

Definition 0.1. Let $X \subseteq \mathbb{R}$. Then...

1. $u \in \mathbb{R}$ is called an upper bound of X if $x \leq u, \forall x \in X$.
2. $l \in \mathbb{R}$ is called a lower bound of X if $x \geq l, \forall x \in X$.

Definition 0.2. Let $X \subseteq \mathbb{R}$ be bounded. Then...

1. $y = \sup(X)$ (supremum of X) if y is an upper bound and, y' is another upper bound, then $y' \geq y$.
2. $z = \inf(X)$ (infimum of X) if z is a lower bound and, z' is another lower bound, then $z' \leq z$.

Also if...

1. $\sup(X) \in X$, then we call it the maximum of X .
2. $\inf(X) \in X$, then we call it the minimum of X .

Definition 0.3. Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Then $B_\epsilon(x) = \{y \in \mathbb{R}^n \mid |x - y| < \epsilon\}$. This is called an ϵ -neighborhood of x .

Definition 0.4. Let $X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then x is called

- interior point of X if $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq X$.
- boundary point of X if $\forall \epsilon > 0, B_\epsilon(x) \cap X \neq \emptyset$ and $B_\epsilon(x) \cap X^c \neq \emptyset$
- exterior point of X if it is an interior point of X^c

Notation: $\overset{\circ}{X}$ = interior of X = set of all interior point of X . δX = boundary of X = set of all boundary points of X

Definition 0.5. X is called open if it only consists of interior points. ($X = \overset{\circ}{X}$)

X is called closed if its complement is open.

$\Rightarrow X$ is open if it contains none of its boundary points.

$\Rightarrow X$ is closed if it contains all of its boundary points

Definition 0.6. (Convergent sequence; limit of sequence). A sequence $i \mapsto a_i$ if points in \mathbb{R}^n converges to $a \in \mathbb{R}^n$ if

$$\forall \epsilon > 0, \exists M \text{ s.t. } m > M \Rightarrow |a_m - a| < \epsilon$$

We then call a the *limit* of the sequence.

Definition 0.7. (Limit of a function). Let X be a subset of \mathbb{R}^n and x_0 a point in \overline{X} ($\overline{X} = X \cup \delta X$). A function $f : X \rightarrow \mathbb{R}^m$ has the limit a at x_0 :

$$\lim_{x \rightarrow x_0} f(x) = a$$

if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x \in X$,

$$|x - x_0| < \delta \Rightarrow |f(x) - a| < \epsilon$$

Definition 0.8. $X \subseteq \mathbb{R}^n$, define the closure of X : $\bar{X} = X \cup \delta X$

Definition 0.9. (Continuous function). Let $X \subset \mathbb{R}^n$. A mapping $f : X \rightarrow \mathbb{R}^m$ is continuous at $x_0 \in X$ iff

$$\lim_{x \rightarrow x_0} f(x) = f(x_0);$$

f is continuous on X if it is continuous at every point of X . Equivalently, $f : X \rightarrow \mathbb{R}^m$ is continuous at $x_0 \in X$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that when $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

Definition 0.10. (Bounded set). A subset $X \in \mathbb{R}^n$ is bounded if it is contained in a ball in \mathbb{R}^n centered at the origin:

$$X \subset B_R(0) \quad \text{for some } R < \infty.$$

Definition 0.11. (Compact set). A nonempty subset $C \subset \mathbb{R}^n$ is compact if it is closed and bounded.

Definition 0.12. (Supremum). ***

Definition 0.13. (Infimum). ***

Definition 0.14. (Minimum value; minimum). ***

Definition 0.15. The directional derivative of f at x in direction \vec{v} gives the rate of change of f as we step into direction \vec{v} . It is defined as

$$\lim_{h \rightarrow 0} \frac{f(x + h\vec{v}) - f(x)}{h}$$

We will see shortly that this evaluates to $[Df(x)]\vec{v}$.

Definition 0.16. (C^p function). A C^p function on $U \subset \mathbb{R}^n$ is a function that is p times continuously differentiable: all of its partial derivatives up to order p exist and are continuous on U .

Definition 0.17. (Newton's method). Let \vec{f} be a differentiable map from U to \mathbb{R}^n , where U is an open subset of \mathbb{R}^n . Newton's method consists of starting with some guesses a_0 for a solution of $\vec{f}(x) = \vec{0}$. Then linearize the equation at a_0 : replace the increment to the function, $\vec{f}(x) - \vec{f}(a_0)$, by a linear function of the increment, $[D\vec{f}(a_0)](x - a_0)$. Now solve the corresponding linear equation:

$$\vec{f}(a_0) + [D\vec{f}(a_0)](x - a_0) = \vec{0}$$

$a_0 = \text{initial guess}$

$$a_{n+1} = a_n - [D\vec{f}(a_n)]^{-1} \vec{f}(a_n)$$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x_n)}$$

Definition 0.18. (Smooth manifold in \mathbb{R}^n). A subset $M \subset \mathbb{R}^n$ is a smooth k -dimensional manifold if locally it is the graph of a C^1 mapping \mathbf{f} expressing $n - k$ variables as functions of other k variables.