

# MA662 – Multivariable Calculus Notes

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## Subsets of $\mathbb{R}$

**Definition 1.1.** Let  $X \subseteq \mathbb{R}$ . Then...

1.  $u \in \mathbb{R}$  is called an upper bound of  $X$  if  $x \leq u, \forall x \in X$ .
2.  $l \in \mathbb{R}$  is called a lower bound of  $X$  if  $x \geq l, \forall x \in X$ .

It is an axiomatic property of  $\mathbb{R}$  that each subset of  $\mathbb{R}$  bounded above has a least upper bound and, likewise, each subset that is bounded below has a greatest lower bound.

**Definition 1.2.** Let  $X \subseteq \mathbb{R}$  be bounded. Then...

1.  $y = \sup(X)$  (supremum of  $X$ ) if  $y$  is an upper bound and,  $y'$  is another upper bound, then  $y' \geq y$ .
2.  $z = \inf(X)$  (infimum of  $X$ ) if  $z$  is a lower bound and,  $z'$  is another lower bound, then  $z' \leq z$ .

Also if...

1.  $\sup(X) \in X$ , then we call it the maximum of  $X$ .
2.  $\inf(X) \in X$ , then we call it the minimum of  $X$ .

**Example:**

$$\begin{array}{llll} X = (0, 1) & \sup(X) = 1 & \inf(X) = 0 & \text{no max, no min} \\ X = [0, 1] & \sup(X) = \max(X) = 1 & \inf(X) = \min(X) = 0 & \end{array}$$

**Proposition 1.3.** If  $X \subseteq \mathbb{R}$ , bounded above, then  $y = \sup(X)$  iff

- (i)  $y$  is an upper bound
- (ii)  $\forall \epsilon > 0, \exists x \in X$  such that  $x > y - \epsilon$

*Proof:* Let  $y = \sup(X)$ .

- (i) is true by definition
- (ii) Suppose  $\exists \epsilon > 0$  such that there is no  $x \in X$  with  $x > y - \epsilon$ . Then  $x \leq y - \epsilon \forall x \in X$ . But that makes  $y - \epsilon < y$  a smaller upper bound of  $X$ , which contradicts  $y = \sup(X)$

Suppose next that (i) and (ii) hold for  $y \in \mathbb{R}$ . We show that  $y = \sup(X)$ . Clearly,  $y$  is an upper bound by (i), so let  $y'$  be a smaller upper bound for the sake of contradiction:  $x \leq y' < y$  for all  $x \in X$ . Now consider  $y - y'$ . Then  $y - \epsilon = y - (y - y') = y' \geq x \forall x \in X$ . This contradicts (ii) because we have found an  $\epsilon > 0$  such that  $\nexists x \in X$  greater than  $y - \epsilon$ .  $\square$

**Proposition 1.4.** Let  $X$  be bounded below.

$$\inf(X) = -\sup(-X)$$

where  $-X = \{-x \mid x \in X\}$

*Proof:* Let  $y = \sup(-X)$ . Then  $y \geq -x \Rightarrow -y \leq x$  for all  $x \in X$ , so  $-y$  is a lower bound for  $X$ . Now assume for the sake of contradiction that  $\exists -y' > -y$ , another lower bound of  $X$ . Then  $-y' \leq x \Rightarrow y' \geq -x$  for all  $x \in X$ . But  $-y' > -y \Rightarrow y' < y$  so  $y \neq \sup(-X)$ . Hence  $\nexists -y'$ , another lower bound of  $X$ .  $\Rightarrow -y = \inf(X) \Rightarrow -\sup(-X) = \inf(X)$   $\square$

**Proposition 1.5.** If  $A, B$  are bounded subsets of  $\mathbb{R}$ . Then  $A \cup B$  is bounded and

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$$

**Subsets  $\mathbb{R}^n$  – Point-set topology**

**Definition 2.1.** Let  $x \in \mathbb{R}^n$  and  $\epsilon > 0$ . Then  $B_\epsilon(x) = \{y \in \mathbb{R}^n \mid |x - y| < \epsilon\}$ . This is called an  $\epsilon$ -neighborhood of  $x$ .

**Definition 2.2.** Let  $X \subseteq \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ . Then  $x$  is called

- interior point of  $X$  if  $\exists \epsilon > 0$  such that  $B_\epsilon(x) \subseteq X$ .
- boundary point of  $X$  if  $\exists \epsilon > 0$ ,  $B_\epsilon(x) \cap X \neq \emptyset$  and  $B_\epsilon(x) \cap X^c \neq \emptyset$
- exterior point of  $X$  if it is an interior point of  $X^c$

Notation:  $\overset{\circ}{X}$  = interior of  $X$  = set of all interior point of  $X$ .  $\delta X$  = boundary of  $X$  = set of all boundary points of  $X$

**Definition 2.3.**  $X$  is called open if it only consists of interior points. ( $X = \overset{\circ}{X}$ )

$X$  is called closed if its complement is open.

$\Rightarrow X$  is open if it contains none of its boundary points.

$\Rightarrow X$  is closed if it contains all of its boundary points

**Exercise 1.5.1, book p.101.** For each of the following subsets, state whether it is open or closed (or both or neither), and say why.

- a.  $\{x \in \mathbb{R} \mid 0 < x \leq 1\}$  as a subset of  $\mathbb{R}$

*Answer:* Neither. 1 is not an interior point of this set and 0 is not an interior point of the complement of the set.

- b.  $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} < 1 \right\}$  as a subset of  $\mathbb{R}^2$

*Answer:* Open. The unit circle (which is the boundary) is not contained within the set.

- c. the interval  $(0, 1]$  as a subset of  $\mathbb{R}$

*Answer:* Neither. Similar to a.

- d.  $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} \leq 1 \right\}$  as a subset of  $\mathbb{R}^2$

- e.  $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$  as a subset of  $\mathbb{R}$ .

- f.  $\{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2 + z^2} = 1 \text{ and } x, y, z \neq 0\}$  as a subset of  $\mathbb{R}^3$

g. the empty set as a subset of  $\mathbb{R}$

**Exercise 1.5.2, book p.101.** For each of the following subsets, state whether it is open or closed (or both or neither), and say why.

a.  $(x, y)$ -plane in  $\mathbb{R}^3$

b.  $\mathbb{R} \subset \mathbb{C}$

c. the line  $x = 5$  in the  $(x, y)$ -plane

d.  $(0, 1) \subset \mathbb{C}$

e.  $\mathbb{R}^n \subset \mathbb{R}^n$

f. the unit sphere in  $\mathbb{R}^3$

**Exercise 1.5.5.** For each of the following subsets of  $\mathbb{R}$  and  $\mathbb{R}^2$ , state whether it is open or closed (or both or neither), and prove it.

a.  $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2 \right\}$

b.  $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid xy \neq 0 \right\}$

c.  $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 0 \right\}$

d.  $\{\mathbb{Q} \subset \mathbb{R}\}$  (the rational numbers)

Recall **Prop 1.5**: If  $A, B$  are bounded subsets of  $\mathbb{R}$ . Then  $A \cup B$  is bounded and

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$$

*Proof:*

- 1 Show that  $x \leq \max\{\sup(A), \sup(B)\}$  for all  $x \in A \cup B$ 
  - Case 1:  $x \in A \Rightarrow x \leq \sup(A) \leq \max\{\sup(A), \sup(B)\}$
  - Case 2:  $x \in B \Rightarrow x \leq \sup(B) \leq \max\{\sup(A), \sup(B)\}$
- 2 Take  $\epsilon > 0$  and consider  $\max\{\sup(A), \sup(B)\} - \epsilon$ 
  - Case 1:  $\max\{\sup(A), \sup(B)\} = \sup A \Rightarrow \exists x \in A$  such that  $x > \sup(A) - \epsilon \Rightarrow x \in A \cup B$  such that  $x > \max\{\sup(A), \sup(B)\} - \epsilon$
  - Case 2:  $\max\{\sup(A), \sup(B)\} = \sup B \Rightarrow$  left to the reader, follows similarly as above.

□

Also recall...

**Exercise 1.5.5.** For each of the following subsets of  $\mathbb{R}$  and  $\mathbb{R}^2$ , state whether it is open or closed (or both or neither), and prove it.

a.  $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2 \right\}$

Answer: Open.

*Proof:* Let  $p \in A$  (annulus).  $1 < |p - 0| < \sqrt{2}$ . To show:  $\exists \epsilon > 0$  s.t. all points in  $B_\epsilon(p)$  are between 1 and  $\sqrt{2}$  from 0. There is such  $\epsilon$ , specifically

$$\epsilon = \frac{1}{2} \cdot \min(\sqrt{2} - |p|, |p| - 1)$$

Now we show that for  $x \in B_\epsilon(p)$ ,  $1 < |x|^2 < 2$ :

WLOG: Consider  $p \in (1, \sqrt{2})$  on the  $x$ -axis. Then the neighborhood of  $p$  is:

$$B_\epsilon(p) = \left\{ \begin{pmatrix} p + r \sin \theta \\ r \sin \theta \end{pmatrix} \mid r \in [0, \epsilon) \right\}$$

$$\begin{aligned} \left| \begin{pmatrix} p + r \sin \theta \\ r \sin \theta \end{pmatrix} \right|^2 &= p^2 + 2pr \cos \theta + r^2 \cos^2 \theta + r^2 \sin^2 \theta \\ &= p^2 + 2pr \cos \theta + r^2 \end{aligned}$$

$$(p - r)^2 = p^2 - 2pr + r^2 \leq p^2 + 2pr \cos \theta + r^2 \leq p^2 + 2pr + r^2 = (p + r)^2$$

$$\text{Since } r < (\sqrt{2} - p), (p + r)^2 < (p + \sqrt{2} - p)^2 = 2$$

$$\text{Also since } r < (p - 1), (p - r)^2 > (p - (p - 1))^2 = 1$$

□

We could also use the triangle inequality:  $|a + b| \leq |a| + |b|$ :

$$|p + r| \leq |p| + |r| < |p| + (\sqrt{2} - |p|) = \sqrt{2}$$

$$|p - r| \geq |p| - |r| > |p| - (|p| - 1) = 1$$

b.  $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid xy \neq 0 \right\}$

Answer: Open.

Proof: Consider  $B_\epsilon(p)$  with  $\epsilon = \frac{1}{2} \min\{|x|, |y|\}$ .

□

c.  $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 0 \right\}$

Answer: Closed.

Proof: Consider the complement,  $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y \neq 0 \right\}$ . Following a similar logic as b, consider  $\epsilon = \frac{x}{2}$ .

□

d.  $\{\mathbb{Q} \subset \mathbb{R}\}$  (the rational numbers)

Answer: Neither.

**Exercise 1.5.3.** Prove the following statements for open subsets of  $\mathbb{R}^n$ :

a. **Any union of open sets is open.**

Proof: Let  $X_i, i \in I$ , be open. Consider  $Y = \bigcup_{i \in I} X_i$ .

To show: each  $y \in Y$  is an interior point of  $Y$ .

Let  $y \in Y$  belong to arbitrary  $X_i$ , for some  $i \in I$ . As  $X_i$  is open,  $y$  is also an interior point of  $X_i$ . So  $\exists \epsilon > 0$  s.t.  $B_\epsilon(y) \subset X_i \subseteq Y \Rightarrow y$  is an interior point of  $Y$ . □

b. **A finite intersection of open sets is open.**

Proof: Consider  $Z = \bigcap_{i=1}^n X_i$ .

To show: each  $z \in Z$  is an interior point of  $Z$ . Since  $z \in Z, z \in X_i$  for  $i = 1, \dots, n$ . Since  $X_i$  is open,  $\exists \epsilon_i > 0 \mid B_{\epsilon_i}(z) \subset X_i$ . As there are finitely many  $i$ , we choose the smallest  $\epsilon = \min\{\epsilon_i \mid i = 1, \dots, n\}$ . Then we have

$$B_\epsilon(z) \subset B_{\epsilon_i}(z) \subset X_i \text{ for all } i = 1, \dots, n$$

Thus  $B_\epsilon(z) \subset Z$ , making  $z$  an interior point of  $Z$ .

□

c. **An infinite intersection of open sets is not necessarily open.** Proof:

$$\bigcap_{n=1}^{\infty} \left\{ x \mid x \in \left( -\frac{1}{n}, \frac{1}{n} \right) \right\} = \{0\}$$

□

**Definition 4.1. (Convergent sequence; limit of sequence).** A sequence  $i \mapsto a_i$  of points in  $\mathbb{R}^n$  *converges* to  $a \in \mathbb{R}^n$  if

$$\forall \epsilon > 0, \exists M \text{ s.t. } m > M \Rightarrow |a_m - a| < \epsilon$$

We then call  $a$  the *limit* of the sequence.

**Proposition 4.2. (Convergence in terms of coordinates).** A sequence  $m \mapsto a_m$  with  $a_m \in \mathbb{R}^n$  converges to  $a$  if and only if each coordinate converges; i.e., if for all  $j$  with  $1 \leq j \leq n$ , the  $j$ th coordinate of  $a_m$  converges to  $a_j$ .

*Proof:*

□

**Proposition 4.3. (Limit of sequence is unique).** If the sequence  $i \mapsto a_i$  of points in  $\mathbb{R}^n$  converges to  $a$  and to  $b$ , then  $a = b$ .

*Proof:* Let the sequence  $i \mapsto a_i$  converge to both  $a$  and  $b$ . Then

$$\forall \epsilon > 0, \exists M_a \wedge M_b \text{ s.t. } m > M_a, m > M_b \Rightarrow |a - a_m| < \frac{\epsilon}{2} \wedge |a_m - b| < \frac{\epsilon}{2}$$

$$\begin{aligned} |a - b| &= |(a - a_m) + (a_m - b)| \leq |a - a_m| + |a_m - b| = \epsilon \\ &\Rightarrow |a - b| = 0 \Rightarrow a = b \end{aligned}$$

□

**Theorem 4.4. (The arithmetic of limits of sequences).** All arithmetics that apply to limits apply here.

**Proposition 4.5. (Sequence in closed set).**

1. Let  $i \mapsto x_i$  be a sequence in a closed set  $C \subset \mathbb{R}^n$  converging to  $x_0 \in \mathbb{R}^n$ . Then  $x_0 \in C$ .
2. Conversely, if every convergent sequence in a set  $C \in \mathbb{R}^n$  converges to a point in  $C$ , then  $C$  is closed. a

**Definition 5.1. (Limit of a function).** Let  $X$  be a subset of  $\mathbb{R}^n$  and  $x_0$  a point in  $\overline{X}$  ( $\overline{X} = X \cup \delta X$ ). A function  $f : X \rightarrow \mathbb{R}^m$  has the limit  $a$  at  $x_0$ :

$$\lim_{x \rightarrow x_0} f(x) = a$$

if for all  $\epsilon > 0$  there exists  $\delta > 0$  such that for all  $x \in X$ ,

$$|x - x_0| < \delta \Rightarrow |f(x) - a| < \epsilon$$

**Proposition 5.2. (Convergence by coordinates).** Suppose

$$U \subset \mathbb{R}^n, \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} : U \rightarrow \mathbb{R}^m$$

**Theorem 5.3. (Limits of functions).** The same rules for traditional limits apply. Additional rules are as follows:

1. Dot product
2. \*\*\*

**Exercise 1.5.14.** State whether the following limits exist, and prove it.

- a.  $\lim_{(x,y) \rightarrow (1,2)} \frac{x^2}{x+y}$
- b.  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sqrt{|x|}y}{x^2+y^2}$
- c. \*\*\*



**Definition 6.1.**  $X \subseteq \mathbb{R}^n$ , define the closure of  $X$ :  $\bar{X} = X \cup \delta X$

**Theorem 6.2.**  $\bar{X}$  is the smallest closed set that contains  $X$ .

*Proof:* If  $X$  is closed, we are done.

Otherwise, assume  $\exists Y \subset \mathbb{R}^n$ ,  $Y$  closed, with

$$X \subsetneq Y \subseteq \bar{X}$$

We show that  $Y = \bar{X}$ : Assume otherwise for the sake of contradiction that that  $\exists z \in \bar{X} - Y \subseteq Y^C$  which is open. Then  $\exists \epsilon > 0$  s.t.  $B_\epsilon(z) \subseteq Y^C$ . Hence  $B_\epsilon(z) \subseteq \mathbb{R}^n - X$ , which contradicts  $x \in \bar{X}$ . Therefore  $\bar{X} - Y = \emptyset$ , so  $Y = \bar{X}$ .  $\square$

**Definition 6.3. (Continuous function).** Let  $X \subset \mathbb{R}^n$ . A mapping  $f : X \rightarrow \mathbb{R}^m$  is continuous at  $x_0 \in X$  if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0);$$

$f$  is continuous on  $X$  if it is continuous at every point of  $X$ . Equivalently,  $f : X \rightarrow \mathbb{R}^m$  is continuous at  $x_0 \in X$  if and only if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that when  $|x - x_0| < \delta$ , then  $|f(x) - f(x_0)| < \epsilon$ .

**Theorem 6.4. (Combining continuous mappings).** Continuous functions are closed under addition, scalar umltiplication, quotients, and compositions.

**Lemma 6.5.** Hence polynomials and rational functions (given that the denominator does not vanish) are continous.

**Exercise 1.5.21.** For the following functions, can you choose a value for  $f$  at  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  to make the function continuous at the origin?

a.  $f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \frac{1}{x^2 + y^2 + 1}$

Answer: Exists.  $f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 1$ .

The limit exists at  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  by substitution.

b.  $f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \frac{\sqrt{x^2 + y^2}}{|x| + |y|^{1/3}}$

Answer: Does not exist.

*Proof:* Approaching  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  from  $\begin{pmatrix} x \\ 0 \end{pmatrix}$  gives  $\lim_{x \rightarrow 0} \frac{\sqrt{x^2}}{|x|} = \lim_{x \rightarrow 0} \frac{|x|}{|x|} = 1$ , whilst approaching

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  from  $\begin{pmatrix} 0 \\ y \end{pmatrix}$  gives  $\lim_{y \rightarrow 0} \frac{\sqrt{y^2}}{|y|^{1/3}} = \lim_{y \rightarrow 0} \frac{y}{y^{1/3}} = \lim_{y \rightarrow 0} y^{2/3} = 0. \Rightarrow \Leftarrow$ .  $\square$

c.  $f\left(\frac{x}{y}\right) = (x^2 + y^2) \ln(x^2 + 2y^2)$

Answer:  $\boxed{f\left(\frac{0}{0}\right) = 0.}$

Proof: Consider

$$g\left(\frac{x}{y}\right) = (x^2 + y^2) \ln(x^2 + y^2)$$

$$g\left(\frac{r}{\theta}\right) = r^2 \ln(r^2) = 2r^2 \ln(r)$$

$$\lim_{r \rightarrow 0} r^2 \ln(r^2) = \lim_{r \rightarrow 0} \frac{2 \ln(r)}{r^{-2}} = \lim_{r \rightarrow 0} \frac{r^{-1}}{-2r^{-3}} = \lim_{r \rightarrow 0} \frac{1}{-2} r^2 = 0$$

Now consider bounding  $f\left(\frac{0}{0}\right)$ .

$$g\left(\frac{x}{y}\right) \leq f\left(\frac{x}{y}\right) \leq 0 \quad \text{for } \left(\frac{x}{y}\right) \text{ sufficiently near } \left(\frac{0}{0}\right)$$

And the squeeze theorem gives that  $\lim_{(x,y) \rightarrow (0,0)} f\left(\frac{x}{y}\right) = 0$ . □

d.  $f\left(\frac{x}{y}\right) = (x^2 + y^2) \ln|x + y|$

Answer:  $\boxed{\text{Limit does not exist.}}$

Proof: Consider approaching  $f\left(\frac{x}{y}\right)$  from  $y = -x$ . We then have

$$\lim_{(x,y) \rightarrow (0,0)} f\left(\frac{x}{y}\right) = \lim_{y \rightarrow 0} 2y^2 \cdot \ln|0| = \infty!$$

□

**Exercise 1.5.16b.** Either show that the limit exists at 0 and find it, or show that it does not exist:

$$f\left(\frac{x}{y}\right) = \frac{\sin(x + y)}{\sqrt{x^2 + y^2}}$$

Answer:  $\boxed{\text{Does not exist.}}$

Proof: Consider approaching  $\left(\frac{0}{0}\right)$  from  $\left(\frac{x}{0}\right)$ . We then have

$$\lim_{(x,y) \rightarrow (0,0)} f\left(\frac{x}{y}\right) = \lim_{x \rightarrow 0} \frac{\sin(x)}{|x|}$$

$$\lim_{x \rightarrow 0^+} \frac{\sin(x)}{|x|} = +1 \quad \text{but} \quad \lim_{x \rightarrow 0^-} \frac{\sin(x)}{|x|} = -1 \neq +1$$

Recall from previously, we were trying to solve:

$$g\left(\frac{x}{y}\right) = (|x| + |y|) \cdot \ln(x^2 + y^4) < 0 \text{ near } \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The solution then follows that

$$(|x| + |y|) \cdot \ln(x^4 + y^4) < g\left(\frac{x}{y}\right) < 0$$

We can use  $lp$ -norms to estimate one of the values in the above function.

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_p = (|x|^p + |y|^p)^{\frac{1}{p}}, p \geq 1 \quad lp\text{-norms}$$

**Definition 7.1. (Bounded set).** A subset  $X \in \mathbb{R}^n$  is bounded if it is contained in a ball in  $\mathbb{R}^n$  centered at the origin:

$$X \subset B_R(0) \quad \text{for some } R < \infty.$$

**Definition 7.2. (Compact set).** A nonempty subset  $C \subset \mathbb{R}^n$  is compact if it is closed and bounded.

**Theorem 7.3. (Convergent subsequence in a compact set).** If a compact set  $C \subset \mathbb{R}^n$  contains a sequence  $i \mapsto x_i$ , then that sequence has a convergent subsequence  $j \mapsto x_{i(j)}$  whose limit is in  $C$ .

**Definition 8.1. (Supremum).**

**Definition 8.2. (Infimum).**

**Definition 8.3. (Minimum value; minimum).**

**Theorem 8.4. (Existence of minima and maxima).** Let  $C \subset \mathbb{R}^n$  be a compact set, and let  $f : C \rightarrow \mathbb{R}$  be a continuous function. Then there exists a point  $\mathbf{a} \in C$  such that  $f(\mathbf{a}) \geq f(\mathbf{x})$  for all  $\mathbf{x} \in C$ , and a point  $\mathbf{b} \in C$  such that  $f(\mathbf{b}) \leq f(\mathbf{x})$  for all  $\mathbf{x} \in C$ .