REAL ANALYSIS: LECTURE 14

OCTOBER 29TH, 2023

1. Preliminaries

Last time we were focused on limits. Specifically, what does

$$\lim_{n\to\infty} a_n = L$$

mean? What we found out: it's hard! Every definition we tried to come up with seemed to not work for some weird edge case, which is why we require a precise definition. Here's a thought exercise:

If $\lim_{n\to\infty} a_n = \frac{1}{3}$, then a_n should be close to $1/3$. To be more precise, Ben noted this means that, for every *large* n, a_n should be close to $1/3$. For example, for every large n, we may expect

$$\left|a_n - \frac{1}{3}\right| < \frac{1}{10},$$

i.e. for every large n we get that a_n is *within* $1/10$ of the limit $(1/3)$. However, this clearly isn't sufficient! We should be able to say the same thing for $1/100$. For every large n, we have

$$\left|a_n - \frac{1}{3}\right| < \frac{1}{100}.$$

Again, this is necessary but not sufficient. And we can say the same thing for every arbitrarily small positive number. As a quick remark, notice that "large" can mean different things. You might need n to be 1000 to ensure a_n is within $1/10$ of $1/3$ (i.e. "large" is 1000), but you might need n to be 1000^3 to ensure a_n is within $1/100$ of $1/3$. In other words, once you specify a "tolerance" ($1/10$ or $1/100$ or $1/100000$), the notion of "large" changes. Here's the main idea of a limit:

No matter what "tolerance" you set, there is some "large" N such that a_n is within that "tolerance" for every $n \geq N$.

And now, the moment you've waited your entire conscious life for, the formal definition of a limit:

Definition (Limit)**.** Given a sequence (a_n), we write

$$\lim_{n\to\infty} a_n = L$$

if and only if, $\forall \epsilon > 0, \exists N \in \mathbb{R}$ s.t.

$$(n > N) \implies |a_n - L| < \epsilon.$$

Remark. Here (a_n) just denotes a_n is a sequence. Ok, let's break down these weird symbols. Let's start with $\forall \epsilon > 0$. In English: choose a tolerance ϵ. This tolerance can be arbitrarily small, i.e. we can get arbitrarily close to the limit L. Once you fix a tolerance ϵ, there must be some "large" number N such that a_n is always within L if $n > N$. Notice this is a really strong condition! We're not saying there's one n such that a_n is within ϵ of L, but rather it's true for *every* $n > N$.

Here was another good point, given by Lexi: why can't $\epsilon = 0$? Aren't there some sequences that actually are within 0 of the limit (i.e. are exactly the limit?). That's true, but this would discard some cases that we intuitively think should have a limit. For example, if we allow $\epsilon = 0$ the following would not have a rigorous limit:

$$\lim_{n\to\infty} (3 + 1/n) = 3.$$

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

But, intuitively this does have a limit (last time I checked it was 3). Let's prove this rigorously. We'll start with some scratchwork and then go on to a rigorous proof.

Scratchwork 1. *Ok, we need to show the definition for a limit it met. Our guess for a limit is* 3*. What we need to do is start by fixing* $\epsilon > 0$*. Now we need to find some* N *s.t.*

$$\begin{aligned}(n > N) &\implies |3 + 1/n - 3| < \epsilon \\ &\iff \frac{1}{n} < \epsilon \\ &\iff n > 1/\epsilon.\end{aligned}$$

Now we've found our N*! Let's use this rigorously.*

Proof. Fix $\epsilon > 0$. Notice that, for every $n > \frac{1}{\epsilon}$,

$$\left|3 + \frac{1}{n} - 3\right| = \frac{1}{n} < \epsilon.$$

$\square$

Here are some things to note. We *always* start with "given $\epsilon > 0$". You have no control over ϵ, and therefore can't do anything with it. Here we never said "let $N = 1/\epsilon$". However it is implicit that "N" is $1/\epsilon$. Also, note that we don't need to find the optimal N, rather that we can find any. For example, $1/\epsilon$ works, but so does $10^{41949124}/\epsilon$. Ok, let's do a harder example:

$$\lim_{n\to\infty} \frac{n+100}{3n+1} = \frac{1}{3.}$$

Let's start by seeing if we can do no work and use the same N (notice, this doesn't actually work!)

Proof. Given $\epsilon > 0$. Notice that, $\forall n > \frac{1}{\epsilon}$,

$$\begin{aligned}\left|\frac{n+100}{3n+1} - \frac{1}{3}\right| &= \left|\frac{3n+300-3n-1}{3(3n+1)}\right| \\ &= \frac{299}{3(3n+1)} \\ &< \frac{299}{3\left(\frac{3}{\epsilon}+1\right)} \\ &< \frac{299}{3\left(\frac{3}{\epsilon}\right)} \\ &= \frac{299}{9}\epsilon.\end{aligned}$$

Hmmm. this is too big. We wanted to show this is $< \epsilon$, but we showed it was $< \frac{299}{9}\epsilon$. We started too early in the sequence, so let's start later. Let's do a take two: $\square$

Here's one that works.

Proof. Given $\epsilon > 0$. Notice that, $\forall n > \frac{1}{100\epsilon}$,

$$\begin{aligned}\left|\frac{n+100}{3n+1} - \frac{1}{3}\right| &= \left|\frac{3n+300-3n-1}{3(3n+1)}\right| \\ &= \frac{299}{3(3n+1)} \\ &< \frac{299}{3\left(\frac{300}{\epsilon}+1\right)} \\ &< \frac{299}{3\left(\frac{300}{\epsilon}\right)} \\ &= \frac{299}{900}\epsilon < \epsilon!\end{aligned}$$

Cool, this way worked! ∎

In general, we showed too different methods here. In the first we did scratchwork and found a good $N = 1/\epsilon$ that worked, and in the second we were lazy. We tried $N = 1/\epsilon$ because, well, why not, and then we iterated on it to figure out what will actually work. This makes our life a lot easier. Maybe the sequence is within ϵ of the limit for every $N \geq 100$. We don't care if we claim it's only true when $N \geq 10000000$, since the end result is still the same.

Ok, now everybody paired up and tried the following example:

$$\lim_{n\to\infty}\left(3+\frac{1}{n}\right)^2 = 9.$$

Here's a (very good) attempt by Miles and Jon.

Proof. Given $\epsilon > 0$. Then $\forall n > \frac{8}{\epsilon}$, we get

$$\begin{aligned}\left|\left(3+\frac{1}{n}\right)^2 - 9\right| &= \frac{6}{n} + \frac{1}{n^2} \\ &= \frac{1}{n}\left(6+\frac{1}{n}\right) \\ &\leq \frac{1}{n}\cdot 7 \\ &\leq \frac{7}{\frac{8}{\epsilon}} = \frac{7}{8}\epsilon < \epsilon.\end{aligned}$$

Notice here we used $6 + \frac{1}{n} < 7$, which is true since $1/n < 1$ ($n \in \mathbb{Z}_{\text{pos}}$). Here's another (slightly quicker) proof.

Given $\epsilon > 0$. Then $\forall n > \frac{7}{\epsilon}$, we get

$$\begin{aligned}\left|\left(3+\frac{1}{n}\right)^2 - 9\right| &= \frac{6}{n} + \frac{1}{n^2} \\ &= \frac{1}{n}\left(6+\frac{1}{n}\right) \\ &\leq \frac{1}{n}\cdot 7 = \frac{7}{n} < \epsilon.\end{aligned}$$

∎

Ok, but what went on behind the scenes here. What is the scratchwork?

Scratchwork 2. *Fix $\epsilon > 0$. We want N s.t.*

$$(n > N) \implies \left| \left(3 + \frac{1}{n}\right)^2 - 9 \right| < \epsilon.$$

This is that same as finding N s.t. if $n > N$ then

$$\frac{6}{n} + \frac{1}{n^2} < \epsilon.$$

This is similar to the proof of $\sqrt{2} \in \mathbb{R}$, where we want to "squeeze" something in this inequality. Since $\frac{1}{n^2} \leq \frac{1}{n}$, it suffices to show

$$\frac{6}{n} + \frac{1}{n} < \epsilon.$$

Why? Well, $\frac{6}{n} + \frac{1}{n^2} < \frac{6}{n} + \frac{1}{n}$, so if

$$\frac{6}{n} + \frac{1}{n} < \epsilon$$

it's surely true that

$$\frac{6}{n} + \frac{1}{n^2} < \frac{6}{n} + \frac{1}{n} < \epsilon.$$

And this is good because

$$\frac{6}{n} + \frac{1}{n} < \epsilon$$

is clearly true if $\frac{7}{n} < \epsilon \iff n > \frac{7}{\epsilon}$.

Gabe asked a great question here: what if we can't (or guess the wrong) limit L. Here's a question:

Is $\lim_{n\to\infty}(3 + 1/n) = 2$?

How we prove the above is False? Well, to prove that it's true we need to show that for every $\epsilon > 0$ there's some large N blah blah blah. So, to show it's *not* true, we need to find a *single* "bad" $\epsilon > 0$, where "bad" means you can't find a large N.

Proof. $\forall n \in \mathbb{Z}_{\text{pos}}$,

$$\begin{aligned} \left|3 + \frac{1}{n} - 2\right| &= \left|1 + \frac{1}{n}\right| \\ &= 1 + \frac{1}{n} > 1. \end{aligned}$$

So, our "bad" $\epsilon > 0$ is 1, since if you fix the threshold 1 you can never get within $\epsilon = 1$ of 2. $\square$

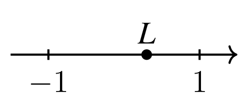
Here's another example:

Proposition 1.

$$\lim_{n\to\infty} (-1)^n \neq L$$

$\forall L \in \mathbb{R}$. *In other words, there is no limit.*

Scratchwork 3. *Here's a picture:*



The problem here is that if there was a limit then

$$|(-1)^n - L|$$

would be small for all large n. Notice it's surely possible for

$$|(-1)^n - L|$$

to be small for some n, but then

$$\left|(-1)^{n+1} - L\right|$$

would be big (L isn't close to both -1 and 1). In particular, that means the sum must be big:

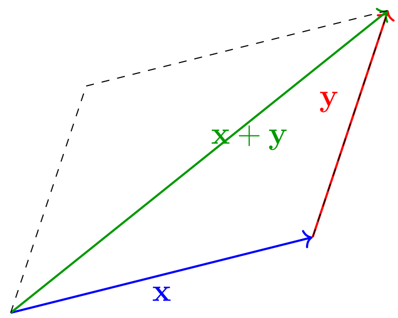
$$|(-1)^n - L| + \left|(-1)^{n+1} - L\right|$$

is big, specifically should be ≥ 2. Before we write down this proof, notice that it requires a key lemma, known as the triangle inequality. Let's write this down and then go to a formal proof:

Proposition 2 (Triangle Inequality). $\forall x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|.$$

We're not going to prove this right now, but here's the main idea:



Remark. Here's what I find to be a good way to think about the triangle inequality. Imagine you are trying to get from point A to point B in the shortest path possible. Notice that going from point A to point C to point B *can never help you*. There's no point in this extra step. This is the essence of the triangle inequality; call the length from A to C by x and the length from C to B by y. Then what I basically just said is

$$|x + y| \text{ i.e. going straight from } A \text{ to } B$$
$$|x| + |y| \text{ i.e. going from } A \text{ to } C \text{ to } B.$$

Thinking about it this way makes it easy to realize that

$$|x + y| \leq |x| + |y|,$$

as it's certainly a shorter path to not have to go through C. Now you might wonder about the case where

$$|x + y| = |x| + |y|.$$

In this analogy, it's when C is on the direct path from A to B. Now the "triangle" part of a triangle inequality is actually a line. Ok, so somehow a special case of a triangle corresponds to a special case of the inequality. How does this continue? Well, if you stare at this and try to imagine triangles in your head, you might start to realize that somehow the angle between $\vec{x}$ and $\vec{y}$ might matter; if the angle is almost π (i.e. a very obtuse triangle) then $|x| + |y|$ is not that much larger than $|x + y|$.

It turns out this intuition is correct, and that the proof of the triangle inequality (along with vast generalizations) is actually not too difficult when you start thinking about the world of vectors (linear algebra flashbacks...). Ok, long remark over. Recall we were going to try to use the triangle inequality to show

$$\lim_{n\to\infty} (-1)^n \neq L$$

for any $L \in \mathbb{R}$.

Thus, by triangle inequality we can bound $|(-1)^n - L| + |(-1)^{n+1} - L|$. Here's a formal proof of the example above.

Proof. Suppose $\lim_{n\to\infty}(-1)^n = L$. Then there's some N s.t. $|(-1)^n - L| < \frac{1}{10}$ for all $n \geq N$. In particular,

$$\left|(-1)^{n+1} - L\right| < \frac{1}{10}$$

if $n \geq N$. Thus, by triangle inequality,

$$\begin{aligned} 2 &= \left|(-1)^n - (-1)^{n+1}\right| \\ &= \left|(-1)^n - L + L - (-1)^{n+1}\right| \\ &\leq |(-1)^n - L| + \left|L - (-1)^{n+1}\right| \\ &< \frac{1}{10} + \frac{1}{10} = 1/5, \end{aligned}$$

a contradiction! $\square$