# REAL ANALYSIS: LECTURE 16

NOVEMBER 2ND, 2023

## 1. Preliminaries

Recall last time we saw that limits "play nice" with the field operations $(+, -, \cdot, \div)$, i.e.

**Proposition 1.** *If $a_n \to A$ and $b_n \to B$, we have*
*(i)* $\lim_{n\to\infty} (a_n + b_n) = A + B$
*(ii)* $\lim_{n\to\infty} (a_n - b_n) = A - B$
*(iii)* $\lim_{n\to\infty} (a_n b_n) = AB$
*(iv)* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ *as long as $B \neq 0$ and $b_n \neq 0$ $\forall n$ large.*

However, $\mathbb{R}$ is much more than just a field—it's a (actually, *the*) complete ordered field! Do limits respect the other structural aspects of $\mathbb{R}$? For example, do limits play nice with order?

### 1.1. Limits and Order.

**Proposition 2.** *Given convergent sequences $(a_n), (b_n)$ s.t. $a_n \leq b_n$ $\forall n$. Then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.*

*Remark.* It's slightly informal to say
$$\lim_{n\to\infty} a_n$$
to denote $L$, where $a_n \to L$. This is because we only defined the notation
$$\lim_{n\to\infty} a_n = L;$$
we never actually defined what $\lim_{n\to\infty} a_n$ means in isolation! Since we proved last time that limits are unique, however, we can now use this informal notation without danger of ambiguity.

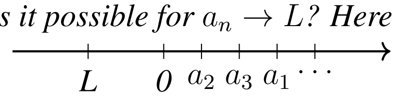
Ok, let's think about proposition 2. It's enough to show (ETS) the following:

**Proposition 3.** *Given convergent $(c_n)$ s.t. $c_n \geq 0$. Then $\lim_{n\to\infty} c_n \geq 0$.*

To prove Proposition 2, we just need to prove Proposition 3 since we can take $c_n := b_n - a_n$ and use what we know about the algebra of limits. The advantage of proving Proposition 3 is that it's simpler: it's phrased in terms of a single sequence, rather than in terms of two different sequences.

Ben proposed a proof of Proposition 3. Here's the idea:

**Scratchwork 1.** *Suppose that $L < 0$. Is it possible for $a_n \to L$? Here's a picture.*




*By this picture it's clear that we'll never be able to get within $|L|$ of the limit! Let's formalize this argument.*

*Proof.* Given $L < 0$. Then, for any $n$,
$$|c_n - L| = c_n - L \geq -L = |L|.$$
In particular, $|c_n - L|$ is *never* $< \frac{|L|}{2}$, so $\lim_{n\to\infty} c_n \neq L$. □

Here's another famous way that limits play with order.

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Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

**Theorem 1** (Squeeze Theorem)**.** *Suppose $(a_n), (b_n), (c_n)$ are sequences s.t. $a_n \leq b_n \leq c_n$ for all $n$. If $a_n \to L$ and $c_n \to L$, then $b_n \to L$.*

*Remark.* Here's how to think about this: if you're given some very complicated sequence $(b_n)$ that you can squeeze in between two simpler sequences $(a_n)$ and $(c_n)$ that converge to the same thing, then you can deduce the limit of the complicated sequence $(b_n)$ without working too hard by just using the Squeeze Theorem.

How do we prove the Squeeze Theorem? Well, we just proved (in Proposition 2) that limits play nice with order. Let's just apply that proposition twice:

$$a_n \leq b_n \implies L \leq \lim_{n\to\infty} b_n \qquad \text{and} \qquad b_n \leq c_n \implies \lim_{n\to\infty} b_n \leq L.$$

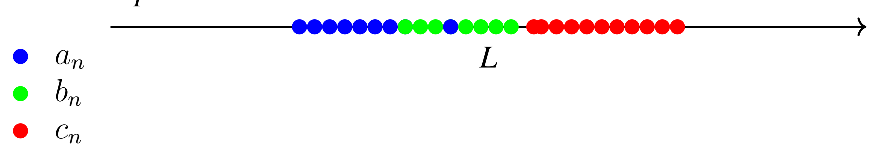
Combining these two statements, we deduce

$$L \leq \lim_{n\to\infty} b_n \leq L,$$

whence $\lim\limits_{n\to\infty} b_n = L$.

Although the above proof seems straightforward, it has a major flaw (observed by Miles): Proposition 2 *requires* the knowledge that $b_n$ converges, which we don't actually know! Secretly, the Squeeze Theorem is asserting that $(b_n)$ converges (at which point the argument we gave above implies it must converge to $L$).

**Scratchwork 2.** *Let's look at a picture:*




*What's our goal here? Well, we're definitely going to start with: Given $\epsilon > 0$. Then we want to do some stuff and get to $|b_n - L| < \epsilon$. Do we know anything close to $L$? Sure: both $a_n$ and $c_n$ are close to $L$ for large $n$, and $b_n$ is trapped between them! Thus $b_n$ must be close to $L$. Let's write this down rigorously.*

*Proof.* Given $\epsilon > 0$. For all large $n$ we have $|c_n - L| < \epsilon$, which is the same as saying

$$-\epsilon < c_n - L < \epsilon.$$

But notice that $b_n - L \leq c_n - L$, which tells us that

$$b_n - L \leq c_n - L < \epsilon. \tag{†}$$

Similarly, for all large $n$ we have

$$b_n - L \geq a_n - L > -\epsilon. \tag{‡}$$

Together, (†) and (‡) imply $|b_n - L| < \epsilon$ for all large $n$. Since $\epsilon > 0$ was arbitrary, we conclude $b_n \to L$ as claimed. □

1.2. **Limits and Completeness.** We've seen that limits play nice with field operations, and also with order. What about with completeness, i.e. with (A13)?

As a warm up, suppose $(a_n)$ is bounded above. Must it converge? Edith says no: $a_n = (-1)^n$ doesn't converge but is bounded. Is there any additional condition on $(a_n)$ that would guarantee convergence? Sean suggests that if $(a_n)$ is eventually increasing and bounded above, then it should converge; Lexi made the more precise conjecture that

$$\lim_{n\to\infty} a_n = \sup\{a_n \mid n \in \mathbb{Z}_{\text{pos}}\}.$$

(Note a few differences between $(a_n)$ and $\{a_n\}$: $\{a_n\}$ is a set, while $(a_n)$ is a sequence (i.e. a function!); $\{a_n\}$ might be finite, while $(a_n)$ is always infinite; $(a_n)$ is ordered, while $\{a_n\}$ is not.) We quickly noted that the Sean-Lexi conjecture can't quite be right: consider the sequence $(a_n)$ with $a_1 = 1000$ and $a_n = 1 - 1/n$ for $n > 1$, which is eventually increasing but has supremum $1000$, which definitely isn't the limit. However, a slight modification of the Sean-Lexi conjecture does turn out to hold:

**Proposition 4.** *If $(a_n)$ is always increasing and $(a_n)$ is bounded above, then*

$$\lim_{n\to\infty} a_n = \sup\{a_n \,:\, n \in \mathbb{Z}_{\text{pos}}\}.$$

**Scratchwork 3.** *Call the supremum $\lambda$. We want to show the $a_n \approx \lambda$, i.e. $|a_n - \lambda| < \epsilon$ for large $n$. Forrest observed that the sequence must get close to $\lambda$ at some point because $\lambda$ is the supremum, and the sequence must stay close to $\lambda$ thereafter because $(a_n)$ is increasing. Let's make this precise.*

*Proof.* Given $\epsilon > 0$. Since $\lambda$ is the least upper bound, $\lambda - \epsilon$ *isn't* an upper bound of $\{a_n\}$. Thus, $\exists N$ s.t.

$$a_N > \lambda - \epsilon.$$

Since $(a_n)$ increasing, $a_n > \lambda - \epsilon$ for any $n > N$. Rewriting we get

$$|a_n - \lambda| = \lambda - a_n < \epsilon$$

for all large $n$. □

Similarly, if $(a_n)$ is always decreasing and is bounded below, then $a_n \to \inf\{a_n \,:\, n \in \mathbb{Z}_{\text{pos}}\}$. You can adapt the proof above (or deduce it as a corollary).

With all of this, we get the following proposition:

**Proposition 5.** *If $(a_n)$ is bounded and monotone, then it converges.*

*Remark.* A sequence $(a_n)$ is monotone is if it's either always increasing or always decreasing. Here, increasing technically should be "always non-decreasing": $a_{n+1} \geq a_n \ \forall n$. Similarly, decreasing means $a_{n+1} \leq a_n \ \forall n$.

What about the converse? Nope. If $(a_n)$ converges it may not be monotone. For example, $\frac{(-1)^n}{n}$ converges and is not monotone. However, the following is true:

**Proposition 6.** *If $(a_n)$ converges, then $(a_n)$ is bounded.*

**Scratchwork 4.** *Here's Sarah's reasoning. Say $a_n \to L$. Then $a_n \approx L$ when $n \geq N$ Let's break into two parts: there's only finitely much stuff before $N$, so it's bounded. And the stuff after $N$ is near $L$, which means it is bounded as well!*

*Proof.* Say $a_n \to L$. Then $\exists N$ s.t. $\forall n \geq N$,

$$L - 1 < a_n < L + 1.$$

Thus, the set

$$\{a_n \,:\, n > N\}$$

is bounded between $L - 1$ and $L + 1$. Now $\{a_n \,:\, n \leq N\}$ is finite, hence bounded. Thus, the entire set $\{a_n : n \in \mathbb{Z}_{\text{pos}}\}$ is bounded, so $(a_n)$ is bounded. □

Putting together our work, we obtain a famous result:

**Theorem 2** (Monotone Convergence Theorem)**.** *Suppose $(a_n)$ is monotone. Then $(a_n)$ converges iff $(a_n)$ is bounded.*