REAL ANALYSIS: LECTURE 7

SEPTEMBER 28TH, 2023

1. Preliminaries

We began with a brief remark about proofs:

Remark. Suppose you wanted to prove that $A \implies B$. There are two general options:

(1) Suppose A. Super smart math stuff... And therefore, by the quantum crypto AI theorem, B. This is a **direct proof**.
(2) Suppose not B. Then equally super smart math... Therefore, not A. This is a **contrapositive proof**.

This is basically saying (if A then B is equivalent to if not B then not A). In fancy math language,

$$(A \implies B) \iff (\neg B \implies \neg A),$$

where $\neg A$ is the *negation of* A, i.e. not A.

Notice that you cannot suppose B and prove A!

This doesn't work out. Here's an example. If you are a math major, then you take real analysis. Notice that the *converse* is: if you take real analysis, you are a math major. This is not true! Plenty of non-math majors take real analysis.

Previously, on real analysis. We discussed induction and proved 1 is the smallest element of $\mathbb{Z}_{\text{pos}}$ and $\mathbb{Z}_{\text{pos}}$ are closed under $+$. Here's a statement similar to induction:

2. Strong Induction

Proposition 1 (Strong Induction). *Suppose $S(n)$ is a sequence of logical assertions, one for each $n \in \mathbb{Z}_{\text{pos}}$, such that*

(i) $S(1)$ is true, and
(ii) $S(k)$ must be true whenever $S(j)$ is true $\forall j \in \mathbb{Z}_{\text{pos}}$ such that $j < k$.

Then, $S(n)$ is true $\forall n \in \mathbb{Z}_{\text{pos}}$.

Notice here you have a *stronger* requirement: rather than just know $S(k-1)$ to imply $S(k)$ (regular induction), we now need $S(j)$ for every $j < k$ to imply $S(k)$.

Let's show an example of using strong induction. We begin with a definition:

Definition (Well Ordered). A set $S \subseteq \mathbb{R}$ is *well ordered* iff every nonempty subset of S has a least element.

Thanks to Jon for the catch that the subset can't be empty. In other words, a set $S \subseteq \mathbb{R}$ is well ordered if you can "order" every nonempty subset of S. For example, $\{1, 2\} \subseteq \mathbb{R}$ is well ordered. Miles gave an example that isn't well ordered. Notice $\mathbb{R} \subseteq \mathbb{R}$, but $\mathbb{R}$ doesn't have a least element.

What about $[0, 1]$? Is it well ordered? Nope. Harry considered subset $(.5, .6) \subseteq [0, 1]$. Notice there is no least element of $.5$. For example, for $\epsilon > 0$, $.5 + \epsilon$ is not the least element, since $.5 + \epsilon/2 < .5 + \epsilon$, and $.5 + \epsilon/2 \in (.5, .6)$. Another example of a set that isn't well ordered is $\mathbb{Z}$; there's no smallest integer.

Proposition 2 ($\mathbb{Z}_{\text{pos}}$ are well ordered). *We proceed by strong induction, which means we need to come up with a sequence of logical assertions and show the relevant conditions hold.*

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

Suppose $A \subseteq \mathbb{Z}_{\text{pos}}$ s.t. A has no least element (our goal is to show $A = \emptyset$). Let $S(n)$ be the logical assertion $n \notin A$. We proved 1 is the least element in $\mathbb{Z}_{\text{pos}}$. So, if $1 \in A$, 1 would be the least element of A. Thus, $1 \notin A$, i.e. $S(1)$ is true.

Suppose $S(j)$ is true for every $j \in \mathbb{Z}_{\text{pos}}$ s.t. $j < k$. Thus, $j \notin A \ \forall j < k$. In other words, $j \in A$ implies $j \geq k$. Suppose, for the sake of contradiction, $k \in A$. Then we know every $j \in A$ has $j \geq k$, which means that k is the least element! This contradictions the definition of A. Thus, $S(k)$ is true. By strong induction, $S(j)$ is true for every positive integer j, which means $A = \emptyset$. Thus, $\mathbb{Z}_{\text{pos}}$ are well ordered!

In general, it makes sense to use strong induction when you must know all of the previous $S(j)$, not just the last piece of information $S(k-1)$.

Let's take a step back. We constructed $\mathbb{R}$, and from there created $\mathbb{Z}_{\text{pos}}, \mathbb{Z}, \mathbb{Q}$. However, we don't really know about relationships between these. Our intuition, on the other hand, understands strong relationships between these sets. For example, we know intuitively that every $x \in \mathbb{R}$ can be approximated pretty well by an integer. Let's prove a formalized version of this:

Proposition 3. *$\forall x \in \mathbb{R}, \exists N \in \mathbb{Z}$ and $\alpha \in [0, 1)$ s.t.*

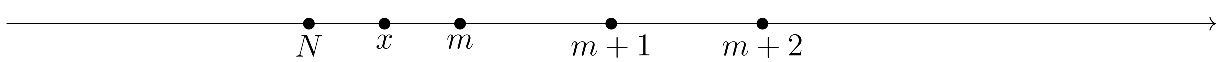
$$x = N + \alpha.$$

Moreover, N and α are uniquely determined by x.

For example, $\pi = 3 + 0.1415926\ldots$ (here $N = 3, \alpha = 0.14159$). Further, the only (N, α) that fulfill the above proposition are 3 and 0.14159; this N, α *uniquely* determined. Intuitively, how can we prove this?

Well, given x, how can we find N and α.

Jenna had a nice argument. Look at all the integer $> x$. By well ordering there's a least integer N'. Then $N' - 1$ is the greatest integer less than x, so $N = N' - 1$. Then we can find α with $\alpha = x - N$. Edith suggest showing it's unique by supposing there are two pairs $(N, \alpha), (M, \beta)$ and using the fact that $|N - M| \geq 1$ to show α or β is too large or too small. Here's a picture:



Ok, let's prove this.

Proof. We'll assume $x \geq 1$, and you'll prove $x < 1$ on your homework. Let

$$J := \{n \in \mathbb{Z}_{\text{pos}} \ : \ n > x\}.$$

Since $\mathbb{Z}_{\text{pos}}$ are well ordered, J has a least element $m \in \mathbb{Z}_{\text{pos}}$. Then set $N := m - 1, \alpha = x - N$. We claim that

(i) $N \in \mathbb{Z}_{\text{pos}}$
(ii) $N \leq x$
(iii) $\alpha \in [0, 1)$

Proof of (i): We know $m \in \mathbb{Z}_{\text{pos}}$. Here's a quick lemma:

Lemma 1. *If $m \in \mathbb{Z}_{\text{pos}}, m - 1 \in \mathbb{Z}_{\text{pos}} \cup \{0\}$.*

Proof. Proved in the book. □

Proof of (i): By the above lemma it suffices to show $N \neq 0$. Since $x \geq 1$, we get $m > x \geq 1$, which means $N = m - 1 > 0 \implies N \neq 0$. So, $N \in \mathbb{Z}_{\text{pos}}$.

Proof of (ii): Notice $m > m - 1 = N \notin J$ but is in $\mathbb{Z}_{\text{pos}}$, which means $N \leq x$.

Proof of (iii) $\alpha = x - N \geq 0$. Also,

$$\begin{aligned} \alpha &= x - N \\ &= x - m + 1. \end{aligned}$$

Since $x < m$, we get $x - m < 0$, which from above tells us $x - m + 1 < 1$. Here, Miles pointed out an issue. To apply well-ordering, we need to know that $J \neq \emptyset$. The fact that there are arbitrary large integers (i.e. J always is nonempty) is something called the *Archimedian Property*. Next time we will prove this property. □