REAL ANALYSIS: LECTURE 24

DECEMBER 7TH, 2023

1. REAL ANALYSIS- THE FINALE

Golly to the gee. It's the last class of Real Analysis (don't cry too much or else you can't read this summary). Just a mere couple of months ago we were all so naive; we didn't even know the real numbers are the only complete ordered field up to isomorphism! Look at us now. It only takes us like 10 minutes to prove $f(x) = 12$ is continuous! Ok, but honestly we learned a lot this semester. Real numbers: super weird. Infinity: also super weird. Limits: a lot more annoying to define than you might think. ϵ: your new favorite Greek letter. Log trivia: Thursday. Real Analysis: cool, but difficult. Ok, enough of this. Let's do some math.

2. IVT

Last time, we were proving the IVT. Let's try to remember what the IVT is.

Idea 1. *Here's Alex's idea. For any $f : [a, b] \to \mathbb{R}$ continuous, $\forall y$ between $f(a), f(b), \exists c \in [a, b]$ s.t. $f(c) = y$.*

This is correct, but IVT is actually stronger. Noah notes it's not that $f : [a, b] \to \mathbb{R}$. Rather, for *every* closed interval that lives in the domain, the above property is true. Namely,

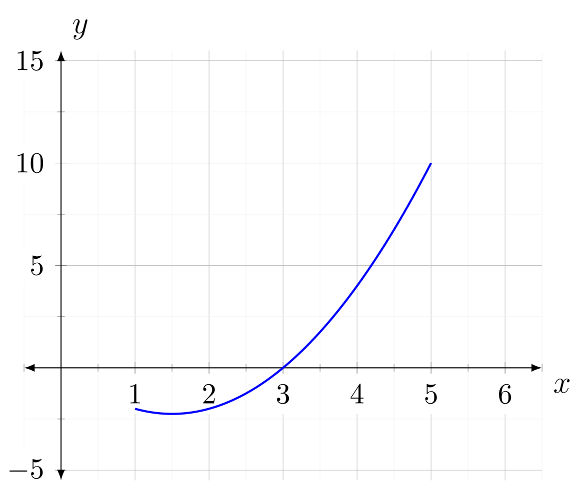
Theorem 1 (IVT)**.** *Given $f : X \to \mathbb{R}$ continuous. For any $[a, b] \subseteq X$ and for any y in between $f(a)$ and $f(b)$, $\exists c \in [a, b]$ s.t. $f(c) = y$.*

Our goal is to deduce IVT from a weaker theorem, called Bolzano's Theorem. Here's the statement again:

Theorem 2 (Bolzano's Theorem)**.** *If $f : X \to \mathbb{R}$ is continuous on $[a, b] \subseteq X$ and*

$$f(a) < 0 < f(b),$$

then there exists $c \in [a, b]$ s.t. $f(c) = 0$.



Here's a picture. Call the function above f and note $f(1) < 0 < f(6)$. Bolzano's Theorem states that there must be some $x \in [1, 6]$ s.t. $f(c) = 0$. Note there may be many c's! Ok, let's prove Bolzano's Theorem by beginning with the following lemma:

Lemma 1. *If $f : [a, b] \to \mathbb{R}$ continuous at $c \in [a, b]$ and $f(c) > 0$, then $\exists \delta > 0$ s.t. $f(x) > 0 \, \forall x \in (c - \delta, c + \delta)$.*

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

Idea 2. *Here's the intuition. If you zoom in on $x = c$, the stuff nearby must be close to $f(c) > 0$, which means the stuff nearby should be > 0 as well. More specifically we can look at some ϵ wiggle-room. Let's prove this.*

Proof. Since f continuous at c, $\exists \delta > 0$ s.t. $|x - c| < \delta \implies |f(x) - f(c)| < \frac{f(c)}{2}$ (here $\frac{f(c)}{2}$ is our choice of ϵ). One of the things this tells us is that

$$f(x) - f(c) > \frac{f(c)}{2}$$
$$f(x) > \frac{f(c)}{2} > 0.$$

□

Remark. Notice Lemma 1 works if you replace all the $f(c) > 0$ with $f(c) < 0$. Why? Well just take a function g where you want to make the claim with $f(c) < 0$. Multiplying everything by -1 gives you a new function $f = -g$, which meets the conditions of Lemma 1 and gives us the desired result.

Let's use Lemma 1 to prove Bolzano's Theorem. Here's the high level idea

Scratchwork 1. *Last time we decided if we defined*

$$c := \sup \underbrace{\{x \in [a,b] \ : \ f(r) < 0 \, \forall r \in [a,x]\}}_{:=\mathcal{A}}$$

then it makes sense that $f(c) = 0$. We can then use trichotomy to prove it. So, what's the problem with $f(c) > 0$. Well, on the one hand we can go slightly left of the supremum and find some d s.t. $f(d) < 0$ (since $d \in \mathcal{A}$). On the other hand, we can use the above lemma and claim that d is close enough to c that both should be positive; namely $f(d) > 0$, which gets us the contradiction. At a high level, this contradiction arises by the fact that there's room to the left (c is not the least upper bound). On the case where $f(c) < 0$, we should be able to move over to the right and still find stuff that lives in $\mathcal{A}$, i.e. c is not an upper bound!

Proof. Define

$$c := \sup \underbrace{\{x \in [a,b] \ : \ f(r) < 0 \, \forall r \in [a,x]\}}_{:=\mathcal{A}}.$$

We proved last class that c exists. To show $f(c) = 0$, it suffices to show $f(c) \not> 0, f(c) \not< 0$. Let's show these two cases.

Case 1. *Suppose, for the sake of contradiction, $f(c) > 0$. By Lemma 1, $\exists \delta > 0$ s.t. $f(x) > 0 \, \forall x \in (c-\delta, c+\delta)$.*

Remark. Here's a subtlety flawed approach. Using the above statement we can deduce that

$$f\left(c - \frac{\delta}{2}\right) > 0,$$

which makes it seem like we're done. However, we don't know that $c - \frac{\delta}{2} \in \mathcal{A}$!.

Ok, let's keep going. Since $c = \sup \mathcal{A}$, we know $(c-\delta, c]$ contains some $d \in \mathcal{A}$. Notice $c \neq d$, since $f(c) > 0$, which means $c \notin \mathcal{A}$ (it doesn't meet the set condition!). But $d \in \mathcal{A}$ means $f(d) < 0$, yet on the other hand $d \in (c-d, c+d)$ implies $f(d) > 0$.

Let's move to the other case:

Case 2. *Suppose, for the sake of contradiction, $f(c) < 0$. By Lemma 1 (actually by our remark afterwards), $\exists \delta > 0$ s.t. $f(x) < 0 \, \forall x \in (c-\delta, c+\delta)$. In particular, $f\left(c + \frac{\delta}{2}\right) < 0$. But also $\exists d \in (c - \delta, c]$ s.t. $d \in \mathcal{A}$. Thus, $f < 0$ on $[a, d]$ and $f < 0$ on $\left[d, c + \frac{\delta}{2}\right]$ since $d \in (c - \delta, c + \delta)$. Thus, $c + \frac{\delta}{2} \in \mathcal{A}$, which means c isn't an upper bound on $\mathcal{A}$.*

Together these cases tell us $f(c) = 0$, and we're done. □

Now let's prove IVT, using Bolzano's Theorem.

Proof. Given $f : X \to \mathbb{R}$ continuous. Pick $[a,b] \subseteq X$. If $f(a) < y < f(b)$, apply Bolzano to $g(x) := f(x) - y$ (here $g(a) < 0 < g(b)$ so Bolzano applies), which gives us $c \in [a,b]$ s.t. $g(c) = 0 \implies f(c) = 0 + y = y$, as desired. On the other hand, if $f(a) > y > f(b)$, apply Bolzano to $h(x) = -f(x) + y$, which gives us $c \in [a,b]$ s.t. $h(c) = 0 \implies f(c) = y - h(c) = y$. □

3. SERIES

Ok, 15 minutes left. Time to move to a new topic: series. Apparently, the following is true:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \cdots = 1.$$

What does this actually mean? How can you add up an infinite amount of things. Here's what this really means. Let

$$S_n := \sum_{n=1}^{N} \frac{1}{2^n}.$$

Then (S_n) is a sequence of what we call *partial sums*, where

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{4} \\ S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\ &\vdots \end{aligned}$$

Then what we're claiming is that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

is *notation* for

$$\lim_{N \to \infty} S_N = 1.$$

More generally,

$$\sum_{n=1}^{N} a_n$$

converges iff the sequences of partial sums

$$S_n := \sum_{n=1}^{N} a_n$$

converges.

Example 1. We proved

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by showing the Cauchy Criterion with telescoping series. This series is known as $\zeta(2)$ (Riemann Zeta Function). It turns out this series converges to $\pi^2/6$.

Example 2. We proved

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. This is known as the *Harmonic Series*.

Example 3. Consider

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

This is known as the *Alternating Harmonic Series*. Let's show a sketch of why this diverges.

Proof. Let $S_N := \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}$. Notice

$$S_{2n} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

is a sum of terms > 0, and in particular the subsequence S_{2n} is monotonically increasing. Also note

$$S_{2n} = S_{2n-1} - \frac{1}{2n} < S_{2n-1}.$$

Further,

$$S_{2n-1} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2n-2} - \frac{1}{2n-1}\right)$$

is 1 subtracted by a bunch of positive terms, which means it's bound above by 1. This $S_{2n} < S_{2n-1} < 1$, which tells us by the MCT that S_{2n} converges, i.e. $S_{2n} \to L$. This implies $\forall \epsilon > 0$,

$$\begin{aligned}
|S_{2n} - L| &< \frac{\epsilon}{2} \ \forall \text{large } N \\
|S_{2n-1} - L| &= |S_{2n} + \frac{1}{2N} - L| \\
&\leq |S_{2n} - L| + \frac{1}{2N} \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Basically the conclusion is that (S_n) converges. It turns out it converges to $\ln(2)$! □

Congrats! Real Analysis: Complete!