overline{W}, b, T, W_{-T})\right| < \varepsilon$, for some previously chosen non negative number ε which may be zero. Our theoretical analysis in section D confirms that such $\hat{j}$ exists with a smaller β in general dictating a smaller $\hat{j}$.

This allows us to define the stationary decision rule $A^*(\overline{W}, b, T, W_{-T}) : S_{\hat{T}}^N \to \{0,1\}$ by $A^*(\overline{W}, b, T, W_{-T}) \equiv A_{\hat{j}}(\overline{W}, b, T, W_{-T})$.

Note that $A^*(\overline{W}, b, T, W_{-T})$ is uniquely identified with the set of parameters chosen at the start of the iterative procedure, $(\alpha, \beta, \gamma_0, \theta, \delta)$ and thus is more precisely be described by: $A^*(\overline{W}, b, T, W_{-T}; \alpha, \beta, \gamma_0, \theta, \delta)$.

1.6 Goodness of fit.

Let us return to the forward time perspective and denote it, once again by t.
Our model agent for whom we have computed the unique $A^*(\ )$ is an abstraction of the actual investor under study. Index this actual investor by i, "his" environment is characterized by the time series $\{W_t^i, A_t^i, r_{p,t}^i, r_{m,t}^i\}_{t=1}^{500}$, where

W_t^i = investor i's true normalized wealth $(W_0^i \equiv 1)$ at the end of trading day t, and $A_t^i = A^*(\ )$ for the particular chosen $(\alpha, \beta, \gamma_0, \theta, \delta)$.

A_t^i = his actual decision taken on day t,

$r_{p,t}^i$ = his portfolio's return during day t; and,

$r_{m,t}$ = the market return on day t (all returns continuously compound); we omit the superscript c for notational convenience.

This information must be transformed into the series $\left\{{}_M\overline{W}_t^i, {}_M b_t^i, {}_M T_t^i, W_{-{}_M T_t^i}\right\}_{t=1}^{500}$ "recognizable" by our model agent (the subscript M denotes model while the superscript i maintains the association with investor i under study). This is accomplished in conjunction with use determination of the model decision $\{{}_M A_t^i\}$ as follows:

(i) ${}_M\overline{W}_0^i = {}_M b_0^i = {}_M W_{-T_0^i} \equiv 1; {}_M T_0^i = 1, \text{and } {}_M A_0^i \equiv 1$, by construction.

(ii) Suppose the model agent is confronted with $\left({}_M\overline{W}_t^i, {}_M b_t^i, {}_M T_t^i, {}_M W_{-T_t^i}\right) \in S_{\hat{T}}^N$ in day t,

and elects decision ${}_M A_t^i = A^*\left({}_M\overline{W}_t^i, {}_M b_t^i, {}_M T_t^i, {}_M W_{-T_t^i}\right)$.

Then,

a. $${}_M\overline{W}_{t+1}^i = \begin{cases} {}_M\overline{W}_t^i e^{r_{M,t+1}} & if\ {}_M A_t^i = 0 \\ W_t^i e^{r_{M,t+1}} & if\ {}_M A_t^i = 1 \end{cases}.$$

Note that W_t^i is the actual investor's wealth at time t, not his model namesake's.

b. $${}_M b_{t+1}^i = \begin{cases} \theta\,{}_M\overline{W}_t^i + (1-\theta)\,{}_M b_t^i & if\ {}_M A_t^i = 0 \\ \delta W_t^i + (1-\delta)\,{}_M b_t^i & if\ {}_M A_t^i = 1 \end{cases}$$

c. $${}_M T_{t+1}^i = \begin{cases} 1 + {}_M T_t^i & if\ {}_M A_t^i = 0 \\ 1 & if\ {}_M A_t^i = 1 \end{cases}$$

d. $${}_M W_{-T_{t+1}^i} = \begin{cases} {}_M W_{-T_t^i} & if\ {}_M A_t^i = 0 \\ W_t^i & if\ {}_M A_t^i = 1 \end{cases}.$$

All quantities are appropriately rounded so that the state vector remains an element of $S_{\hat{T}}^N$ for all time periods t. In this way, the sequence $\{{}_M A_t^i\}_{t=1}^{500}$ is generated. It is then compared to the

investor's actual sequence of decisions $\{A_t^i\}$ according to the following goodness-of-fit criterion, GOF_1^i.

$$GOF_1^i = GOF_1^i(\alpha,\beta,\gamma_0,\theta,\delta) = \sum_{t=1}^{500}\left(A_t^i - {}_M A_t^i\right)^2.$$

For investor i, the parameter set $(\alpha^*,\beta^*,\gamma_0{}^*,\theta^*,\delta^*)_i$ that gives the minimum GOF_1^i, solves:

$$\min_{(\alpha,\beta,\gamma_0,\theta,\delta)\in A\times B\times G\times T\times D} GOF_1^i$$

For individual i in a particular subset I of interest, each with $(\alpha^*,\beta^*,\gamma_0{}^*,\theta^*,\delta^*)_i$, the corresponding frequency distribution can be constructed over $A\times B\times G\times T\times D$.

1.7 A second, less computationally intensive perspective.

The infinite-horizon investor whose behavior in a slice-out-of-time is to be matched by a similarly infinite horizon model agent is quite intensive numerically since the $A^*(\ )$ must be defined on the entire state space $S_{\hat{T}}^N$. In contrast, if we hypothesize a finitely lived agent, lifetime $\hat{T}$, working backwards to solve his finite horizon dynamic program, we are free to recognize that, as we iterate backwards, fewer and fewer possible $\overline{W}, b, T, \text{and } W_{-T}$ could actually have occurred when starting from $\overline{W}_0 = W_0 = b_0 = W_{-T_0} \equiv 1$. As a result, at any time j (backward time) we are free to consider only those vectors in $S_{\hat{T}}^N$ that also satisfy:

a) $e^{(\mu^*-2\sigma^*)(\hat{T}-j)} \le \overline{W}_j \le e^{(\mu^*+2\sigma^*)(\hat{T}-j)}$

b) $e^{(\mu^*-2\sigma^*)(\hat{T}-j)} \le W_j \le e^{(\mu^*+2\sigma^*)(\hat{T}-j)}$

c) $e^{(\mu^*-2\sigma^*)(\hat{T}-j)} \le W_{-T_j} \le e^{(\mu^*+2\sigma^*)(\hat{T}-j)}$

Inequalities a) and b) imply:

d) $e^{(\mu^*-2\sigma^*)(\hat{T}-j)} \le b_j \le e^{(\mu^*+2\sigma^*)(\hat{T}-j)}$

In addition,

e) $T_j < \hat{T}_{-j}$.

Accordingly, $A_j(\ )$ and $J_j(\ )$ need only be defined over

$S_j^N = S_{\hat{T}}^N \cap \left\{\left(\overline{W}_j, b_j, T_j, W_{-T_j}\right): \text{a), b), c), d), and e) are satisfied}\right\}$, with the results that the number of calculations declines exponentially with increasing j.

There is, however, no free lunch. In the case where we solved for the decision rule of an infinite horizon model agent, the backward induction procedure would end when successive decision rules satisfied our convergence criterion. In actual practice, no more than twenty iterations have been required. In contrast, and notwithstanding the reduced state space, by

regarding the model agent as finitely lived, the full set of decision rules $\left\{A_j : j=1,2,...,\hat{T};\ A_j : S_j^N \to \{0,1\}\right\}$ must be computed. In the present context $\hat{T}=500$.

Setting these considerations aside the above set of decision rules is then applied in reverse order, corresponding to forward time[4]. They are applied in the context of the evolution of the true environment under study, with the model agents' actions then compared to the actions of the investor under study in a manner identical to that described in section 1.5.

Such a procedure effetely presumes the investor had no investment history prior to the initiation of the data set, and will have no investment history post $\hat{T}$, a somewhat less satisfactory interpretation than the "slice-out-of-time" story considered earlier.

Program 2: The Myopic Investor Case.

With myopic investors, there is no reason to employ backward recursion / dynamic programming to determine an investor's optimal decision rule since the investor ignores the consequences of his current choice on his expected future decisions or potential welfare. As a result, there is also no need to compute the investor's optimal decision rule at each time period for every possible combination of values of the economy's state variables. Accordingly, the number of required calculations is much smaller with the implication that the fineness of the approximating grid may be made extremely high.

The required input data is largely the same as in the more general case of Program 1:

(1) $\left\{r_{m,t}^c\right\}, \hat{\mu}_m^c, \hat{\sigma}_m^c$ (the actual sample daily realizations of market returns, continuously compounded, and their associated statistical summary);

(2) $\left\{r_{p,t}^c\right\}, \{W_t\}, \hat{\mu}_p^c, \hat{\sigma}_p^c, \hat{\rho}_{r_{m,t}^c, r_{p,t}^c}, \hat{\beta}_{r_{m,t}^c, r_{p,t}^c}$ (the actual daily portfolio returns and normalized wealth series for the specific investor under study and his return series statistical summary. These return series are also continuously compounded); the particular parameter set $\{\alpha, \theta, \delta, \gamma_0\}$ assigned to the model's stand-in investor, and N, the number of wealth partition elements. This latter quantity directly determines the fineness of the investor's approximating conditional wealth density partition.

Also unchanged from Program 1 is the model investor's period t state vector, $\left(\overline{W}_t, b_t, T_t, W_{-T_t}\right)$, with the same general interpretation assigned to each term save that the time index "t" unambiguously measures only forward time. Note furthermore that each term represents a

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[4] That is, rule $A_{\hat{T}}(\ )$ is applied in period $t=1$, and, in general rule A_j is applied in period $\hat{T}-j$.

quantity actually "experienced" by the investor in the following sense. We do not know in truth why the investor may elect to check his portfolio on some particular day. But according to our abstraction of his decision process, the state vector describes those quantities which the investor would actually seek to know over the course of the two year sample period. W_{-T_t} will, for instance, always exclusively denote the investor's actual wealth at some previous date $t-T_t$, and never a potential wealth realization. By further analogy to Program 1,

(A) $\overline{W}_{t+1} = \begin{cases} e^{r^c_{m,t+1}} \overline{W}_t \text{ if the investor does not look in } t \\ e^{r^c_{m,t+1}} W_t \text{ if the investor does look in } t \text{ and,} \end{cases}$

(B) $b_{t+1} = \begin{cases} \theta \overline{W}_t + (1-\theta) b_t \text{ if the investor does not look in } t \\ \delta W_t + (1-\delta) b_t \quad \text{if the investor does look in } t. \end{cases}$

In both (A) and (B), W_t will represent an actual wealth experience of the investor under study. All these remarks are to emphasize that it is unnecessary to compute the optimal decision rule for every possible state vector $\left(\overline{W}'_t, b'_t, T'_t, W'_{-T_t}\right)$ that might potentially arise in period t.

As previously, $t=0$ in the model corresponds to the first actual date at which the individual investor under study did actually examine his portfolio.

We thus define:

$\left(\overline{W}_0, b_0, T_0, W_{-T_0}\right) \equiv (1,1,0,1)$, and

$A^*\left(\overline{W}_0, b_0, T_0, W_{-T_0}\right) = A_0^* \equiv 1$, representing "a look".

Continuing forward to period t, suppose the model's stand-in investor is confronted with the state vector $\left(\overline{W}_t, b_t, T_t, W_{-T_t}\right)$. His optimal decision $A^*\left(\overline{W}_t, b_t, T_t, W_{-T_t}\right) \in \{0,1\}$ in this simplified setting is computed according to the evaluation of

$\max_{A_t^* \in \{0,1\}} \left\{ E_{W_t/\overline{W}_t, W_{-T_t}} u^L(W_t, b_t), u^{DL}\left(\overline{W}_t, b_t\right) \right\}$ with

$$A_t^* = \begin{cases} 1 \; if \; E_{W_t/\overline{W}_t, W_{-T_t}} u^L(W_t, b_t) > u^{DL}\left(\overline{W}_t, b_t\right) \\ 0 \; otherwise \end{cases}$$

In more detail,

$$E_{W_t/\overline{W}_t, W_{-T_t}} u^L(W_t, b_t) = \sum_{W_t(j) \in P_{T_t}(N)} u^L(W_t(j), b_t) prob\left(W_t(j) \mid \overline{W}_t, W_{-T_t}\right)$$

$$= \sum_{W_1(j)\in P^N_{[W^L_{\hat{T}},W^u_{\hat{T}}]}} u^L(W_1(j),b_1)\cdot\frac{1}{\sqrt{2\pi}\sqrt{1-\hat{\rho}^2}\,\hat{\sigma}^c_m}\cdot e^{-\frac{1}{2}\left[\frac{\ln\left(\frac{W_1(j)}{W_{-T_1}}\right)-T_1\hat{\mu}^c_p-\left(\hat{\sigma}^c_p/\hat{\sigma}^c_m\right)\hat{\rho}^c_p\left(\ln\left(\frac{\overline{W}_1}{W_{-T_1}}\right)-T_1\hat{\mu}^c_m\right)}{\sqrt{1-\hat{\rho}^2_p}\,\hat{\sigma}^c_m}\right]^2},$$

Note the absence of any requirement to estimate the investor's unconditional return distribution. The possible values of his true wealth at period t, the set $\{W_t(j)\}_j$, are approximated by the $N+1$ (inclusive of end points) elements of the equal spaced partition of $\left[W_{-T_t}e^{T_t\left(\hat{\mu}^c_p-2\hat{\sigma}^c_p\right)}, W_{-T_t}e^{\hat{\mu}^c_p+2\hat{\sigma}^c_p}\right]$.

We denote this "conditional" partition by $P_{T_t}(N)$, which is recreated and modified on a period by period basis. It remains to describe the evolution of the "stand-in" investor's stat vector:

a. Suppose the decision $A^*\left(\overline{W}_t,b_t,T_t,W_{-T_t}\right)=1$. Then $\left(\overline{W}_{t+1},b_{t+1},T_{t+1},W_{-T_{t+1}}\right)$ satisfies

 (i) $\overline{W}_{t+1}=e^{r^c_{m,t+1}}W_t$, where W_t is the specific investor-under-study's actual wealth (what he actually sees) and $r^c_{m,t+1}$ is the actually observed market return in period $t+1$.

 (ii) $W_{-T_{t+1}}=W_t$

 (iii) $T_{t+1}=1$

 (iv) $b_{t+1}=\delta W_t+(1-\delta)b_t$

b. Suppose $A^*\left(\overline{W}_t,b_t,T_t,W_{-T_t}\right)=0$; then $\left(\overline{W}_{t+1},b_{t+1},T_{t+1},W_{-T_{t+1}}\right)$ satisfies:

 (i) $\overline{W}_{t+1}=e^{r^c_{m,t+1}}\overline{W}_t$

 (ii) $W_{-T_{t+1}}=W_{-T_t}$

 (iii) $T_{t+1}=T_t+1$

 (iv) $b_{t+1}=\theta\overline{W}_t+(1-\delta)b_t$

The result of this recursive process to create a sequence of artificial decision $\{A^*_t\}$ which are then matched to the actual decisions of the specific investor under consideration in the exact same manner as detailed in Program 1.