

Math 1010: Analysis *Lecture Notes*

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These are lecture notes for “Math 1010: Analysis: Functions of One Variable” taught at BROWN UNIVERSITY by Javi Gómez Serrano in the Spring of 2023.

These notes are taken by Jiahua Chen with gracious help and input from classmates. Please direct any mistakes/errata to me via [email](#), or feel free to pull request the notes repository (<https://github.com/jchen/math1010-notes>).

Notes last updated March 21, 2023.

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References

- [Rud76] W. Rudin. *Principles of Mathematical Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1976.

§10 March 14, 2023

By default, the quiz will be next Tuesday.

§10.1 Continuity, *continued*

Theorem 10.1

$f : X \rightarrow Y$ is continuous if and only if $\forall O \subset Y$ open, the preimage $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$ is open.

Proof. We proved the right implication last time. We'll work on the left implication.

Suppose $\forall O \in Y, f^{-1}(O)$. Let $x_0 \in X, \epsilon > 0$. Then $B_\epsilon(f(x_0))$ is open in Y .

Then $f^{-1}(B_\epsilon(f(x_0)))$ is open in X . Since $x_0 \in f^{-1}(B_\epsilon(f(x_0)))$, $\exists \delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0)))$.

This implies that if $d(x, x_0) < \delta$, then $x \in f^{-1}(B_\epsilon(f(x_0)))$, then $d_Y(f(x), f(x_0)) < \epsilon$. Then f is continuous at x_0 . \square

Corollary 10.2

$f : X \rightarrow Y$ is continuous in X if and only if for all $C \subset Y$ closed, $f^{-1}(C)$ is closed.

Proof. If C is closed, C^C is open, then $f^{-1}(C^C)$ is open, then $f^{-1}(C^C)^C$ is closed.

We claim $f^{-1}(C) = f^{-1}(C^C)^C$. If $x \in f^{-1}(C)$ iff $f(x) \in C$ iff $x \notin f^{-1}(C^C)$ iff $x \in f^{-1}(C^C)^C$.

This gives a biconditional between the second condition in [theorem 10.1](#) and [corollary 10.2](#). \square

Proposition 10.3

Let X be a metric space. Let $f, g : X \rightarrow \mathbb{R}$ be continuous at x_0 . If $\alpha \in \mathbb{R}$, the following are continuous at x_0 :

- (1) $f + g, fg, \alpha f$.
- (2) f/g as long as $g(x_0) \neq 0$.

We'll state this without proof.

Theorem 10.4

Let X, Y, Z be metric spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ with f continuous at x_0 and g continuous at $f(x_0)$. Then $g \circ f$ is continuous at x_0 .

Proof. Let $\epsilon > 0$. Since g is continuous, $\exists \delta' > 0$ such that if indeed $d_Y(y, f(x_0)) < \delta'$, then $d_Z(g(y), (g \circ f)(x_0)) < \epsilon$.

Then by continuity of f , $\exists \delta > 0$ such that if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \delta'$ then $d_Z((g \circ f)(x), (g \circ f)(x_0)) < \epsilon$. Therefore $g \circ f$ is continuous. \square

Theorem 10.5

If $f : X \rightarrow Y$ is continuous, $E \subset X$ compact, then $f(E) = \{f(x) \mid x \in E\}$ is compact.

Proof. Let \mathcal{C} be an open cover of $f(E)$. Consider $\mathcal{C}' = \{f^{-1}(O) \mid O \in \mathcal{C}\}$.

Every element of \mathcal{C}' is open because f is continuous. \mathcal{C}' covers E since all elements of $f(E)$ are covered by \mathcal{C} . So \mathcal{C}' is an open cover of E .

Then there exists a finite subcover $\{f^{-1}(O_1), \dots, f^{-1}(O_n)\}$ of E . We claim that $\{O_1, \dots, O_n\}$ is a finite subcover of $f(E)$. Let $y \in f(E)$ so $y = f(x)$ for some $x \in E$. $\exists O_i$ such that $x \in f^{-1}(O_i)$. Then $y = f(x) \in O_i$.

So this is a finite subcover of E . \square

Corollary 10.6

Let $f : X \rightarrow Y$ which is continuous, and $E \subset X$ compact, then f is bounded on E .

Proof. We know that $f(E)$ is compact by above, and if a set is compact it is bounded. \square

Corollary 10.7 (Extreme Value Theorem)

Let $f : X \rightarrow \mathbb{R}$ be continuous, and some $E \subset X$ compact.

Then f attains a maximum on E . That is, $\exists x_0 \in E$ such that $f(x_0) \geq f(x) \forall x \in E$. This statement also applies to the minimum.

Proof. $f(E)$ is closed and bounded, so f contains its supremum, say $y = \sup(f(E))$. Since $y \in f(E)$, $\exists x$ such that $f(x) = y$. That is such an x_0 . \square

Definition 10.8 (Uniform Continuity)

We say that $f : X \rightarrow Y$ is uniformly continuous if given $\epsilon > 0$, $\exists \delta > 0$ such that if x, y satisfy

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon.$$

We note that δ does not depend on x, y (normal continuity has δ depend on x, y).

Theorem 10.9

If $f : X \rightarrow Y$ continuous, and X is compact, then f is uniformly continuous.

§11 March 16, 2023

Midterm will be the Thursday after Spring Break, quiz next Tuesday. Homework will be posted today, hopefully²⁹

§11.1 Uniform Continuity

Theorem 11.1

If $f : X \rightarrow Y$ is continuous, and X is compact, then f is uniformly continuous.

The intuition is that we'll use the compactness of X to take some open subcover, taking the minimum of the finite subcover.

Proof. Let $\epsilon > 0$. For each $x \in X$, by continuity, $\exists \delta_x > 0$ such that if $d_X(x, x') < \delta_x$, then $d_Y(f(x), f(y)) < \frac{\epsilon}{2}$.

Consider the collection

$$\mathcal{C} = \left\{ B_{\frac{\delta_x}{2}}(x) \mid x \in X \right\}$$

which is an open cover of X .

By compactness, exists a finite subcover

$$\left\{ B_{\frac{\delta_{x_1}}{2}}(x_1), \dots, B_{\frac{\delta_{x_n}}{2}}(x_n) \right\}$$

Let $\delta = \min \left\{ \frac{\delta_{x_1}}{2}, \dots, \frac{\delta_{x_n}}{2} \right\}$. We claim that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \epsilon$.

For each x , $\exists i$ such that $d_X(x, x_i) < \frac{\delta_{x_i}}{2}$. Then

$$\begin{aligned} d_X(x_i, y) &\leq d_X(x, y) + d_X(x, x_i) \\ &< \delta + \frac{\delta_{x_i}}{2} \leq \delta_{x_i} \end{aligned}$$

But then

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(y)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

which is as desired. □

²⁹Or the graders will, verbatim, 'be strangled'.

Definition 11.2 (Connectedness)

A metric space X is disconnected if $\exists O_1, O_2$, non-empty open sets such that $X = O_1 \cup O_2$, and $O_1 \cap O_2 = \emptyset$. If X is not disconnected, we say that X is connected.

Example 11.3

A line or line segment is connected.

Two open intervals who share a limit point is disconnected.

Two disjoint sets is disconnected.

The set $(x, \sin(\frac{1}{x}))$ is connected.

Theorem 11.4

Let X, Y be metric spaces, and $f : X \rightarrow Y$ continuous. If X is connected, then $f(X)$ is connected.

Proof. By contraposition. Assume $\exists O_1, O_2$ disconnected such that $f(X) = O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$.

Take $U_1 = f^{-1}(O_1)$ and $U_2 = f^{-1}(O_2)$. U_1, U_2 are open by continuity.

We show that $U_1 \cap U_2 = \emptyset$. If $x \in U_1 \cap U_2$, then $f(x) \in O_1, f(x) \in O_2$ which is a contradiction.

U_1, U_2 are nonempty since O_1, O_2 are nonempty. Then U_1, U_2 are disconnected. \square

Theorem 11.5 (Intermediate Value Theorem)

Let $f : [a, b] \rightarrow \mathbb{R}$ continuous. $\forall L \in \mathbb{R}$ such that $f(a) < L < f(b)$. Then $\exists c \in (a, b)$ such that $f(c) = L$.³⁰

Proof. Since $[a, b]$ is connected and f is continuous, then $f([a, b])$ is connected.

Let $O_1 = (-\infty, L) \cap f([a, b])$, $O_2 = (L, \infty) \cap f([a, b])$.

Which are non-empty, open in $f([a, b])$, and disjoint. Then $O_1 \cup O_2 \neq f([a, b])$. So, $L \in f([a, b])$ had better be true. \square

³⁰This follows the normal high school definition of continuity, ‘you can draw it without lifting your pen’.

§11.2 Discontinuities

Definition 11.6 (Discontinuities)

The following definitions:

- (1) x_0 is a removable discontinuity if f if $\lim_{x \rightarrow x_0} f(x)$ exists but is different from $f(x_0)$.
- (2) x_0 is a simple discontinuity if $\lim_{x \rightarrow x_0^-} f(x)$, $\lim_{x \rightarrow x_0^+} f(x)$ exist but disagree.
- (3) x_0 may be a discontinuity of a different type.

Example 11.7

Take

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

§12 March 21, 2023

§12.1 Differentiation

From now on, we fix $f : \mathbb{R} \rightarrow \mathbb{R}$, or from an interval to another interval. We'll mostly be dealing with real-valued functions.

Definition 12.1

Let $f : (a, b) \rightarrow \mathbb{R}$. We say f is differentiable at $x_0 \in (a, b)$ if

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In that case, we write $f'(x_0)$ for this limit.

Proposition 12.2

Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable at x_0 . Then f is continuous at x_0 .

Proof.

$$\begin{aligned} \lim_{x \rightarrow x_0} f(x) &= f(x_0) + \lim_{x \rightarrow x_0} + \lim_{x \rightarrow x_0} (f(x) - f(x_0)) (x - x_0) \\ &= f(x_0) + \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \rightarrow x_0} (x - x_0) \\ &= f(x_0) + f'(x_0) \cdot 0 = f(x_0) \end{aligned}$$

implies $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ implies f is continuous at x_0 . □

Remark 12.3. The converse of this statement is not true. Continuous does not imply differentiable!

Example 12.4

$f(x) = |x|$ is continuous at 0 but not differentiable.

Let's focus at $x = 0$.

$$\lim_{x \rightarrow 0} \frac{|x| - |0|}{x - 0} = \lim_{x \rightarrow 0} \operatorname{sgn}(x)$$

which doesn't exist.

Proposition 12.5

Let $f, g : (a, b) \rightarrow \mathbb{R}$ be differentiable functions at x . Then the following are differentiable and satisfy:

- (1) Sum rule: $(f + g)'(x) = f'(x) + g'(x)$.
- (2) Product rule: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.
- (3) Quotient rule: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$.

Theorem 12.6 (Chain Rule)

Let $f : (a, b) \rightarrow (c, d)$ and $g : (c, d) \rightarrow \mathbb{R}$ both be differentiable functions at $x_0, f(x_0)$ respectively. Then $(g \circ f)$ is differentiable at x_0 with derivative

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. We do so in cases.

- (i) Suppose exists sequence $(x_n) \in (a, b)$ such that $x_n \rightarrow x_0$ if $f(x_n) = f(x_0)$ but $x_n \neq x_0$ for infinitely many n .

Then

$$\begin{aligned} f'(x_0) &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{n \rightarrow \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = 0 \end{aligned}$$

and

$$\begin{aligned} (g \circ f)' &= \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= \lim_{n \rightarrow \infty} \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} = 0 \end{aligned}$$

- (ii) Every $(x_0) \in (a, b)$ such that $x_n \rightarrow x_0$, $x_n \neq x_0$ has $f(x_n) = f(x_0)$ finitely many times.

By continuity, $f(x_n) \rightarrow f(x_0)$. Then,

$$\begin{aligned}
 (g \circ f)'(x_0) &= \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\
 &= \lim_{n \rightarrow \infty} \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{g(f(x_n)) - g(f(x_0))}{f(x_n) - f(x_0)} \right) \left(\frac{f(x_n) - f(x_0)}{x_n - x_0} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{g(f(x_n)) - g(f(x_0))}{f(x_n) - f(x_0)} \right) \cdot \lim_{n \rightarrow \infty} \left(\frac{f(x_n) - f(x_0)}{x_n - x_0} \right) \\
 &= g'(f(x_0)) \cdot f'(x_0)
 \end{aligned}$$

which covers all cases. □

Example 12.7

Let

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

The derivative at $x = 0$ is

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \rightarrow 0} \frac{f(h)}{h}$$

We try to squeeze this limit,

$$-h = -\frac{h^2}{h} \leq \frac{f(h)}{h} \leq \frac{h^2}{h} = h$$

then

$$\lim_{h \rightarrow 0} -h \leq \lim_{h \rightarrow 0} \frac{f(h)}{h} \leq \lim_{h \rightarrow 0} h$$

which forces $\lim_{h \rightarrow 0} \frac{f(h)}{h} = 0$.

This gives $f'(x) = 0$, and

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

so $f'(x)$ is continuous. We say that $f \in \mathcal{C}^1$ (continuous and differentiable). However, the second derivative of f might not exist at $x = 0$.