REAL ANALYSIS: LECTURE 6

SEPTEMBER 25TH, 2023

1. Preliminaries

2:35 on Monday- no better time for Real Analysis. Last time we introduced the last axiom (A13), which we will call the **completeness axiom**.

Axiom 13: If $\emptyset \neq S \subseteq \mathbb{R}$ has an upper bound (i.e. $\exists u \in \mathbb{R}$ s.t. $u \geq x \; \forall x \in S$), then $\exists \alpha \in \mathbb{R}$ s.t. $\alpha \geq x \; \forall x \in S$ (i.e. α is an upper bound) and $\forall \beta$ s.t. $\beta \geq x \; \forall x \in S$, then $\alpha \leq \beta$ (i.e. for every upper bound β, we have $\alpha \leq \beta$: α is the least upper bound). We denote this α as a **supremum** of S, denoted $\sup S$.

Here, Jenna helped with the condition that $S \neq \emptyset$, which doesn't work since everything is an upper bound (or lower bound!) on $\emptyset$.

So, why do we care? Well, meta-analytically we roughly know axioms $1-12$ ensures $\mathbb{R}$ is either $\mathbb{Q}$ or $\mathbb{R}$. So, the claim here is that $\mathbb{Q}$ doesn't satisfy (A13). Miles gave us the following example:

$$S := \{x \in \mathbb{Q} \;:\; x < \sqrt{2}\}.$$

Let's show S satisfies (A13)'s condition but doesn't have a supremum. Notice $S \neq \emptyset$ and is bounded above (for example 20). Thus, by (A13) $\sup S$ exists, but $\sup S$ should be $\sqrt{2}$, which we know doesn't exist in $\mathbb{Q}$.

Remark. Informally, we call (A13) the completeness axiom because the real numbers have no "holes". $\mathbb{Q}$ therefore doesn't abide by this axiom because it does have holes!

Ali raised a question about if $\sqrt{2}$ doesn't exist in $\mathbb{Q}$, couldn't there be another supremum that's not $\sqrt{2}$ but does abide by the conditions? Here's some brief reasoning. Suppose $x \in \mathbb{Q}$ is the supremum of S. Then there's a point in between (midpoint!) that is also an upper bound. So, for any upper bound of S in $\mathbb{Q}$, we can find another one smaller.

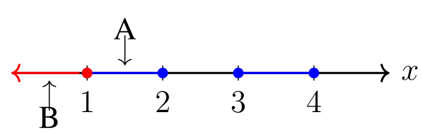
Here's a proposition from the book:

Proposition 1. *If* $\sup S$ *exists, then it is unique.*

2. Greatest Lower Bound

What about the "opposite" of a least upper bound. Maybe we need an analogous axiom like this: if $\emptyset \subseteq S \in \mathbb{R}$ has a lower bound, then $\exists \alpha \in \mathbb{R}$ s.t. (1) α is a lower bound of S, and (2) $\alpha \geq \beta$ for every β that's a lower bound of S. We call α the **infimum** of S, i.e. $\alpha = \inf S$.

Turns out we don't need a new axiom, we can simply deduce this! Alexia gave an intuitive argument: given a set A, the infimum of A should be the supremum of the "stuff below A". Well, formally the stuff below A is the set of lower bounds of A. Ok game plan: given a set A that we think should have an infimum, we also think the infimum should be the supremum of the set of lower bounds (call it B) on A.



Let's formalize this.

Proof. Given a set $\emptyset \neq S \subseteq \mathbb{R}$, define

$$B := \{x \in \mathbb{R} \;:\; x \text{ is a lower bound on } S\}$$

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

to be the set of lower bounds on S. Notice B is nonempty by hypothesis, and now we need to know that B has an upper bound. Harry suggested any $x \in S$, which we know exists since $S \neq \emptyset$. Any $x \in S$ is an upper bound since by definition any $b \in B$ is a lower bound on S, which in particular means $b \leq x$.

By (A13), $\exists \sup B \in \mathbb{R}$, which we'll call β. The claim is that $\beta = \inf S$. To check this, we need to show that (1) β is a lower bound on S, and (2) β is the greatest lower bound.

Here's Yana idea for (1). Pick $x \in S$. By definition, x is an upper bound on B. Since β is the least upper bound on B, $\beta \leq x$, which implies $\beta \leq x \; \forall x \in S$, i.e. β is a lower bound on S.

Now we need to show (2). Pick $l \in \mathbb{R}$ a lower bound on S. We want to show $\beta \geq l$. Since l is a lower bound, $l \in B$, which by definition of supremum (in particular β is an upper bound on B) tells us that $\beta \geq l$. $\square$

Ok, that was a lot. Here's the big picture. When you want to prove that something is a supremum (infimum) you must show it is an upper (lower) bound and that it is the least upper (greatest lower) bound.

Great! These 13 axioms intuitively are enough, so let's start moving on to different number systems, i.e. can we define $\mathbb{Z}$ and $\mathbb{Q}$?

3. Defining Integers

The claim is that to define $\mathbb{Z}$ (all integers), it's enough to define $\mathbb{Z}_{\text{pos}} := \{1, 2, \dots\}$. Here's an attempt by Miles:

$$\mathbb{Z}_{\text{pos}} := \{1, 1+1, 1+1+1, \dots\}.$$

This is a good idea, but the main problem is that the $\dots$ is tough to define here. Here's a formal attempt:

Definition (Successor Set)**.** A *successor set* is any $S \subseteq \mathbb{R}$ s.t.

$$1 \in S$$
$$(n \in S) \implies (n+1 \in S).$$

So, are we done? Meta-analytically, no. $\mathbb{Z}_{\text{pos}}$ is a successor set, but so is $\mathbb{N} := \mathbb{Z}_{\text{pos}} \cup \{0\}$ (Harry), $\mathbb{R}$ (Jenna), $\mathbb{Z}$ (Jon), and many many more.

But how can we use this definition to define the positive integers?

Sean noted that $\mathbb{Z}_{\text{pos}}$ is the **smallest successor set**, which Alex formalized to meaning the intersection of all possible successor sets. Sanity check: does this work? One way is to check if the intersection of two successor sets is a successor sets, which you can probably convince yourself is true if you sit down and think about it for a bit. Great! Now we can get a formal definition:

Definition (Positive Integers)**.**

$$\mathbb{Z}_{\text{pos}} := \bigcap_{\substack{S \subseteq \mathbb{R} \\ S \text{ is a successor set}}} S.$$

In other words, the positive integers are the intersection of all successor sets.

M was wondering how we can actually take this abstract notion of $\mathbb{Z}_{\text{pos}}$ and extend to $\mathbb{Z}$ and $\mathbb{Q}$. Lexi suggested the following:

$$\mathbb{Z} := \mathbb{Z}_{\text{pos}} \cup \{x \in \mathbb{R} \ : \ -x \in \mathbb{R}_{\text{pos}}\} \cup \{0\}.$$

Additionally,

$$\mathbb{Q} := \left\{ \frac{a}{b} \ : \ a \in \mathbb{Z}, b \in \mathbb{Z}_{\text{pos}} \right\}.$$

Let's take a step back. The successor set definition looks a lot like mathematical induction. Let's actually prove this as a valid method of proof:

Proposition 2 (Induction)**.** *Given a sequence of logical assertions $S(n)$, one for each $n \in \mathbb{Z}_{\text{pos}}$ s.t.*

(1) $S(1)$ is true
(2) $S(n)$ is true implies $S(n+1)$ is true.

Then $S(n)$ is true for every $n \in \mathbb{Z}_{\text{pos}}$.

Proof. Let $A := \{n \in \mathbb{Z}_{\text{pos}} \,:\, S(n) \text{ is true.}\}$. The goal here is to show that $A = \mathbb{Z}_{\text{pos}}$, i.e. $S(n)$ is true for every positive integer n. To do so we will show $A \subseteq \mathbb{Z}_{\text{pos}}$ and $\mathbb{Z}_{\text{pos}} \subseteq A$. Note $A \subseteq \mathbb{Z}_{\text{pos}}$, since we defined it as a set of positive integers satisfying some condition.

Thus, it suffices to show $\mathbb{Z}_{\text{pos}} \subseteq A$. Notice $1 \in A$ (we're given $S(1)$) and for every $n \in A$ that $n + 1 \in A$. In other words, A is a successor set! The positive integers are defined to be the intersection of all successor sets, which means it's a subset of any other successor set, in particular A. □

Ok, by proving induction we have proved that one is able to utilize induction as a method of proof. Let's show this:

Theorem 1. 1 *is the smallest positive integer.*

Proof. We proceed by induction. Let $S(n)$ be the logical assertion that $n \geq 1$. Notice $S(1)$ is true, since $1 \geq 1$.

Assume $S(n)$ is true for some positive integer n. This tells us that $n \geq 1$, which implies

$$n + 1 \geq n$$
$$n + 1 \geq 1.$$

Technically, we only know $n+1 \geq n$ since $1 > 0$ (our first theorem!) and adding n to both sides. By induction $S(n)$ is true $\forall n \in \mathbb{Z}_{\text{pos}}$, which means $1 \leq n \ \forall n \in \mathbb{Z}_{\text{pos}}$. □

Here's another proof by induction:

Proposition 3. $\mathbb{Z}_{\text{pos}}$ *are closed under* $+$.

Proof. Let $S(n)$ be the logical assertion that $n + m \in \mathbb{Z}_{\text{pos}} \ \forall m \in \mathbb{Z}_{\text{pos}}$. Notice $S(1)$ is true because $\mathbb{Z}_{\text{pos}}$ is a successor set. Suppose $S(n)$ is true. Then $n + m \in \mathbb{Z}_{\text{pos}} \ \forall m \in \mathbb{Z}_{\text{pos}}$. Since we are working in a successor set, $n + m + 1 \in \mathbb{Z}_{\text{pos}}$, which tells us that $(n + 1) + m \in \mathbb{Z}_{\text{pos}} \ \forall m \in \mathbb{Z}_{\text{pos}}$ (i.e. $S(n + 1)$ is true).

By induction $S(n)$ holds for all $n \in \mathbb{Z}_{\text{pos}}$, which gives us the desired result. □