# REAL ANALYSIS: LECTURE 4

SEPTEMBER 18TH, 2023

## 1. Preliminaries

So, where were we. Last class we began listing properties (axioms) of $\mathbb{R}$, and we wish to continue until we *uniquely define* $\mathbb{R}$. To recap, here's the axioms we've written already:

(A1) $\exists + : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
(A2) $+$ is associative
(A3) $+$ is commutative
(A4) $\exists$ additive identity, denoted $0$
(A5) $\exists$ additive inverses. We'll denote the additive inverse of $x$ by $-x$. Notice the word "the" is only allowed since we proved additive inverses are unique.

We call any set $G$ with an operation $+$ that satisfies (A1)-(A5) is called an **abelian group**. You will study this much more in abstract algebra. In fact, (A6) - (A10) is basically the exact repeat for an operation $\cdot$ (multiplication).

(A6) $\exists \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
(A7) $\cdot$ is associative
(A8) $\cdot$ is commutative
(A9) $\exists$ multiplicative identity, denoted $1$ (with $1 \neq 0$: otherwise $\mathbb{R}$ could be $0$).
(A10) For every $x \neq 0, \exists$ a multiplicative inverse. We'll denote the multiplicative inverse of $x$ by $x^{-1}$ (or $\frac{1}{x}$).

Finally, we have a property connecting $+$ and $\cdot$, know as the distributive property:

(A11) Distributive property: $x \cdot (y + z) = x \cdot y + x \cdot z$.

Any set $F$ with two operations $+$ and $\cdot$ that satisfy (A1) - (A11) is called a **field**.

Here Emily asked an interesting question: does (A11) tell us that (meta-analytically) multiplication is repeated addition, i.e. that $2 \cdot 2 = 2 \cdot (1 + 1) = 2 \cdot 1 + 2 \cdot 1 = 2 + 2$. In fact, it tells us even more! For example, $\pi \cdot e$ still somehow retains this connection with addition, though the intuition isn't as clear.

Ok, let's use these axioms to prove something analytically:

**Proposition 1.** *If $S$ satisfies (A1) - (A11) (i.e. if $S$ is a field), then*

$$x \cdot 0 = 0 \ \forall x \in S$$

Take 1: (False Proof!)

*Proof.* Suppose $\exists x \in S$ s.t. $x \cdot 0 \neq 0$. Choose $y \in S$. The

$$\begin{aligned} x \cdot (y + -y) &\neq 0 && \text{for some } y \in S \\ x \cdot y + x \cdot (-y) &\neq 0 && \text{by (A11)} \\ x \cdot y + -x \cdot y &\neq 0 && \text{hmm...} \end{aligned}$$

but $xy + -(xy) = 0$ by definition of additive inverse. Ideally this would be the contradiction we wanted, but Annie realized that we can't assume (and never proved) $-(x \cdot y) = x \cdot -y$. In other words, we want to be able to "pull out" the negative sign, i.e. that $-y = -1 \cdot y$. Ok, let's try again. ☐

Take 2.0 (Yana's Correct Proof!):

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

*Proof.*

$$\begin{aligned}
x \cdot 0 &= x(0+0) && \text{since } 0 = 0+0 \\
x \cdot 0 &= x \cdot 0 + x \cdot 0 && \text{by distributing} \\
-(x \cdot 0) + x \cdot 0 &= (x \cdot 0 + x \cdot 0) + -(x \cdot 0) && \text{by adding additive inverses} \\
0 &= (x \cdot 0 + x \cdot 0) + -(x \cdot 0) && \text{definition of additive inverses} \\
0 &= (x \cdot 0) + (x \cdot 0 + -(x \cdot 0)) && \text{by associativity} \\
0 &= x \cdot 0 + 0 && \text{by definition of additive inverse} \\
0 &= x \cdot 0 && \text{by definition of additive identity.}
\end{aligned}$$

□

Notice that we use distributivity. Intuitively we *had* to use distributivity, since distributivity connects $+$ and $\cdot$, and this proposition does the same. Here's an example of something that doesn't fulfill (A11) that also doesn't fulfill the above proposition:

*Example* 1. Consider $S = \{0, 1\}$ where $+$ is addition mod 2 and $\cdot$ is such that

(1) $0 \cdot 0 = 1$
(2) $0 \cdot 1 = 0$
(3) $1 \cdot 0 = 0$
(4) $1 \cdot 1 = 1$

It turns out $S$ as defined satisfies (A1) - (A10) but *fails* (A11) (check this yourself!). However here $0 \cdot 0 = 1 \neq 0$, which means the above proposition doesn't hold.

So, did we define $\mathbb{R}$. Nope, there's still lots of stuff that satisfy (A1)-(A11) (i.e. $\mathbb{R}$ isn't the only field).

Examples of Fields:

(1) $\mathbb{R}$ under usual $+, \cdot$
(2) $\mathbb{Q}$ under usual $+, \cdot$
(3) $\mathbb{Z}$ (mod 2) under $+, \cdot$ mod 2
(4) $\mathbb{Z}$ (mod 7) under $+, \cdot$ mod 7

Here $\mathbb{Z}$ (mod 7) means the set $\mathbb{Z}$ (mod 7) $= \{0, 1, 2, 3, 4, 5, 6\}$ and $a + b$ (mod 7) means add $a + b$ normally (if $a = 6, b = 4$ then $a + b = 10$) then subtract 7 until you get to something in the set $10 - 7 = 3 \in \mathbb{Z}$ (mod 7). Multiplication modulo 7 is exactly analogous. If $a = 6, b = 4$ then $a \cdot b = 24$ then subtract 7 three times to get to $24 - 7 - 7 - 7 = 3 \in \mathbb{Z}$ (mod 7), which means $a \cdot b$ (mod 7) $= 3$. Ok, so we're not done yet. Let's keep going.

## 2. Order Axiom

Recall our intuition about the next axioms has to do with relations between two arbitrary numbers. Specifically, given $x, y \in \mathbb{R}$, one of $x$ or $y$ is at least as large as the other. Formally, there is a **trichotomy**: exactly one of

(1) $x > y$
(2) $x = y$
(3) $x < y$

holds. We can't literally use this as (A12) since we have no idea what $>$ means, but let's try to capture it.

*Intuition* 1. What do $>$ and $<$ mean? Here's some initial thoughts:

(1) $x > y \iff x - y$ is positive
(2) $x < y \iff x - y$ is negative (i.e. $y - x$ is positive)
(3) $x = y \iff x - y = 0$.

This indicates that is suffices to define **positive**. Let's do this.

Axiom 12 (A12): $\exists \mathbb{P} \subseteq \mathbb{R}$ s.t.

(i) $\mathbb{P}$ is closed under $+$ and $\cdot$

(ii) Trichotomy: $\forall x \in \mathbb{R}$ exactly one of the following hold

  (a) $x \in \mathbb{P}$

  (b) $-x \in \mathbb{P}$

  (c) $x = 0$

We can extend this to establish some notation. If $x \in \mathbb{P}$ then $x$ is positive. If $-x \in \mathbb{P}$ then $x$ is negative. If $x > 0$ then $x$ is positive.

Now, our first theorem. Get ready for some real math:

**Theorem 1.** $1 > 0$.

Ok, before jumping into a proof let's brainstorm:

*Intuition* 2. Lexi suggested it's possible to use the fact that $1$ (something we know *isn't* $0$). Thus $1 \in \mathbb{P}$ or $-1\mathbb{P}$. Edith also noticed that, for every positive number $x$, $1 \cdot x \in \mathbb{P}$.

Miles used a lemma to complete the proof:

*Proof.* Here's a lemma we will take for granted for now.

**Lemma 1.** $-1 \cdot -1 = -(-1)$.

With this, let's try to prove $1 > 0$. Suppose $-1 \in \mathbb{P}$. By the above lemma, $-1 \cdot -1 = -(-1) = 1$ (since additive inverses are unique!). Since $\mathbb{P}$ is closed, $-1 \cdot -1 = 1 \in \mathbb{P}$, which contradicts trichotomy (more than one hold). The lemma is proved in the book. $\square$

We claim that $\mathbb{Z} \pmod 2$ doesn't satisfy (A12), i.e. it doesn't have order.