Definition 0.1. (Bounds). Let $X \subseteq \mathbb{R}$. Then. . .

1. $u \in \mathbb{R}$ is called an upper bound of X if $x \leqslant u,\ \forall x \in X$.
2. $l \in \mathbb{R}$ is called a lower bound of X if $x \geqslant l,\ \forall x \in X$.

Definition 0.2. (Extremum). Let $X \subseteq \mathbb{R}$ be bounded. Then. . .

1. $y = \sup(X)$ (supremum of X) if y is an upper bound and, y' is another upper bound, then $y' \geqslant y$.
2. $z = \inf(X)$ (infinum of X) if z is an lower bound and, z' is another lower bound, then $z' \leqslant z$.

Also if. . .

1. $\sup(X) \in X$, then we call it the maximum of X.
2. $\inf(X) \in X$, then we call it the minimum of X.

Definition 0.3. (Neighborhood). Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Then $B_\epsilon(x) = \{y \in \mathbb{R}^n \mid |x - y| < \epsilon\}$. This is called an ϵ-neighborhood of x.

Definition 0.4. (Classification of points). Let $X \subseteq \mathbb{R}^n$ and $x \in R^n$. Then x is called

- interior point of X if $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq X$.
- boundary point of X if $\forall \epsilon > 0$, $B_\epsilon(x) \cap X \neq \emptyset$ and $B_\epsilon(x) \cap X^c \neq \emptyset$
- exterior point of X if it is an interior point of X^c

Notation: X = interior of X = set of all interior point of X. δX = boundary of X = set of all boundary points of X

Definition 0.5. (Closure). X is called open if it only consists of interior points. ($X = X$)
X is called closed if its complement is open.
$\Rightarrow$ X is open if it contains none of its boundary points.
$\Rightarrow$ X is closed if it contains all of its boundary points

Definition 0.6. (Convergent sequence; limit of sequence). A sequence $i \mapsto \mathbf{a}_i$ if points in $\mathbb{R}^n$ converges to $\mathbf{a} \in \mathbb{R}^n$ if

$$\forall \epsilon > 0,\ \exists M \text{ s.t. } m > M \Rightarrow |\mathbf{a}_m - \mathbf{a}| < \epsilon$$

We then call $\mathbf{a}$ the limit of the sequence.

Definition 0.7. (Limit of a function). Let X be a subset of $\mathbb{R}^n$ and $\mathbf{x}_0$ a point in $\overline{X}$ (note $\overline{X} = X \cup \delta X$). A function $\mathbf{f} : X \to \mathbb{R}^m$ has the limit $\mathbf{a}$ at $\mathbf{x}_0$:

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{a}$$

if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall \mathbf{x} \in X$,

$$|\mathbf{x} - \mathbf{x}_0| < \delta \Rightarrow |\mathbf{f}(\mathbf{x}) - \mathbf{a}| < \epsilon$$

Related Prop: *If a function has a limit, it is unique.*

Definition 0.8. (Closure). $X \subseteq \mathbb{R}^n$, define the closure of X: $\overline{X} = X \cup \delta X$

Definition 0.9. (Continuous function). Let $X \subset \mathbb{R}^n$. A mapping $f : X \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0 \in X$ iff

$$\lim_{x \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0);$$

$\mathbf{f}$ is continuous on X if it is continuous at every point of X. Equivalently, $\mathbf{f} : X \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0 \in X$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that when $|\mathbf{x} - \mathbf{x}_0| < \delta$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| < \epsilon$.

Definition 0.10. (Bounded set). A subset $X \in \mathbb{R}^n$ is bounded if it is contained in a ball in $\mathbb{R}^n$ centered at the origin:

$$X \subset B_R(0) \quad \text{for some } R < \infty.$$

Definition 0.11. (Compact set). A nonempty subset $C \subset \mathbb{R}^n$ is compact if it is closed and bounded.

Definition 0.12. (Derivative). Let U be an open subset of $\mathbb{R}$, and let $f : U \to \mathbb{R}$ be a function. Then f is differentiable at $a \in U$ with derivative $f'(a)$ if the limit

$$f'(a) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{1}{h}(f(a+h) - f(a)) \quad \text{exists}$$

Definition 0.13. (Derivatives in $\mathbb{R}^n$). Let $U \subset \mathbb{R}^n$ be an open subset and let $\mathbf{f} : U \to \mathbb{R}^m$ be a mapping; let $\mathbf{a}$ be a point in U. If there exists a linear transformation (represented by matrix) $[\mathbf{Df}(\mathbf{x})] \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{\vec{h} \to 0} \frac{1}{|\vec{h}|}(\mathbf{f}(\mathbf{x} + \vec{h}) - \mathbf{f}(\vec{\mathbf{x}})) - [\mathbf{Df}(\mathbf{x})]\vec{h} = \vec{0}$$

then $\mathbf{f}$ is differentiable at $\mathbf{a}$, and $[\mathbf{Df}(\mathbf{x})]$ is unique and is the derivative of $\mathbf{f}$ at $\mathbf{a}$.

Definition 0.14. (Partial derivative). The right-hand side of # is called the partial derivative of f (with respect to the ith variable evaluated at $\mathbf{x}$):

$$D_i f(\mathbf{x}) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{1}{h}\left(f\begin{pmatrix} x_1 \\ \vdots \\ x_i + h \\ \vdots \\ x_n \end{pmatrix} - f\begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix}\right)$$

(Given such limit exists, of course). Therefore, we can calculate it by considering x_i the only variable, and holding all other components constant.

Definition 0.15. (Jacobian matrix). The Jacobian matrix of a function $\mathbf{f} : U \subset \mathbb{R}^n \to \mathbb{R}^m$ [i.e. $\mathbf{f}(\mathbf{a}) = (f_1(\mathbf{a}), \dots, f_n(\mathbf{a}))$] is the $m \times n$ matrix composed of the n partial derivatives of $\mathbf{f}$ evaluated at $\mathbf{a}$:

$$[\mathbf{Jf}(\mathbf{a})] \stackrel{\text{def}}{=} \begin{bmatrix} D_1 f_1(\mathbf{a}) & \cdots & D_n f_1(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ D_1 f_m(\mathbf{a}) & \cdots & D_n f_m(\mathbf{a}) \end{bmatrix}$$

Definition 0.16. (Directional derivatives). The directional derivative of $\mathbf{f}$ at $\mathbf{x}$ in direction $\vec{\mathbf{v}}$ gives the rate of change of f as we step into direction $\vec{v}$. It is defined as

$$\lim_{h \to 0} \frac{\mathbf{f}(\mathbf{x} + h\vec{\mathbf{v}}) - \mathbf{f}(\mathbf{x})}{h}$$

We will see shortly that this evaluates to $[\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}$.

Definition 0.17. (C^p function). A C^p *function* on $U \subset \mathbb{R}^n$ is a function that is p times continuously differentiable: all of its partial derivatives up to order p exist and are continuous on U.

Definition 0.18. (Newton's method). Let $\vec{f}$ be a differentiable map from U to $\mathbb{R}^n$, where U is an open subset of $\mathbb{R}^n$. Newton's method consists of starting with some guesses $\mathbf{a}_0$ for a sollution of $\vec{f}(\mathbf{x}) = \vec{0}$. Then linearize the equation at $\mathbf{a}_0$: replace the increment to the function, $\vec{f}(\mathbf{x}) - \vec{f}(\mathbf{a}_0)$, by a linear function of the increment, $[\mathbf{D}\vec{f}(\mathbf{a}_0)](\mathbf{x} - \mathbf{a}_0)$. Now solve the corresponding linear equation:

$$\vec{f}(\mathbf{a}_0) + [\mathbf{D}\vec{f}(\mathbf{a}_0)](\mathbf{x} - \mathbf{a}_0) = \vec{0}$$

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$$\begin{aligned} a_0 &= \text{initial guess} \\ a_{n+1} &= a_n - [\mathbf{D}\vec{f}(a_n)]^{-1}\mathbf{f}(a_n) \\ x_{n+1} &= x_n - \frac{\mathbf{f}(x)}{\mathbf{f}'(x_n)} \end{aligned}$$

Definition 0.19. (Smooth manifold in $\mathbb{R}^n$). A subset $M \subset \mathbb{R}^n$ is a smooth k-dimensional manifold if locally it is the graph of a C^1 mapping $\mathbf{f}$ expressing $n - k$ variables as functions of other k variables.