# REAL ANALYSIS: LECTURE 10

OCTOBER 12TH, 2023

## 1. Preliminaries

Let me guess. Right now you're shocked, perhaps even crying hysterically: it's already Real Analysis lecture 10??? Oh how the time flies by. It seems like just yesterday we were talking about axioms and elephants and $\sqrt{2}$.

Ok, snap back to reality. Last time, we proved there are rationals between any two (distinct) real numbers. Here's a formal description:

**Proposition 1.** *Any nonempty open interval contains a rational number.*

Here's the precise definition of an open interval:

$$(\alpha, \beta) := \{x \in \mathbb{R} \,:\, \alpha < x < \beta\}.$$

To ensure this open interval is nonempty, we require $\alpha < \beta$. Here's exactly what we proved last time. For all sufficiently large $n \in \mathbb{Z}_{\text{pos}}$,

$$\frac{\lfloor n\alpha + 1 \rfloor}{n} \in (\alpha, \beta).$$

This actually gives us *infinite* many rationals in $(\alpha, \beta)$. Here's a corollary:

**Corollary 1.** *Any nonempty open interval contains an irrational number, i.e. some $x \in \mathbb{R} \setminus \mathbb{Q}$.*

*Proof.* The proof is in the book, but it's basically a neat trick. Take any interval, slide it back (i.e. subtract everything) by $\sqrt{2}$. By above there's some rational number in this interval. Slide the interval back, and this rational number we know exists now becomes irrational. $\square$

Propositions of the form Proposition 1 are common. In fact, the terminology is to say Proposition 1 means that $\mathbb{Q}$ is *dense* in $\mathbb{R}$. Corollary 1 means that $\mathbb{R} \setminus \mathbb{Q}$ is *dense* in $\mathbb{R}$. Here's the definition of *dense*:

**Definition.** A set $S \subseteq T \subseteq \mathbb{R}$ is *dense* in $T$ iff between every two elements of $T$ there's an element of $S$.

We're now going to move into a completely new area of real analysis!

## 2. Sizes of Sets

Georg Cantor studied Fourier Analysis, and a specific proof he did led him to the conclusion that, for some weird reason, there's *no* way to write

$$\mathbb{R} = \{x_1, x_2, \dots\}.$$

This makes no sense! How else could you define $\mathbb{R}$ other than listing its points? This led him to a certain path to start thinking much more carefully about infinity.

Here's a thought experiment. Imagine a theater, and people with tickets waiting to get in. Are there more people or more seats in the theater?

Ben suggests count the number of seats, count the number of people, and compare. Matt suggests that we let people in and sit down. There are three possibilities:

i There are empty seats at the end (more seats than people)
ii Every seat has exactly one person and every person is seated (equal)
iii People left standing with seats filled (more people than seats)

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Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

Let's take a step back. Given two sets $A, B$ (set of people and set of seats), how can we compare the size? You can either directly count the number of elements, or you can compare (like how we did in Matt's suggestion). The problem with infinite sets is that we can *only* use the second approach, since you can't count to infinity.

Here's another question. What's bigger? $\mathbb{Z}_{\text{pos}}$ or $10\mathbb{Z}_{\text{pos}}$:

$$\mathbb{Z}_{\text{pos}} := \{1, 2, 3, 4, 5, 6, 7, \ldots\}$$
$$10\mathbb{Z}_{\text{pos}} = \{10, 20, 30, 40, \ldots\}.$$

On the one hand, $10\mathbb{Z}_{\text{pos}}$ is a strict subset of $\mathbb{Z}_{\text{pos}}$ (there's stuff in $\mathbb{Z}_{\text{pos}}$ that's not in $10\mathbb{Z}_{\text{pos}}$ and everything in $10\mathbb{Z}_{\text{pos}}$ is in $\mathbb{Z}_{\text{pos}}$). But, on the other hand, we can perfectly *pair up elements* from $\mathbb{Z}_{\text{pos}}$ and $10\mathbb{Z}_{\text{pos}}$. If we match up $x$ and $10x$, everything matches perfectly (just like having a bunch of people each matched perfectly with a distinct theater ticket).

Both perspectives are very reasonable, but the latter notion turns out to be much more flexible (and is the one Cantor and us will be using). Here's an example. Consider

$$\{1.5, 2.5, 3.5, 4.5, \ldots\}$$
$$\{10, 20, 30, 40, \ldots\}.$$

Intuitively shifting things by $.5$ shouldn't influence the size, but we can no longer think of the subset example.

Let's mathematize this.

**Definition** (Size of Sets)**.** Given sets $A, B$, we say they have the same *size* (denoted $A \approx B$) iff $\exists f : A \to B$ s.t. $\forall b \in B, \exists! a \in A$ s.t. $f(a) = b$.

Thanks to Blakeley, Noah, and Ben for helping with this definition.

We would verify $A \approx B$ by finding a function $f : A \to B$ s.t.

i $\forall b \in B$, there's *at least* one $a \in A$ s.t. $f(a) = b$.
ii $\forall b \in B$, there's *at most* one $a \in A$ s.t. $f(a) = b$.

One good way to show injectivity is that, if $f(a) = f(b)$, then $a = b$. One good way to show surjectivity is to take some arbitrary element $b \in B$ and find some $a \in A$ s.t. $f(a) = b$.

*Remark.* Let's think about conditions (i) and (ii). We start with a function $f : A \to B$. Condition (i) says anything you choose in $B$ is actually an output of something in $A$. Condition (ii) says that two things in $A$ can't be mapped to the same thing in $B$.

Any function $f$ that satisfies condition (i) is known as *surjective*, and any that satisfies condition (ii) is known as *injective*. Functions that are simultaneously injective and surjective (i.e. perfect matches) are known as *bijective* (and vice-versa).

Thus, here's a way to restate our definition of sizes of sets. Given two sets $A, B$, we get $A \approx B$ iff $\exists f : A \to B$ that is a bijection.

## 3. Meta-Analytic Examples

Let $f : \mathbb{Z} \to \mathbb{Z}$ s.t. $x \mapsto x^2$. Is $f$ surjective? No. Annie pointed out that nothing maps to $6$, meaning there are outputs that aren't hit. Is $f$ injective. No. Notice $1 = f(1) = f(-1)$, so it's not the case that $f(a) = f(b)$ implies $a = b$.

Here's another example. Let $g : \mathbb{Z} \to \mathbb{Z}_{\text{pos}}$ defined by $x \mapsto x^2$. Is it surjective? No. Is it injective? Also no. However, $h : \mathbb{Z}_{\text{pos}} \to \mathbb{Z}$ define by $x \mapsto x^2$ is still not surjective, but it is injective! Notice that still nothing maps to $6$ (so not surjective), but now we don't have this problem of $f(1) = f(-1)$.

Now let's do some examples of comparing sets. Let's show $\mathbb{Z}_{\text{pos}} \approx 10\mathbb{Z}_{\text{pos}}$. To do so we need to find a bijection. Annie suggested $f : \mathbb{Z}_{\text{pos}} \to 10\mathbb{Z}_{\text{pos}}$ defined by $x \mapsto 10x$. Consider $a \in 10\mathbb{Z}_{\text{pos}}$. Notice $a = 10b$ for some $b$ by definition of $10\mathbb{Z}_{\text{pos}}$. Then $f(b) = 10b = a$, so $f$ is surjective. Suppose $f(x) = f(y)$. Then $10x = 10y$, which implies that $x = y$. Thus, $f$ is injective. Since $f$ is both an injective and a surjective, it's a bijection, meaning $\mathbb{Z}_{\text{pos}} \approx 10\mathbb{Z}_{\text{pos}}$.

What about $\mathbb{Z}_{\mathrm{pos}}$ and $\mathbb{Z}$?

$$\mathbb{Z}_{\mathrm{pos}} = \{1, 2, 3, 4, 5, \dots\}$$
$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Sean suggests that we can map negatives to even numbers, $0$ to $1$, and positives to odds. If you think it through, you can make this work.