Math 1010: Analysis *Lecture Notes*

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These are lecture notes for "Math 1010: Analysis: Functions of One Variable" taught at Brown University by Javi Gómez Serrano in the Spring of 2023.

These notes are taken by Jiahua Chen with gracious help and input from classmates. Please direct any mistakes/errata to me via email, or feel free to pull request the notes repository (`https://github.com/jchen/math1010-notes`).

Notes last updated January 31, 2023.

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References

[Rud76] W. Rudin. *Principles of Mathematical Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1976.

§1 January 26, 2023

§1.1 Introductions

Instructor: Javi Gómez Serrano. Email: javier_gomez_serrano@brown.edu.

Text: *Principles of Mathematical Analysis*[1], W. Rudin [Rud76]. We will cover select chapters of this book. If the pace is too fast, bring it up.

Structure: Biweekly HW with problems from the book and extra problems. Homework posted Thursdays. Roughly 6 homework assignments (maybe 5, or 7). Worst homework will be dropped.

Midterm and final, in class (TBA). There will additionally be 2 quizzes (20 minutes each), which should be extremely easy. There are 2 questions from the quizzes: one will literally be a problem from the homework, and one will be a definition.

Grading potentially returned by Tuesday following submission. Grading is as follows: 30% problem sets, 10% quizzes, 25% midterm, 35% final. There will be a curve.

Office Hours: Kassar 314, Monday 5-7pm.

§1.2 Fundamentals

§1.2.1 Sets

Definition 1.1 (Set)

One can think of a set as a collection of objects.

If a is an object, and A is a set. $a \in A$ means that a is a member of A.

Definition 1.2 (Subsets)

If A, B are sets, we say that A is a subset of B (we write $A \subset B$[a]) whenever $a \in A \implies a \in B$.

A is a proper subset of B if $A \subset B$ and $A \neq B$.

[a]Some also write $A \subseteq B$.

Remark 1.3. $\emptyset$ is the set with no elements. $\emptyset$ is a subset of *every* set.

[1]Find online using *appropriate channels*.

Definition 1.4 (Union)
If A, B are sets, $A \cup B$ is the union of A and B.

$$A \cup B \overset{\text{def}}{=} \{a \mid a \in A \text{ or } a \in B\}$$

Definition 1.5 (Intersection)
If A, B are sets, $A \cap B$ is the intersection of A and B.

$$A \cap B \overset{\text{def}}{=} \{a \mid a \in A \text{ and } a \in B\}$$

We can generalize this to an arbitrary number of sets:

Definition 1.6 (Generalization of Union/Intersection)
If $\mathcal{C}$ is a collection of sets (possibly infinite). Then

$$\bigcup_{A \in \mathcal{C}} A \overset{\text{def}}{=} \{a \mid a \in A \text{ for some } a \in \mathcal{C}\}$$

and

$$\bigcap_{A \in \mathcal{C}} A \overset{\text{def}}{=} \{a \mid a \in A \text{ for all } A \in \mathcal{C}\}$$

Definition 1.7 (Disjunction)
We say that A and B are disjoint if $A \cap B = \emptyset$.

Proposition 1.8 (Distribution over intersection and union)
The following is true:

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof of statement 1. We prove that the sets on the left are contained in the sets on the right, and the sets on the right are contained on the sets on the left. That is, $X \subset Y$ and $Y \subset X \Rightarrow X = Y$.

Case $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$: Let $a \in A \cap (B \cup C)$. Then $a \in A$ and $a \in (B \cup C)$. So $a \in B$ or $a \in C$.

Suppose $a \in B$, then $a \in A \cap B$ so $a \in (A \cap B) \cup (A \cap C)$. Suppose $a \in C$, then $a \in A \cap B$ so still $a \in (A \cap B) \cup (A \cap C)$.

Regardless, this is precisely the set on the right.

Case $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$**:** Let $a \in (A \cap B) \cup (A \cap C)$. Then $a \in (A \cap B)$ or $a \in (A \cap C)$.

If $a \in (A \cap B)$, then $a \in A$ and $a \in B$, so $a \in A$ and $a \in B \cup C$, so $a \in A \cap (B \cup C)$.

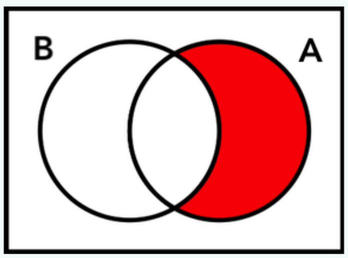
If $a \in (A \cap C)$, then $a \in A$ and $a \in C$, so $a \in A$ and $a \in B \cup C$, so still $a \in A \cap (B \cup C)$.

Again, regardless, $a \in A \cap (B \cup C)$.

Having proven containment in both directions, we conclude the sets on the left and right are equal. $\square$

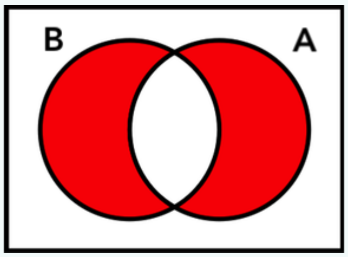
Definition 1.9 (Set Difference)
We define the difference $A \setminus B$ as

$$A \setminus B = \{a \mid a \in A, a \notin B\}.$$



Definition 1.10 (Symmetric Difference)
We define the symmetric difference

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$



Definition 1.11 (Set Product)
We define the product of A and B as

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

§1.2.2 Relations

Definition 1.12 (Relations)
A relation R between sets A and B is any subset of $A \times B$.

If $(a, b) \in R$, we will think of a "being related" to b.

For now, consider $A = B$. We define a special set of relations.

Definition 1.13 (Reflexive/Symmetric/Transitive Relations)
We have the following definitions:

1. R is reflexive iff $(a, a) \in R, \forall a \in A$.[a]
2. R is symmetric iff whenever $(a_1, a_2) \in R$, then $(a_2, a_1) \in R$.
3. R is transitive iff whenever $(a_1, a_2) \in R$, and $(a_2, a_3) \in R$, then $(a_1, a_3) \in R$.

[a] $\forall$ is to say 'for all'.

Definition 1.14 (Equivalence Relation)
A relation R that satisfies all conditions 1-3 in definition 1.13 is called an equivalence relation.

Definition 1.15 (Equivalence Class)
Given an element $a \in A$ with relation R, we write

$$[a]_R \stackrel{\text{def}}{=} \{a' \in A \mid (a, a') \in R\}$$

as the equivalence class of a (the set of all elements that are 'related' to a).

Remark 1.16. Instead of writing $(a, a') \in R$, we write $a \sim_R a'$ or $a \sim a'$.

Proposition 1.17 (Distinct Equivalence Classes are Disjoint)
If R be an equivalence relation on A. Suppose $a_1, a_2 \in A$. Then, either

$$[a_1]_R = [a_2]_R \quad \text{or} \quad [a_1]_R \cap [a_2]_R = \emptyset$$

Proof. We show that these statements cannot happen simultaneously, and that one or the other *must* be true.

The former is as follows: notice $a_1 \in [a_1]_R$ because $a_1 \sim a_1$ by R reflexive. Since $[a_1]_R = [a_2]_R$, then $a_1 \in [a_1]_R \cap [a_2]_R$. This implies that the intersection $[a_1]_R \cap [a_2]_R$ is nonempty, which is a contradiction.

We now show that at least one has to happen. If $a_1 \in [a_1]_R \cap [a_2]_R = \emptyset$, we are done. Otherwise, if $a_1 \in [a_1]_R \cap [a_2]_R \neq \emptyset$, then $\exists$[2] $a \in A$ such that $a \in [a_1]_R$ and $a \in [a_2]_R$.

We claim WLOG $[a]_R = [a_1]_R$. Suppose $a' \in [a]_R$, then $(a, a') \in R$. Since $(a, a') \in R$ and $(a_1, a) \in R$ (due to $a \in [a_1]_R$), transitivity gives $(a_1, a') \in R$ so $a' \in [a_1]_R$. Let's otherwise suppose $a' \in [a_1]_R$, so $(a_1, a') \in R$. Reflexivity gives $(a, a_1) \in R$ ($(a_1, a) \in R$ by $a \in [a_1]_R$), and by transitivity $(a, a_1) \in R$ and $(a_1, a') \in R$ gives $(a, a') \in R$ so $a' \in [a]_R$.[3]

By symmetry, $[a]_R = [a_2]_R$, so $[a_1]_R = [a]_R = [a_2]_R$ which is as desired. □

Essentially, what we have shown, is that we can partition a set according to equivalence classes. All equivalence classes are either going to be the same or different. This is, equivalence relations form a *partition* of a set. We can construct a set of partitions (disjoint subsets) of a set S. That is, "equivalence classes partition A."

Definition 1.18 (Partition)
A partition of A is a collection $\mathcal{P}$ of subsets of A such that

1. $A = \bigcap_{S \in \mathcal{P}} S$,
2. $S_i \cap S_j = \emptyset$ if $S_i \neq S_j$ for each $S_i, S_j \in \mathcal{P}$.

With this definition, if R is an equivalence relation, then $\mathcal{C}_R = \{[a]_R \mid a \in A\}$ is a partition of A.

We'll use all of this to define a function (which is the whole purpose of this course).

§1.2.3 Functions

[2] $\exists$ is to denote 'there exists'.
[3] Wow, this is quite dense but the idea is to chain reflexivity and transitivity to get everything related to one another.

Definition 1.19 (Function)

Let A, B be sets and f be a relation between A and B.

We say f is a function from A to B and we write $f : A \to B$ iff the following holds:

1. $\forall a \in A$, there exists *at least* one $b \in B$ s.t. $(a, b) \in f$. (all elements have an image)
2. $\forall a \in A$, there exists *at most* one $b \in B$ s.t. $(a, b) \in f$. (all elements have only one image)

We write $f(a) = b$, and b is the image of a by f. We call A the domain of f, and B the codomain of f[a].

[a] I like to denote B the codomain and the actual set of images the *image* of f. And 'range' is just ambiguous it seems.

§2 January 31, 2023

§2.1 Properties of Functions

We introduce some properties of functions:

Definition 2.1 (Injectivity)

A function $f : A \to B$ is injective (one-to-one) if

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

"No two elements in A map to the same element in B."

Definition 2.2 (Surjectivity)

A function $f : A \to B$ is surjective (onto) if

$$\forall b \in B, \exists a \in A, \text{ s.t. } f(a) = b.$$

"Every element in B has a preimage in A."

Definition 2.3 (Functional Composition)

Suppose $f : A \to B$ and $g : B \to C$ are functions, then $(g \circ f) : A \to C$ with

$$(g \circ f)(a) := g(f(a)).$$

Proposition 2.4

Let $f : A \to B$, $g : B \to C$ be functions, then if f, g are both injective, then so is $(g \circ f)$.

The same holds if they are surjective.

Proof of injective part. Let $a, a' \in A$ be such that

$$(g \circ f)(a) = (g \circ f)(a')$$
$$g(f(a)) = g(f(a'))$$

With g injective, therefore

$$f(a) = f(a')$$

With f injective, we then have

$$a = a'$$

Therefore, $(g \circ f)$ is injective. □

Proof of surjective part. We want to show that $\forall c \in C$, $\exists a \in A$ such that $g(f(a)) = (g \circ f)(a) = c$.

Since g is surjective, $\exists b \in B$ s.t. $g(b) = c$.

Since f is surjective, $\exists a \in A$ such that $f(a) = b$.

Now, plugging in, $g(f(a)) = g(b) = c$. Therefore, exists such a pre-image $a \in A$ for every $c \in C$ under $(g \circ f)$. Hence, $(g \circ f)$ is surjective. □

Definition 2.5 (Bijectivity)

If $f : A \to B$ is injective and surjective, we call f a bijection.

Theorem 2.6 (Existence of Inverse)

Let $f : A \to B$, then $\exists f^{-1} : B \to A$ such that

$$f^{-1} \circ f = \mathrm{id}_A$$
$$f \circ f^{-1} = \mathrm{id}_B$$

where

$$\mathrm{id}_A : A \to A, \quad \forall a \in A, \mathrm{id}_A(a) = a$$
$$\mathrm{id}_B : B \to B, \quad \forall b \in B, \mathrm{id}_B(b) = b$$

if and only if f is a bijection.

Proof.

$\Longleftarrow$: Suppose $f : A \to B$ is a bijection. Then, we define $f^{-1} \subset B \times A$ as

$$f^{-1} = \{(b, a) \mid (a, b) \in f\}$$

It's clear that this is a relation. We check that f^{-1} is a function.

That is, we need to check that $\forall b \in B$, there exists one and only one $a \in A$ such that $(b, a) \in f^{-1}$.

In other words, taking the definition of f^{-1}, we have to check that $\forall b \in B$, $\exists!$[4]$a \in A$ such that $(a, b) \in f$. This is exactly f being bijective.

Let's now show that

$$f^{-1} \circ f = \mathrm{id}_A \tag{2.1}$$
$$f \circ f^{-1} = \mathrm{id}_B \tag{2.2}$$

Let's first show eq. (2.1), suffices to show $f^{-1}(f(a)) \stackrel{?}{=} a, \forall a$.

$\forall a \in A$, $\exists b \in B$ such that $f(a) = b$, so $(a, b) \in f$. By definition, $(b, a) \in f^{-1}$ so $f^{-1}(b) = a$. Then $f^{-1}(f(a)) = f^{-1}(b) = a$.

We do eq. (2.2) very similarly.

$\Longrightarrow$: Suppose $f : A \to B$ and g[5] $: B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. We show that f is a bijection.

We first show injectivity. Suppose $f(a_1) = f(a_2)$, applying g on both sides gives

$$\begin{gathered} g(f(a_1)) = g(f(a_2)) \\ \mathrm{id}_A(a_1) = \mathrm{id}_A(a_2) \\ a_1 = a_2 \end{gathered}$$

This gives us injectivity.

We then show surjectivity. Let $b \in B$, we claim $\exists a \in A$ such that $f(a) = b$. Take $a := g(b)$, then $f(a) = f(g(b)) = b$. We've found such an a such that $f(a) = b$, giving us surjectivity.

In both directions, this gives us the bijection. $\square$

[4] $\exists!$ as 'exists exactly one'.

[5] We use g to reduce confusion with f^{-1}

Remark 2.7. Let $f : A \to B$ be a bijection, with A and B finite. We have that A has n elements $\iff$ B has n elements. That is, A and B have equal cardinality.

This extends to arguments on infinite sets too, like between $\mathbb{Z}$ and $\mathbb{Q}$. $\mathbb{Q}$ and $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{R}$ have different cardinalities and hence don't have bijections.

If $f : A \to B$ is not a bijection, we cannot define an inverse in the same way as we did before. But, we can consider the inverse image.

Definition 2.8 (Inverse Image)
Given function $f : A \to B$, $C \subseteq B$ we define the inverse image of C as

$$f^{-1}(C) := \{a \in A \mid f(a) \in C\}.$$

"The set of $a \in A$ that take me somewhere in C."

If $C = \{b\}$, we get all elements mapped to b. That is,

$$f^{-1}(\{b\}) = \{a \in A \mid f(a) = b\}$$

§2.2 Cardinality

Definition 2.9 (Cardinality)
If A, B are sets, we say that A and B are equivalent ($A \simeq B$) if there exists a bijective $f : A \to B$. This is an equivalence relation.

The cardinality of a set A ($\#(A)$) is the equivalence class of A under this relation:

$$\#(A) := \{B \mid A \simeq B\}.$$

Remark 2.10. When A is finite with n elements, this is an abuse of notation. $\#(A)$ is the set of all sets with n elements, or we also say the 'cardinality' of A is simply n.

Example 2.11
Consider $\mathbb{Z}$, the set of integers. Consider $2\mathbb{Z}$, the set of even integers. They belong to the same cardinality. Even though they are infinite sets, they are the 'same level' of infinite[a].

Specifically, we have bijection $f : \mathbb{Z} \to 2\mathbb{Z}$ with $x \mapsto 2x$.

[a] Even though we might think that $2\mathbb{Z} \subseteq \mathbb{Z}$.

§2.3 Natural Numbers

We'll assume the Peano Axioms, which are the following:

A. $\exists \mathbb{N}$, an element $1 \in \mathbb{N}$.

B. a function $s : \mathbb{N} \to \mathbb{N}$ satisfying

(1) $\forall n \in \mathbb{N}, s(n) \neq 1$,

(2) s is injective,

(3) if $\exists S$, such that $1 \in S \subset \mathbb{N}$, and S satisfies that whenever $n \in S$, then $s(n) \in S$, this implies $S = \mathbb{N}$.

Why do we need condition (3)?

Example 2.12
Picking the half integers

$$\left\{\frac{n}{2} \mid n \in \mathbb{N}\right\}$$

satisfies the first two conditions but not the third.

s is called the *successor* function.

Proposition 2.13
For every $n \in \mathbb{N}$, $s(n) \neq n$.

Proof. Let

$$S = \{n \in \mathbb{N} \mid s(n) \neq n\}.$$

Clearly, $1 \in S$ by axiom (1). Suppose $n \in S$ for some n. We claim that $s(n) \in S$. By the injectivity of s, $s(n) \neq n$ so $s(s(n)) \neq s(n)$.

By (3), $S = \mathbb{N}$ which gives $s(n) \neq n, \forall n$. We can never loop when we keep applying s. □

Definition 2.14 (Natural Number)
We'll call $s(1) = 2, s(2) = 3, \ldots$. These are the natural numbers.