REAL ANALYSIS: LECTURE 8

OCTOBER 2ND, 2023

1. Preliminaries

Recall that last time we were approximating real numbers by integers. Specifically, we were proving the following:

Proposition 1. *$\forall x \in \mathbb{R}, \exists N \in \mathbb{Z}$ and $\alpha \in [0,1)$ s.t.*

$$x = N + \alpha.$$

Moreover, N, α are uniquely determined by x.

Proof. Last time we went through this in more depth, but here's where we were. We'll only consider $x \geq 1$. Let

$$J := \{n \in \mathbb{Z}_{\text{pos}} \ : \ n > x\}$$

be the set of positive integers greater than x. Since $\mathbb{Z}_{\text{pos}}$ well ordered, J has a least element m. Let $N := m - 1$ and $\alpha = x - N$. We proved last time that

(1) $N \in \mathbb{Z}_{\text{pos}}$
(2) $\alpha \in [0,1)$
(3) $x = N + \alpha$ (trivially).

Thus, we've shown the existence of a solution, so let's now prove uniqueness. Suppose

$$x = N + \alpha = M + \beta,$$

where $N, M \in \mathbb{Z}_{\text{pos}}$ and $\alpha, \beta \in [0,1)$. Notice the goal here is to show $N = M$ and $\alpha = \beta$. Without loss of generality (WLOG), let $M \geq N$. Then

$$M - N = \alpha - \beta.$$

Since M, N are integers with $M \geq N$ we get $M - N \in \mathbb{Z}_{\text{pos}} \cup \{0\}$ (proved in chapter 6 but intuitively it's certainly true). However, Emily noted $\alpha - \beta < 1$ since $\alpha < 1$ and $-\beta \leq 0$, so $\alpha - \beta < 1 - 0 = 1$. Thus, $M - N = 0$, since 1 is the least positive integer, i.e. $\alpha - \beta < 1$ means that the only integer $\alpha - \beta$ can be is 0. Thus, $\alpha - \beta = M - N = 0$, which implies $M = N$ and $\alpha = \beta$, which satisfies for uniqueness. □

This proof seems fine, but it all relies on the well ordering of J, which is only true **as long as J is nonempty**. But we never proved that J is nonempty! We'll now prove a formal version of this, known as the **Archimedian Property of $\mathbb{R}$**.

2. Archimedian Property

Theorem 1 (Archimedian Property)**.** *$\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}_{\text{pos}}$ s.t. $n > x$. In other words, there are arbitrarily large positive integers.*

After combining pieces of proofs from Edith, Jenna, Sarah, Blakeley, Harry, and more, we arrived at the following:

Proof. Suppose, for the sake of contradiction, that there exists $x \in \mathbb{R}$ s.t. $x \geq n \, \forall n \in \mathbb{Z}_{\text{pos}}$. Then by (A13) the existence of an upper bound means $\omega := \sup(\mathbb{Z}_{\text{pos}}) \in \mathbb{R}$. Notice $\omega - 1 < \omega$ is *not* an upper bound on $\mathbb{Z}_{\text{pos}}$. Thus, by definition there's some $H \in \mathbb{Z}_{\text{pos}}$ s.t. $H > \omega - 1$. Yet, $H + 1 \in \mathbb{Z}_{\text{pos}}$ by closure of positives, and $H > \omega$, which contradicts ω is an upper bound on the positive integers. □

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

It turns out it is absolutely necessary to use (A13) to prove the Archimedian Property, as there are examples of ordered fields (satisfy A1- A12) that do not satisfy the Archimedian Property. Here's a corollary, also known as the Archimedian Property:

Proposition 2. *$\forall \epsilon > 0, \exists n \in \mathbb{Z}_{\text{pos}}$ s.t. $\frac{1}{n} < \epsilon$. In other words, there are arbitrarily small integers.*

Proof. Fix $\epsilon > 0$. Notice $\frac{1}{\epsilon} \in \mathbb{R}$. By the previous Archimedian Property, $\exists n > 1/\epsilon$, which rearranges to $\frac{1}{n} < \epsilon$. Notice that the problem with $\epsilon < 0$ is that rearranging the inequality would require flipping the inequality, which Ben pointed out. □

Here's the 3rd Archimedian Property, which is probably at least 2 Archimedian Properties too many...

Proposition 3. *$\forall \epsilon > 0, \forall x \in \mathbb{R}, \exists n \in \mathbb{Z}_{\text{pos}}$ s.t. $n\epsilon > x$.*

We're now going to move on to our first Real Analysis theorem!

3. Really Real Real Analysis

We have previously proved that $\sqrt{2}$ is not rational. However, we never proved that $\sqrt{2} \in \mathbb{R}$.

Theorem 2. *$\exists \alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.*

Let's start with some scratchwork.

Scratchwork 1. *Intuitively, $\alpha = \pm\sqrt{2}$. Forrest suggested the following. Let $A := \{x \in \mathbb{R} \, : \, x^2 \leq 2\}$.*

By trichotomy, exactly one of $\alpha^2 = 2, \alpha^2 > 2, \alpha^2 < 2$ is possible. Our general strategy is to show the latter two aren't possible. Then we get $\alpha^2 = 2$, i.e. we win!

Suppose $\alpha^2 > 2$. Intuitively we should be slightly to the right of $\sqrt{2}$, which means there's some wiggle room. Thus, there should be some element in the middle such that if we square that element we are still > 2, which contradicts α being the supremum. Formally, the idea is that $(\alpha - \text{tiny})^2 > 2$.

What do we mean by tiny? Well one way we know how to find arbitrarily small positive integers is to find some $n \in \mathbb{Z}_{\text{pos}}$ s.t. $(\alpha - \frac{1}{n})^2 > 2$. Rearranging this requires

$$\begin{aligned} \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} &> 2 \\ \alpha^2 - 2 &> \frac{2\alpha}{n} - \frac{1}{n^2} \\ \alpha^2 - 2 &= \frac{1}{n}\left(2\alpha - \frac{1}{n}\right), \end{aligned}$$

which means we win if we can find some n s.t.

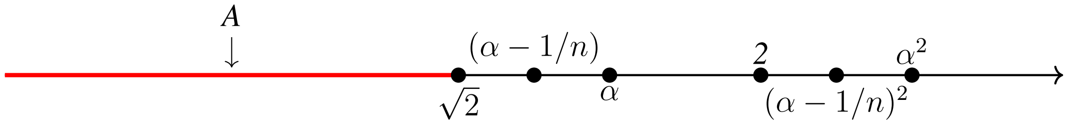
$$\frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha - \frac{1}{n}}.$$

We can't immediately use Archimedian Property here since the RHS depends on n. Remember, we don't need to be exact here; let's just bump n huge:

Jenna suggested using

$$\begin{aligned} \frac{1}{n} &< \frac{\alpha^2 - 2}{2\alpha} \\ &< \frac{\alpha^2 - 2}{2\alpha - \frac{1}{n}}. \end{aligned}$$

So, we can just find some $n \in \mathbb{Z}_{\text{pos}}$ s.t. $\frac{1}{n} < \frac{\alpha^2-2}{2\alpha}$ (which exists by Archimedian Property!) and then backtrack these steps and be good. Here's a picture of this whole thing:



Proof. Let $A := \{x \in \mathbb{R} \ : \ x^2 \leq 2\}$. Define $\alpha := \sup A$, which exists by (A13). Suppose $\alpha^2 > 2$. By Archimedian Property, $\exists n \in \mathbb{Z}_{\text{pos}}$ s.t.

$$n > \frac{2\alpha}{\alpha^2 - 2}.$$

Thus,

$$\begin{aligned} \frac{1}{n} &< \frac{\alpha^2 - 2}{2\alpha} \\ \frac{1}{n} &< \frac{\alpha^2 - 2}{2\alpha - \frac{1}{n}} \\ \frac{1}{n}\left(2\alpha - \frac{1}{n}\right) &< \alpha^2 - 2 \\ \frac{2\alpha}{n} - \frac{1}{n^2} &< \alpha^2 - 2 \\ \implies \left(\alpha - \frac{1}{n}\right)^2 &> 2, \end{aligned}$$

which implies $\alpha - 1/n$ is an upper bound of A. However, $\alpha - 1/n < \alpha$, which contradicts α being the least upper bound of A. Thus, $\alpha^2 \not> 2$. $\square$