REAL ANALYSIS: LECTURE 20

NOVEMBER 20TH, 2023

1. Preliminaries

Wowza. Real Analysis Part 20. Double digits for double digits. Cool. Math time. Last time, we introduced and explained metric spaces:

Definition (Metric)**.** Given $X \neq \emptyset$, a *metric* on X is a function $d : X \times X \to \mathbb{R}$ s.t.

(i) $d(x, y) = d(y, x) \; \forall x, y \in X$
(ii) $d(x, y) = 0 \iff x = y$
(iii) $d(x, y) + d(y, z) \geq d(x, z)$ (*triangle inequality*)

A metric space is then a nonempty set X equipped with a metric d: (X, d). Our motivation to define metric spaces last class was to generalize sequences and convergence. Then again, why should we care about this?

Well, we often want to measure distance between two objects that aren't necessarily points in space or numbers. For example, Ben offered the example of words. How similar are different words? What about species, colors, languages, etc? Metric spaces often can offer a precise way to determine this.

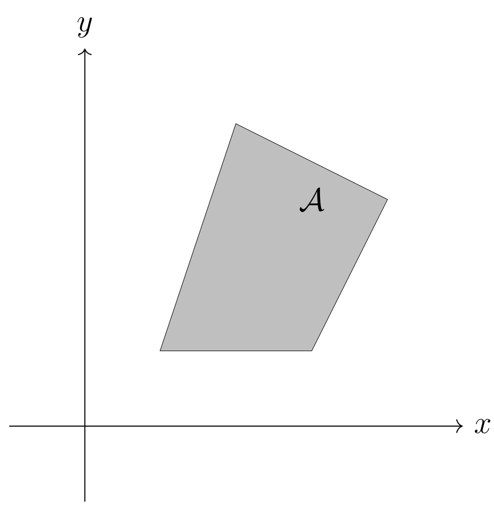
Last time, we looked at many metrics on the same set. Here's an interesting note:

Remark. Any nonempty set X is a metric space with respect to the *discrete metric*:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

1.1. **Topology of Metric Spaces.** In $\mathbb{R}$, we have *open intervals*, *closed intervals*, and neither. For example $(1, 4)$ is an open interval, $[0, 3]$ is a closed interval, and $(3, 4]$ is neither. Can we generalize the notion of open/closed sets?

Example 1. Let's try to do this in $\mathbb{R}^2$.



Most people agree that this $A \subseteq \mathbb{R}^2$ is *closed*, since the boundaries of A are included in A. What if some, but not all of the boundary was included? This is probably neither. Finally, what if none of the boundary was included? Intuitively, this is open.

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

What is our intuition telling us here? Denote the boundary of A by ∂A. Then we have the following definition:

Definition. $\mathcal{A}$ is *closed* iff $\partial\mathcal{A} \subseteq \mathcal{A}$. Additionally, $\mathcal{A}$ is open iff $\partial\mathcal{A} \cap A = \emptyset$.

There's something really weird here.

What does boundary mean?

Lexi pointed out that not all sets $\mathcal{A} \subseteq \mathbb{R}^2$ (or most other sets) have an obvious boundary. For a random subset $\mathcal{A} \subseteq \mathbb{R}^2$, what is $\partial\mathcal{A}$. Let's look back at our set $\mathcal{A}$ and determine what distinguishes boundary points from regular points. Let's brainstorm.

Idea 1. *Edith suggested looking in a short radius around each point. A short radius around boundary points would get us outside $\mathcal{A}$. Jenna noted that you can look at random points not even close to anything in $\mathcal{A}$, and by this definition it would be considered a boundary point. This idea is really close, but not quite there.*

Idea 2. *Alex wants to add to Edith's working definition. If $x \in \partial\mathcal{A}$, you can take a small (ϵ) step and end up outside of $\mathcal{A}$ and can take a small step in a different direction and end up inside of $\mathcal{A}$.*

Idea 3. *Miles has a cautionary example. What about a single point in space. Let's think about this, say in $\mathbb{R}$ for now. Is this single point open or closed? Well, it should be closed. Note that $\{p\}$ can be written as the closed interval $[p, p]$, with boundary $\{p\}$, which is contained within the original set (our definition for closed!).*

Let's combine this all. Here's the intuition:

Intuition 1. *First of all, what's the problem with Edith's/Alex's definition. The main idea is that a "different direction" isn't so clear in arbitrary metric spaces. Let's fix this.*

p is in the boundary of $\mathcal{A}$ iff no matter how close you zoom in on p, you can always "see" points in $\mathcal{A}$ and points not in $\mathcal{A}$.

What do we mean by "zoom in". Well, we can look at all the points in the space that are within ϵ of p, i.e. in an ϵ-neighborhood. What do we mean by see points both in and not in $\mathcal{A}$. Well, there should be an intersection between $\mathcal{A}$ and this ϵ-neighborhood (i.e. points in $\mathcal{A}$), and there should be an intersection between $\mathcal{A}^c$ and this ϵ-neighborhood (i.e. points not in $\mathcal{A}$).

Now let's provide a formal definition.

Definition (Boundary of $\mathcal{A}$)**.** Given a metric space (X, d), $p \in \partial\mathcal{A}$ iff $\forall\epsilon > 0$,

$$\{x \in X \ : \ d(x, p) < \epsilon\} \cup \mathcal{A} \neq \emptyset$$
$$\{x \in X \ : \ d(x, p) < \epsilon\} \cup \mathcal{A}^c \neq \emptyset.$$

This ϵ-neighborhood set has a special name:

Definition (Ball)**.** The *open ball* of radius r around $p \in X$ is

$$\mathcal{B}_r(p) := \{x \in X \ : \ d(x, p) < r\}.$$

Remark. The *closed ball* would be the say with the $<$ replaced with $\leq$. We'll mostly focus on the open ball, and if you see the word "ball" on its own we are referring to the open ball.

Remark. Note that calling this a ball really only is to hang on to the intuition that we might find if we think about the ball in $\mathbb{R}^2$ or $\mathbb{R}^3$ under the Euclidean distance metric. This "ball" doesn't need to "look spherical" (whatever that really means...).

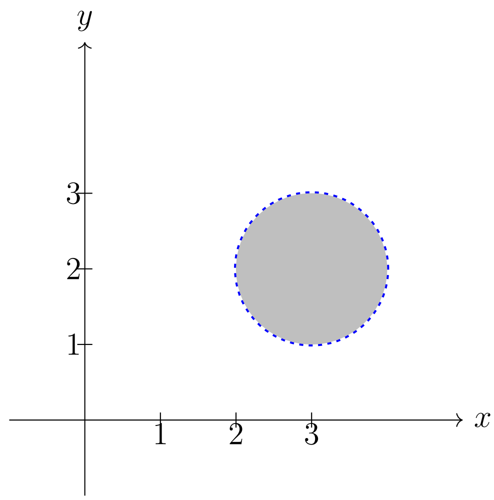
You might find it curious that we use the notation ∂ for the partial derivative and for the boundary. Here's some intuition why. Notice that the area of the circle is πr^2, and that the derivative of the area (meta-analytically $\partial(\pi r^2) = 2\pi r$ is the circumference (meta-analytically the boundary). This extends much much further:

Theorem 1 (Generalized Stokes Theorem)**.**
$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega.$$
Leo explained this with big words and Justin's plebeian mind was, as the kids say, lost in the sauce.

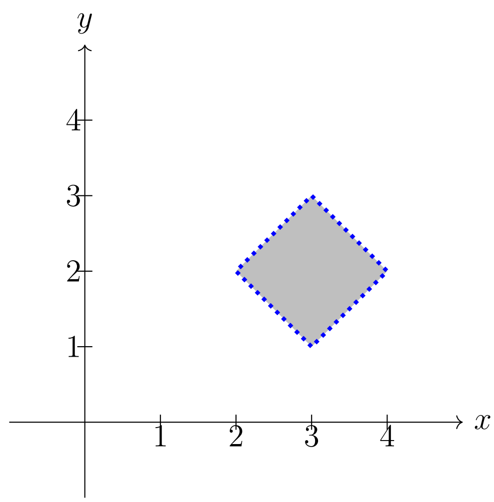
To be clear, you guys don't need to know this theorem, it just has cool symbols and stuff.

1.2. **Examples.**

Example 2. In $\mathbb{R}^2$ with respect to the Euclidean metric, let's look at $\mathcal{B}_1((3,2))$. Here's what this looks like:



Example 3. In $\mathbb{R}^2$ with respect to the Taxicab metric, which is defined by
$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|.$$
Let's look at $\mathcal{B}_1((3,2))$. Here's what this looks like:



If you stare at this long enough, you can convince yourself that the dotted boundary are the set of points exactly 1 away under the taxicab metric.

Example 4. $\mathbb{R}^2$ w.r.t. the discrete metric. Let's look at $\mathcal{B}_1((3,2))$. We're looking at the open ball, so we're looking at all things that are strictly less than 1 away from $(3,2)$. Everything here is exactly 1 away, except for $(3,2)$. Thus, $\mathcal{B}_1((3,2)) = \{(3,2)\}$; it's literally just that point!

Example 5. In $\mathbb{R}_{\geq 0}$, w.r.t the Euclidean Metric. What is $\mathcal{B}_2(1)$? Intuitively, on the left hand side we go to 0 but can't go further, and on the right hand side we go to 3 but can't quite get there. In other words, $\mathcal{B}_2(1) = [0,3)$ is open! Here's the point, being open is *context dependent*; this set $[0,3)$ is not open in $\mathbb{R}$ under the Euclidean metric, but it is open in $\mathbb{R}_{\geq 0}$ under the Euclidean metric.

Remark. We never proved that the Euclidean metric is indeed a metric on $\mathbb{R}_{\geq 0}$. If you look at our properties of a metric, notice that all of them will work as you remove points. Since we know that the Euclidean metric is a metric on $\mathbb{R}$ and $\emptyset \neq \mathbb{R}_{\geq 0} \subseteq \mathbb{R}$, therefore the Euclidean metric is a metric on $\mathbb{R}_{\geq 0}$.

Example 6. In $\mathbb{R}$ w.r.t. Euclidean metric. There is no boundary of $\mathbb{R}$, and it's pretty easy to see that $\mathbb{R}$ meets the definition for being both open and closed. Sets like this have a funny name: $\mathbb{R}$ is a *clopen set*. The $\emptyset$ is another clopen set, and there are cases with other non-trivial clopen sets.