REAL ANALYSIS: LECTURE 9

OCTOBER 5TH, 2023

1. Real Real Analysis Theorem

Last time, we were in the process of proving the following:

Theorem 1. *$\exists \alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.*

Proof. Let

$$A := \{x \in \mathbb{R} \,:\, x^2 < 2\}$$

and set $\alpha := \sup A \in \mathbb{R}$, since its existence is guaranteed by the fact that $1 \in A$ and 27 is an upper bound on A. Here's a lemma:

Lemma 1. *If $\alpha^2 > 2$, then $\exists \epsilon > 0$ s.t. $(\alpha - \epsilon)^2 > 2$.*

Before proving this lemma, let's deduce $\alpha^2 \not> 2$ from assuming this claim. If $\alpha^2 > 2$, then by the above lemma $\exists \epsilon > 0$ s.t. $\alpha - \epsilon$ would be an upper bound on A, which contradict that $\alpha - \epsilon < \alpha$ is the *least* upper bound. Thus, $\alpha^2 \not> 2$ as long as we can prove this lemma, which we'll do now:

Proof. Suppose $\alpha^2 > 2$. By the Archimedian Property, $\exists n \in \mathbb{Z}_{\text{pos}}$ s.t. $n > \frac{2\alpha}{\alpha^2 - 2}$, which implies

$$\begin{aligned}
\frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha} &< \frac{\alpha^2 - 2}{2\alpha - 1/n} \\
\frac{2\alpha}{n} - \frac{1}{n^2} &< \alpha^2 - 2 \\
\alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} &> 2 \\
\left(\alpha - \frac{1}{n}\right)^2 &> 2.
\end{aligned}$$

□

We can use an analogous idea on the other side:

Lemma 2. *If $\alpha^2 < 2$, then $\exists \epsilon > 0$ s.t. $(\alpha + \epsilon)^2 < 2$.*

After some scratchwork, a couple of proof attempts, and some work by Edith, Lexi, Emily, Sean, Aidan, Miles, Jon, Forrest, and more, we get the following:

Proof. Suppose $\alpha^2 < 2$. By Archimedian Property, pick $n \in \mathbb{Z}_{\text{pos}}$ s.t. $\frac{1}{n} < \frac{2-\alpha^2}{3\alpha}$. Recall $\alpha = \sup A$ and that $1 \in A$, which implies $\alpha \geq 1$. Thus, $\alpha \geq 1 \geq \frac{1}{m}$ $\forall m \in \mathbb{Z}_{\text{pos}}$ (the fact that we are using this inequality means that we probably could have just used $2\alpha + 1$ in the denominator, but who cares??). Using this inequality we get

$$\begin{aligned}
1/n &< \frac{2 - \alpha^2}{3\alpha} = \frac{2 - \alpha^2}{2\alpha + \alpha} \\
1/n &< \frac{2 - \alpha^2}{2\alpha + 1/n}.
\end{aligned}$$

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

Thus, we get

$$\frac{2\alpha}{n} + \frac{1}{n^2} < 2 - \alpha^2$$
$$\left(\alpha + \frac{1}{n}\right)^2 < 2.$$

Fixing $\epsilon = \frac{1}{n}$ we have found some ϵ s.t. $(\alpha + \epsilon) < 2$. □

Given the above lemma, we are essentially done. Assume $\alpha^2 < 2$. By choosing ϵ as above, we get that $\alpha + \epsilon \in A$, which means that $\alpha + \epsilon > \alpha$ is *not* an upper bound. This contradicts α being the least *upper bound*. This is a contradiction that tells us $\alpha^2 \not< 2$. By trichotomy, it must be that $\alpha^2 = 2$, and we are done! □

Here's a good thing to remember about inequalities: If you know $a < b$ and $a < c$, you have no idea about b vs. c! In particular, if you claim $a < b$ and also know $c < b$, you cannot claim anything about the relative size of a and c.

Here's a generalization- Theorem 7.5 in the book:

Theorem 2. $\forall n, k \in \mathbb{Z}_{\text{pos}}, \exists \sqrt[n]{k} \in \mathbb{R}$. *In other words, nth roots exist.*

Remark. Leo will have his own take on a proof of this, which he'll post on the website soon. This is a more complicated proof, but it's essentially the same ideas.

2. Rationals between Reals

Q: Given $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. We know there's a real number between them, but does there exist $q \in \mathbb{Q}$ s.t.

$$\alpha < q < \beta.$$

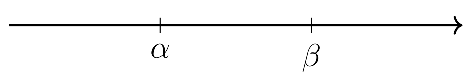
Notice you cannot simply average $\frac{\alpha+\beta}{2}$, since α, β might not be "nice".

Proposition 1. *For any nonempty interval $(\alpha, \beta) \subseteq \mathbb{R}$ (i.e. pick $\alpha, \beta \in \mathbb{R}$ s.t. $\alpha < \beta$). Then*

$$\mathbb{Q} \cap (\alpha, \beta) \neq \emptyset,$$

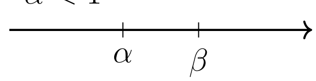
i.e. there is a rational number between them.

Scratchwork 1. *Let's look at a picture.*

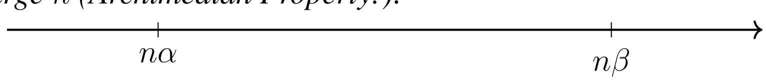


Annie noted that if $\alpha, \beta \in \mathbb{Q}$ this is easy! We can just take the midpoint, argue it's between, and argue it's a fraction. Lexi noted that if $\beta - \alpha > 1$, there should be some integer in (α, β). So, if there's a "long interval" we should be good.

Alex had the following idea: suppose $\beta - \alpha < 1$



Then multiply by some large n (Archimedian Property!):

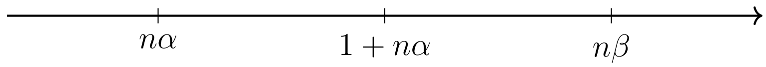


Now $n\beta - n\alpha > 1$, which means there should be some integer m in between $n\alpha$ and $n\beta$, which means that

$$n\alpha < m < n\beta \tag{2.1}$$
$$\alpha < \frac{m}{n} < \beta, \tag{2.2}$$

and we've found our rational number!

But this assume we can find an integer m. How can we do this? Well we know that



Using this idea and the trick of using floor functions (to turn a real into an integer), we'll be able to work this out. Let's see a formal proof.

Proof. Given $\alpha, \beta \in \mathbb{R}$ s.t. $\alpha < \beta$. By the Archimedian Property, $\exists n \in \mathbb{Z}_{\text{pos}}$ s.t. $n > \frac{1}{\beta - \alpha}$. Thus,

$$n(\beta - \alpha) > 1 \implies n\beta > 1 + n\alpha.$$

Consider $\lfloor 1 + n\alpha \rfloor \in \mathbb{Z}$. We know

$$\lfloor 1 + n\alpha \rfloor \leq 1 + n\alpha < \lfloor 1 + n\alpha \rfloor + 1.$$

This works for arbitrary n, α. Notice that taking the right inequality and subtracting 1 gets us

$$n\alpha < \lfloor 1 + n\alpha \rfloor .$$

This, combined with the left inequality, tells us that

$$n\alpha < \lfloor 1 + n\alpha \rfloor \leq 1 + n\alpha.$$

Finally, $1 + n\alpha < n\beta$ (the interval is greater than 1). All together we get

$$n\alpha < \lfloor 1 + n\alpha \rfloor \leq 1 + n\alpha < n\beta,$$

which means that $\lfloor 1 + n\alpha \rfloor$ is in between $n\alpha$ and $n\beta$. Overall, we get

$$\frac{\lfloor 1 + n\alpha \rfloor}{n} \in (\alpha, \beta) \cap \mathbb{Q}.$$

$\square$