MA662 – Multivariable Calculus Notes

Jiahua Chen

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Subsets of $\mathbb{R}$

Definition 1.1. Let $X \subseteq \mathbb{R}$. Then...

1. $u \in \mathbb{R}$ is called an upper bound of X if $x \leqslant u$, $\forall x \in X$.
2. $l \in \mathbb{R}$ is called a lower bound of X if $x \geqslant l$, $\forall x \in X$.

It is an axiomatic property of $\mathbb{R}$ that each subset of $\mathbb{R}$ bounded above has a least upper bound and, likewise, each subset that is bounded below has a greatest lower bound.

Definition 1.2. Let $X \subseteq \mathbb{R}$ be bounded. Then...

1. $y = sup(X)$ (supremum of X) if y is an upper bound and, y' is another upper bound, then $y' \geqslant y$.
2. $z = inf(X)$ (infinum of X) if z is an lower bound and, z' is another lower bound, then $z' \leqslant z$.

Also if...

1. $sup(X) \in X$, then we call it the maximum of X.
2. $inf(X) \in X$, then we call it the minimum of X.

Example:
$X = (0,1)$ $\quad sup(X) = 1 \quad inf(X) = 0 \quad$ no max, no min
$X = [0,1]$ $\quad sup(X) = max(X) = 1 \quad inf(X) = min(X) = 0$

Proposition 1.3. If $X \subseteq \mathbb{R}$, bounded above, then $y = sup(X)$ iff

(i) y is an upper bound

(ii) $\forall \epsilon > 0, \exists x \in X$ such that $x > y - \epsilon$

Proof: Let $y = sup(X)$.

(i) is true by definition

(ii) Suppose $\exists \epsilon > 0$ such that there is no $x \in X$ with $x > y - \epsilon$.
Then $x \leqslant y - \epsilon \forall x \in X$. But that makes $y - \epsilon < y$ a smaller upper bound of X, which contradicts $y = sup(X)$

Suppose next that (i) and (ii) hold for $y \in \mathbb{R}$. We show that $y = sup(X)$. Clearly, y is an upper bound by (i), so let y' be a smaller upper bound for the sake of contradiction: $X \leqslant y' < y$ for all $x \in X$. Now consider $= y - y'$. Then $y - \epsilon = y - (y - y') = y' \geqslant x \forall x \in X$. This contradicts (ii) because we have found an $\epsilon > 0$ such that $\nexists x \in X$ greater than $y - \epsilon$. $\square$

Proposition 1.4. Let X be bounded below.

$$inf(X) = -sup(-X)$$

where $-X = \{-x \mid x \in X\}$

Proof: Let $y = sup(-X)$. Then $y \geqslant -x \Rightarrow -y \leqslant x$ for all $x \in X$, so $-y$ is a lower bound for X. Now assume for the sake of contradiction that $\exists -y' > -y$, another lower bound of X. Then $-y' \leqslant x \Rightarrow y' \geqslant -x$ for all $x \in X$. But $-y' > -y \Rightarrow y' < y$ so $y \neq sup(-X)$. Hence $\nexists - y'$, another lower bound of X. $\Rightarrow -y = inf(X) \Rightarrow -sup(-X) = inf(X)$ $\square$

Proposition 1.5. If A, B are bounded subsets of $\mathbb{R}$. Then $A \cup B$ is bounded and

$$sup(A \cup B) = max\{sup(A), sup(B)\}$$

Subsets $\mathbb{R}^n$ – Point-set topology

Definition 2.1. Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Then $B_\epsilon(x) = \{y \in \mathbb{R}^n \mid |x-y| < \epsilon\}$. This is called an ϵ-neighborhood of x.

Definition 2.2. Let $X \subseteq \mathbb{R}^n$ and $x \in R^n$. Then x is called

- interior point of X if $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq X$.
- boundary point of X if $\exists \epsilon > 0$, $B_\epsilon(x) \cap X \neq \emptyset$ and $B_\epsilon(x) \cap X^c \neq \emptyset$
- exterior point of X if it is an interior point of X^c

Notation: $\mathring{X}$ = interior of X = set of all interior point of X. δX = boundary of X = set of all boundary points of X

Definition 2.3. X is called open if it only consists of interior points. ($X = \mathring{X}$)
X is called closed if its complement is open.
$\Rightarrow$ X is open if it contains none of its boundary points.
$\Rightarrow$ X is closed if it contains all of its boundary points

Exercise 1.5.1, book p.101. For each of the following subsets, state whether it is open or closed (or both or neither), and say why.

a. $\{x \in \mathbb{R} \mid 0 < x \leqslant 1\}$ as a subset of $\mathbb{R}$
Answer: $\boxed{\text{Neither.}}$ 1 is not an interior point of this set and 0 is not an interior point of the complement of the set.

b. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} < 1\right\}$ as a subset of $\mathbb{R}^2$
Answer: $\boxed{\text{Open.}}$ The unit circle (which is the boundary) is not contained within the set.

c. the interval $(0,1]$ as a subset of $\mathbb{R}$
Answer: $\boxed{\text{Neither.}}$ Similar to a.

d. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} \leqslant 1\right\}$ as a subset of $\mathbb{R}^2$

e. $\{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$ as a subset of $\mathbb{R}$.

f. $\{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2+z^2} = 1 \text{ and } x,y,z \neq 0\}$ as a subset of $\mathbb{R}^3$

g. the empty set as a subset of $\mathbb{R}$

Exercise 1.5.2, book p.101. For each of the following subsets, state whether it is open or closed (or both or neither), and say why.

a. (x, y)-plane in $\mathbb{R}^3$

b. $\mathbb{R} \subset \mathbb{C}$

c. the line $x = 5$ in the (x, y)-plane

d. $(0, 1) \subset \mathbb{C}$

e. $\mathbb{R}^n \subset \mathbb{R}^n$

f. the unit sphere in $\mathbb{R}^3$

Exercise 1.5.5. For each of the following subsets of $\mathbb{R}$ and $\mathbb{R}^2$, state whether it is open or closed (or both or neither), and prove it.

a. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\right\}$

b. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid xy \neq 0\right\}$

c. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 0\right\}$

d. $\{\mathbb{Q} \subset \mathbb{R}\}$ (the rational numbers)

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Recall **Prop 1.5**: If A, B are bounded subsets of $\mathbb{R}$. Then $A \cup B$ is bounded and

$$sup(A \cup B) = max\{sup(A), sup(B)\}$$

Proof:

1 Show that $x \leqslant max\{sup(A), sup(B)\}$ for all $x \in A \cup B$
Case 1: $x \in A \Rightarrow x \leqslant sup(A) \leqslant max\{sup(A), sup(B)\}$
Case 2: $x \in B \Rightarrow x \leqslant sup(B) \leqslant max\{sup(A), sup(B)\}$

2 Take $\epsilon > 0$ and consider $max\{sup(a), sup(B)\} - \epsilon$
Case 1: $max\{sup(A), sup(B)\} = supA \Rightarrow \exists x \in A$ such that $x > sup(A) - \epsilon \Rightarrow x \in A \cup B$ such that $x > max\{sup(A), sup(B)\} - \epsilon$
Case 2: $max\{sup(A), sup(B)\} = supB \Rightarrow$ *left to the reader, follows similarly as above.*

□

Also recall...

Exercise 1.5.5. For each of the following subsets of $\mathbb{R}$ and $\mathbb{R}^2$, state whether it is open or closed (or both or neither), and prove it.

a. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\right\}$

Answer: $\boxed{\text{Open.}}$

Proof: Let $p \in A$(nnulus). $1 < |p - 0| < \sqrt{2}$. To show: $\exists \epsilon > 0$ s.t. all points in $B_\epsilon(p)$ are between 1 and $\sqrt{2}$ from 0. There is such ϵ, specifically

$$\epsilon = \frac{1}{2} \cdot min(\sqrt{2} - |p|, |p| - 1)$$

Now we show that for $x \in B_\epsilon(p)$, $1 < |x|^2 < 2$:
WLOG: Consider $p \in (1, \sqrt{2})$ on the x-axis. Then the neighborhood of p is:

$$B_\epsilon(p) = \left\{\begin{pmatrix} p + r\sin\theta \\ r\sin\theta \end{pmatrix} \middle| r \in [0, \epsilon)\right\}$$

$$\begin{aligned} \left|\begin{pmatrix} p + r\sin\theta \\ r\sin\theta \end{pmatrix}\right|^2 &= p^2 + 2pr\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta \\ &= p^2 + 2pr\cos\theta + r^2 \end{aligned}$$

$$(p - r)^2 = p^2 - 2pr + r^2 \leqslant p^2 + 2pr\cos\theta + r^2 \leqslant p^2 + 2pr + r^2 = (p + r)^2$$

$$\text{Since } r < (\sqrt{2} - p), (p + r)^2 < (p + \sqrt{2} - p)^2 = 2$$

$$\text{Also since } r < (p - 1), (p - r)^2 > (p - (p - 1))^2 = 1$$

$\square$

We could also use the triangle inequality: $|a+b| \leqslant |a| + |b|$:

$$|p+r| \leqslant |p| + |r| < |p| + (\sqrt{2} - |p|) = \sqrt{2}$$

$$|p-r| \geqslant |p| - |r| > |p| - (|p| - 1) = 1$$

b. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid xy \neq 0\right\}$
Answer: $\boxed{\text{Open.}}$
Proof: Consider $B_\epsilon(p)$ with $\epsilon = \frac{1}{2} min\{|x|, |y|\}$. $\square$

c. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 0\right\}$
Answer: $\boxed{\text{Closed.}}$
Proof: Consider the complement, $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y \neq 0\right\}$. Following a similar logic as b, consider $\epsilon = \frac{x}{2}$. $\square$

d. $\{\mathbb{Q} \subset \mathbb{R}\}$ (the rational numbers)
Answer: $\boxed{\text{Neither.}}$

Exercise 1.5.3. Prove the following statements for open subsets of $\mathbb{R}^n$:

a. **Any union of open sets is open.**
Proof: Let X_i, $i \in I$, be open. Consider $Y = \bigcup_{i \in I} X_i$.
To show: each $y \in Y$ is an interior point of Y.
Let $y \in Y$ belong to arbitrary X_i, for some $i \in I$. As X_i is open, y is also an interior point of X_i. So $\exists \epsilon > 0$ s.t. $B_\epsilon(y) \subset X_i \subseteq Y \Rightarrow y$ is an interior point of Y. $\square$

b. **A finite intersection of open sets is open.**
Proof: Consider $Z = \bigcap_{i=I}^{n} X_i$.
To show: each $z \in Z$ is an interior point of Z. Since $z \in Z$, $z \in X_i$ for $i = 1, \ldots, n$. Since X_i is open, $\exists \epsilon_i > 0 \mid B_{\epsilon_i}(z) \subset X_i$. As there are finitely many i, we choose the smallest $\epsilon = min\{\epsilon_i \mid i = 1, \ldots, n\}$. Then we have

$$B_\epsilon(z) \subset B_{\epsilon_i}(z) \subset X_i \text{ for all } x = 1, \ldots, n$$

Thus $\mathcal{B}_\epsilon(z) \subset Z$, making z an interior point of Z. $\square$

c. **An infinite interesection of open sets is not necessarily open.** *Proof:*

$$\bigcap_{n=1}^{\infty} \left\{ x \,\middle|\, x \in \left(-\frac{1}{n}, \frac{1}{n}\right)\right\} = \{0\}$$

$\square$

Definition 4.1. (Convergent sequence; limit of sequence). A sequence $i \mapsto \boldsymbol{a}_i$ if points in $\mathbb{R}^n$ *converges* to $\boldsymbol{a} \in \mathbb{R}^n$ if

$$\forall \epsilon > 0, \exists M \text{ s.t. } m > M \Rightarrow |\boldsymbol{a}_m - \boldsymbol{a}| < \epsilon$$

We then call $\boldsymbol{a}$ the *limit* of the sequence.

Proposition 4.2. (Convergence in terms of coordinates). A sequence $m \mapsto \boldsymbol{a}_m$ with $\boldsymbol{a}_m \in \mathbb{R}^n$ converges to $\boldsymbol{a}$ if and only if each coordinate converges; i.e., if for all j with $1 \leqslant j \leqslant n$, the jth coordinate of $\boldsymbol{a}_m$ converges to $\boldsymbol{a}_j$.

Proof:

□

Proposition 4.3. (Limit of sequence is unique). If the sequence $i \mapsto \boldsymbol{a}_i$ of points in $\mathbb{R}^n$ converges to $\boldsymbol{a}$ and to $\boldsymbol{b}$, then $\boldsymbol{a} = \boldsymbol{b}$.

Proof: Let the sequence $i \mapsto \boldsymbol{a}_i$ converge to both $\boldsymbol{a}$ and $\boldsymbol{b}$. Then

$$\forall \epsilon > 0, \exists M_a \wedge M_b \text{ s.t. } m > M_a, m > M_b \Rightarrow |\boldsymbol{a} - \boldsymbol{a}_m| < \frac{\epsilon}{2} \wedge |\boldsymbol{a}_m - \boldsymbol{b}| < \frac{\epsilon}{2}$$

$$|\boldsymbol{a} - \boldsymbol{b}| = |(\boldsymbol{a} - \boldsymbol{a}_m) + (\boldsymbol{b}_m - \boldsymbol{b})| \leqslant |\boldsymbol{a} - \boldsymbol{a}_m| + |\boldsymbol{a}_m - \boldsymbol{b}| = \epsilon$$
$$\Rightarrow |\boldsymbol{a} - \boldsymbol{b}| = 0 \Rightarrow \boldsymbol{a} = \boldsymbol{b}$$

□

Theorem 4.4. (The arithmetic of limits of sequences). All arithmetics that apply to limits apply here.

Proposition 4.5. (Sequence in closed set).

1. Let $i \mapsto \boldsymbol{x}_i$ be a sequence in a closed set $C \subset \mathbb{R}^n$ converging to $\boldsymbol{x}_0 \in \mathbb{R}^n$. Then $\boldsymbol{x}_0 \in C$.
2. Conversely, if every convergent sequence in a set $C \in \mathbb{R}^n$ converges to a point in C, then C is closed. a

Definition 5.1. (Limit of a function). Let X be a subset of $\mathbb{R}^n$ and $\boldsymbol{x}_0$ a point in $\overline{X}$ ($\overline{X} = X \cup \delta X$). A function $\boldsymbol{f} : X \to \mathbb{R}^m$ has the limit $\boldsymbol{a}$ at $\boldsymbol{x}_0$:

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{a}$$

if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $\boldsymbol{x} \in X$,

$$|\boldsymbol{x} - \boldsymbol{x}_0| < \delta \Rightarrow |\boldsymbol{f}(\boldsymbol{x}) - a| < \epsilon$$

Proposition 5.2. (Convergence by coordinates). Suppose

$$U \subset \mathbb{R}^n, \quad \boldsymbol{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} : U \to \mathbb{R}^m$$

Theorem 5.3. (Limits of functions). The same rules for traditional limits apply. Additional rules are as follows:

1. Dot product
2. ***

Exercise 1.5.14. State whether the following limits exist, and prove it.

a. $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2}{x+y}$

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt{|x|}y}{x^2+y^2}$

c. ***

Definition 6.1. $X \subseteq \mathbb{R}^n$, define the closure of X: $\overline{X} = X \cup \delta X$

Theorem 6.2. $\overline{X}$ is the smallest closed set that contains X.

Proof: If X is closed, we are done.
Otherwise, assume $\exists Y \subset \mathbb{R}^n$, Y closed, with

$$X \subsetneqq Y \subseteq \overline{X}$$

We show that $Y = \overline{X}$: Assume otherwise for the sake of contradiction that that $\exists x \in \overline{X}$ but $x \notin Y$. $x \in \overline{X} \Rightarrow x \in X$ or $x \in \delta X$. If $x \in X$ then $x \in Y$ as $X \subset Y$. Thus $x \in \delta X$. Define sequence in X that converges to x, by **Proposition 4.5** we have that Y is open. Hence we have achieved a contradiction $\Rightarrow\!\Leftarrow$. $\square$