Trigonometry and a Sequence of Polynomials
(Chebyshev Polynomials)

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Introduction

What are Chebyshev polynomials?

- Chebyshev polynomials rewrite $\cos(nx)$ into a polynomial or formula in terms of $\cos(x)$.

For example...

$$\cos(4x) = 8\cos^4(x) - 8\cos^2(x) + 1$$

We denote the Chebyshev polynomial of $\cos(nx)$ as $T_n(\cos(x))$, a function on $\cos(x)$, and we call this the n^{th} Chebyshev polynomial. From now on, we let $u = \cos(x)$. Hence...

$$\cos(4x) = T_4(u) = 8u^4 - 8u^2 + 1$$

The First Few Terms of T_n

$$
\begin{aligned}
T_0(u) &= 1 \\
T_1(u) &= u \\
T_2(u) &= 2u^2 - 1 \\
T_3(u) &= 4u^3 - 3u \\
T_4(u) &= 8u^4 - 8u^2 + 1 \\
T_5(u) &= 16u^5 - 20u^3 + 5u \\
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T_7(u) &= 64u^7 - 112u^5 + 56u^3 - 7u \\
T_8(u) &= 128u^8 - 256u^6 + 160u^4 - 32u^2 + 1 \\
T_9(u) &= 256u^9 - 576u^7 + 432u^5 - 120u^3 + 9u
\end{aligned}
$$

Any conjectures?

Noticeable Patterns

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- Leading coefficients are powers of 2.

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- T_n is of degree n.

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- For even n, the ending coefficients are ± 1
- Terms are alternating.

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- The coefficient of the u^{n-4} in T_n is $n \cdot s$, where $-s$ is the coefficient of the u^{n-5} in T_{n-3}.

Recursive Formula

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Conjecture

$$T_n(u) = 2u \cdot T_{n-1}(u) - T_{n-2}(u)$$

Proof of Recursive Formula

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$$T_n(u) = 2u \cdot T_{n-1}(u) - T_{n-2}(u)$$

Proof.

Recall that...

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

What we want is $\cos(nx)$ expressed using previous terms, so we let $a = (n-1)x$ and $b = x$ and thus $a + b = nx$

$$\cos((n-1)x + x) = \cos((n-1)x)\cos(x) - \sin((n-1)x)\sin(x)$$

$$\sin((n-1)x)\sin(x) = \cos((n-1)x)\cos(x) - \cos((n-1)x + x)$$

Proof of Recursive Formula

Using the product-to-sum formula for $\sin(a)\sin(b)$, we get...

$$\sin(nx - x)\sin(x) = \cos((n-1)x)\cos(x) - \cos((n-1)x + x)$$

$$\frac{\cos(nx - 2x) - \cos(nx)}{2} = \cos((n-1)x)\cos(x) - \cos((n-1)x + x)$$

$$\cos(nx - 2x) - \cos(nx) = 2\cos((n-1)x)\cos(x) - 2\cos(nx)$$

$$\cos(nx) = 2\cos((n-1)x)\cos(x) - \cos((n-2)x)$$

$$T_n(u) = 2T_{n-1}(u) \cdot u - T_{n-2}(u) \qquad \blacksquare$$

General Formula

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Conjecture

$$T_n(u) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[(u^2 - 1)^k \cdot u^{n-2k} \cdot \binom{n}{2k} \right]$$

Computing $T_5(u)$ with the General Formula

$$T_n(u) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[(u^2 - 1)^k \cdot u^{n-2k} \cdot \binom{n}{2k} \right]$$

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Computing $T_5(u)$ with the General Formula

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General Formula #2

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Conjecture

$$T_n(u) = \frac{1}{2}\left[\frac{1}{(u+\sqrt{u^2-1})^n} + \frac{1}{(u-\sqrt{u^2-1})^n}\right]$$

Explaining General Formula #2

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Proof.

We define a generating function for $T_n(u)$...

$$\text{Let } f(x) = \sum_{n=0}^{\infty} T_n x^n.$$

We use our recursive formula and multiply $f(x)$ by $2u$, then consider $2uxf(x) - f(x)$.

Explaining General Formula #2

Stuff cancels! We get

$$f(x) = \frac{1-ux}{1-2ux+x^2}.$$

We then use partial fractions, letting be p and q be the roots of the quadratic $1-2ux+x^2$:

$$\frac{A}{x-p} + \frac{B}{x-q} = \frac{1-ux}{1-2ux+x^2}.$$

If we make the substitution $A' = -Ap$ and $B' = -Bq$, then we can instead solve

$$A' \cdot \left(\frac{1}{1-\frac{x}{p}}\right) + B' \cdot \left(\frac{1}{1-\frac{x}{q}}\right) = \frac{1-ux}{1-2ux+x^2} = f(x).$$

We can rewrite the LHS with geometric series and equate coefficients!

Explaining General Formula #2

It follows that

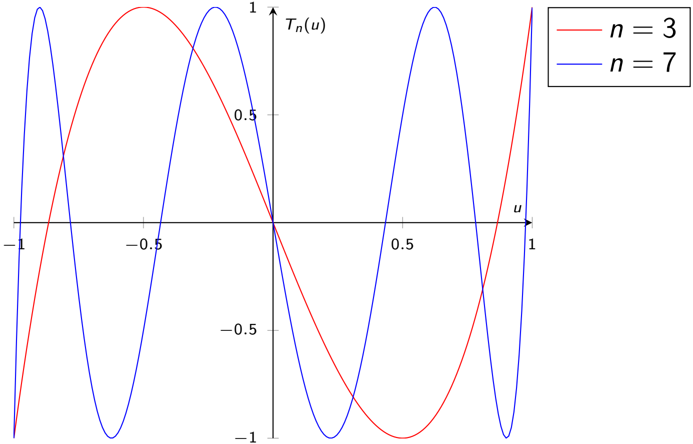
$$A' \cdot \left(\frac{1}{p}\right)^k + B' \cdot \left(\frac{1}{q}\right)^k = T_k.$$

We set $k = 0$ and $k = 1$ to give us equations. We can express p and q in terms of u using the quadratic formula: recall that $(x-p)(x-q) = 1 - 2ux + x^2$, so $p = u + \sqrt{u^2-1}, q = u - \sqrt{u^2-1}$. After some work, we get $A' = B' = \frac{1}{2}$, so our general formula becomes

$$T_n(u) = \frac{1}{2}\left[\frac{1}{(u+\sqrt{u^2-1})^n} + \frac{1}{(u-\sqrt{u^2-1})^n}\right]$$

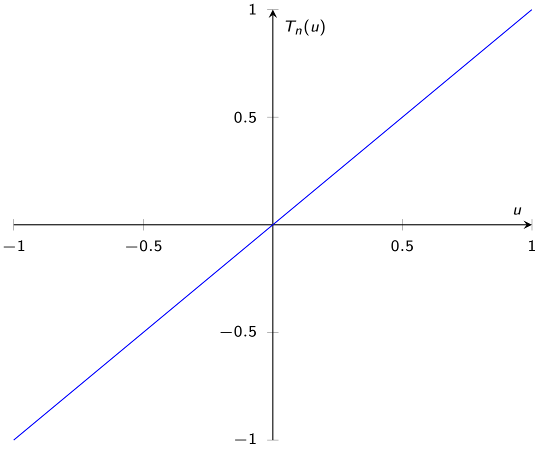
Exploring Graphs of T_n

What happens when we look at the graph of $T_n(u)$?



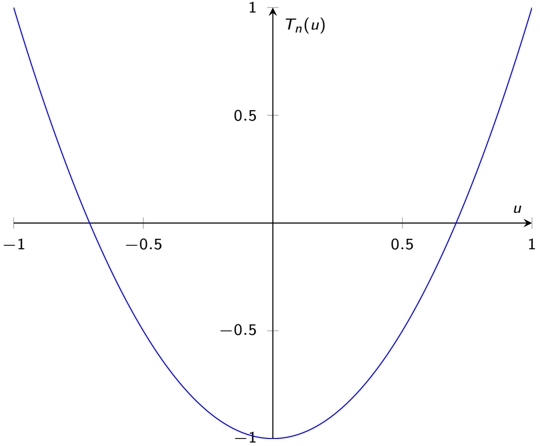
The First Few Graphs of T_n

$$T_1(u) = u$$



The First Few Graphs of T_n

$$T_2(u) = 2u^2 - 1$$



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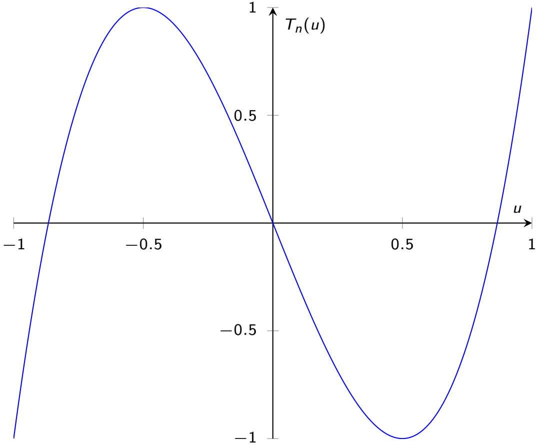
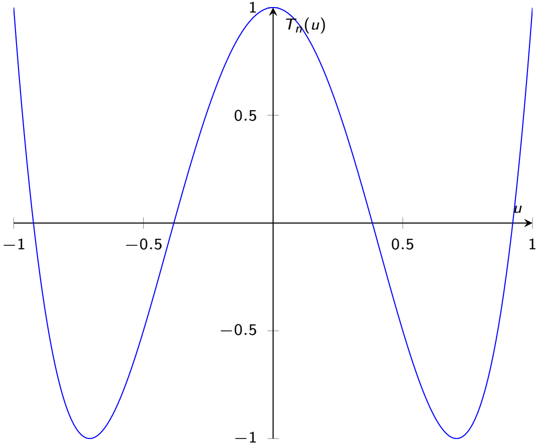


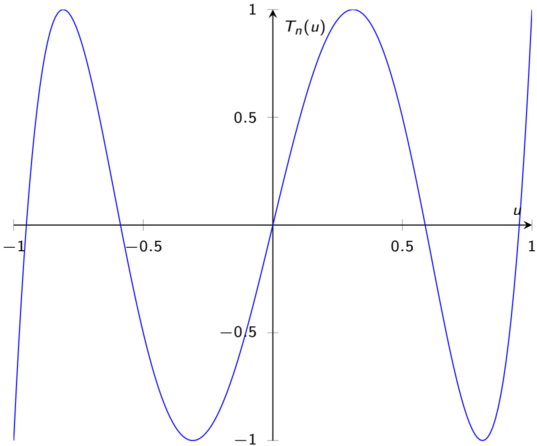
The First Few Graphs of T_n

$$T_4(u) = 8u^4 - 8u^2 + 1$$



The First Few Graphs of T_n

$$T_5(u) = 16u^5 - 20u^3 + 5u$$



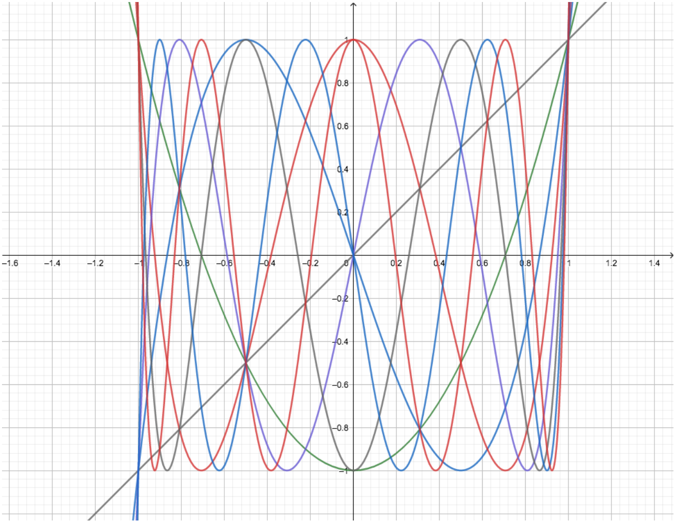
Initial Observations of Graphs of T_n

$$T_n(u) \text{ is an even function} \iff n \text{ is even}$$

$$T_n(u) \text{ is an odd function} \iff n \text{ is odd}$$

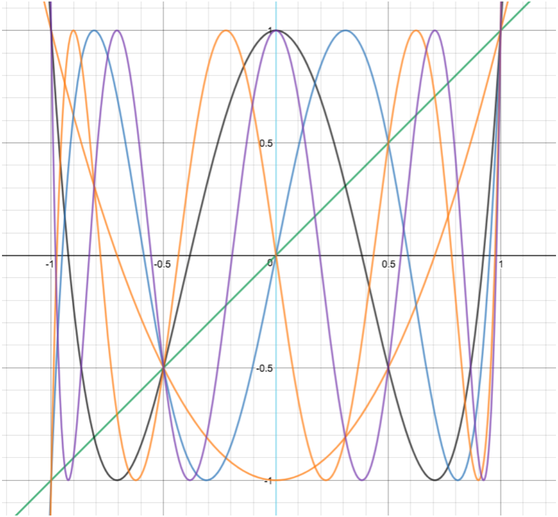
An Interesting Observation

$$T_1(u),\ T_2(u),\ T_3(u),\ T_4(u),\ T_5(u),\ T_6(u),\ T_7(u),\ T_8(u)$$



Intersection $(-0.5, -0.5)$

$$T_1(u),\ T_2(u),\ T_4(u),\ T_5(u),\ T_7(u),\ T_8(u) \qquad (3 \nmid n)$$



Explaining $(-0.5, -0.5)$

Conjecture

$$n \in \mathbb{N} \wedge 3 \nmid n \implies T_n(-0.5) = -0.5$$

In other words, $T_n(u)$ *passes through point* $(-0.5, -0.5)$

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Proof.

$$3 \nmid n \implies n = 3k \pm 1$$

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$$T_n(u) = T_{3k\pm1}(\cos x) = \cos\left((3k \pm 1)x\right)$$

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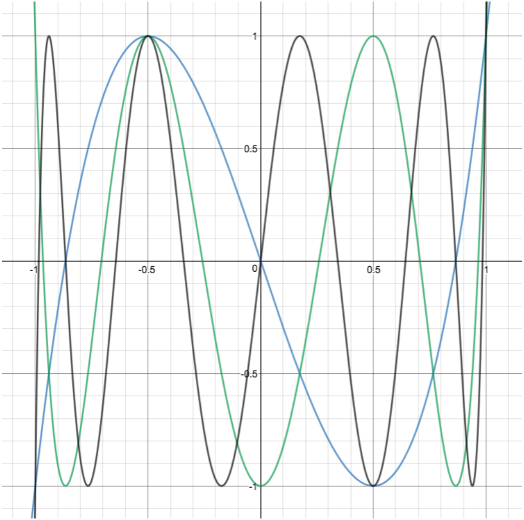
$$= \cos\left(\pm 2k\pi \pm \frac{2\pi}{3}\right) = \cos\left(\pm\frac{2\pi}{3}\right) = -0.5 \qquad \blacksquare$$

Intersection $(-0.5, 1)$

$$T_3(u),\ T_6(u),\ T_9(u) \qquad (3|n)$$



Explaining $(-0.5, 1)$

Conjecture

$$n \in \mathbb{N},\ T_{3n}(-0.5) = 1$$

In other words, $T_{3n}(u)$ *passes through point* $(-0.5, 1)$

Explaining $(-0.5, 1)$

Conjecture

$$n \in \mathbb{N},\ T_{3n}(-0.5) = 1$$

In other words, $T_{3n}(u)$ *passes through point* $(-0.5, 1)$

Proof.

$$T_{3n}(-0.5) = \cos\left(3n \cdot \pm\frac{2\pi}{3}\right) = \cos(\pm 2n \cdot \pi) = 1 \qquad \blacksquare$$

Intersections of $T_n(u)$ and $T_m(u)$

Definition

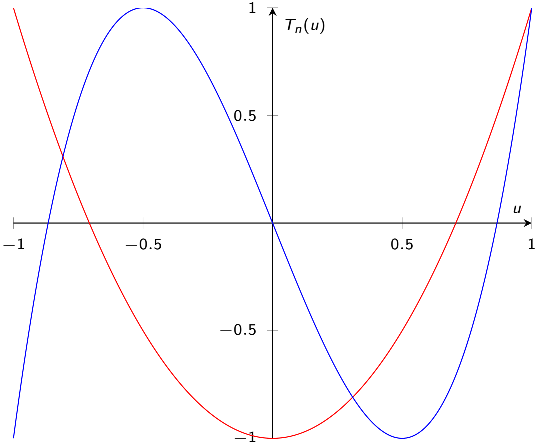
We define $I(n, m)$ as the number of intersections of the graphs of $T_n(u)$ and $T_m(u)$. In other words, the number of solutions to $T_n(u) = T_m(u)$.

Question

Can we predict $I(n, m)$ given n, m?

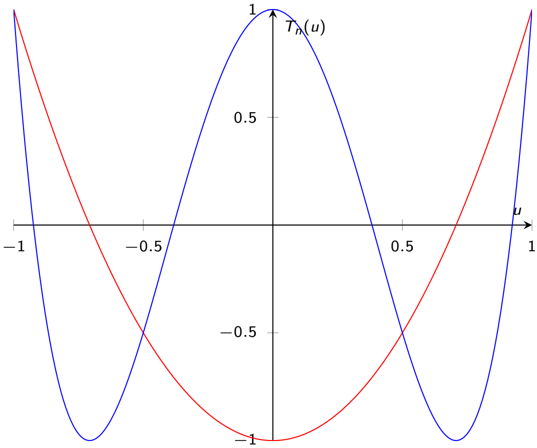
Some Numerical Examples of $I(n, m)$

$$I(2,3) = 3$$



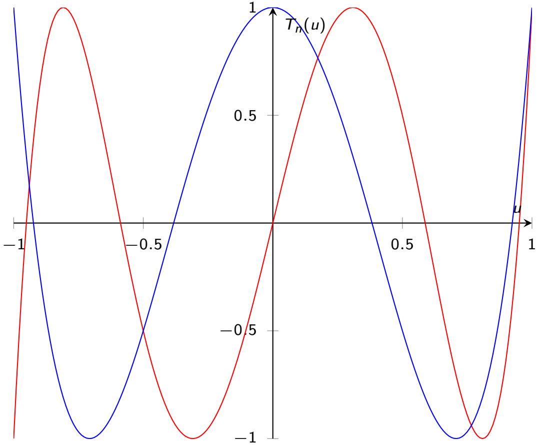
Some Numerical Examples of $I(n, m)$

$$I(2, 4) = 4$$



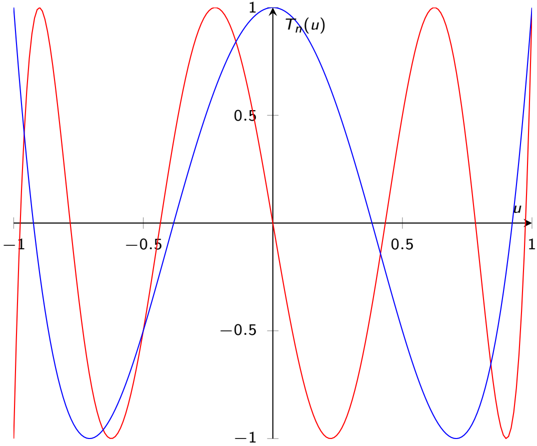
Some Numerical Examples of $I(n,m)$

$$I(4,5) = 5$$



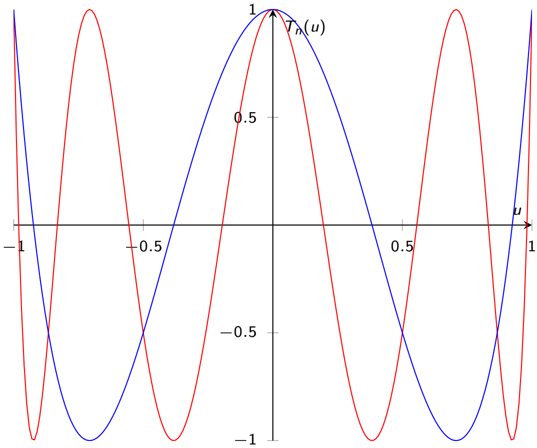
Some Numerical Examples of $I(n, m)$

$$I(4, 7) = 7$$



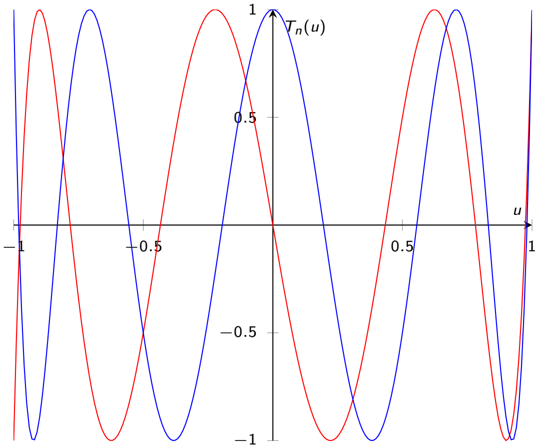
Some Numerical Examples of $I(n, m)$

$$I(4, 8) = 7$$



Some Numerical Examples of $I(n, m)$

$$I(7, 8) = 8$$



Some Numerical Examples of $I(n, m)$

n	2	2	4	4	4	7		2	10	5	3
m	3	4	5	7	8	8	$\cdots$	10	12	10	9
$I(n, m)$	3	4	5	7	7	8		9	12	8	7

$T_n(u)$ in $\mathbb{Z}_p[u]$

Question

What **structures** *do the coefficients of $T_n(u)$ exhibit modulo prime p? (In other words, what do we get when we observe the coefficients of $T_n(u)$ in $\mathbb{Z}_p[u]$?)*

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$$= 16u^5 + 0u^4 - 20u^3 + 0u^2 + 5u + 0$$

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$$T_5(u) = 0 + 5u + 0u^2 - 20u^3 + 0u^4 + 16u^5$$

$T_n(u)$ in $\mathbb{Z}_p[u]$

which all yields...

$$
\begin{aligned}
T_0(u) &= 1 \\
T_1(u) &= 0 + u \\
T_2(u) &= -1 + 0u + 2u^2 \\
T_3(u) &= 0 - 3u + 0u^2 + 4u^3 \\
T_4(u) &= 1 + 0u - 8u^2 + 0u^3 + 8u^4 \\
T_5(u) &= 0 + 5u + 0u^2 - 20u^3 + 0u^4 + 16u^5 \\
T_6(u) &= -1 + 0u + 18u^2 + 0u^3 - 48u^4 + 0u^5 + 32u^6
\end{aligned}
$$

$\cdots$

$T_n(u)$ in $\mathbb{Z}_p[u]$

Now, since the coefficient terms are now matching, we extrapolate the coefficients and place them into a table...

n	u^0	u^1	u^2	u^3	u^4	u^5	u^6
0	1						
1	0	1					
2	-1	0	2				
3	0	-3	0	4			
4	1	0	-8	0	8		
5	0	5	0	-20	0	16	
6	-1	0	18	0	-48	0	32

$T_n(u)$ in $\mathbb{Z}_p[u]$

Taken mod 7...

n	u^0	u^1	u^2	u^3	u^4	u^5	u^6
0	1						
1	0	1					
2	6	0	2				
3	0	4	0	4			
4	1	0	6	0	1		
5	0	5	0	1	0	2	
6	6	0	4	0	1	0	4

$T_n(u)$ in $\mathbb{Z}_p[u]$

More terms. . .

n																	
0	1																
1	0	1															
2	6	0	2														
3	0	4	0	4													
4	1	0	6	0	1												
5	0	5	0	1	0	2											
6	6	0	4	0	1	0	4										
7	0	0	0	0	0	0	0	1									
8	1	0	3	0	6	0	3	0	2								
9	0	2	0	6	0	5	0	5	0	4							
10	6	0	1	0	6	0	0	0	1	0	1						
11	0	3	0	3	0	0	0	2	0	5	0	2					
12	1	0	5	0	0	0	0	0	3	0	2	0	4				
13	0	6	0	0	0	0	0	5	0	1	0	2	0	1			
14	6	0	0	0	0	0	0	0	0	0	0	0	0	0	2		
15	0	6	0	0	0	0	0	2	0	6	0	5	0	6	0	4	
16	1	0	5	0	0	0	0	0	4	0	5	0	3	0	3	0	1

$T_n(u)$ in $\mathbb{Z}_p[u]$

The previous table seems to have **more** 0s than usual, meaning that many of these terms are fully divisible by $p = 7$. This leads us to. . .

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What determines the appearance of these 'degenerate' 0 terms, and what structure do they exhibit?

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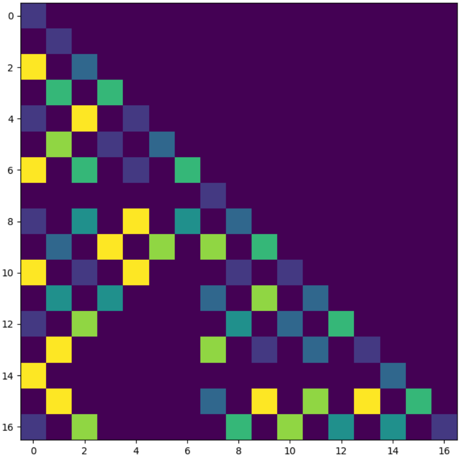
Question

What determines the appearance of these 'degenerate' 0 terms, and what structure do they exhibit?

To see this on a larger scale, we interpolate these values as colour values on a grid. Up to $T_{16}(u)$ and taken mod 7 yields...

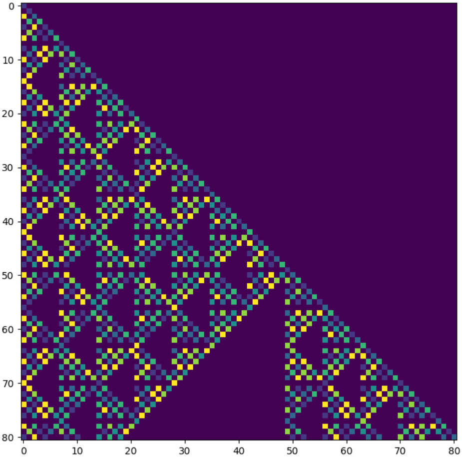
$T_n(u)$ in $\mathbb{Z}_p[u]$



$T_n(u)$ in $\mathbb{Z}_p[u]$

What about more?

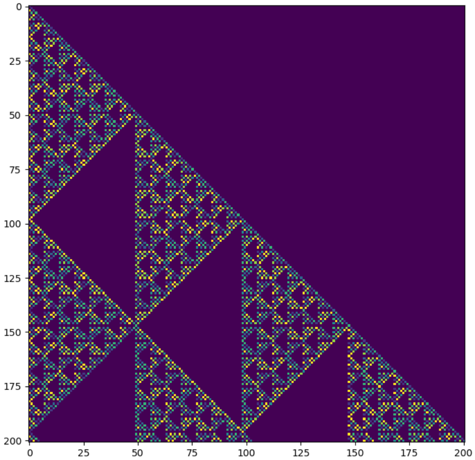
$T_n(u)$ in $\mathbb{Z}_p[u]$



$T_n(u)$ in $\mathbb{Z}_p[u]$

Even more?

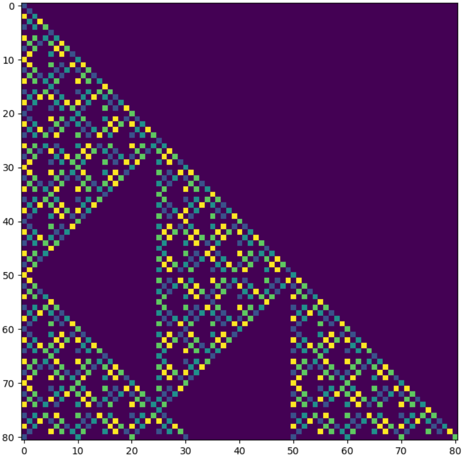
$T_n(u)$ in $\mathbb{Z}_p[u]$



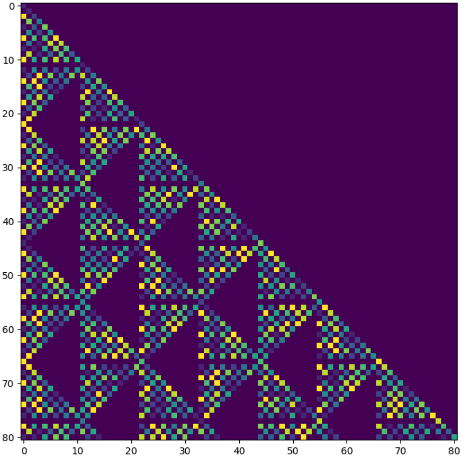
$T_n(u)$ in $\mathbb{Z}_p[u]$

What about some other primes p?

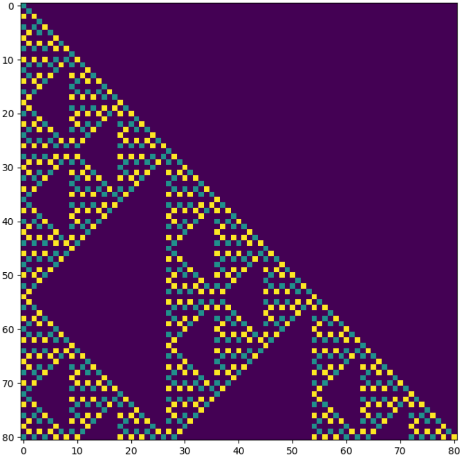
$T_n(u)$ in $\mathbb{Z}_p[u]$, $p = 5$



$T_n(u)$ in $\mathbb{Z}_p[u]$, $p = 11$



$T_n(u)$ in $\mathbb{Z}_p[u]$, $p = 3$



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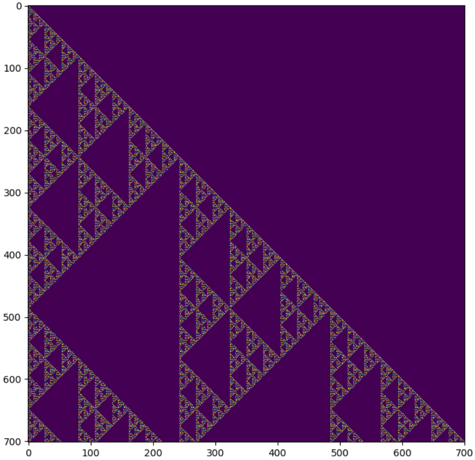

What next?

Continuations

- Let $S_n(u) = \frac{\sin(nx)}{\sin x}$, where $u = \cos x$.

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 2u \\ S_3 &= 4u^2 - 1 \\ S_4 &= 8u^3 - 4u \\ S_5 &= 16u^4 - 12u^2 + 1 \\ S_6 &= 32u^5 - 32u^3 + 6u \end{aligned}$$

We discover that similarly, $S_n(u) = 2S_{n-1}(u) \cdot u - S_{n-2}(u)$

What next?

We also discover that...

Conjecture

$$n \in \mathbb{N} \; odd \implies \sin(nx) = (-1)^{\frac{n-1}{2}} \cdot T_n(\sin x)$$

Conjecture

$$n \in \mathbb{N} \; even \implies \sin(nx) = (-1)^{\frac{n}{2}-1} \cdot \cos x \cdot S_n(\sin x)$$

What next?

- Let $A_n(v) = \tan(nx)$, where $v = \tan(x)$. We have an explicit formula for this...

Conjecture

$$A_n(v) = \frac{\displaystyle\sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j \cdot \binom{n}{2j+1} \cdot v^{2j+1}}{\displaystyle\sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \cdot \binom{n}{2j} \cdot v^{2j}}$$

Thank You! (Panda per Arya's request)