Math 1010: Analysis *Lecture Notes*

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Spring 2023

These are lecture notes for "Math 1010: Analysis: Functions of One Variable" taught at Brown University by Javi Gómez Serrano in the Spring of 2023.

These notes are taken by Jiahua Chen with gracious help and input from classmates. Please direct any mistakes/errata to me via email, or feel free to pull request the notes repository (`https://github.com/jchen/math1010-notes`).

Notes last updated February 7, 2023.

Contents

References

[Rud76] W. Rudin. *Principles of Mathematical Analysis.* International series in pure and applied mathematics. McGraw-Hill, 1976.

§1 January 26, 2023

§1.1 Introductions

Instructor: Javi Gómez Serrano. Email: javier_gomez_serrano@brown.edu.

Text: *Principles of Mathematical Analysis*[1], W. Rudin [Rud76]. We will cover select chapters of this book. If the pace is too fast, bring it up.

Structure: Biweekly HW with problems from the book and extra problems. Homework posted Thursdays. Roughly 6 homework assignments (maybe 5, or 7). Worst homework will be dropped.

Midterm and final, in class (TBA). There will additionally be 2 quizzes (20 minutes each), which should be extremely easy. There are 2 questions from the quizzes: one will literally be a problem from the homework, and one will be a definition.

Grading potentially returned by Tuesday following submission. Grading is as follows: 30% problem sets, 10% quizzes, 25% midterm, 35% final. There will be a curve.

Office Hours: Kassar 314, Monday 5-7pm.

§1.2 Fundamentals

§1.2.1 Sets

Definition 1.1 (Set)
One can think of a set as a collection of objects.

If a is an object, and A is a set. $a \in A$ means that a is a member of A.

Definition 1.2 (Subsets)
If A, B are sets, we say that A is a subset of B (we write $A \subset B$[2]) whenever $a \in A \implies a \in B$.

A is a proper subset of B if $A \subset B$ and $A \neq B$.

Remark 1.3. $\emptyset$ is the set with no elements. $\emptyset$ is a subset of *every* set.

[1]Find online using *appropriate channels*.
[2]Some also write $A \subseteq B$.

Definition 1.4 (Union)
If A, B are sets, $A \cup B$ is the union of A and B.

$$A \cup B \overset{\text{def}}{=} \{a \mid a \in A \text{ or } a \in B\}$$

Definition 1.5 (Intersection)
If A, B are sets, $A \cap B$ is the intersection of A and B.

$$A \cap B \overset{\text{def}}{=} \{a \mid a \in A \text{ and } a \in B\}$$

We can generalize this to an arbitrary number of sets:

Definition 1.6 (Generalization of Union/Intersection)
If $\mathcal{C}$ is a collection of sets (possibly infinite). Then

$$\bigcup_{A \in \mathcal{C}} A \overset{\text{def}}{=} \{a \mid a \in A \text{ for some } a \in \mathcal{C}\}$$

and

$$\bigcap_{A \in \mathcal{C}} A \overset{\text{def}}{=} \{a \mid a \in A \text{ for all } A \in \mathcal{C}\}.$$

Definition 1.7 (Disjunction)
We say that A and B are disjoint if $A \cap B = \emptyset$.

Proposition 1.8 (Distribution over intersection and union)
The following is true:

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof of statement 1. We prove that the sets on the left are contained in the sets on the right, and the sets on the right are contained on the sets on the left. That is, $X \subset Y$ and $Y \subset X \Rightarrow X = Y$.

Case $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$: Let $a \in A \cap (B \cup C)$. Then $a \in A$ and $a \in (B \cup C)$. So $a \in B$ or $a \in C$.

Suppose $a \in B$, then $a \in A \cap B$ so $a \in (A \cap B) \cup (A \cap C)$. Suppose $a \in C$, then $a \in A \cap B$ so still $a \in (A \cap B) \cup (A \cap C)$.

Regardless, this is precisely the set on the right.

Case $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$**:** Let $a \in (A \cap B) \cup (A \cap C)$. Then $a \in (A \cap B)$ or $a \in (A \cap C)$.

If $a \in (A \cap B)$, then $a \in A$ and $a \in B$, so $a \in A$ and $a \in B \cup C$, so $a \in A \cap (B \cup C)$.

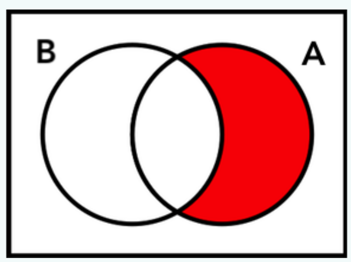
If $a \in (A \cap C)$, then $a \in A$ and $a \in C$, so $a \in A$ and $a \in B \cup C$, so still $a \in A \cap (B \cup C)$.

Again, regardless, $a \in A \cap (B \cup C)$.

Having proven containment in both directions, we conclude the sets on the left and right are equal. $\square$

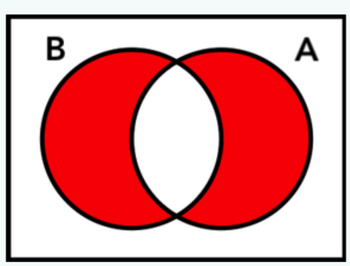
Definition 1.9 (Set Difference)
We define the difference $A \setminus B$ as

$$A \setminus B = \{a \mid a \in A, a \notin B\}.$$



Definition 1.10 (Symmetric Difference)
We define the symmetric difference

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$



Definition 1.11 (Set Product)
We define the product of A and B as

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

§1.2.2 Relations

Definition 1.12 (Relations)
A relation R between sets A and B is any subset of $A \times B$.

If $(a, b) \in R$, we will think of a "being related" to b.

For now, consider $A = B$. We define a special set of relations.

Definition 1.13 (Reflexive/Symmetric/Transitive Relations)
We have the following definitions:

1. R is reflexive iff $(a, a) \in R, \forall a \in A$.[3]
2. R is symmetric iff whenever $(a_1, a_2) \in R$, then $(a_2, a_1) \in R$.
3. R is transitive iff whenever $(a_1, a_2) \in R$, and $(a_2, a_3) \in R$, then $(a_1, a_3) \in R$.

Definition 1.14 (Equivalence Relation)
A relation R that satisfies all conditions 1-3 in definition 1.13 is called an equivalence relation.

Definition 1.15 (Equivalence Class)
Given an element $a \in A$ with relation R, we write

$$[a]_R \stackrel{\text{def}}{=} \{a' \in A \mid (a, a') \in R\}$$

as the equivalence class of a (the set of all elements that are 'related' to a).

Remark 1.16. Instead of writing $(a, a') \in R$, we write $a \sim_R a'$ or $a \sim a'$.

[3] $\forall$ is to say 'for all'.

Proposition 1.17 (Distinct Equivalence Classes are Disjoint)
If R be an equivalence relation on A. Suppose $a_1, a_2 \in A$. Then, either

$$[a_1]_R = [a_2]_R \quad \text{or} \quad [a_1]_R \cap [a_2]_R = \emptyset$$

Proof. We show that these statements cannot happen simultaneously, and that one or the other *must* be true.

The former is as follows: notice $a_1 \in [a_1]_R$ because $a_1 \sim a_1$ by R reflexive. Since $[a_1]_R = [a_2]_R$, then $a_1 \in [a_1]_R \cap [a_2]_R$. This implies that the intersection $[a_1]_R \cap [a_2]_R$ is nonempty, which is a contradiction.

We now show that at least one has to happen. If $a_1 \in [a_1]_R \cap [a_2]_R = \emptyset$, we are done. Otherwise, if $a_1 \in [a_1]_R \cap [a_2]_R \neq \emptyset$, then $\exists$[4] $a \in A$ such that $a \in [a_1]_R$ and $a \in [a_2]_R$.

We claim WLOG $[a]_R = [a_1]_R$. Suppose $a' \in [a]_R$, then $(a, a') \in R$. Since $(a, a') \in R$ and $(a_1, a) \in R$ (due to $a \in [a_1]_R$), transitivity gives $(a_1, a') \in R$ so $a' \in [a_1]_R$. Let's otherwise suppose $a' \in [a_1]_R$, so $(a_1, a') \in R$. Reflexivity gives $(a, a_1) \in R$ ($(a_1, a) \in R$ by $a \in [a_1]_R$), and by transitivity $(a, a_1) \in R$ and $(a_1, a') \in R$ gives $(a, a') \in R$ so $a' \in [a]_R$.[5]

By symmetry, $[a]_R = [a_2]_R$, so $[a_1]_R = [a]_R = [a_2]_R$ which is as desired. □

Essentially, what we have shown, is that we can partition a set according to equivalence classes. All equivalence classes are either going to be the same or different. This is, equivalence relations form a *partition* of a set. We can construct a set of partitions (disjoint subsets) of a set S. That is, "equivalence classes partition A."

Definition 1.18 (Partition)
A partition of A is a collection $\mathcal{P}$ of subsets of A such that

1. $A = \bigcap_{S \in \mathcal{P}} S$,
2. $S_i \cap S_j = \emptyset$ if $S_i \neq S_j$ for each $S_i, S_j \in \mathcal{P}$.

With this definition, if R is an equivalence relation, then $\mathcal{C}_R = \{[a]_R \mid a \in A\}$ is a partition of A.

We'll use all of this to define a function (which is the whole purpose of this course).

§1.2.3 Functions

[4] $\exists$ is to denote 'there exists'.
[5] Wow, this is quite dense but the idea is to chain reflexivity and transitivity to get everything related to one another.

Definition 1.19 (Function)
Let A, B be sets and f be a relation between A and B.

We say f is a function from A to B and we write $f : A \to B$ iff the following holds:

1. $\forall a \in A$, there exists *at least* one $b \in B$ s.t. $(a, b) \in f$. (all elements have an image)

2. $\forall a \in A$, there exists *at most* one $b \in B$ s.t. $(a, b) \in f$. (all elements have only one image)

We write $f(a) = b$, and b is the image of a by f. We call A the domain of f, and B the codomain of f[6].

§2 January 31, 2023

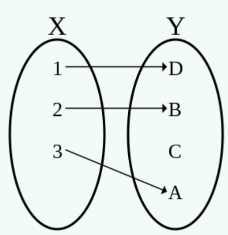
§2.1 Properties of Functions

We introduce some properties of functions:

Definition 2.1 (Injectivity)
A function $f : A \to B$ is injective (one-to-one) if

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

"No two elements in A map to the same element in B."



(Using X and Y, injective but not surjective.)

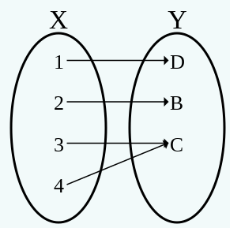
[6] I like to denote B the codomain and the actual set of images the *image* of f. And 'range' is just ambiguous it seems.

Definition 2.2 (Surjectivity)

A function $f : A \to B$ is surjective (onto) if

$$\forall b \in B, \exists a \in A, \text{ s.t. } f(a) = b.$$

"Every element in B has a preimage in A."



(Using X and Y, surjective but not injective.)

Definition 2.3 (Functional Composition)

Suppose $f : A \to B$ and $g : B \to C$ are functions, then $(g \circ f) : A \to C$ with

$$(g \circ f)(a) := g(f(a)).$$

Proposition 2.4

Let $f : A \to B$, $g : B \to C$ be functions, then if f, g are both injective, then so is $(g \circ f)$.

The same holds if they are surjective.

Proof of injective part. Let $a, a' \in A$ be such that

$$(g \circ f)(a) = (g \circ f)(a')$$
$$g(f(a)) = g(f(a'))$$

With g injective, therefore

$$f(a) = f(a')$$

With f injective, we then have

$$a = a'$$

Therefore, $(g \circ f)$ is injective. $\square$

Proof of surjective part. We want to show that $\forall c \in C$, $\exists a \in A$ such that $g(f(a)) = (g \circ f)(a) = c$.

Since g is surjective, $\exists b \in B$ s.t. $g(b) = c$.

Since f is surjective, $\exists a \in A$ such that $f(a) = b$.

Now, plugging in, $g(f(a)) = g(b) = c$. Therefore, exists such a pre-image $a \in A$ for every $c \in C$ under $(g \circ f)$. Hence, $(g \circ f)$ is surjective. $\square$

Definition 2.5 (Bijectivity)
If $f : A \to B$ is injective and surjective, we call f a <u>bijection</u>.

Theorem 2.6 (Existence of Inverse)
Let $f : A \to B$, then $\exists f^{-1} : B \to A$ such that

$$\begin{aligned} f^{-1} \circ f &= \mathrm{id}_A \\ f \circ f^{-1} &= \mathrm{id}_B \end{aligned}$$

where

$$\begin{aligned} \mathrm{id}_A : A \to A, &\quad \forall a \in A, \mathrm{id}_A(a) = a \\ \mathrm{id}_B : B \to B, &\quad \forall b \in B, \mathrm{id}_B(b) = b \end{aligned}$$

if and only if f is a bijection.

Proof.

$\Longleftarrow$: Suppose $f : A \to B$ is a bijection. Then, we define $f^{-1} \subset B \times A$ as

$$f^{-1} = \{(b, a) \mid (a, b) \in f\}$$

It's clear that this is a relation. We check that f^{-1} is a function.

That is, we need to check that $\forall b \in B$, there exists one and only one $a \in A$ such that $(b, a) \in f^{-1}$.

In other words, taking the definition of f^{-1}, we have to check that $\forall b \in B$, $\exists!$[7] $a \in A$ such that $(a, b) \in f$. This is exactly f being bijective.

Let's now show that

$$f^{-1} \circ f = \mathrm{id}_A \tag{2.1}$$

$$f \circ f^{-1} = \mathrm{id}_B \tag{2.2}$$

[7] $\exists!$ as 'exists exactly one'.

Let's first show eq. (2.1), suffices to show $f^{-1}(f(a)) \stackrel{?}{=} a, \forall a$.

$\forall a \in A$, $\exists b \in B$ such that $f(a) = b$, so $(a, b) \in f$. By definition, $(b, a) \in f^{-1}$ so $f^{-1}(b) = a$. Then $f^{-1}(f(a)) = f^{-1}(b) = a$.

We do eq. (2.2) very similarly.

$\Longrightarrow$: Suppose $f : A \to B$ and g[8] $: B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. We show that f is a bijection.

We first show injectivity. Suppose $f(a_1) = f(a_2)$, applying g on both sides gives

$$\begin{aligned} g(f(a_1)) &= g(f(a_2)) \\ \mathrm{id}_A(a_1) &= \mathrm{id}_A(a_2) \\ a_1 &= a_2 \end{aligned}$$

This gives us injectivity.

We then show surjectivity. Let $b \in B$, we claim $\exists a \in A$ such that $f(a) = b$. Take $a := g(b)$, then $f(a) = f(g(b)) = b$. We've found such an a such that $f(a) = b$, giving us surjectivity.

In both directions, this gives us the bijection. $\square$

> **Remark 2.7.** Let $f : A \to B$ be a bijection, with A and B finite. We have that A has n elements $\iff$ B has n elements. That is, A and B have equal cardinality.
>
> This extends to arguments on infinite sets too, like between $\mathbb{Z}$ and $\mathbb{Q}$. $\mathbb{Q}$ and $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{R}$ have different cardinalities and hence don't have bijections.

If $f : A \to B$ is not a bijection, we cannot define an inverse in the same way as we did before. But, we can consider the inverse image.

Definition 2.8 (Inverse Image)

Given function $f : A \to B$, $C \subseteq B$ we define the inverse image of C as

$$f^{-1}(C) := \{a \in A \mid f(a) \in C\}.$$

"The set of $a \in A$ that take me somewhere in C."

If $C = \{b\}$, we get all elements mapped to b. That is,

$$f^{-1}(\{b\}) = \{a \in A \mid f(a) = b\}$$

[8] We use g to reduce confusion with f^{-1}

§2.2 Cardinality

Definition 2.9 (Cardinality)

If A, B are sets, we say that A and B are equivalent ($A \simeq B$) if there exists a bijective $f : A \to B$. This is an equivalence relation.

The cardinality of a set A ($\#(A)$) is the equivalence class of A under this relation:

$$\#(A) := \{B \mid A \simeq B\}.$$

Remark 2.10. When A is finite with n elements, this is an abuse of notation. $\#(A)$ is the set of all sets with n elements, or we also say the 'cardinality' of A is simply n.

Example 2.11

Consider $\mathbb{Z}$, the set of integers. Consider $2\mathbb{Z}$, the set of even integers. They belong to the same cardinality. Even though they are infinite sets, they are the 'same level' of infinite[9].

Specifically, we have bijection $f : \mathbb{Z} \to 2\mathbb{Z}$ with $x \mapsto 2x$.

§2.3 Natural Numbers

We'll assume the Peano Axioms, which are the following:

A. $\exists \mathbb{N}$, an element $1 \in \mathbb{N}$.

B. a function $s : \mathbb{N} \to \mathbb{N}$ satisfying

(1) $\forall n \in \mathbb{N}, s(n) \neq 1$,

(2) s is injective,

(3) if $\exists S$, such that $1 \in S \subset \mathbb{N}$, and S satisfies that whenever $n \in S$, then $s(n) \in S$, this implies $S = \mathbb{N}$.

Why do we need condition (3)?

Example 2.12

Picking the half integers

$$\left\{\frac{n}{2} \mid n \in \mathbb{N}\right\}$$

[9] Even though we might think that $2\mathbb{Z} \subseteq \mathbb{Z}$.

which satisfies the first two conditions but not the third.

s is called the *successor* function.

Proposition 2.13

For every $n \in \mathbb{N}$, $s(n) \neq n$.

Proof. Let

$$S = \{n \in \mathbb{N} \mid s(n) \neq n\}.$$

Clearly, $1 \in S$ by axiom (1). Suppose $n \in S$ for some n. We claim that $s(n) \in S$. By the injectivity of s, $s(n) \neq n$ so $s(s(n)) \neq s(n)$.

By (3), $S = \mathbb{N}$ which gives $s(n) \neq n, \forall n$. We can never loop when we keep applying s. □

Definition 2.14 (Natural Number)

We'll call $s(1) = 2, s(2) = 3, \ldots$. These are the natural numbers.

§3 February 2, 2023

§3.1 Natural Numbers, *continued*

Recall: we defined the natural numbers $\mathbb{N}$. We had a *successor* function s where $2 = s(1)$, $3 = s(2)$, and so on...

Definition 3.1

We define

a. For any $n \in \mathbb{N}$, $n + 1 := s(n)$.

b. For every $m, n \in \mathbb{N}$ $n + s(m) := s(n + m)$.

Proposition 3.2

We can check that

1. $\forall m, n \in \mathbb{N}$, $m + n = n + m$.
2. $\forall n, m, r \in \mathbb{N}$,
$$m + (n + r) = (m + n) + r$$
3. $\forall n, m \in \mathbb{N}$, $n + m \neq N$.[10]
4. s is a bijection from $\mathbb{N}$ to $\mathbb{N} \setminus \{1\}$.

Proof of 4. We note that by the Peano axioms, s doesn't map any element to 1. Then, $s : \mathbb{N} \to \mathbb{N} \setminus \{1\}$.

We show that s is bijective. Consider

$$S = \{1\} \cup \{s(n) \mid n \in \mathbb{N}\}.$$

Suffices to show that $S = \mathbb{N}$. We know that $1 \in S$ by construction, and if $n \in S$, $n \in \mathbb{N}$, $s(n) \in S$ by construction, then by axiom (3), $S = \mathbb{N}$.

Hence if $k \neq 1$, $\exists n$ such that $k = s(n)$. Since s is injective by axiom (2), s is bijective. □

Now that we have a bijection, we can define predecessors:

Definition 3.3 (Predecessor)

We can define $n - 1$ for $n \neq 1$ by saying that $n - 1$ is the element such that

$$(n - 1) + 1 = n.$$

Definition 3.4 (Ordering)

We say that $m \leq n$ for $m, n \in \mathbb{N}$ if either $m = n$ or $m + a = n$ for some $a \in \mathbb{N}$.

Moreover, "$\leq$" is a partial order that satisfies $\forall m, n \in \mathbb{N}$, $m \leq n$ or $n \leq m$ (which makes $\leq$ a total order).

[10] We note that $0 \notin \mathbb{N}$.

Definition 3.5 (Partial Order)
A partial order R is a relation such that

1. aRa,
2. $aRb, bRa \Rightarrow a = b$,
3. $aRb, bRc \Rightarrow aRc$.

Definition 3.6 (Multiplication)
We define multiplication by

$$\forall n \in \mathbb{N}, \qquad n \cdot 1 = n$$
$$n \cdot s(m) = n + n \cdot m.$$

Proposition 3.7
Our definition of multiplication satisfies the following: $\forall n, m, r, s \in \mathbb{N}$,

1. *(Distributivity)* $n \cdot (m + r) = n \cdot m + n \cdot r$
2. *(Commutativity)* $n \cdot m = m \cdot n$
3. *(Associativity)* $(n \cdot m) \cdot r = n \cdot (m \cdot r)$
4. If $n < m$ and $r \leq s$, then $r \cdot n < s \cdot m$.

§3.2 Cardinality & Natural Numbers

For each $n \in \mathbb{N}$, consider

$$J_n = \{m \in \mathbb{N} \mid m \leq n\}.$$

That is, $J_1 = \{1\}$, $J_2 = \{1, 2\}$.

Proposition 3.8
For $n \geq 1$,

$$J_{n+1} = J_n \cup \{n + 1\}.$$

Proof. We prove by showing subset inclusion in both directions:

$\supset$: Let $k \in J_n \cup \{n+1\}$. If $k = n+1$, $k \in J_{n+1}$. If $k \in J_n$, then $k \leq n$ and since $n \leq n+1 \Rightarrow k \leq n+1 \Rightarrow k \in J_{n+1}$.

$\subset$: Let $k \in J_{n+1}$. If $k \in J_n$, we are done. If $k \notin J_n$, then $k \not\leq n \Rightarrow k \geq n+1$. But $k \in J_{n+1} \Rightarrow k \leq n+1$. So necessarily, $k = n+1$.

Which is as desired. □

Definition 3.9 (Cardinality, *again*)

We say that a set A has cardinality[11] n if

$$A \simeq {}^{12} J_n{}^{13}.$$

In this case, we say A is finite and write $\#(A) = n$.

If A is not related to any J_n, we say that A is infinite.

Lemma 3.10

Let A, B be finite[14] sets. If $A \subset B$, then $\#(A) \leq \#(B)$.

Proof. Define $f : A \to B$ with $a \mapsto a$. f is injective by definition.

This is a bijection with a subset of B, therefore the cardinality has to satisfy $\#(A) \leq \#(B)$. □

Theorem 3.11

$\forall n \in \mathbb{N}$,

$$\#(J_n) < \#(J_{n+1}) < \#(\mathbb{N})$$

Proof. If we replace the first $<$ by $\leq$, then this is easy by the previous lemma.

We assume we already know[15]

$$\#(J_n) \neq \#(J_{n+1}), \forall n, \tag{3.1}$$

we want to show that $\#(J_n) \neq \#(\mathbb{N}), \forall n$.

[11] This is unfortunately an abuse of notation.
[12] We use $\sim$ to denote a relation, and $\simeq$ to denote a bijection.
[13] This is in a model that assumes $0 \in \mathbb{N}$, unfortunately, which is a bit confusing.
[14] They don't need to be finite, but then the comparison of cardinalities is a bit wishy-washy.
[15] Note from me: why can't we claim $\#(J_n) = n$ by definition of cardinality?

Assume there exists some n such that

$$\#(\mathbb{N}) = \#(J_n)$$

Then we have

$$\#(J_{n+1}) \leq \#(\mathbb{N}) = \#(J_n)$$

which is a contradiction. □

Proof of eq. (3.1). We induct on n.

Base case $n = 1$. $J_1 = \{1\}$ and $J_2 = \{1, 2\}$. Then $\#(J_1) < \#(J_2)$, since we can't define a bijection.

Inductive step. Suppose that $\#(J_n) \neq \#(J_{n+1})$. We wish to prove $\#(J_{n+1}) \neq \#(J_{n+2})$. By contraposition, assume there exists a bijection $f : J_{n+1} \to J_{n+2}$. Then, $\exists k \in J_{n+1}$ such that $f(k) = n + 2$. Define $h : J_{n+1} \to J_{n+1}$ by

$$h(m) := \begin{cases} m & \text{if } m \notin \{k, n+1\} \\ m+1 & \text{if } m = k \\ k & \text{if } m = n+1 \end{cases}$$

Let $\hat{f} = f \circ h : J_{n+1} \to J_{n+2}$, a bijection. $\hat{f}$ maps $n + 1 \mapsto n + 2$. Defining $g(x) = \hat{f}(x)$ for $x \in J_n$ ($\hat{f}$ restricted to J_n so $g : J_n \to J_{n+1}$) completes the contraposition.

Which gives the general statement of eq. (3.1). □

Corollary 3.12

If $A \simeq \mathbb{N}$, A is infinite.

We say in this case that A is <u>countable</u>.

If A is infinite and $\#(A) \neq \#(\mathbb{N})$[16], then we say that A is <u>uncountable</u>.

Proof. If A were finite, then $A \simeq J_n$ for some n which is impossible by theorem 3.11. □

Example 3.13

$\mathbb{N}$ is countable, $\mathbb{Q}$ is countable, $\mathbb{R}$ is uncountable.

Corollary 3.14

Let S be an infinite subset of $\mathbb{N}$. Then S is countably infinite.

[16]That is, $A \not\simeq \mathbb{N}$.

Definition 3.15 (Countability)
We say that A is countable if A is finite or countable infinite.

§4 February 7, 2023

§4.1 Natural Numbers, *continued*

Theorem 4.1
Let S be an infinite subset of $\mathbb{N}$. Then S is countably infinite.

Proof. The objective is to construct a bijection between S and $\mathbb{N}$. We will do this inductively.

Let $f(1)$ be the smallest element[17] of S for $f : \mathbb{N} \to S$.

Assuming we have defined $f(1), f(2), \ldots, f(n)$, we define $f(n+1)$ to be the smallest element of $S \setminus \{f(1), f(2), \ldots, f(n)\}$.

By construction, f is bijective. $\square$

Recall definition 3.15:

Definition (Countability)
We say that A is countable if it is finite or countably infinite.

Theorem 4.2
Let $\mathcal{C}$ be a countable collection[18] of countable sets.

Then the union $\displaystyle\bigcup_{A \in \mathcal{C}} A$ is countable.

We can do something like this:

[17]Within resonable context, we assume that $\mathbb{N}$ is well-ordered. That is, every subset has a least element.
[18]A 'collection' usually refers to a collection of sets.

$$\begin{tikzcd}
x_{0,0} \arrow[d] & x_{0,1} \arrow[r] & x_{0,2} \arrow[dl] & x_{0,3} \arrow[r] & x_{0,4} \arrow[dl] & \cdots \\
x_{1,0} \arrow[ur] & x_{1,1} \arrow[dl] & x_{1,2} \arrow[ur] & x_{1,3} \arrow[dl] & \vdots & \\
x_{2,0} \arrow[d] & x_{2,1} \arrow[ur] & x_{2,2} \arrow[dl] & \vdots & & \\
x_{3,0} \arrow[ur] & x_{3,1} \arrow[dl] & \vdots & & & \\
x_{4,0} & \vdots & & & & \\
\vdots & & & & &
\end{tikzcd}$$

Proof. Again, we try to construct a bijection f from $\mathbb{N}$. We list all the elements of $\bigcup_{A \in \mathcal{C}} A$ in a sequence. This will give $f(1), f(2), \ldots$ and so on.

Since each A is countable, we may list its elements. The collection $\mathcal{C}$ is also countable. We can arrange every row to be an element of $\mathcal{C}$.

$$\begin{array}{c|cccc}
A_1 & a_1 & a_2 & a_3 & \cdots \\
A_2 & b_1 & b_2 & b_3 & \cdots \\
A_3 & c_1 & c_2 & \cdots & \\
A_4 & d_1 & \cdots & & \\
\vdots & \vdots & \ddots & &
\end{array}$$

We then count the elements using diagonals (or a zig-zag), and remove duplicates[19]. □

§4.2 Before We Get to the Real Numbers

Let's begin by proving the following theorem.

Theorem 4.3
$\sqrt{2}$ is not rational.

Proof. Assume not, that we can write $\sqrt{2} = \frac{m}{n}$, for $m, n \in \mathbb{Z}$, $N \neq 0$ and m, n coprime.

Then, $2n^2 = m^2$. So, m is even. Writing $m = 2m'$, $2n^2 = 4m'^2$, which in particular implies n is even, which is a contradiction.

It had better be, then, that $\sqrt{2}$ is irrational. □

[19]This is a bit wishy-washy, but we're only removing elements here so we remain in the 'countable' territory.

We now show more, namely that we can get arbitrarily close to irrational numbers by rational numbers[20].

Theorem 4.4

If $q \in \mathbb{Q}$ satisfies $0 < q^2 < 2$, then we can find $\bar{q} \in \mathbb{Q}$ such that $q^2 < \bar{q}^2 < 2$. Similarly if $r \in \mathbb{Q}$ satisfies $2 < r^2$, then we can find $\bar{r} \in \mathbb{Q}$ such that $2 < \bar{r}^2 < r^2$.

Proof. We'll just prove the first part; the second part is similar.

Suppose $q > 0$, define
$$\bar{q} = q + \frac{2 - q^2}{2 + q}.$$
then $\bar{q} > q$ and
$$\begin{aligned}
\bar{q}^2 - 2 &= q^2 + \left(\frac{2-q^2}{q+2}\right)^2 + \frac{2q(2-q^2)}{2+q} - 2 \\
&= \frac{(q^2-2)(q+2)^2 + (2-q^2)^2 + (2-q^2)2q(2+q)}{(q+2)^2} \\
&= \frac{q^2-2}{(q+2)^2}\left[\cancel{q^2} + 4 + \cancel{4q} + \cancel{q^2} - 2 - \cancel{2q(2+q)}\right] \\
&= \frac{2(q^2-2)}{(q+2)^2}
\end{aligned}$$
which is surely negative. ☐

Corollary 4.5

The set
$$\left\{q \in \mathbb{Q} \mid q^2 < 2\right\}$$
does not have a largest element.

Definition 4.6 (Upper Bound)

If A is a set with a partial ordering $\leq$, we say that $a \in A$ is an upper bound for a subset $B \subset A$ if $b \leq a\ \forall b \in B$.

We say that a is a least upper bound for B if whenever a' is an upper bound for B, then $a \leq a'$.

Lemma 4.7

If a is a least upper bound for B, then a is unique.

[20]This is to say, $\mathbb{Q}$ is *dense* in $\mathbb{R}$.

Proof. Assume a, a' are both least upper bounds for B. By definition, both a, a' are upper bounds. WLOG, if a is a least upper bound and a' an upper bound, $a \leq a'$. Similarly, $a' \leq a$. Hence, $a = a'$. $\square$

Now that we have proved uniqueness, we can give a a name:

Definition 4.8 (Supremum & Infimum)
We define $\sup B := a$ when a is the least upper bound of B.

Similarly, $\inf B := a$ when a is the greatest lower bound of B.

Definition 4.9 (Least-Upper-Bound Property)
Let A be a totally ordered set with order $\leq$. We say that A has the least upper bound property if each nonempty subset of $B \subset A$ with an upper bound in A has a least upper bound in A.

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§4.3 The Real Numbers

Definition 4.10

A field F is a set with two operations: $+, \cdot$.

The $+$ operation satisfies:

Closure. $x, y \in F \Rightarrow x + y \in F$.

Commutativity. $x + y = y + x$, $\forall x, y \in F$.

Associatifity. $(x + y) + z = x + (y + z)$, $\forall x, y, z \in F$.

Identity. $\exists 0$ such that $0 + x = x \ \forall x \in F$.

Inverses. $\forall x \in F$, $\exists - x \in F$ such that $x + (-x) = 0$.

And a $\cdot$ operation that satisfies:

Closure. $x, y \in F \Rightarrow x \cdot y \in F$.

Commutativity. $x \cdot y = y \cdot x$, $\forall x, y \in F$.

Associatifity. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $\forall x, y, z \in F$.

Identity. $\exists 1 \neq 0$ such that $1 \cdot x = x$, $\forall x \in F$.

Inverse. $\forall x \neq 0$, $\exists x^{-1} \in F$ such that $x \cdot x^{-1} = 1$

Together, we also have

Distribution. $\forall x, y, z \in F$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

In other words, a field F is a commutative ring with multiplicative inverse.

Definition 4.11

An ordered field is a field F which is an ordered set such that

1. $x + y < x + z$ if $x, y, z \in F$ and $y < z$.
2. $x \cdot y > 0$ if $x, y \in F$ and $x > 0$, $y > 0$.

Example 4.12

$\mathbb{Q}, \mathbb{R}$ are ordered fields. $\mathbb{N}, \mathbb{Z}$ are not fields.

Theorem 4.13 (Archimedian Property)

Given $x, y \in \mathbb{R}$, $x > 0$. Then $\exists n \in \mathbb{Z}$ such that $nx > y$.

Proof. We argue by contradiction. Assume such n doesn't exists.

Then take set

$$A = \{nc \mid n \in \mathbb{N}\}$$

which is bounded by y.

Therefore, if it has supremum[21] s, $s - x < s$, so $s - x$ is not an upper bound of A (otherwise s is not a supremum). Then, $\exists a \in A$ (otherwise $s - x$ would have been an upper bound) such that $a \geq s - x$, which means $\exists m \in \mathbb{N}$ such that $mx > s - x$. This implies that $s < (m+1)x$ but $m + 1 \in A$, which is a contradiction. □

Corollary 4.14

Let $x, y \in \mathbb{R}$ with $x < y$. Then $\exists q \in \mathbb{Q}$ such that $x < q < y$. This is to say, "in between any two real numbers lies rational numbers."

Proof. By theorem 4.13 applied to $y - x$ and 1. Then $\exists n \in \mathbb{Z}$ such that

$$n(y - x) > 1$$

and we can also find $m_1, m_2 \in \mathbb{N}$ such that

$$m_1 > nx, m_2 > -nx$$

by the Archimedian property again. which implies

$$-m_2 < nx < m_1$$

Then, $\exists m \in \mathbb{N}$ such that $m - 1 \leq nx < m$[22]. Which implies that

$$nx < m \leq 1 + nx < ny$$

which then gives

$$x < \frac{m}{n} < y$$

which is as desired. $\frac{m}{n}$ is precisely a rational between real x and y. □

[21] $\mathbb{R}$ has the least upper bound property.

[22] By contradiction, if there is no such m we incur the previous statement

Theorem 4.15

For every $x \in \mathbb{R}$ with $x > 0$, $n > 0 \in \mathbb{N}$, $\exists! y > 0$ such that $y^n = x$.

Proof. We prove uniqueness. Suppose by contradiction that WLOG $y_1 < y_2$ such that $y_1^n = x$ and $y_2^n = x$. Contradiction by $y_1^n < y_2^n$.

We now start the proof for existence. Let

$$E = \{t \mid t^n < x\}$$

If $t = \frac{x}{x+1}$, then $t < x$ and $0 < t < 1$. So $t^n < t < x$ so E is not empty.

If $t > 1 + x$, $t^n \geq t > x$. We've found an upper bound of E. Then, $\exists y = \sup E$.

We'll leave it here that E has a supremum, and continue next time. $\square$