MA662: Multivariable Calculus

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These are the course notes for Multivariable Calculus (**MA662**) at Hotchkiss taught by Dr. Weiss. These notes were last updated March 4, 2019. Any sections denoted with asterisks (`***`) are currently incomplete, and I will update them when I get to those. Although the notes are my own documentation, I've appropriated Krit's[1] sectional headings (and also occasionally some of his notes when I have partially missing sections).

[1] Krit's version of these notes are *** `here`

Contents

1 Topology

1.1 Subsets of $\mathbb{R}$

January 7, 2019

We start with an in-depth exploration of topology first in single-dimensional reals.

Definition 1.1. (Bounds). Let $X \subseteq \mathbb{R}$. Then...

1. $u \in \mathbb{R}$ is called an upper bound of X if $x \leqslant u$, $\forall x \in X$.
2. $l \in \mathbb{R}$ is called a lower bound of X if $x \geqslant l$, $\forall x \in X$.

It is an axiomatic property of $\mathbb{R}$ that each subset of $\mathbb{R}$ bounded above has a least upper bound and, likewise, each subset that is bounded below has a greatest lower bound.

Definition 1.2. (Extremum). Let $X \subseteq \mathbb{R}$ be bounded. Then...

1. $y = \sup(X)$ (supremum of X) if y is an upper bound and, y' is another upper bound, then $y' \geqslant y$.
2. $z = \inf(X)$ (infinum of X) if z is an lower bound and, z' is another lower bound, then $z' \leqslant z$.

Also if...

1. $\sup(X) \in X$, then we call it the maximum of X.
2. $\inf(X) \in X$, then we call it the minimum of X.

Example:

$X = (0, 1)$ $\quad$ $\sup(X) = 1$ $\quad$ $\inf(X) = 0$ $\quad$ no max, no min

$X = [0, 1]$ $\quad$ $\sup(X) = \max(X) = 1$ $\quad$ $\inf(X) = \min(X) = 0$

Proposition 1.3. If $X \subseteq \mathbb{R}$, bounded above, then $y = \sup(X)$ iff

(i) y is an upper bound

(ii) $\forall \epsilon > 0, \exists x \in X$ such that $x > y - \epsilon$

Proof. Let $y = \sup(X)$.

(i) is true by definition

(ii) Suppose $\exists\epsilon > 0$ such that there is no $x \in X$ with $x > y - \epsilon$.
Then $x \leqslant y - \epsilon \forall x \in X$. But that makes $y - \epsilon < y$ a smaller upper bound of X, which contradicts $y = \sup(X)$

Suppose next that (i) and (ii) hold for $y \in \mathbb{R}$. We show that $y = \sup(X)$. Clearly, y is an upper bound by (i), so let y' be a smaller upper bound for the sake of contradiction: $X \leqslant y' < y$ for all $x \in X$. Now consider $\epsilon = y - y'$. Then $y - \epsilon = y - (y - y') = y' \geqslant x \forall x \in X$. This contradicts (ii) because we have found an $\epsilon > 0$ such that $\nexists x \in X$ greater than $y - \epsilon$. □

Proposition 1.4. Let X be bounded below.

$$\inf(X) = -\sup(-X) \tag{1.1}$$

where $-X = \{-x \mid x \in X\}$

Proof. Let $y = \sup(-X)$. Then $y \geqslant -x \Rightarrow -y \leqslant x$ for all $x \in X$, so $-y$ is a lower bound for X. Now assume for the sake of contradiction that $\exists - y' > -y$, another lower bound of X. Then $-y' \leqslant x \Rightarrow y' \geqslant -x$ for all $x \in X$. But $-y' > -y \Rightarrow y' < y$ so $y \neq \sup(-X)$. Hence $\nexists - y'$, another lower bound of X.
$\Rightarrow -y = \inf(X) \Rightarrow -\sup(-X) = \inf(X)$ □

Proposition 1.5. If A, B are bounded subsets of $\mathbb{R}$. Then $A \cup B$ is bounded and

$$\sup(A \cup B) = \max\left\{\sup(A), \sup(B)\right\} \tag{1.2}$$

1.2 Topology in $\mathbb{R}^n$

January 8, 2019

Definition 1.6. (Neighborhood). Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Then

$$B_\epsilon(x) = \{y \in \mathbb{R}^n \mid |x - y| < \epsilon\} \tag{1.3}$$

This is called an ϵ-neighborhood of x.

Definition 1.7. (Classification of points). Let $X \subseteq \mathbb{R}^n$ and $x \in R^n$. Then x is called

- interior point of X if $\exists\epsilon > 0$ such that $B_\epsilon(x) \subseteq X$.
- boundary point of X if $\forall\epsilon > 0$, $B_\epsilon(x) \cap X \neq \emptyset$ and $B_\epsilon(x) \cap X^c \neq \emptyset$

- exterior point of X if it is an interior point of X^c

Notation: X = interior of X = set of all interior point of X. δX = boundary of X = set of all boundary points of X

Definition 1.8. (Closure). X is called open if it only consists of interior points. X is called closed if its complement is open.
$\Rightarrow$ X is open if it contains none of its boundary points.
$\Rightarrow$ X is closed if it contains all of its boundary points.

Exercise 1.5.1, book p.101. For each of the following subsets, state whether it is open or closed (or both or neither), and say why.

a. $\{x \in \mathbb{R} \mid 0 < x \leqslant 1\}$ as a subset of $\mathbb{R}$
Answer: $\boxed{\text{Neither.}}$ 1 is not an interior point of this set and 0 is not an interior point of the complement of the set.

b. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} < 1\right\}$ as a subset of $\mathbb{R}^2$
Answer: $\boxed{\text{Open.}}$ The unit circle (which is the boundary) is not contained within the set.

c. the interval $(0, 1]$ as a subset of $\mathbb{R}$
Answer: $\boxed{\text{Neither.}}$ Similar to a.

d. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} \leqslant 1\right\}$ as a subset of $\mathbb{R}^2$
Answer: $\boxed{\text{Closed.}}$ This is the unit circle on the plane, and the boundary point set $x^2 + y^2 = 1$ is wholly contained within this subset.

e. $\{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$ as a subset of $\mathbb{R}$.
Answer: $\boxed{\text{Closed.}}$ Both boundary points, 0 and 1 are contained within this set.

f. $\{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2+z^2} = 1 \text{ and } x, y, z \neq 0\}$ as a subset of $\mathbb{R}^3$.
Answer: $\boxed{\text{Closed.}}$ This constitutes its own boundary points, where every (x, y, z)'s nbhd has intersections with both the set and the complement of the set. It is the unit spherical shell in 3-dimensions.

g. the empty set as a subset of $\mathbb{R}$.
Answer: $\boxed{\text{Both open and closed.}}$ Its complement, the set of all real numbers, contains all of its boundary points (of which it has none) and contains none of its boundary points (of which it has none).

Exercise 1.5.2, book p.101. For each of the following subsets, state whether it is open or closed (or both or neither), and say why.

a. (x, y)-plane in $\mathbb{R}^3$

b. $\mathbb{R} \subset \mathbb{C}$

c. the line $x = 5$ in the (x, y)-plane

d. $(0, 1) \subset \mathbb{C}$

e. $\mathbb{R}^n \subset \mathbb{R}^m$

f. the unit sphere in $\mathbb{R}^3$

Exercise 1.5.5. For each of the following subsets of $\mathbb{R}$ and $\mathbb{R}^2$, state whether it is open or closed (or both or neither), and prove it.

a. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\right\}$

b. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid xy \neq 0\right\}$

c. $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 0\right\}$

d. $\{\mathbb{Q} \subset \mathbb{R}\}$ (the rational numbers)

January 10, 2019

Recall **Prop 1.5**: If A, B are bounded subsets of $\mathbb{R}$. Then $A \cup B$ is bounded and

$$\sup(A \cup B) = \max\left\{\sup(A), \sup(B)\right\}$$

Proof. 1 Show that $x \leqslant \max\left\{\sup(A), \sup(B)\right\}$ for all $x \in A \cup B$
xr Case 1: $x \in A \Rightarrow x \leqslant \sup(A) \leqslant \max\left\{\sup(A), \sup(B)\right\}$
Case 2: $x \in B \Rightarrow x \leqslant \sup(B) \leqslant \max\left\{\sup(A), \sup(B)\right\}$

2 Take $\epsilon > 0$ and consider $\max\left\{\sup(a), \sup(B)\right\} - \epsilon$
Case 1: $\max\left\{\sup(A), \sup(B)\right\} = \sup A \Rightarrow \exists x \in A$ such that $x > \sup(A) - \epsilon \Rightarrow$ $x \in A \cup B$ such that $x > \max\left\{\sup(A), \sup(B)\right\} - \epsilon$
Case 2: $\max\left\{\sup(A), \sup(B)\right\} = \sup B \Rightarrow$ *left to the reader, follows similarly as above.*

$\square$

Also recall. . . **Exercise 1.5.5.** For each of the following subsets of $\mathbb{R}$ and $\mathbb{R}^2$, state whether it is open or closed (or both or neither), and prove it.

a. $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2 \right\}$
Answer: $\boxed{\text{Open.}}$

Proof. Let $p \in A$(nnulus). $1 < |p - 0| < \sqrt{2}$. To show: $\exists \epsilon > 0$ s.t. all points in $B_\epsilon(p)$ are between 1 and $\sqrt{2}$ from 0. There is such ϵ, specifically

$$\epsilon = \frac{1}{2} \cdot \min(\sqrt{2} - |p|, |p| - 1)$$

Now we show that for $x \in B_\epsilon(p)$, $1 < |x|^2 < 2$:
WLOG: Consider $p \in (1, \sqrt{2})$ on the x-axis. Then the neighborhood of p is:

$$B_\epsilon(p) = \left\{ \begin{pmatrix} p + r\sin\theta \\ r\sin\theta \end{pmatrix} \middle| \, r \in [0, \epsilon) \right\}$$

$$\begin{aligned} \left| \begin{pmatrix} p + r\sin\theta \\ r\sin\theta \end{pmatrix} \right|^2 &= p^2 + 2pr\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta \\ &= p^2 + 2pr\cos\theta + r^2 \end{aligned}$$

$$(p - r)^2 = p^2 - 2pr + r^2 \leqslant p^2 + 2pr\cos\theta + r^2 \leqslant p^2 + 2pr + r^2 = (p + r)^2$$

$$\text{Since } r < (\sqrt{2} - p), (p + r)^2 < (p + \sqrt{2} - p)^2 = 2$$

$$\text{Also since } r < (p - 1), (p - r)^2 > (p - (p - 1))^2 = 1$$

$\square$

We could also use the triangle inequality ($|a+b| \leqslant |a|+|b|$ and $|a-b| \geqslant ||a|-|b||$):

$$|p + r| \leqslant |p| + |r| < |p| + (\sqrt{2} - |p|) = \sqrt{2}$$

$$|p - r| \geqslant |p| - |r| > |p| - (|p| - 1) = 1$$

b. $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid xy \neq 0 \right\}$
Answer: $\boxed{\text{Open.}}$

Proof. Consider $B_\epsilon(p)$ with $\epsilon = \frac{1}{2}\min\{|x|, |y|\}$. $\square$

c. $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 0 \right\}$
Answer: $\boxed{\text{Closed.}}$

Proof. Consider the complement, $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y \neq 0\right\}$. Following a similar logic as *b*, consider $\epsilon = \frac{x}{2}$. ☐

d. $\{\mathbb{Q} \subset \mathbb{R}\}$ (the rational numbers)
Answer: $\boxed{\text{Neither.}}$

Exercise 1.5.3. Prove the following statements for open subsets of $\mathbb{R}^n$:

a. **Any union of open sets is open.**

Proof. Let X_i, $i \in I$, be open. Consider $Y = \bigcup_{i \in I} X_i$.
To show: each $y \in Y$ is an interior point of Y.
Let $y \in Y$ belong to arbitrary X_i, for some $i \in I$. As X_i is open, y is also an interior point of X_i. So $\exists \epsilon > 0$ s.t. $B_\epsilon(y) \subset X_i \subseteq Y \Rightarrow y$ is an interior point of Y. ☐

b. **A finite intersection of open sets is open.**

Proof. Consider $Z = \bigcap_{i=I}^{n} X_i$.
To show: each $z \in Z$ is an interior point of Z. Since $z \in Z$, $z \in X_i$ for $i = 1, \ldots, n$. Since X_i is open, $\exists \epsilon_i > 0 \mid B_{\epsilon_i}(z) \subset X_i$. As there are finitely many i, we choose the smallest $\epsilon = \min\{\epsilon_i \mid i = 1, \ldots, n\}$. Then we have

$$B_\epsilon(z) \subset B_{\epsilon_i}(z) \subset X_i \text{ for all } x = 1, \ldots, n$$

Thus $B_\epsilon(z) \subset Z$, making z an interior point of Z. ☐

c. **An infinite interesection of open sets is not necessarily open.**

Proof.

$$\bigcap_{n=1}^{\infty} \left\{x \,\middle|\, x \in \left(-\frac{1}{n}, \frac{1}{n}\right)\right\} = \{0\}$$

☐

1.3 Sequences and Limits

January 14, 2019

Definition 1.9. (Convergent sequence; limit of sequence). A sequence $i \mapsto \mathbf{a}_i$ if points in $\mathbb{R}^n$ converges to $\mathbf{a} \in \mathbb{R}^n$ if

$$\forall \epsilon > 0,\ \exists M \text{ s.t. } m > M \Rightarrow |\mathbf{a}_m - \mathbf{a}| < \epsilon \tag{1.4}$$

We then call $\mathbf{a}$ the limit of the sequence.

Proposition 1.10. (Convergence in terms of coordinates). A sequence $m \mapsto \mathbf{a}_m$ with $\mathbf{a}_m \in \mathbb{R}^n$ converges to $\mathbf{a}$ if and only if each coordinate converges; i.e., if for all j with $1 \leqslant j \leqslant n$, the jth coordinate of $\mathbf{a}_m$ converges to $\mathbf{a}_j$, the jth coordinate of the limit $\mathbf{a}$.

Proof. (p.88) The gist of the proof is to find sufficiently large M for given ϵ, in this case we set $M = max\,\{M_i\}$ which guarantees that we stay within the error. □

Proposition 1.11. (Limit of sequence is unique). If the sequence $i \mapsto \mathbf{a}_i$ of points in $\mathbb{R}^n$ converges to $\mathbf{a}$ and to $\mathbf{b}$, then $\mathbf{a} = \mathbf{b}$.

Proof. Let the sequence $i \mapsto \mathbf{a}_i$ converge to both $\mathbf{a}$ and $\mathbf{b}$. Then

$$\forall \epsilon > 0, \exists M_a \wedge M_b \text{ s.t. } m > M_a, m > M_b \Rightarrow |\mathbf{a} - \mathbf{a}_m| < \frac{\epsilon}{2} \wedge |\mathbf{a}_m - \mathbf{b}| < \frac{\epsilon}{2}$$

$$|\mathbf{a} - \mathbf{b}| = |(\mathbf{a} - \mathbf{a}_m) + (\mathbf{b}_m - \mathbf{b})| \leqslant |\mathbf{a} - \mathbf{a}_m| + |\mathbf{a}_m - \mathbf{b}| = \epsilon$$

$$\Rightarrow |\mathbf{a} - \mathbf{b}| = 0 \Rightarrow \mathbf{a} = \mathbf{b}$$

□

Theorem 1.12. (The arithmetic of limits of sequences). All arithmetics that apply to limits apply here.

Proposition 1.13. (Sequence in closed set).

1. Let $i \mapsto \mathbf{x}_i$ be a sequence in a closed set $C \subset \mathbb{R}^n$ converging to $\mathbf{x}_0 \in \mathbb{R}^n$. Then $\mathbf{x}_0 \in C$.
2. Conversely, if every convergent sequence in a set $C \in \mathbb{R}^n$ converges to a point in C, then C is closed.

January 15, 2019

Definition 1.14. (Limit of a function). Let X be a subset of $\mathbb{R}^n$ and $\mathbf{x}_0$ a point in $\overline{X}$ (note $\overline{X} = X \cup \delta X$). A function $\mathbf{f} : X \to \mathbb{R}^m$ has the limit $\mathbf{a}$ at $\mathbf{x}_0$:

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{a} \tag{1.5}$$

if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall \mathbf{x} \in X$,

$$|\mathbf{x} - \mathbf{x}_0| < \delta \Rightarrow |\mathbf{f}(\mathbf{x}) - \mathbf{a}| < \epsilon \tag{1.6}$$

Related Prop: *If a function has a limit, it is unique.*

Proposition 1.15. (Convergence by coordinates). Suppose

$$U \subset \mathbb{R}^n, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} : U \to \mathbb{R}^m, \text{ and } \mathbf{x}_0 \in \overline{U} \tag{1.7}$$

Then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \text{ iff } \lim_{\mathbf{x} \to \mathbf{x}_0} f_i(\mathbf{x}) = a_i, i = 1, \ldots, m \tag{1.8}$$

The above proposition basically states that for a multi-dimensional function, with each coordinate a function that has a limit, the limit of the multi-dimensional function is simply the individual limits as its coordinates.

Theorem 1.16. (Limits of functions). The same rules for traditional limits apply: addition, multiplication, division. Additional rules are as follows:

1. The dot product
$$\lim_{\mathbf{x} \to \mathbf{x}_0} (\mathbf{f} \cdot \mathbf{g})(\mathbf{x}) = \lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) \cdot \lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{g}(\mathbf{x})$$
2. The limit of the product of two functions, one of whose limit evaluates to 0 and another which is bounded, will be 0 (see textbook p.95, there are nuances to this rule).

Exercise 1.5.14. State whether the following limits exist, and prove it.

a. $\displaystyle \lim_{(x,y) \to (1,2)} \frac{x^2}{x + y}$

Answer: $\boxed{\text{Exists.}}$ We can simply evaluate the function at the given point. The polynomial and non-diminishing quotient nature of the function guarantee its existence.

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt{|x|}y}{x^2+y^2}$

Answer: $\boxed{\text{Does not exist.}}$ It intuitively makes sense as the power in the denominator outweigh the power in the numerator. We can prove this by approaching this function and showing that it is unbounded. Let us approach this from $y = x$:

$$\lim_{y,x\to 0} \frac{\sqrt{|x|}x}{x^2+x^2} = \lim_{y,x\to 0} \frac{x^{3/2}}{2x^2} = \lim_{y,x\to 0} \frac{1}{2x^{1/2}} = \infty \quad (!)$$

January 17, 2019

Definition 1.17. (Closure). $X \subseteq \mathbb{R}^n$, define the closure of X: $\overline{X} = X \cup \delta X$

Theorem 1.18. $\overline{X}$ is the smallest closed set that contains X.

Proof. If X is closed, we are done.
Otherwise, assume $\exists Y \subset \mathbb{R}^n$, Y closed, with

$$X \subsetneqq Y \subseteq \overline{X}$$

We show that $Y = \overline{X}$: Assume otherwise for the sake of contradiction that that $\exists z \in \overline{X} - Y \subseteq Y^C$ which is open. Then $\exists \epsilon > 0$ s.t. $B_\epsilon(z) \subseteq Y^C$. Hence $B_\epsilon(z) \subseteq R^n - X$, which contradicts $x \in \overline{X}$. Therefore $\overline{X} - Y = \emptyset$, so $Y = \overline{X}$. ◻

1.4 Continuity

Definition 1.19. (Continuous function). Let $X \subset \mathbb{R}^n$. A mapping $f : X \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0 \in X$ iff

$$\lim_{x\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0); \tag{1.9}$$

$\mathbf{f}$ is continuous on X if it is continuous at every point of X. Equivalently, $\mathbf{f} : X \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0 \in X$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that when $|\mathbf{x} - \mathbf{x}_0| < \delta$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| < \epsilon$.

Theorem 1.20. (Combining continuous mappings). Continuous functions are closed under addition, scalar multiplication, division, and compositions.

Lemma 1.21. Hence polynomials and rational functions (given that the denominator does not vanish) are continous.

Exercise 1.5.21. For the following functions, can you choose a value for f at $\binom{0}{0}$ to make the function continuous at the origin?

a. $f\binom{x}{y} = \dfrac{1}{x^2+y^2+1}$

Answer: $\boxed{\text{Exists. } f\binom{0}{0} = 1.}$

The limit exists at $\binom{0}{0}$ by substitution.

b. $f\binom{x}{y} = \dfrac{\sqrt{x^2+y^2}}{|x|+|y|^{1/3}}$

Answer: $\boxed{\text{Does not exist.}}$

Proof. Approaching $\binom{0}{0}$ from $\binom{x}{0}$ gives $\lim_{x\to 0}\dfrac{\sqrt{x^2}}{|x|} = \lim_{x\to 0}\dfrac{|x|}{|x|} = 1$, whilst approaching $\binom{0}{0}$ from $\binom{0}{y}$ gives $\lim_{y\to 0}\dfrac{\sqrt{y^2}}{|y|^{1/3}} = \lim_{y\to 0}\dfrac{y}{y^{1/3}} = \lim_{y\to 0} y^{2/3} = 0$. $\Rightarrow\Leftarrow$. □

c. $f\binom{x}{y} = (x^2+y^2)\ln(x^2+2y^2)$

Answer: $\boxed{f\binom{0}{0} = 0.}$

Proof. Consider

$$g\binom{x}{y} = (x^2+y^2)\ln(x^2+y^2)$$

$$g\binom{r}{\theta} = r^2\ln(r^2) = 2r^2\ln(r)$$

$$\lim_{r\to 0} r^2\ln(r^2) = \lim_{r\to 0}\frac{2\ln(r)}{r^{-2}} = \lim_{r\to 0}\frac{r^{-1}}{-2r^{-3}} = \lim_{r\to 0}\frac{1}{-2}r^2 = 0$$

Now consider bounding $f\binom{0}{0}$.

$$g\binom{x}{y} \leqslant f\binom{x}{y} \leqslant 0 \quad \text{for } \binom{x}{y} \text{ sufficiently near } \binom{0}{0}$$

And the squeeze theorem gives that $\lim_{(x,y)\to(0,0)} f\binom{x}{y} = 0$. □

d. $f\binom{x}{y} = (x^2+y^2)\ln|x+y|$

Answer: $\boxed{\text{Limit does not exist.}}$

Proof. Consider approaching $f\begin{pmatrix}x\\y\end{pmatrix}$ from $y = -x$. We then have

$$\lim_{(x,y)\to(0,0)} f\begin{pmatrix}x\\y\end{pmatrix} = \lim_{y\to 0} 2y^2 \cdot \ln|0| = \infty \quad (!)$$

$\square$

Exercise 1.5.16b. Either show that the limit exists at 0 and find it, or show that it does not exist:

$$f\begin{pmatrix}x\\y\end{pmatrix} = \frac{\sin(x+y)}{\sqrt{x^2+y^2}}$$

Answer: $\boxed{\text{Does not exist.}}$

Proof. Consider approaching $\begin{pmatrix}0\\0\end{pmatrix}$ from $\begin{pmatrix}x\\0\end{pmatrix}$. We then have

$$\lim_{(x,y)\to(0,0)} f\begin{pmatrix}x\\y\end{pmatrix} = \lim_{x\to 0} \frac{\sin(x)}{|x|}$$

$$\lim_{x\to 0^+} \frac{\sin(x)}{|x|} = +1 \quad \text{but} \quad \lim_{x\to 0^-} \frac{\sin(x)}{|x|} = -1 \neq +1$$

$\square$

January 19, 2019

Recall from previously, we were trying to solve the following limit:

$$g\begin{pmatrix}x\\y\end{pmatrix} = (|x|+|y|)\cdot\ln(x^2+y^4) < 0 \text{ near } \begin{pmatrix}0\\0\end{pmatrix}$$

The solution consists of bounding our function below with a lesser function that still tends to 0:

$$\lim_{(x,y)\to\vec{0}} (|x|+|y|)\cdot\ln(x^4+y^4) < \lim_{(x,y)\to\vec{0}} (|x|+|y|)\cdot\ln(x^2+y^4) < \vec{0}$$

We can use lp-norms to estimate one of the values in the above function

$$\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|_p = (|x|^p + |y|^p)^{\frac{1}{p}}, p \geqslant 1 \quad \text{lp-norms}$$

due to the fact that

$$(x^4+y^4)^{\frac{1}{4}} < |x|+|y|$$

thus

$$\lim_{(x,y)\to\vec{0}} (x^4+y^4)^{\frac{1}{4}} \cdot \ln(x^4+y^4) < \lim_{(x,y)\to\vec{0}} (|x|+|y|) \cdot \ln(x^2+y^4) < \vec{0}$$

We can substitute u into this function as follows:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = f(x^4+y^4) = f(u) = u^{\frac{1}{4}} \cdot \ln(u)$$

$$\lim_{u\to 0} f(u) = \lim_{u\to 0} u^{\frac{1}{4}} \cdot \ln(u) = \lim_{u\to 0} \frac{\ln(u)}{u^{-\frac{1}{4}}} \overset{L'H}{=} \lim_{u\to 0} \frac{u^{-1}}{-\frac{1}{4}u^{-5/4}} = \lim_{u\to 0} -4u^{\frac{1}{4}} = 0$$

We sandwich the original function from both sides:

$$\vec{0} < \lim_{(x,y)\to\vec{0}} (|x|+|y|) \cdot \ln(x^2+y^4) < \vec{0}$$

$$\lim_{(x,y)\to\vec{0}} g\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

1.5 Bound and Compactness

Definition 1.22. (Bounded set). A subset $X \in \mathbb{R}^n$ is bounded if it is contained in a ball in $\mathbb{R}^n$ centered at the origin:

$$X \subset B_R(0) \quad \text{for some } R < \infty \tag{1.10}$$

Definition 1.23. (Compact set). A nonempty subset $C \subset \mathbb{R}^n$ is compact if it is closed and bounded.

Theorem 1.24. (Convergent subsequence in a compact set). If a compact set $C \subset \mathbb{R}^n$ contains a sequence $i \mapsto \mathbf{x}_i$, then that sequence has a convergent subsequence $j \mapsto \mathbf{x}_{i(j)}$ whose limit is in C.

January 22, 2019

Theorem 1.25. (Existence of minima and maxima). Let $C \subset \mathbb{R}^n$ be a compact subset, and let $f : C \to \mathbb{R}$ be a continuous function. Then there exists a point $\mathbf{a} \in C$ such that $f(\mathbf{a}) \geqslant f(\mathbf{x})$ for all $\mathbf{x} \in C$, and a point $\mathbf{b} \in C$ such that $f(\mathbf{b}) \leqslant f(\mathbf{x})$ for all $\mathbf{x} \in C$.

Proof. *Detailed in textbook p.109.* ☐

Theorem 1.26. (Mean value theorem). If $f : [a, b] \to \mathbb{R}$ is continuous, and f is differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1.11}$$

The below theorem naturally followed the ***mean value theorem*** *as one of the five big theorems, but seems dissociated with the topics at hand. It's a nice thing to know though.*

Theorem 1.27. (Fundamental theorem of algebra). Let

$$p(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_0 \tag{1.12}$$

be a polynomial of degree $k > 0$ with complex coefficients (recall that real numbers are a subset of complex numbers). Then p has a root: there exists a complex number z_0 such that $p(z_0) = 0$.

Corollary 1.28. Such polynomial $p(z)$ has k roots.

˜This concludes the first chapter, the test will be on the following topics: topology, sets, limits, supremum, infimum, continuity, compactness and boundedness.

1.6 Topology Test Review

February 21, 2019

1. Consider the function
$$f(x) = \frac{x}{2} + x^2 \sin\frac{1}{x}$$
Show whether it is possible to define $f(0)$ so that $f(x)$ is continuous everywhere.

   Proof. Yes, we set $f(0) = 0$. The limit exists as $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$, as x^2 tends to 0 and $\sin\frac{1}{x}$ is wholly bounded. □

2. Show that any finite union of compact sets is compact. Give a counterexample to show that an infinite union of compact sets does not need to be compact. ***

3. ***

4. ***

2 Derivatives

2.1 Abstract

Replace a complicated nonlinear equation by a linear one with the understanding that the results only hold approximately in a small neighborhood around a point $p \in \mathbb{R}^n$ but that the error vanishes faster than the distance to p.

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - f'(x) \cdot h}{h} = 0 \tag{2.1}$$

or formally:

Definition 2.1. (Derivative). Let U be an open subset of $\mathbb{R}$, and let $f : U \to \mathbb{R}$ be a function. Then f is differentiable at $a \in U$ with derivative $f'(a)$ if the limit

$$f'(a) \overset{\text{def}}{=} \lim_{h \to 0} \frac{1}{h} \left(f(a+h) - f(a) \right) \quad \text{exists} \tag{2.2}$$

For $f : \mathbb{R}^n \to \mathbb{R}^m$, we are looking for a function $\mathbf{Df}(\mathbf{x}_0) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{\vec{h} \to 0} \frac{\left\{ f(\mathbf{x} + \vec{h}) - f(\mathbf{x}) \right\} - \left\{ [\mathbf{Df}(\mathbf{x}_0)]\vec{h} \right\}}{|\vec{h}|} = 0 \tag{2.3}$$

As a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{Df}(\mathbf{x}_0)$ has a matrix which is called the Jacobian of f at $\mathbf{x}_0$: $[\mathbf{Df}(\mathbf{x}_0)] = [\mathbf{Jf}(\mathbf{x}_0)]$. $[\mathbf{Df}(\mathbf{x}_0)]\vec{h}$ is referred to as the directional derivative.

The Jacobian and actual derivative matrix is in a bit of a grey area, where it is hard to differentiate the individual usages. In general, the Jacobian matrix is simply the matrix of the partial derivatives, regardless of whether or not the function is differentiable. If the function happens to be differentiable, then the derivative matrix matches the Jacobian matrix.

2.2 Derivatives in $\mathbb{R}^n$

January 22, 2019

Let $f : \mathbb{R}^n \to \mathbb{R}^m$

Goal: local linearization of f with error going to zero sufficiently fast.

Definition 2.2. (Derivatives in $\mathbb{R}^n$). Let $U \subset \mathbb{R}^n$ be an open subset and let $\mathbf{f} : U \to \mathbb{R}^m$ be a mapping; let $\mathbf{a}$ be a point in U. If there exists a linear transformation (represented by a matrix) $[\mathbf{Df}(\mathbf{x})] \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{\vec{h}\to 0} \frac{1}{|\vec{h}|}(\mathbf{f}(\mathbf{x}+\vec{h}) - \mathbf{f}(\vec{\mathbf{x}})) - [\mathbf{Df}(\mathbf{x})]\vec{h} = \vec{0} \tag{2.4}$$

then $\mathbf{f}$ is differentiable at $\mathbf{a}$, and $[\mathbf{Df}(\mathbf{x})]$ is unique and is the derivative of $\mathbf{f}$ at $\mathbf{a}$.

If we know that $[\mathbf{Df}(\mathbf{x})]$ exists, we can calculate its matrix $[\mathbf{Jf}(\mathbf{x})]$ (Jacobian matrix) by evaluating $[\mathbf{Df}(\mathbf{x})]$ on the standard basis vectors.

We know that

$$\begin{aligned}
0 &= \lim_{|h|\to 0} \frac{1}{|h\vec{e}_i|}\left(f(\mathbf{x}+h\vec{e}_i) - f(\mathbf{x}) - [\mathbf{Df}(\mathbf{x})](h\vec{e}_i)\right) \\
&= \lim_{|h|\to 0} \frac{1}{|h|}\left(f(\mathbf{x}+h\vec{e}_i) - f(\mathbf{x}) - h[\mathbf{Df}(\mathbf{x})](\vec{e}_i)\right) \\
&= \lim_{h\to 0} \frac{f(\mathbf{x}+h\vec{e}_i) - f(\mathbf{x})}{h} - [\mathbf{Df}(\mathbf{x})](\vec{e}_i) \\
\Rightarrow [\mathbf{Df}(\mathbf{x})]\vec{e}_i &= \lim_{h\to 0} \frac{f(\mathbf{x}+h\vec{e}_i) - f(\mathbf{x})}{h} \qquad (\#)
\end{aligned}$$

Definition 2.3. (Partial derivative). The right-hand side of # is called the partial derivative of f (with respect to the ith variable evaluated at $\mathbf{x}$):

$$D_i f(\mathbf{x}) \stackrel{\text{def}}{=} \lim_{h\to 0} \frac{1}{h}\left(f\begin{pmatrix} x_1 \\ \vdots \\ x_i + h \\ \vdots \\ x_n \end{pmatrix} - f\begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix}\right) \tag{2.5}$$

(Given such limit exists, of course). Therefore, we can calculate it by considering x_i the only variable, and holding all other components constant.

This limit is essentially the ith row in $[\mathbf{Df}]$ or $[\mathbf{Jf}]$.

There are a variety of notations for this derivative:

- $D_i f(x)$
- $D_x f(\mathbf{x}), D_y f(\mathbf{x}), D_z f(\mathbf{x})$
- $\frac{\delta f}{\delta x}, \frac{\delta f}{\delta x_2} \ldots$
- $f_x, f_y \ldots$

Let's backtrack a bit for a full definition of the Jacobian:

Definition 2.4. (Jacobian matrix). The Jacobian matrix of a function $\mathbf{f} : U \subset \mathbb{R}^n \to \mathbb{R}^m$ [i.e. $\mathbf{f}(\mathbf{a}) = (f_1(\mathbf{a}), \ldots, f_n(\mathbf{a}))$] is the $m \times n$ matrix composed of the n partial derivatives of $\mathbf{f}$ evaluated at $\mathbf{a}$:

$$[\mathbf{Jf}(\mathbf{a})] = \left[\mathbf{J}f\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right] \stackrel{\text{def}}{=} \begin{bmatrix} D_1 f_1(\mathbf{a}) & \cdots & D_n f_1(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ D_1 f_m(\mathbf{a}) & \cdots & D_n f_m(\mathbf{a}) \end{bmatrix} \tag{2.6}$$

Example:

$$\begin{aligned} f\begin{pmatrix} x \\ y \end{pmatrix} &= \sin(x^2 + y^3) \\ D_x f\begin{pmatrix} x \\ y \end{pmatrix} &= \cos(x^2 + y^3) \cdot 2x \\ D_y f\begin{pmatrix} x \\ y \end{pmatrix} &= \cos(x^2 + y^3) \cdot 3y^2 \\ \left[Df\begin{pmatrix} x \\ y \end{pmatrix}\right] &= \cos(x^2 + y^3)[2x \quad 3y^2] \end{aligned}$$

Warning: The Jacobian matrix is only the matrix of the derivative if the function is actually differentiable!

(**Preview:** We will see shortly that f is differentiable if all its partials exist and are continuous. This gives us a pathway to prove whether a function is differentiable at a point, by manually proving that its partial derivatives match up to the Jacobian matrix.)

This **Df** gives us the rate of change in the axes, if we want to find the directional rate of change in any direction, we have to use a direction derivative.

Definition 2.5. (Directional derivatives). The directional derivative of $\mathbf{f}$ at $\mathbf{x}$ in direction $\vec{\mathbf{v}}$ gives the rate of change of f as we step into direction $\vec{\mathbf{v}}$. It is defined as

$$\lim_{h \to 0} \frac{\mathbf{f}(\mathbf{x} + h\vec{\mathbf{v}}) - \mathbf{f}(\mathbf{x})}{h} \tag{2.7}$$

We will see shortly that this evaluates to $[\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}$ given the function is differentiable at $\mathbf{x}$. We can exploit this fact to prove differentiability at point $\mathbf{x}$ by showing that $[\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}$ matches up to the directional derivative definition.

Exercise 1.7.4. Using the definition, check whether the following functions are differentiable at 0.

a. $f(x) = |x|^{3/2}$
Answer: $\boxed{\text{Exists.}}$

b. $f(x) = \begin{cases} x \cdot \ln|x| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
Answer: $\boxed{\text{Does not exist.}}$

c. $f(x) = \begin{cases} x/\ln|x| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
Answer: $\boxed{\text{Exists.}}$

January 28, 2019

Proposition 2.6. If $U \subset \mathbb{R}^n$ is open, and $\mathbf{f} : U \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in U$, then all directional derivatives of $\mathbf{f}$ at $\mathbf{a}$ exist, and the directional derivative in the direction $\vec{\mathbf{v}}$ is given by the formula

$$[\mathbf{Df}(\mathbf{a})]\vec{\mathbf{v}} = \lim_{h \to 0} \frac{\mathbf{f}(\mathbf{a} + h\vec{\mathbf{v}}) - \mathbf{f}(\mathbf{a})}{h} \tag{2.8}$$

Proof. *Datailed in textbook p.130 (Proposition 1.7.14)* □

2.3 Rules for calculating derivatives

(A lot of them are surprisingly similar to what we're used to seeing in Calculus BC!)

1. If $\mathbf{f} : U \to \mathbb{R}^m$ is a constant function, then $\mathbf{f}$ is differentiable, and its derivative is $[0]$.

2. If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then it is differentiable everywhere, and its derivative at all points $\mathbf{a}$ is $\mathbf{f}$, *i.e.*, $[\mathbf{Df}(\mathbf{a})]\vec{\mathbf{v}} = \mathbf{f}(\mathbf{v})$.

3. If $f_1, \ldots, f_m : U \to \mathbb{R}$ are m scalar valued functions differentiable at $\mathbf{a}$, then so is $\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$ with derivative

$$[\mathbf{Df}(\mathbf{a})]\vec{\mathbf{v}} = \begin{bmatrix} [\mathbf{D}f_1(\mathbf{a})]\vec{\mathbf{v}} \\ \vdots \\ [\mathbf{D}f_m(\mathbf{a})]\vec{\mathbf{v}} \end{bmatrix} \tag{2.9}$$

The same applies the other direction, if $\mathbf{f}$ is differentiable with derivative $[\mathbf{Df}(\mathbf{a})]$, then its coordinate components f_i are the ith row of the entire derivative matrix.

4. Given differentiable $\mathbf{f}, \mathbf{g}$ at $\mathbf{a}$, then

$$[\mathbf{D}(\mathbf{f}+\mathbf{g})(\mathbf{a})] = [\mathbf{Df}(\mathbf{a})] + [\mathbf{Dg}(\mathbf{a})] \tag{2.10}$$

5. Given differentiable $f : U \to \mathbb{R}, \mathbf{g} : U \to \mathbb{R}^m$ at $\mathbf{a}$, then

$$[\mathbf{D}(f\mathbf{g})(\mathbf{a})]\vec{\mathbf{v}} = \underbrace{f(\mathbf{a})}_{\mathbb{R}} \underbrace{[\mathbf{Dg}(\mathbf{a})]\vec{\mathbf{v}}}_{\mathbb{R}^m} + \underbrace{([\mathbf{D}f(\mathbf{a})]\vec{\mathbf{v}})}_{\mathbb{R}} \underbrace{\mathbf{g}(\mathbf{a})}_{\mathbb{R}^m} \tag{2.11}$$

6. Some variant of the quotient rule. This is too complicated... (see textbook p.138)

7. Given differentiable $\mathbf{f} : U \to \mathbb{R}^m, \mathbf{g} : U \to \mathbb{R}^m$ at $\mathbf{a}$, then

$$[\mathbf{D}(\mathbf{f}\cdot\mathbf{g})(\mathbf{a})]\vec{\mathbf{v}} = \underbrace{\mathbf{f}(\mathbf{a})}_{\mathbb{R}^m} \cdot \underbrace{[\mathbf{Dg}(\mathbf{a})]\vec{\mathbf{v}}}_{\mathbb{R}^m} + \underbrace{[\mathbf{Df}(\mathbf{a})]\vec{\mathbf{v}}}_{\mathbb{R}^m} \cdot \underbrace{\mathbf{g}(\mathbf{a})}_{\mathbb{R}^m} \tag{2.12}$$

Theorem 2.7. (Chain rule). Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open sets, let $\mathbf{g} : U \to V$ and $\mathbf{f} : V \to \mathbb{R}^p$ be mappings, and let $\mathbf{a}$ be a point of U. If $\mathbf{g}$ is differentiable at $\mathbf{a}$ and $\mathbf{f}$ is differentiable at $\mathbf{g}(\mathbf{a})$, then the composition $\mathbf{f}\circ\mathbf{g}$ is differentiable at $\mathbf{a}$, and its derivative is given by

$$\mathbf{D}[(\mathbf{f}\circ\mathbf{g})(\mathbf{a})] = [\mathbf{Df}(\mathbf{g}(\mathbf{a}))] \circ [\mathbf{Dg}(\mathbf{a})] \tag{2.13}$$

January 29, 2019

Theorem 2.8. (Mean value theorem for functions of several variables). Let $U \subset \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}$ be differentiable, and let the segment $[\mathbf{a}, \mathbf{b}]$ joining $\mathbf{a}$ to $\mathbf{b}$ be contained in U. Then there exists $\mathbf{c}_0 \in [\mathbf{a}, \mathbf{b}]$ such that

$$f(\mathbf{b} - (\mathbf{a}) = [\mathbf{D}f(\mathbf{c}_0)](\vec{\mathbf{b}-\mathbf{a}}) \tag{2.14}$$

2.4 Differentiability

Definition 2.9. (C^p function). A C^p *function* on $U \subset \mathbb{R}^n$ is a function that is p times continuously differentiable: all of its partial derivatives up to order p exist and are continuous on U.

Exercise 1.9.1. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \dfrac{x^4 + y^4}{x^2 + y^2} & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

is differentiable at every point of $\mathbb{R}^2$.

2.5 Exercises

Exercise 1.9.2. Show that for

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \dfrac{3x^2 y - y^3}{x^2 + y^2} & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

all directional derivatives exist, but that f is not differentiable at the origin.

January 31, 2019

Recall **Exercise 1.9.2 a.** Show that for

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \dfrac{3x^2 y - y^3}{x^2 + y^2} & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

all directional derivaties exist, but that f is not differentiable at the origin.

Proof.

$$D_x f\begin{pmatrix}0\\0\end{pmatrix} = \lim_{h\to 0}\frac{f\begin{pmatrix}0+h\\0\end{pmatrix}-0}{h}$$
$$= \lim_{h\to 0}\frac{3h^2\cdot 0-0^3}{h(h^2+0^2)} = 0$$

$$D_y f\begin{pmatrix}0\\0\end{pmatrix} = \lim_{h\to 0}\frac{f\begin{pmatrix}0\\0+h\end{pmatrix}-0}{h}$$
$$= \lim_{h\to 0}\frac{0-h^3}{h(0^2+h^2)} = -1$$

$$\mathbf{Jf}\begin{pmatrix}0\\0\end{pmatrix} = \begin{bmatrix}0 & -1\end{bmatrix}$$

$$\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix} \qquad \lim_{h\to 0}\frac{f(0+h\vec{v})-f(0)}{h} = \lim_{h\to 0}\frac{3h^2v_1^2hv_2-h^3v_2^2}{h(h^2v_1+h^2v_2^2)} = \frac{3v_1^2v_2-v_2^3}{v_1^2+v_2^2}$$

$$\Rightarrow \text{ all directional derivatives exist}$$

If f is differentiable, the directional derivative is $\left[\mathbf{Jf}\begin{pmatrix}0\\0\end{pmatrix}\right]\vec{v} = -v_2$, so setting $\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$ gives a contradiction, hence $f\begin{pmatrix}x\\y\end{pmatrix}$ is not differentiable at the origin. □

The digest of the above proof is that the Jacobian matrix was first calculated using the limit definition of the partial derivatives. Then the directional derivative was calculated using the definition of the directional derivative. In this process, a contradiction was achieved by showing that $[\mathbf{Jf}(\mathbf{a})]\vec{\mathbf{v}} = [\mathbf{Df}(\mathbf{a})]\vec{\mathbf{v}} \neq \lim_{h\to 0}\frac{f(0+h\vec{v})-f(0)}{h}$, violating **Proposition 10.1**.

Exercise 1.9.2b. Show that

$$g\begin{pmatrix}x\\y\end{pmatrix} = \begin{cases}\dfrac{x^2y}{x^4+y^2} & \text{if } \begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}\\ 0 & \text{if } \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\end{cases}$$

has directional derivatives at every point but is not continuous.

Proof. The directional derivative is given by:

$$\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix} \qquad \lim_{h\to 0}\frac{g(0+h\vec{v})-g(0)}{h} = \lim_{h\to 0}\frac{h^2v_1^2\cdot hv_2}{h(h^4v_1^4+h^2v_2^2)} = \lim_{h\to 0}\frac{v_1^2v_2}{h^2v_1^4+v_2^2} = \frac{v_1^2}{v_2}$$

Yet $g\begin{pmatrix}0\\0\end{pmatrix} \neq \lim_{\mathbf{x}\to(0,0)} g(\mathbf{x})$. In fact, the limit does not exist. Approaching the limit from $y = 0$ gives

$$\lim_{x\to 0} \frac{x^2 \cdot 0}{x^4 + 0^2} = \lim_{x\to 0} \frac{0}{x^4} = 0$$

But approaching the limit from $y = x^2$ gives

$$\lim_{y=x^2\to 0} \frac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2} \qquad \Rightarrow\!\Leftarrow!$$

□

Exercise 1.9.2c. Show that

$$h\begin{pmatrix}x\\y\end{pmatrix} = \begin{cases} \dfrac{x^2 y}{x^6 + y^2} & \text{if } \begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} \\ 0 & \text{if } \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \end{cases}$$

has directional derivatives at every point but is not bounded in a neighborhood of $\mathbf{0}$.

Proof. *** □

Exercise 1.8.11. Show that if $f\begin{pmatrix}x\\y\end{pmatrix} = \varphi\left(\dfrac{x+y}{x-y}\right)$ for some differentiable function $\varphi : \mathbb{R} \mapsto \mathbb{R}$, then

$$xD_x f + yD_y f = 0$$

Proof.

$$xD_x f = xD_x \varphi\left(\frac{x+y}{x-y}\right) = x \cdot \varphi'\left(\frac{x+y}{x-y}\right) \cdot \frac{(x-y)-(x+y)}{(x-y)^2} = \varphi\left(\frac{x+y}{x-y}\right) \frac{-2xy}{(x-y)^2}$$

$$yD_y f = yD_y \varphi\left(\frac{x+y}{x-y}\right) = x \cdot \varphi'\left(\frac{x+y}{x-y}\right) \cdot \frac{(x-y)-(-1)(x+y)}{(x-y)^2} = \varphi\left(\frac{x+y}{x-y}\right) \frac{2xy}{(x-y)^2}$$

□

Just for the future, we might want to switch to a new coordinate system. We'll have to use the chain rule in these cases. For example:

$$x = r\cos\theta \quad y = r\sin\theta$$

$$\mathbf{D}_\theta f\begin{pmatrix}X(r,\theta)\\Y(r,\theta)\end{pmatrix}$$

$$\mathbf{D}_r f \begin{pmatrix} X(r,\theta) \\ Y(r,\theta) \end{pmatrix}$$

$$\mathbf{D} f \begin{pmatrix} X(r,\theta) \\ Y(r,\theta) \end{pmatrix}$$

Consider f as the "outside" function, and $h : \begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$. Recall

$$\mathbf{D}[(\mathbf{f} \circ \mathbf{g})(\mathbf{a})] = [\mathbf{D}\mathbf{f}(\mathbf{g}(\mathbf{a}))] \circ [\mathbf{D}\mathbf{g}(\mathbf{a})]$$

We then get

$$\mathbf{D}h = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\mathbf{D}f = [\mathbf{D}_x f(r,\theta) \quad \mathbf{D}_y f(r,\theta)]$$

$$[\mathbf{D}f][\mathbf{D}h] = [\mathbf{D}_x f(r,\theta) \quad \mathbf{D}_y f(r,\theta)] \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Exercise 1.8.10 (p.144).

2.6 Newton's Method

February 5, 2019

Definition 2.10. (Newton's method). Let $\vec{f}$ be a differentiable map from U to $\mathbb{R}^n$, where U is an open subset of $\mathbb{R}^n$. Newton's method consists of starting with some guesses $\mathbf{a}_0$ for a sollution of $\vec{f}(\mathbf{x}) = \vec{0}$. Then linearize the equation at $\mathbf{a}_0$: replace the increment to the function, $\vec{f}(\mathbf{x}) - \vec{f}(\mathbf{a}_0)$, by a linear function of the increment, $[\mathbf{D}\vec{f}(\mathbf{a}_0)](\mathbf{x} - \mathbf{a}_0)$. Now solve the corresponding linear equation:

$$\vec{f}(\mathbf{a}_0) + [\mathbf{D}\vec{f}(\mathbf{a}_0)](\mathbf{x} - \mathbf{a}_0) = \vec{0} \tag{2.15}$$

***

$$\begin{aligned} a_0 &= \text{initial guess} \\ a_{n+1} &= a_n - [\mathbf{D}\vec{f}(a_n)]^{-1}\mathbf{f}(a_n) \\ x_{n+1} &= x_n - \frac{\mathbf{f}(x)}{\mathbf{f}'(x_n)} \end{aligned}$$

Example:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos x + y - 1.1 \\ x + \cos(x+y) - 0.9 \end{pmatrix}$$

2.7 Inverse and Implicit Function Theorems

February 9, 2019

Given $f : \mathbb{R}^n \to \mathbb{R}^m$, is there a neighbourhood $U \subseteq \mathbb{R}^m$ with a function $g : U \to \mathbb{R}^n$ such that $f \circ g = g \circ f - \mathrm{id}$?

Theorem 2.11. (Inverse Function Theorem). If a mapping $\mathbf{f}$ is continuously differentiable, and its derivative is invertible at some point $\mathbf{x}_0$, then $\mathbf{f}$ is locally invertible, with differentiable inverse, in some neighborhood of the point $\mathbf{f}(\mathbf{x}_0)$.

Given an equation $F(x_1, \ldots, x_n) = 0$, where $F : \mathbb{R}^n \to \mathbb{R}^m$. Is there a neighbourhood $U \subseteq \mathbb{R}^n$ so that some of the x_i are functions of the others?

Theorem 2.12. (Implicit Function Theorem). Let $U \subset \mathbb{R}^n$ be open and $\mathbf{c}$ a point in U. Let $\mathbf{F} : U \to \mathbb{R}^{n-k}$ be a C^1 mapping such that $\mathbf{F}(\mathbf{c}) = 0$ and $[\mathbf{DF}(\mathbf{c})]$ is onto. Then the system of linear equations $[\mathbf{DF}(\mathbf{c})](\vec{\mathbf{x}}) = \vec{0}$ has $n - k$ pivotal (passive) variables and k nonpivotal (active) variables, and there exists a neighborhood of bmc in which $\mathbf{F} = 0$ implicitly defines the $n - k$ passive variable as a function $\mathbf{g}$ of the k active variables.

Example:

$$\boxed{\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 1}$$

$$\mathbf{DF}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

$$C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \mathbf{DF}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix}$$

$\Rightarrow x$ is pivotal
$\Rightarrow x$ is a function of y
$\Rightarrow y$ cannot be pivotal
$\Rightarrow y$ is not a function of x

$$\mathbf{DF}\begin{pmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} = \begin{bmatrix} 2\sqrt{\frac{1}{2}} & 2\sqrt{\frac{1}{2}} \end{bmatrix}$$

$\Rightarrow$ both x and y can be pivotal
$\Rightarrow x = x(y)$

or $y = y(x)$ in some nbhd of $\begin{pmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix}$

Exercise 2.10.1. Does the inverse function theorem guarantee that the following functions are locally invertible with differentiable inverse?

a. $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2y \\ -2x \\ y^2 \end{pmatrix}$ at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\mathbf{DF}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2xy & x^2 \\ -2 & 0 \\ 0 & 2y \end{bmatrix}$

$\mathbf{DF}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix}$

which isn't invertible.

b. $F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x^2y\\-2x\end{pmatrix}$ at $\begin{pmatrix}1\\1\end{pmatrix}$
$\mathbf{D}F\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}2xy & x^2\\-2 & 0\end{bmatrix}$
$\mathbf{D}F\begin{pmatrix}1\\1\end{pmatrix} = \begin{bmatrix}2 & 1\\-2 & 0\end{bmatrix}$
to which an inverse exists thus this function is invertible.

Exercise 2.10.5. Using direct computation, determine where $y^2 + y + 3x + 1 = 0$ defines y implicitly as a function of x.

Without directly computing, we attempt to use the implicit function theorem:

$$F(x, y) = y^2 + y + 3x + 1 = 0$$

$$\mathbf{D}F(x, y) = [3 \quad 2y + 1]$$

$$y \neq -\frac{1}{2}$$

Alternately, we can compute it directly:

$$y^2 + y + \frac{1}{4} + \frac{3}{4} + 3x = 0$$

$$(y + \frac{1}{2})^2 = -3x - \frac{3}{4}$$

$$y = -\frac{1}{2} \pm \sqrt{-3x - \frac{3}{4}}$$

February 11, 2019

Exercise 14 (another book). Using the notation of the preceding exercise, let A, B be sets in $\mathbb{R}$. Show that:

1. $A^\circ \subseteq A$.

   Proof. $\forall a \in A^\circ$, a is an interior point of A, that is $\exists \epsilon > 0$ s.t. $B_\epsilon(a) \subseteq A$. $\forall \epsilon, a \in B_\epsilon(a) \subseteq A \Rightarrow a \in A \Rightarrow A^\circ \subseteq A$ □

2. $(A^\circ)^\circ = A^\circ$

   Proof. $(A^\circ)^\circ = \{x \in A^\circ | \exists \epsilon > 0 \text{ with } B_\epsilon(x) \subseteq A^\circ\}$.

   To show that i. $(A^\circ)^\circ \subseteq A^\circ$, ii. $A^\circ \subseteq (A^\circ)^\circ$

   i. Follows as above ($(A^\circ)^\circ \subseteq A^\circ$)

ii. $\forall a \in A^\circ, \exists \epsilon > 0$ s.t. $B_\epsilon(a) \subseteq A$. Assuse $\forall \epsilon' > 0$, $B_{\epsilon'}(a) \cap (A^\circ)^C \neq \emptyset$, so $\exists a' \in B_{\epsilon'}(a) \cap (A^\circ)^C \Rightarrow \exists a' \in B_{\epsilon'}(a) \wedge \exists a' \in (A^\circ)^C$, let $\epsilon' = \epsilon$, so $a \in B_\epsilon(a) \subseteq A^\circ$ but also $a \in (A^\circ)^C \Rightarrow\!\Leftarrow$

Hence $(A^\circ)^\circ = A^\circ$. □

3. $(A \cap B)^\circ = A^\circ \cap B^\circ$

Proof. i. $\forall x \in (A \cap B)^\circ, \exists \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq A \cap B \Rightarrow B_\epsilon(x) \subseteq A, B \Rightarrow x \in A^\circ, B^\circ \Rightarrow x \in A^\circ \cap B^\circ$

ii. $\forall x \in A^\circ \cap B^\circ \Rightarrow x \in A^\circ, B^\circ \Rightarrow \exists \epsilon > 0, \epsilon' > 0$ s.t. $B_\epsilon(x) \subseteq A \wedge$ WLOG let $\epsilon' > \epsilon \Rightarrow B_\epsilon(x) \subseteq B_{\epsilon'}(x) \subseteq B \Rightarrow B_\epsilon(x) \subseteq A \cap B \Rightarrow x \in (A \cap B)^\circ$

□

4. $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

Proof. WLOG let $x \in A^\circ \Rightarrow \exists \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq A \subseteq A \cup B \Rightarrow x \in (A \cup B)^\circ$ □

February 12, 2019

Exercise 2.10.9. Does the system of equations $x+y+\sin(xy) = a$ and $\sin(x^2+y) = 2a$ have a solution for sufficiently small a?

Answer: $\boxed{\text{Yes}}$

Proof. We rewrite the equation to be $F(x, y, a) = \begin{cases} x + y + \sin(xy) - a & = 0 \\ \sin(x^2 + y) - 2a & = 0 \end{cases}$

$$\mathbf{D}F(x, y, a) = \begin{bmatrix} 1 + y\cos(xy) & 1 + x\cos(xy) & -1 \\ 2x\cos(x^2 + y) & \cos(x^2 + y) & -2 \end{bmatrix}$$

$$\mathbf{D}F\vec{c} = \mathbf{D}F\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

x and y are pivotal so x and y both exist. □

3 k-Manifolds in $\mathbb{R}^n$

3.1 Introduction

Idea: In BC Calculus, the main object of study was "functions". This is too restrictive as many objects that are smooth (have a best linear approximation at each point) are not graphs of functions globally; e.g. circle, spiral, etc.

Since the derivative only tells us about the local properties of a set of points, it suffices to ask that the set is the graph of a diffible function in in some nbhd of every point. A point set in $\mathbb{R}^n$ that is locally the graph of some C^1-function $\mathbb{R}^k \to \mathbb{R}^{n-k}$ is called a k-manifold in $\mathbb{R}^n$.

Definition 3.1. (Smooth manifold in $\mathbb{R}^n$). A subset $M \subset \mathbb{R}^n$ is a smooth k-dimensional manifold if locally it is the graph of a C^1 mapping $\mathbf{f}$ expressing $n-k$ variables as functions of other k variables.

There are two important ways to define a manifold:

(i) By equation (e.g. $x^2 + y^2 - 1 = 0$)

(ii) By parametrization: $\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad t \in (0, 2\pi)$

Theorem 3.2. (Showing that a locus is a smooth manifold).

1. Let $U \in \mathbb{R}^n$ be open, and let $\mathbf{F} : U \to \mathbb{R}^{n-k}$ be a C^1 mapping. Let M be a subset of $\mathbb{R}^n$ such that

$$M \cap U = \{\mathbf{z} \in U \mid \mathbf{F}(\mathbf{z}) = \mathbf{0}\} \tag{3.1}$$

   If $[\mathbf{DF}(\mathbf{z})]$ is onto for every $\mathbf{z} \in M \cap U$, then $M \cap U$ is a smooth k-dimensional manifold embedded in $\mathbb{R}^n$. If every $\mathbf{z} \in M$ is in such a U, then M is a k-dimensional manifold.

2. Conversely, if M is a smooth k-dimensional manifold embedded in $\mathbb{R}^n$, then every point $\mathbf{z} \in M$ has a neighborhood $U \in \mathbb{R}^n$ such that there exists a C^1 mapping $\mathbf{F} : U \to \mathbb{R}^{n-k}$ with $[\mathbf{DF}(\mathbf{z})]$ onto and $M \cap U = \{\mathbf{y} \mid \mathbf{F}(\mathbf{y}) = \mathbf{0}\}$.

3.2 Parametrization

February 18, 2019

Consider the unit circle in $\mathbb{R}^2$ (a manifold).

$$x^2 + y^2 = 1$$

A parametrization of this could be $\gamma(t) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, t \in (0, 2\pi)$ [2]

Definition 3.3. (Parametrization of a manifold). A parametrization of a k-manifold $M \subset \mathbb{R}^n$ is a mapping $\gamma : U \subset \mathbb{R}^k \to M$ satisfying the following conditions:

1. U is open.
2. γ is C^1, one to one, and onto M.
3. $[\mathbf{D}\gamma(\mathbf{u})]$ is one to one for every $\mathbf{u} \in U$.

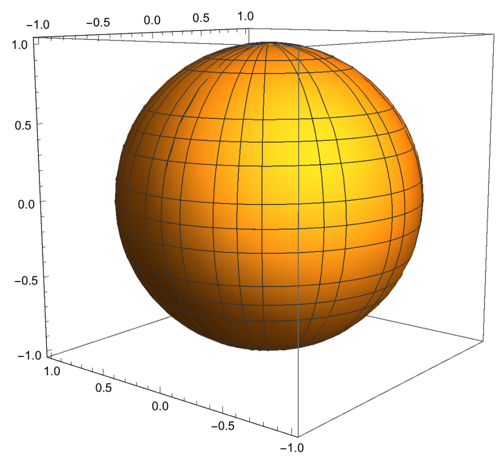
Example: This is the famous parametrization of the unit sphere in $\mathbb{R}^3$ by latitude φ and longitude θ.

$$\gamma : \begin{pmatrix}\theta\\ \varphi\end{pmatrix} \mapsto \begin{pmatrix}\cos\theta\cos\varphi\\ \sin\theta\cos\varphi\\ \sin\varphi\end{pmatrix} \tag{3.2}$$

We can visualize this in Mathematica using

```
In[•]:= ParametricPlot3D[{Cos[θ] Cos[ϕ], Sin[θ] Cos[ϕ], Sin[ϕ]}, {θ, -4, 4}, {ϕ, -4, 4}]
```

Out[•]=



Exercise 3.1.11.

a. Find a parametrization for the union X of the lines through the origin and a point of the parametrized curve $t \mapsto \begin{pmatrix}t\\ t^2\\ t^3\end{pmatrix}$.

Answer: $\boxed{t, u \mapsto \begin{pmatrix}ut\\ ut^2\\ ut^3\end{pmatrix}}$

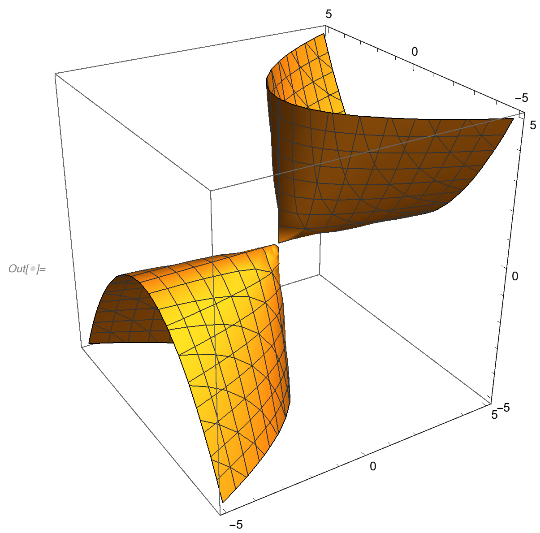
[2]The open interval is of no significance, it goes away when integrating or working with it.

b. Find an equation for the closure $\overline{X}$ of X. Is $\overline{X}$ exactly X?

Answer: $\boxed{\overline{x} : \frac{y}{x} = \frac{z}{y} \text{ or } y^2 = xz}$ This is the limit as t approaches ∞ and there is one line that is never traced out.

In[◦]:=
```
ContourPlot3D[{y * y == x * z}, {x, -5, 5}, {y, -5, 5}, {z, -5, 5}]
```

Out[◦]=



February 20, 2019

c. Show that $\overline{X} - \{0\}$ is a smooth surface.

$$\overline{X} : xz = y^2$$

$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xz - y^2 = 0$$

$$\mathbf{DF} = \begin{bmatrix} z & -2y & x \end{bmatrix}$$

is onto except for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

d. Show that the map $\begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} r(1+\sin\theta) \\ r\cos\theta \\ r(1-\sin\theta) \end{pmatrix}$ is another parametrization of $\overline{X}$. In this form you should have no trouble giving a name to the surface $\overline{X}$.

Proof. Recall $\overline{X} = xz = y^2$, indeed:

$$r(1+\sin\theta)\cdot r(1-\sin\theta) = r^2(1-\sin^2\theta) = r^2\cos^2\theta = (r\cos\theta)^2$$

Now is this onto? Assume otherwise, that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a point with $xz = y^2$, are there $\begin{pmatrix} r \\ \theta \end{pmatrix}$ such that:

$$\begin{aligned} x &= r(1 + \sin\theta) \\ y &= r\cos\theta \\ z &= r(1 - \sin\theta) \end{aligned}$$

*** $\square$

e. Relate $\overline{X}$ to the set of noninvertible symmetric 2×2 matrices.

$$xz - y^2 = \det \begin{pmatrix} x & y \\ y & z \end{pmatrix} = 0$$

Exercise 3.1.21.

a. Is there a set of $\begin{pmatrix} \theta \\ \varphi \end{pmatrix} \in \mathbb{R}^2$ such that $\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \cos\varphi + 1 \\ 0 \\ \sin\varphi \end{pmatrix}$ are distance 2 apart a smooth curve?

b. At what points is this set locally a graph of φ as a function of θ? At what points is it locally a graph of θ as a function of φ.

3.3 Tangent Spaces

February 21, 2019

As we saw, a k-manifold in $\mathbb{R}^n$ can be thought of as the solution to an equation $F(x) = 0$ or the image of a parametrization $\gamma : \mathbb{R}^k \to \mathbb{R}^n$.

Since both F and γ are assumed to be C^1, their derivatives give a local linear approximation of the manifold which we call the "tangent space".

The linear equivalent of "$F(x) = 0$" is $ker[\mathbf{D}F(x)]$, and the linear equivalent of "$im\,\gamma(u)$" is "$im[\mathbf{D}\gamma(u)]$".

The tangent space at a point $\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{pmatrix} \in M$ is given by equation

$$\underbrace{\mathbf{y} - \mathbf{y}_0}_{\text{change in output}} = [\mathbf{Df}(\mathbf{x}_0)] \underbrace{\mathbf{x} - \mathbf{x}_0}_{\text{change in input}} \tag{3.3}$$

We use $\dot{\mathbf{x}}$ and $\dot{\mathbf{y}}$ to denote increments to $\mathbf{x}$ and $\mathbf{y}$ respectively, so $\dot{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$ and $\dot{\mathbf{y}} = \mathbf{y} - \mathbf{y}_0$.

Formally...

Definition 3.4. (Tangent space to a manifold). Let $M \in \mathbb{R}^n$ be a k-dimensional manifold. The tangent space to M at $\mathbf{z}_0 \overset{\text{def}}{=} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{y}_0 \end{pmatrix}$, denoted $T_{\mathbf{z}_0}M$, is the graph of the linear transformation $[\mathbf{Df}(\mathbf{x}_0)]$.

Naturally, this prompts the question...how do we actually compute the tangent spaces to manifolds. If a manifold is defined by an equation, we can see it as the null space of the derivative of a function $\mathbf{F}(\mathbf{z}) = \mathbf{0}$.

Theorem 3.5. (Tangent space to a manifold given by equation). If $\mathbf{F}(\mathbf{z}) = \mathbf{0}$ describes a manifold M, and $[\mathbf{DF}(\mathbf{z}_0)]$ is onto for some $\mathbf{z}_0 \in M$, then the tangent space $T_{\mathbf{z}_0}M$ is the kernel of $[\mathbf{DF}(\mathbf{z}_0)]$:

$$T_{\mathbf{z}_0}M = \ker[\mathbf{DF}(\mathbf{z}_0)] \tag{3.4}$$

Remark. We might be used to seeing functions in the form $\mathbf{f}(\mathbf{x}) = \mathbf{y}$, which we have to ensure we rewrite as $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{x}) - \mathbf{y} = 0$ in order to find the tangent space this way.

Theorem 3.6. (Tangent space of manifold given by parametrization). Let $U \in \mathbb{R}^k$ be open, and let $\gamma : U \to \mathbb{R}^n$ be a parametrization of a manifold M. Then

$$T_{\gamma(\mathbf{u})}M = \operatorname{im}[\mathbf{D}\gamma(\mathbf{u})] \tag{3.5}$$

Example: Tangent spaces to the unit circle

1.

$$F\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 1$$

$$\mathbf{D}F\begin{pmatrix} x \\ y \end{pmatrix} = [2x \quad 2y]$$

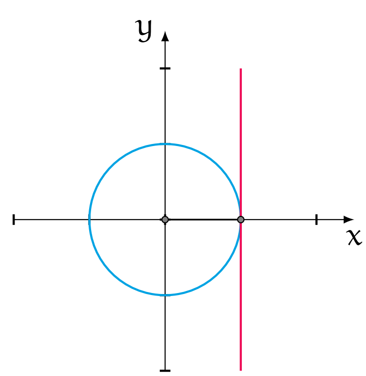
$$\ker[2x \quad 2y] = \left\{ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; [2x \quad 2y]\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 0 \right\}$$

If $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$[2 \quad 0]\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 0$$

$$\Rightarrow 2\dot{x} = 0$$

$$\Rightarrow \dot{x} = 0$$



The <u>tangent line</u> will be the kernel of $\mathbf{D}F$ w.r.t the x-y-coordinate system:

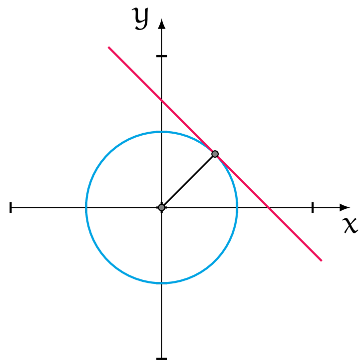
$$\boxed{x - 1 = 0}$$

If $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix}$, then $\ker \mathbf{D}F = \ker \begin{bmatrix} \sqrt{2} & \sqrt{2} \end{bmatrix}$, so we solve

$$\sqrt{2}\dot{x} + \sqrt{2}\dot{y} = 0$$

$$\dot{y} = -\dot{x} \qquad \text{(tangent space)}$$

$$\left(y - \sqrt{\frac{1}{2}}\right) = -\left(x - \sqrt{\frac{1}{2}}\right) \qquad \text{(tangent line)}$$



In general, for $\begin{pmatrix} x \\ y \end{pmatrix}$ on the circle, the tangent space to the circle at point $\begin{pmatrix} x \\ y \end{pmatrix}$ is given by

$$\ker \begin{bmatrix} 2x & 2y \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \,\middle|\, x\dot{x} + y\dot{y} = 0 \right\}$$

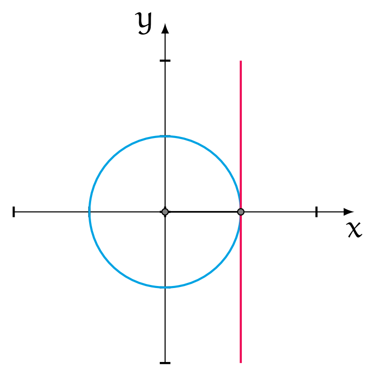
2. A parametrization of the unit circle is given by

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, t \in (0, 2\pi)$$

$$\mathbf{D}\gamma(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

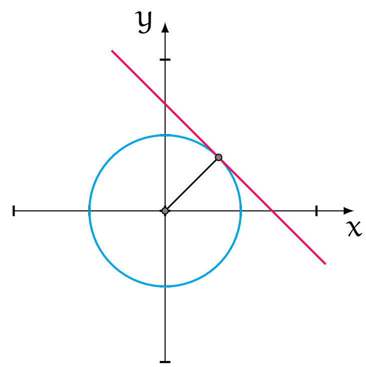
Each $t \in (0, 2\pi)$ gives a point on the circle with the tangent space spanned by $\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$.

$t = 0 : \mathbf{D}\gamma(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$$t = \frac{\pi}{4} : [\mathbf{D}\gamma(\frac{\pi}{4})] = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\Rightarrow \text{span}\left(\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$



Definition 3.7. (Tangent space to a manifold). ***

Theorem 3.8. (Tangent space to a manifold given by equations). If $\mathbf{F}(\mathbf{z}) = \mathbf{0}$ describes ***

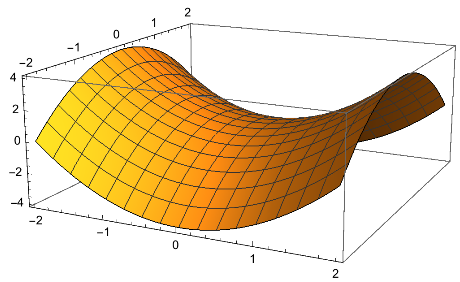
Proposition 3.9. (Tangent space of a manifold given by parametrization). ***

Exercise 3.2.4. For each of the following functions f and points $\begin{pmatrix} a \\ b \end{pmatrix}$, state whether there is a tangent plane to the graph of f at the point $\begin{pmatrix} a \\ b \\ f\begin{pmatrix} a \\ b \end{pmatrix} \end{pmatrix}$. If there is such a tangent plane, find its equation, and compute the intersection of the tangent plane with the graph.

a. $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 - y^2$ at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

```
Plot3D[x^2 - y^2, {x, -2, 2}, {y, -2, 2}]
```

Out[∘]=



The graph of $z = f\begin{pmatrix} x \\ y \end{pmatrix}$ is parametrized by

$$\gamma\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix} = \begin{pmatrix} u \\ v \\ u^2 - v^2 \end{pmatrix}$$

$$\mathbf{D}\gamma\begin{pmatrix} u \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{bmatrix}$$

$$\mathbf{D}\gamma\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\mathrm{im}\left[\mathbf{D}\gamma\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = \left\{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \mid \dot{x} = s, \dot{y} = t, \dot{z} = 2s - 2t\right\}$$

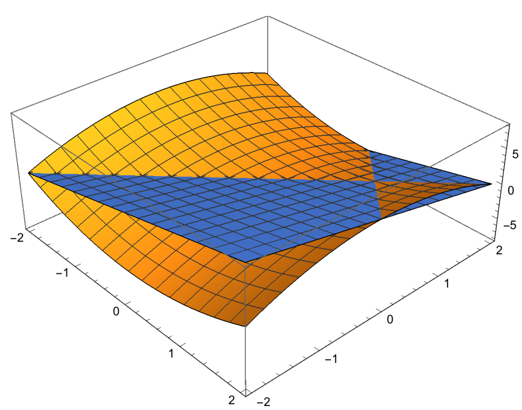
$$\text{tangent plane} = \left\{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \mid \dot{x} = 1 + s, \dot{y} = 1 + t, \dot{z} = 2s - 2t\right\}$$

We can check this using ***

This is the tangent plane graphed:

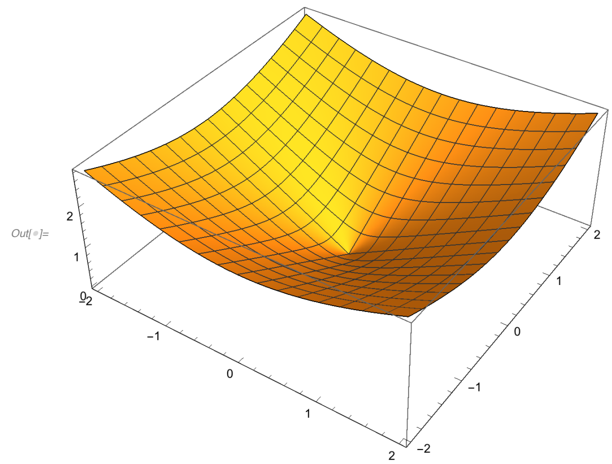
```
Plot3D[{x^2 - y^2, 2 x - 2 y}, {x, -2, 2}, {y, -2, 2}]
```

Out[∘]=



b. $f\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{x^2 + y^2}$ at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

```
Plot3D[Sqrt[x^2 + y^2], {x, -2, 2}, {y, -2, 2}]
```



$$F\begin{pmatrix}x\\y\\z\end{pmatrix} = z^2 - x^2 - y^2 = 0$$

$$\left[\mathbf{D}F\begin{pmatrix}x\\y\\z\end{pmatrix}\right] = \begin{bmatrix}-2x & -2y & 2z\end{bmatrix}$$

$$\left[\mathbf{D}F\begin{pmatrix}0\\0\\0\end{pmatrix}\right] = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$$

Evidently, this matrix is not onto so there does not exist a tangent plane at point $\begin{pmatrix}0\\0\\0\end{pmatrix}$.

c. Same function as above but at point $\begin{pmatrix}1\\-1\end{pmatrix}$

$$\left[\mathbf{D}F\begin{pmatrix}x\\y\\z\end{pmatrix}\right] = \begin{bmatrix}-2x & -2y & 2z\end{bmatrix}$$

$$\left[\mathbf{D}F\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}\right] = \begin{bmatrix}-2 & 2 & 2\sqrt{2}\end{bmatrix}$$

$$\ker\left[\mathbf{D}F\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}\right] = \left\{\begin{pmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{pmatrix} \,\middle|\, -2\dot{x} + 2\dot{y} + 2\sqrt{2}\dot{z} = 0\right\}$$

$$-\dot{x} + \dot{y} + \sqrt{2}\dot{z} = 0$$

$$-(x-1) + (y+1) + \sqrt{2}(z - \sqrt{2}) = 0$$

$$\text{tangent plane}: -x + y + \sqrt{2}z = 0$$

Some review of old material

February 25, 2019

Exercise 1.8.9. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any differentiable function. Show that the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = y\varphi(x^2 - y^2)$$

satisfies the equation

$$\frac{1}{x}D_1 f\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{y}D_2 f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{y^2} f\begin{pmatrix} x \\ y \end{pmatrix}$$

Proof.

$$\begin{aligned} D_1 &= 2xy\varphi'(x^2 - y^2) \\ D_2 &= \varphi(x^2 - y^2) + 2y^2\varphi'(x^2 - y^2) \end{aligned}$$

*** □

Exercise 1.34. Consider the function defined in $\mathbb{R}^2$ and given by the formula.

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

a. Show that both partial derivatives exist everywhere.

Proof. Because f is a symmetric function, and x and y are interchangeable, it suffices to merely calculate one partial derivative. This is the partial derivative for $x, y \neq 0$

$$D_x f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{(x^2 + y^2)(1) - (xy)(2x)}{(x^2 + y^2)^2} = \frac{x^2 + y^2 - 2x^2y}{x^4 + 2x^2y^2 + y^4}$$

The partial derivative at $(0, 0)$ is

$$\lim_{h \to 0} \frac{f\begin{pmatrix} 0 + h \\ 0 \end{pmatrix} - f\begin{pmatrix} 0 \\ 0 \end{pmatrix}}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

Thus the partial derivatives exist everywhere. □

b. Where is f differentiable? f is not continuous.

$$\lim_{y=0,x\to 0} f\begin{pmatrix}x\\y\end{pmatrix} = 0 \qquad \lim_{x=y\to 0} f\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{2} \qquad \Rightarrow\!\Leftarrow!$$

Exercise 2.10.15.

a. Show that the mapping $F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}e^x + e^y\\e^x + e^{-y}\end{pmatrix}$ is locally invertible at every point $\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$.

Proof. By the Inverse Function Theorem *** (14.1), F is locally invertible at $\mathbf{x} = \begin{pmatrix}x\\y\end{pmatrix}$ if it derivative is invertible at $\mathbf{x}$. $[JF(\mathbf{x})]$ *** □

b. If $F(\mathbf{a}) = \mathbf{b}$, what is the derivative of F^{-1} at $\mathbf{b}$

Exercise 2.31.

a. True or false? The equation $\sin(xyz) = z$ expresses x implicitly as a differentiable function of y an z near the point $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}\pi/2\\1\\1\end{pmatrix}$.

Proof.

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = \sin(xyz) - z$$

$$Df\begin{pmatrix}x\\y\\z\end{pmatrix} = [yz\cos(xyz) \quad xz\cos(xyz) \quad xy\cos(xyz) - 1] = [0 \quad 0 \quad -1]$$

Because the x column is non-pivotal, this is false. □

b. True or false? z can be expressed implicitly as a differentiable function of x and y near the same point of the same function.

Proof. The third column, the z column, is pivotal, so z can be expressed as a function of x and y. □

February 26, 2019

Exercise 3.2.6.

1. Show that the subset $X \subset \mathbb{R}^4$ where

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0 \text{ and } x_1 + 2x_2 + 3x_3 + 4x_4 = 4$$

is a manifold in $\mathbb{R}^4$ in a neighborhood of the point $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

Proof.

$$F_1 = x_1^2 + x_2^2 - x_3^2 - x_4^2$$
$$F_2 = x_1 + 2x_2 + 3x_3 + 4x_4 - 4$$

$$F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 - x_3^2 - x_4^2 \\ x_1 + 2x_2 + 3x_3 + 4x_4 - 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{D}F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 2x_1 & 2x_2 & -2x_3 & -2x_4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\mathbf{D}F\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 2 & 0 & -2 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Which are linearly independent, which shows that this is onto, and so it is a smooth manifold locally at $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. □

2. What is the tangent space to X at $\mathbf{p}$?

$$T_pM = \ker \begin{bmatrix} 2 & 0 & -2 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \overset{rref}{=} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\dot{x}_1 - \dot{x}_3 = 0 \Rightarrow \dot{x}_1 = \dot{x}_3$$

$$\dot{x}_2 + 2\dot{x}_3 + 2\dot{x}_4 = 0 \Rightarrow \dot{x}_2 = -2\dot{x}_3 - 2\dot{x}_4$$

$$\ker[\cdots] = \left\{ \left[\begin{matrix} \dot{x}_3 \\ -2\dot{x}_3 - 2\dot{x}_4 \\ \dot{x}_3 \\ \dot{x}_4 \end{matrix} \right] \middle| \dot{x}_i \in \mathbb{R} \right\} = span\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

3. What pair of variables do the equations above not express as functions of the other two?

4. Is the entire set of X a manifold?

Proof. We attempt to find a counterexample, so we want all the columns of

$$\mathbf{D}F\begin{pmatrix}\vdots\end{pmatrix} = \begin{bmatrix} 2x_1 & 2x_2 & -2x_3 & -2x_4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

to be linearly dependent, which means that

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{-x_3}{3} = \frac{-x_4}{4}$$

whilst still satisfying the original equation

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0 \text{ and } x_1 + 2x_2 + 3x_3 + 4x_4 = 4$$

There isn't a point where we can solve for x_i, so $\mathbf{D}F$ is always onto, and X is a manifold. □

3.4 Optimization

March 2, 2019

Recall the Taylor polynomial at $x = 0$:

$$f(x) \approx \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

And in general the Taylor polynomial at $x = a$:

$$f(x) \approx \frac{f(a)}{a!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

If $f'(a) = 0$, the behavior of f near a is determined by the quadratic term (if $f''(a) \neq 0$)

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, if f is at least C^2, we can write at $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$:

$$\begin{aligned} f\begin{pmatrix}x\\y\end{pmatrix} \approx\ & \underbrace{f\begin{pmatrix}0\\0\end{pmatrix}}_{\text{absolute term}} + \underbrace{D_x f\begin{pmatrix}0\\0\end{pmatrix} x + D_y\begin{pmatrix}0\\0\end{pmatrix} y}_{\text{linear term}} \\ & + \frac{1}{2!}\underbrace{\left(D_{xx} f\begin{pmatrix}0\\0\end{pmatrix} x^2 + D_{xy} f\begin{pmatrix}0\\0\end{pmatrix} xy + D_{yx} f\begin{pmatrix}0\\0\end{pmatrix} xy + D_{yx} f\begin{pmatrix}0\\0\end{pmatrix} yx + D_{yy} f\begin{pmatrix}0\\0\end{pmatrix} y^2\right)}_{\text{quadratic term}} \\ & = f\begin{pmatrix}0\\0\end{pmatrix} + \left[Df\begin{pmatrix}0\\0\end{pmatrix}\right]\begin{bmatrix}x\\y\end{bmatrix} + \frac{1}{2!}\begin{bmatrix}x & y\end{bmatrix}\underbrace{\begin{bmatrix} D_{xx} f\begin{pmatrix}0\\0\end{pmatrix} & D_{xy} f\begin{pmatrix}0\\0\end{pmatrix} \\ D_{yx} f\begin{pmatrix}0\\0\end{pmatrix} & D_{yy} f\begin{pmatrix}0\\0\end{pmatrix}\end{bmatrix}}_{\left[Hf\begin{pmatrix}0\\0\end{pmatrix}\right] \text{ (Hessian matrix)}}\begin{bmatrix}x\\y\end{bmatrix} \end{aligned} \tag{3.6}$$

In turns out that in higher order partials, the order of differentiation does not matter:

$$D_{xy} f = D_{yx} f \quad \text{wherever they exist}$$

$$\begin{aligned}
f \begin{pmatrix} x \\ y \end{pmatrix} &= x^2 \cdot \sin(x_y^2) \\
D_x f \begin{pmatrix} x \\ y \end{pmatrix} &= 2x \cdot \cos(x + y^2) \cdot 2y - x^2 \sin(x + y^2) \cdot 2y \\
&= 4xy \cos(x + y^2) - 2x^2 y \sin(x + y^2) \\
D_y f \begin{pmatrix} x \\ y \end{pmatrix} &= 2x^2 y \cdot \cos(x + y^2) \\
D_{yx} f \begin{pmatrix} x \\ y \end{pmatrix} &= 2y(2x \cdot \cos(x + y^2) - x^2 \sin(x + y^2))
\end{aligned}$$

If $Hf \begin{pmatrix} a \\ b \end{pmatrix}$ is positive definite: $f \begin{pmatrix} x \\ y \end{pmatrix} \approx \lambda_1 x^2 + \lambda_2 y^2$ for $\lambda_{1,2} > 0$, then $\begin{pmatrix} a \\ b \end{pmatrix}$ is a local minimum.

If $Hf \begin{pmatrix} a \\ b \end{pmatrix}$ is negative definite, $\begin{pmatrix} a \\ b \end{pmatrix}$ is a local max.

If $Hf \begin{pmatrix} a \\ b \end{pmatrix}$ is indefinite, we have a "saddle point" at $\begin{pmatrix} a \\ b \end{pmatrix}$.

For semi-definite matrixes $Hf \begin{pmatrix} a \\ b \end{pmatrix}$, we get local mins/max, but they may not be unique.

Exercise 3.6.1.

a. Show that $f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + xy + z^2 - \cos y$ has a critical point at the origin.

   We calculate the Jacobian $[Jf(\vec{\mathbf{v}})] = [2x + y \quad x + \sin(y) \quad 2z]$. The second partials are: ***